Quantized Symmetric Powers

R. M. Green & R. J. Marsh

1 Introduction

Let $\mathbf{g} = sl_n(\mathbb{C})$ be the Lie algebra of $n \times n$ matrices over $\mathbb{C}$ of trace zero, and let U be the q-analogue of its universal enveloping algebra defined by Drinfel'd [2] and Jimbo [4]. According to [7, 3.5.6, 6.2.3 & 6.3.4], for each dominant weight μ in the weight lattice of $\mathbf{g}$ there is an irreducible, finite-dimensional highest weight U-module $V(\mu)$ with highest weight μ. Kashiwara [5] and Lusztig [7, 14.4.12] have independently shown the existence of a certain canonical basis $\mathbf{B}(\mu)$ for $V(\mu)$. In [5], Kashiwara proves the existence of a certain *crystal basis* associated with $V(\mu)$ using certain operators, and defines a graph (the *crystal graph*) which encodes how these operators act on the crystal basis. Using this, he defines $\mathbf{B}(\mu)$. In §2 we define a certain subspace of the r-th tensor power of the basic module for U, for each integer $r \geq 1$, and show that it is a module for U, with highest weight $(r, 0, 0, \ldots, 0)$. (We use the same numbering as [1, Planche I]). In §3 we study the irreducible finite-dimensional highest weight module for U with highest weight $(r, 0, 0, \ldots, 0)$, and find its canonical basis in terms of monomials in U acting on its highest weight vector. We then bring all of this together by showing that the module in §2 is isomorphic to the module studied in §3, and as a result we have two nice descriptions of the canonical basis for this module. Using these we determine how the generators of U act on this basis.

We use the treatment in [7, §§1-3]. Let $I = \{1, 2, \ldots, n-1\}$. Let $\mathbf{g}$ be a semisimple Lie algebra of type A_{n-1}, with root system Φ, simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, and Killing form $(\ ,\ )$. Let $h_1, h_2, \ldots, h_{n-1}$ be a basis for a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, satisfying $(h_i, h) = \alpha_i^*(h)$ for all h in $\mathbf{h}$ and all $i \in I$. Let Y be the $\mathbb{Z}$-lattice spanned by $h_1, h_2, \ldots, h_{n-1}$. Let $\omega_1, \omega_2, \ldots, \omega_{n-1}$ be the fundamental weights of $\mathbf{g}$, defined by $\omega_i(h_j) = \delta_{ij}$, and let X be the $\mathbb{Z}$-lattice spanned by them (the weight lattice). Define $i \cdot j$ to be -1 if $i \neq j$ in I and 2 if $i = j \in I$. (So $i \cdot j$ is a scalar multiple of (α_i, α_j).) Then $(I, \cdot)$ is a Cartan datum as in [7, 1.1.1]. Note that we have $(i \cdot i)/2 = 1$ for all $i \in I$. For $h \in Y$ and $\mu \in X$, define $\langle h, \mu \rangle$ to be $\mu(h)$. Define an imbedding of I into Y by $i \mapsto h_i$ and into X by $i \mapsto \alpha_i$ for all $i \in I$. We then have a root datum of type $(I, \cdot)$ as in [7, 2.2.1], with $\langle h_i, \alpha_j \rangle = \alpha_j(h_i) = A_{ij}$ the corresponding Cartan matrix.

Let $\mathbb{Q}(v)$ be the field of rational functions in an indeterminate v, and $\mathcal{A} \subseteq \mathbb{Q}(v)$ the ring $\mathbb{Z}[v, v^{-1}]$. For $N, M \in \mathbb{N}$ we define the following (which all lie in $\mathcal{A}$):

$$[N] = \frac{v^N - v^{-N}}{v - v^{-1}}, \quad [N]! = [N][N-1]\cdots[1], \quad \begin{bmatrix} M \\ N \end{bmatrix} = \frac{[M]!}{[N]![M-N]!}.$$

These are referred to as quantized integers, quantized factorials and quantized binomial coefficients, respectively. If v is specialized to 1 they specialize to the usual integers, factorials and binomial coefficients.

We define the quantized enveloping algebra U corresponding to the above data (as in [7, 3.1.1 & 33.1.5]) to be the $\mathbb{Q}(v)$-algebra U with generators $1, E_1, E_2, \ldots, E_{n-1}$, $F_1, F_2, \ldots, F_{n-1}$, and K_h for $h \in Y$, subject to the relations: (for each $i, j \in I$ and $h, h' \in Y$)

$$K_0 = 1,$$

$$K_h K_{h'} = K_{h+h'},$$

$$K_h E_i = v^{\alpha_i(h)} E_i K_h,$$

$$K_h F_i = v^{-\alpha_i(h)} F_i K_h,$$

$$E_i F_i - F_i E_i = \frac{K_i - K_i^{-1}}{v - v^{-1}},$$

$$E_i F_j - F_j E_i = 0, \quad i \neq j,$$

$$\sum_{\substack{p+p'=1-A_{ij} \\ p,p' \geq 0}} (-1)^{p'} \begin{bmatrix} 1 - A_{ij} \\ p' \end{bmatrix} E_i^p E_j E_i^{p'} = 0, \quad i \neq j,$$

$$\sum_{\substack{p+p'=1-A_{ij} \\ p,p' \geq 0}} (-1)^{p'} \begin{bmatrix} 1 - A_{ij} \\ p' \end{bmatrix} F_i^p F_j F_i^{p'} = 0, \quad i \neq j,$$

(where, for $i \in I$, we put $K_i = K_{h_i}$ and $K_i^{-1} = K_{-h_i}$).

Note that U is in fact generated by the elements E_i, F_i, K_i and K_i^{-1} for $i \in I$. We make the following definitions (see [7, 3.1.1 & 3.1.13]). For $M \in \mathbb{N}$, and $i \in I$, we put $E_i^{(M)} = E_i^M/[M]!$, and $F_i^{(M)} = F_i^M/[M]!$, which are called *divided powers*. Let $U_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of U generated by the elements $E_i^{(N)}, F_i^{(N)}, K_h$ for $i \in I$, $N \in \mathbb{N}$ and $h \in Y$. It is called the *integral form* of U. Let U^+ be the $\mathbb{Q}(v)$-subalgebra of U generated by the E_i, $i \in I$, and $U_{\mathcal{A}}^+$ the $\mathcal{A}$-subalgebra of U generated by $E_i^{(N)}$, $i \in I, N \in \mathbb{N}$. Let U^- be the $\mathbb{Q}(v)$-subalgebra of U generated by the F_i, $i \in I$, and $U_{\mathcal{A}}^-$ the

$\mathcal{A}$-subalgebra of U generated by $F_i^{(N)}$, $i \in I, N \in \mathbb{N}$. Let U^0 be the $\mathbb{Q}(v)$-subalgebra generated by the K_h, $h \in Y$.

Let $X^+ \subseteq X$ be the set of dominant weights, i. e. those of the form $\mu_1\omega_1+\mu_2\omega_2+\cdots+\mu_{n-1}\omega_{n-1} \in X$ where $\omega_1, \omega_2, \ldots, \omega_{n-1}$ are the fundamental weights of $\mathbf{g}$ and $\mu_1, \mu_2, \ldots, \mu_{n-1} \in \mathbb{N}$. If L is a U-module, $x \in L$ and $\mu = \mu_1\omega_1 + \mu_2\omega_2 + \cdots + \mu_{n-1}\omega_{n-1} \in X$, we say that x has weight μ (or weight $(\mu_1, \mu_2, \ldots, \mu_{n-1})$), if $K_h x = v^{\mu(h)}x$ for all $h \in Y$. We call the subspace of L consisting of all of the elements of weight μ the *μ-weight space* of L. As in [7, 3.4.1], we restrict our attention to U-modules which are direct sums (as $\mathbb{Q}(v)$-vector spaces) of their weight spaces. We say that $x \in L$, $x \neq 0$, is a *highest* (respectively, *lowest*) weight vector if x has weight μ, for some $\mu \in X$, $E_i x = 0$ (respectively, $F_i x = 0$) for each $i \in I$ and $U^- x = L$ (respectively, $U^+ x = L$). Such a vector is uniquely determined up to a non-zero scalar multiple. We say that L is a highest weight module with highest weight μ if it contains a highest weight vector of weight μ. Let $\mu = \mu_1\omega_1 + \mu_2\omega_2 + \cdots + \mu_{n-1}\omega_{n-1}$ be a dominant weight. We follow the construction in [7, 3.4.5 & 3.5.6]. Let J be the left ideal of U generated by the elements E_i for $i \in I$ and the elements $K_\mu - v^{(\mu(h))}$ for $h \in Y$. Then the map from U^- to U/J taking $x \in U^-$ to $x + J$ is a $\mathbb{Q}(v)$-vector space isomorphism, which can be used to transfer the left U-module structure of U/J to U^-. The resulting U-module we denote by $M(\mu)$; it is the *Verma module.* Let $T(\mu)$ be the left ideal of $M(\mu)$ (as a $\mathbb{Q}(v)$-algebra) generated by the elements $F_i^{\mu_i+1}$, for $i \in I$, and let $V(\mu)$ be the quotient module $M(\mu)/T(\mu)$. Then, by [7, 6.2.3 & 6.3.4], $V(\mu)$ is an irreducible, finite-dimensional highest weight U-module with highest weight μ, unique up to isomorphism. We fix x_1 as the image of $1 \in M(\mu)$ under the natural map from $M(\mu)$ to $V(\mu)$. Then x_1 is a highest weight vector for $V(\mu)$. It is known that $V(\mu)$ is the direct sum of its weight spaces (see [7, 3.4.1 & 3.5.6]). We also write $V(\mu)_{\mathcal{A}} = U_{\mathcal{A}}^- x_1$, the integral form of $V(\lambda)$ (see [7, 19.3.1]). By [7, 19.3.2], $V(\lambda)_{\mathcal{A}}$ is a $U_{\mathcal{A}}$-module.

The algebra U is also equipped with a coassociative comultiplication, $\Delta : U \to U \otimes U$, which has the following effect on the generators:

$$\begin{aligned} \Delta(E_i) &= 1 \otimes E_i + E_i \otimes K_i, \\ \Delta(F_i) &= K_i^{-1} \otimes F_i + F_i \otimes 1, \\ \Delta(K_h) &= K_h \otimes K_h. \end{aligned}$$

The map Δ is also an algebra homomorphism. We put $\Delta^{(2)} = \Delta(\Delta \otimes 1)$, etc., so that for $r \in \mathbb{N}$, $r \geq 1$, $\Delta^{(r-1)}$ is an algebra homomorphism from U to $U^{\otimes r}$. Thus if M is a U-module, we can make $M^{\otimes r}$ into a U-module via

$$u.(m_1 \otimes \cdots \otimes m_r) = \Delta^{(r-1)}(u)(m_1 \otimes \cdots \otimes m_r).$$

It can be checked by a simple inductive argument that the effect of $\Delta^{(r-1)}$ on the generators E_i, F_i, K_h, is as follows:

$$\begin{aligned}
\Delta^{(r-1)}(E_i) &= (1 \otimes \cdots \otimes 1 \otimes E_i) + (1 \otimes \cdots \otimes 1 \otimes E_i \otimes K_i) + \cdots \\
&\qquad \cdots + (E_i \otimes K_i \otimes \cdots \otimes K_i), \\
\Delta^{(r-1)}(F_i) &= (K_i^{-1} \otimes \cdots \otimes K_i^{-1} \otimes F_i) + (K_i^{-1} \otimes \cdots \otimes K_i^{-1} \otimes F_i \otimes 1) + \cdots \\
&\qquad \cdots + (F_i \otimes 1 \otimes \cdots \otimes 1), \\
\Delta^{(r-1)}(K_h) &= K_h \otimes \cdots \otimes K_h.
\end{aligned}$$

We shall need the following definition of the Kashiwara operators (see [5, 2.2]):

Definition 1.1 Suppose that M is an integrable U-module. Fix $i \in [1, n-1]$. Any element $x \in M$ can be written uniquely $x = \sum_{0 \leq k \leq k'} F_i^{(k)} x_{k,k'}$, where the $x_{k,k'}$ satisfy $E_i x_{k,k'} = 0$ and $K_i x_{k,k'} = v^{k'} x_{k,k'}$. We then set $\widetilde{F}_i(x) = \sum_{0 \leq k \leq k'} F_i^{(k+1)} x_{k,k'}$ and $\widetilde{E}_i(x) = \sum_{1 \leq k \leq k'} F_i^{(k-1)} x_{k,k'}$.

Following Kashiwara [5, 2.3.1] we make the following definition:

Definition 1.2 Suppose that M is an integrable U-module. Let R be the ring of rational functions in v regular at $v^{-1} = 0$. A pair $(\mathcal{L}, B)$ is called a *crystal basis* of M if the following conditions hold:

(1) $\mathcal{L}$ is a free R-submodule of M, and $\mathbb{Q}(v) \otimes_R \mathcal{L} \cong M$.

(2) B is a basis of the $\mathbb{Q}$-vector space $\mathcal{L}/v^{-1}\mathcal{L}$.

(3) $\mathcal{L} = \bigoplus_\nu \mathcal{L}_\nu$ and $B = \sqcup_\nu B_\nu$, where M_ν is the ν-weight space of M, $\mathcal{L}_\nu = \mathcal{L} \cap M_\nu$, and $B_\nu = B \cap (\mathcal{L}_\nu / v^{-1} \mathcal{L}_\nu)$.

(4) $\widetilde{F}_i \mathcal{L} \subseteq \mathcal{L}$, $\widetilde{E}_i \mathcal{L} \subseteq \mathcal{L}$, where $\widetilde{E}_i$ and $\widetilde{F}_i$ are the Kashiwara operators (see Definition 1.1).

(5) $\widetilde{F}_i B \subseteq B \cup \{0\}$ and $\widetilde{E}_i B \subseteq B \cup \{0\}$, for all $i \in [1, n-1]$.

(6) For $b_1, b_2 \in B$ and $i \in [1, n-1]$, $b_1 = \widetilde{E}_i b_2$ if and only if $b_2 = \widetilde{F}_i b_1$.

We use (6) to draw the corresponding crystal graph, which is an indication of how the Kashiwara operators act on the crystal basis. There is one vertex corresponding to each element of B. If b_1 and

b_2 are as in (6), we draw an edge from the vertex corresponding to b_1 to the vertex corresponding to b_2, with the arrow from b_1 to b_2, and the label i on it.

Theorem 1.3 *Suppose that μ is a dominant weight and $V(\mu)$ is the irreducible finite-dimensional highest weight U-module with highest weight μ, and highest weight vector x_1, as defined above. Let $L(\mu)$ be the R-submodule generated by the vectors of the form $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}x_1$ and let $B(\mu)$ be the subset of $L(\mu)/v^{-1}L(\mu)$ consisting of the non-zero images of these vectors under the natural projection $L(\mu)\to L(\mu)/v^{-1}L(\mu)$. Then $(L(\mu),B(\mu))$ is a crystal basis of $V(\mu)$, unique up to an automorphism of $V(\mu)$.*

Proof: See [5, 2.6]. □

We also have:

Theorem 1.4 *Suppose $(\mathcal{L}_j,B_j)$ is a crystal basis for an integrable U-module M_j, for $j=1,2$. Set $\mathcal{L}=\mathcal{L}_1\otimes\mathcal{L}_2\subseteq M_1\otimes M_2$ and $B=\{b_1\otimes b_2 \ : \ b_j\in B_j, \ j=1,2\}\subseteq\mathcal{L}/v^{-1}\mathcal{L}$. Then $(\mathcal{L},B)$ is a crystal basis of $M_1\otimes M_2$.*

Proof: See [5, 2.4]. □

Following Kashiwara and Nakashima (see [6, 2.1.2]), we make the following definition:

Definition 1.5 Fix $i\in[1,n-1]$. Suppose that M is an integrable U-module with crystal basis $(\mathcal{L},B)$, and $b_1,b_2\in B$. Suppose further that $\widetilde{E}_ib_1=\widetilde{F}_ib_2=0$ and $\widetilde{F}_ib_1=b_2$. We say that b_1 is *of type u_+ for i* and that b_2 is *of type u_- for i*. If $b\in B$ and $\widetilde{E}_ib=\widetilde{F}_ib=0$, we say that b is *of type u_0 for i*.

Proposition 1.6 *Fix $i\in[1,n-1]$. Suppose that M_j, $j=1,2,\ldots,r$, are integrable U-modules, and that $(\mathcal{L}_j,B_j)$ is a crystal basis for each M_j, with $b_j\in B_j, j=1,2,\ldots,r$. Suppose also that each b_j is of type u_+, u_- or u_0 for i. Then the following procedure describes how $\widetilde{E}_i$ and $\widetilde{F}_i$ act on $b_1\otimes b_2\otimes\cdots\otimes b_r$, an element of a crystal basis for $M_1\otimes M_2\otimes\cdots\otimes M_r$:*

(a) Rewrite the tensor formally by replacing each b_j with u_+ if it is of type u_+ for i, or with u_- or u_0 similarly.

(b) Neglect u_0 and also any pair $u_+ \otimes u_-$. Repeatedly remove pairs of the form $u_+ \otimes u_-$ until there are no more left. Rewrite the tensor without these elements, and call this new tensor t. We call this the i-reduced form*of b.*

(c) To apply $\widetilde{E}_i$: Change the rightmost u_- in t to u_+. (If there is no u_- in the i-reduced form, then $\widetilde{E}_i$ acts as zero on the original crystal basis element.)

To apply $\widetilde{F}_i$: Change the leftmost u_+ in t to u_-. (If there is no u_+ in the i-reduced form, then $\widetilde{F}_i$ acts as zero on the original crystal basis element.)

(d) Return to t all of the elements deleted in (b). Then replace each u_+, u_- or u_0 with the b_j it was originally. This is well-defined except when a u_+ or u_- has been changed as in step (b). In the case when a u_- has been replaced with a u_+, suppose that originally the u_- was b_j. Replace the u_+ in t with $\widetilde{E}_i b_j$. Similarly if a u_+ has been replaced with a u_- and the u_+ originally was b_j, replace the u_- in t with $\widetilde{F}_i b_j$.

Proof: See Remarks 2.1.2 and 2.1.3 in [6]. □

Let $^-$ be the $\mathbb{Q}$-algebra automorphism from U to U taking E_i to E_i, F_i to F_i, and K_h to K_{-h}, for each $i \in [1, n-1]$ and $h \in Y$, and v to v^{-1} (see [7, 3.1.12]). There is an induced automorphism (also denoted $^-$) of any module $V(\mu)$ for U defined by $\overline{ux_1} = \overline{u}x_1$ for any $u \in U^-$ (see [7, 19.3.4]).

Theorem 1.7 *Suppose $V(\mu)$, $B(\mu)$ and $L(\mu)$ are as in Theorem 1.3. Then for each $b \in B(\mu)$ there is a unique $\widetilde{b} \in L(\mu)$ such that $\widetilde{b} \mapsto b$ under the canonical projection $L(\mu) \to L(\mu)/v^{-1}L(\mu)$ and $\widetilde{b} \in L(\mu) \cup \overline{L(\mu)}$. Furthermore, the set $\{\widetilde{b} \ : \ b \in B(\mu)\}$ forms a $\mathbb{Q}(v)$-basis $\mathbf{B}(\mu)$ for $V(\mu)$ (the canonical basis).*

Proof: See [5, §0] □

From any Coxeter system with a finite Coxeter group W, we can construct the so called *Hecke algebra*, denoted by $\mathcal{H}(W)$. This is a $\mathbb{Q}(v)$-algebra with basis elements T_w parametrized by $w \in W$, and the following multiplication rules, for any elements s of length 1 in W:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w); \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

In this paper, we take W to be the symmetric group on r letters, and s to be a simple transposition $(p, p+1)$. We identify the indeterminate q with v^2.

The following properties of Hecke algebras will be used in this paper.

Lemma 1.8 *Let $d \in W$, and let $s(i_1) \cdots s(i_t)$ be a reduced expression for d (i.e. one of minimal length). Then $T_d = T_{s(i_1)} \cdots T_{s(i_t)}$ and*

$$T_d . \left(\sum_{w \in W} T_w \right) = q^{\ell(d)} . \left(\sum_{w \in W} T_w \right).$$

Proof: The first assertion follows by a very easy induction. The proof of the second assertion is by induction on d, and the base case follows by splitting the sum into

$$\sum_{w : \ell(sw) < \ell(w)} T_w + \sum_{w : \ell(sw) > \ell(w)} T_w . \ \square$$

Let V be the $\mathbb{Q}(v)$-vector space with basis $(e_1, \ldots, e_n)$. Then $T^r(V) = V^{\otimes r}$ can be made into a U–$\mathcal{H}$ bimodule as follows. We first make V into a left U-module via

$$\begin{aligned} E_i e_{i+1} &= e_i, \\ E_i e_j &= 0 \quad (\text{for } j \neq i+1), \\ F_i e_i &= e_{i+1}, \\ F_i e_j &= 0 \quad (\text{for } j \neq i), \\ K_i e_i &= v e_i, \\ K_i e_{i+1} &= v^{-1} e_{i+1}, \\ K_i e_j &= e_j \quad (j \neq i, i+1). \end{aligned}$$

We now make $T^r(V)$ into a left U-module via the homomorphism $\Delta^{(r-1)}$. Following [3], we also make $T^r(V)$ into a right $\mathcal{H}$-module via

$$e_{i_1} \otimes \cdots \otimes e_{i_r} T_{p,p+1} := \begin{cases} v e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r} & \text{if } i_p < i_{p+1}; \\ v^2 e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_p = i_{p+1}; \\ v e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r} + (v^2 - 1) e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_p > i_{p+1}. \end{cases}$$

2 The Quantized Symmetric Powers

Definitions 2.1 Fix $r \in \mathbb{N},\ r \geq 1$. Let $\mathbf{i}$ be an r-tuple $(i_1, \ldots, i_r)$ satisfying $1 \leq i_1 \leq \cdots \leq i_r \leq n$. For each $i \in [1, n]$, let λ_i be the multiplicity of i in $(i_1, \ldots, i_r)$. Thus λ is a composition of r. Let

W_λ be the corresponding parabolic subgroup of $\mathcal{S}_r$ generated by all simple transpositions $(p, p+1)$ satisfying $i_p = i_{p+1}$, and let $\mathcal{D}_\lambda$ be the associated set of distinguished right coset representatives.

We define the element $[e_{i_r} \vee \cdots \vee e_{i_1}] \in V^{\otimes r}$ to be

$$v^{-m_{\mathbf{i}}} \sum_{\sigma \in \mathcal{D}_\lambda} v^{\ell(\sigma)} e_{i_{\sigma 1}} \otimes \cdots \otimes e_{i_{\sigma r}},$$

where $m_{\mathbf{i}} := |\{1 \leq a < b \leq r : i_a < i_b\}|$, i.e. the length of the longest element in $\mathcal{D}_\lambda$.

We call the span of all such elements $[e_{i_r} \vee \cdots \vee e_{i_1}]$ the *r-th quantized symmetric power* of the vector space V, and denote it by $S_q^r(V)$.

Note. This quantized symmetric power can also be obtained as a quotient of $V^{\otimes r}$ (see [8, p166]). Consider the subspace I_r of $V^{\otimes r}$ spanned by the elements of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_s} \otimes e_{i_{s+1}} \otimes \cdots \otimes e_{i_r} - v^{-1} e_{i_1} \otimes \cdots e_{i_{s+1}} \otimes e_{i_s} \otimes \cdots \otimes e_{i_r},$$

where $i_s < i_{s+1}$. Then it is easy to see that I_r is a U-submodule of $V^{\otimes r}$. The quotient $V^{\otimes r}/I_r$ is isomorphic as a U-module to $S_q^r(V)$. (It is an irreducible finite-dimensional highest weight module for U with highest weight $(r, 0, 0, \ldots, 0)$. We shall see in Lemma 3.2 that $S_q^r(V)$ is also such a module, so they must be isomorphic.)

Lemma 2.2 *The vector space $S_q^r(V)$ coincides with the vector space S spanned by all vectors*

$$e_{j_1} \otimes \cdots \otimes e_{j_r}.x_W,$$

where $x_W = \sum_{w \in \mathcal{S}_r} T_w$, and the elements j_a are integers between 1 and n inclusive. (Note that there is no monotonicity assumption on the sequence $(j_1, \ldots, j_r)$.)

Proof: We first claim that S is spanned by all vectors of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_r}.x_W,$$

where the i_a are integers between 1 and n inclusive, and $i_1 \leq \cdots \leq i_r$. To see this, consider an arbitrary vector

$$e_{j_1} \otimes \cdots \otimes e_{j_r}.x_W,$$

as given in the statement of the Lemma. Let $\mathbf{i} = (i_1, \ldots, i_r)$ be a rearrangement of $(j_1, \ldots, j_r)$ satisfying $i_1 \leq \cdots \leq i_r$. Let W_λ be the parabolic subgroup corresponding to the sequence $\mathbf{i}$, and let $\mathcal{D}_\lambda$ be the associated set of distinguished right coset representatives. Then there exists $w \in \mathcal{D}_\lambda$, satisfying $i_{s.w} = j_s$ for all $1 \leq s \leq r$. It follows from Lemma 1.8 that

$$e_{i_1} \otimes \cdots \otimes e_{i_r}.T_w = v^{\ell(w)} e_{j_1} \otimes \cdots \otimes e_{j_r}.$$

Since $T_w.x_W = q^{\ell(w)}.x_W$, we find that $e_{j_1} \otimes \cdots \otimes e_{j_r}.x_W$ is a multiple of $e_{i_1} \otimes \cdots \otimes e_{i_r}.x_W$. It now follows that $S = S_q^r(V)$ because if $i_1 \leq \cdots \leq i_r$, then

$$e_{i_1} \otimes \cdots \otimes e_{i_r}.x_W = (e_{i_1} \otimes \cdots \otimes e_{i_r}.x_{W_\lambda}). \sum_{\sigma \in \mathcal{D}_\lambda} T_\sigma$$

and

$$e_{i_1} \otimes \cdots \otimes e_{i_r}.x_{W_\lambda} = P_\lambda(q).e_{i_1} \otimes \cdots \otimes e_{i_r},$$

where P_λ is the Poincaré polynomial associated with the parabolic subgroup W_λ. □

We now show that $S_q^r(V)$ is a U-module.

Lemma 2.3 *(i) The vector space $S_q^r(V)$ is a U-module.*

(ii) Assume $i_1 \leq i_2 \leq \cdots \leq i_r$. Then there is only one natural basis vector $e_{j_1} \otimes \cdots \otimes e_{j_r}$ of $V^{\otimes r}$ occurring with nonzero coefficient in the expression for $[e_{i_r} \vee \cdots \vee e_{i_1}]$ which satisfies $j_1 \geq \cdots \geq j_r$. We call such a basis vector the leading term of $[e_{i_r} \vee \cdots \vee e_{i_1}]$. It occurs with coefficient 1.

(iii) The elements $[e_{i_r} \vee \cdots \vee e_{i_1}]$, where $i_1 \leq \cdots \leq i_r$, form a basis for $S_q^r(V)$.

Proof: We start by proving (i). Using Lemma 2.2 it is enough to show that S is a U-module, via the usual comultiplication action. We know that $V^{\otimes r}$ is a U-module. The result now follows from the fact that the actions of U and $\mathcal{H}$ on $V^{\otimes r}$ commute (see [3]). We have:

$$u.\left(e_{j_1} \otimes \cdots \otimes e_{j_r}.x_W\right) = \left(u.e_{j_1} \otimes \cdots \otimes e_{j_r}\right).x_W.$$

The right-hand side is now clearly an element of S, as required.

For the proof of (ii), consider $\mathbf{i} = (i_1, \ldots, i_r)$ where $i_1 \leq \cdots \leq i_r$ as usual. Denote the associated parabolic subgroup and set of distinguished coset representatives by W_λ and $\mathcal{D}_\lambda$ respectively. It is immediate that

$$e_{i_1} \otimes \cdots \otimes e_{i_r}. \sum_{\sigma \in \mathcal{D}_\lambda} T_w = \sum_{\sigma \in \mathcal{D}_\lambda} v^{\ell(\sigma)} e_{i_{\sigma 1}} \otimes \cdots \otimes e_{i_{\sigma r}}.$$

Denote by m the length of the longest element of $\mathcal{D}_\lambda$, and multiply both sides of the above equation by v^{-m}. The result (ii) follows because all the terms on the right are multiples of distinct basis elements.

To prove (iii), we only need prove independence, because the spanning property comes from the definition of quantized symmetric power. We observe that distinct elements of the form $[e_{i_r} \vee \cdots \vee e_{i_1}]$ have distinct leading terms, and each term $e_{j_1} \otimes \cdots \otimes e_{j_r}$ where $j_1 \geq \cdots \geq j_r$ corresponds to a unique element of this form (i.e. $[e_{j_1} \vee \cdots \vee e_{j_r}]$). The independence of the spanning set is a direct consequence of this. □

Using Lemma 2.3, we can explicitly calculate the coefficients of the matrices corresponding to the left actions of E_i, F_i and K_i on $S_q^r(V)$.

Let $1 \leq i_1 \leq \cdots \leq i_r \leq n$, and write $\mathbf{i} = (i_1, \ldots, i_r)$. Consider the basis element $e_{\mathbf{i}} = [e_{i_r} \vee \cdots \vee e_{i_1}]$ of $S_q^r(V)$. Let λ_j be the multiplicity with which j occurs in the sequence $(i_1, \ldots, i_r)$.

If $\lambda_j > 0$, we define $\mathbf{i}_j^-$ to be the (monotone increasing) r-tuple which is the same as $\mathbf{i}$ except that the rightmost occurrence of j has been replaced by a $j+1$.

If $\lambda_{j+1} > 0$, we define $\mathbf{i}_j^+$ to be the (monotone increasing) r-tuple which is the same as $\mathbf{i}$ except that the leftmost occurrence of $j+1$ has been replaced by a j.

Examples

Consider $n = r = 4$ and $\mathbf{i} = (1, 2, 2, 4)$. Then $\mathbf{i}_2^- = (1, 2, 3, 4)$, $\mathbf{i}_3^+ = (1, 2, 2, 3)$ and $\mathbf{i}_2^+$ is not defined.

Lemma 2.4 *Maintain the above notation. Then the following identities hold:*

(i)

$$F_j.e_{\mathbf{i}} = \begin{cases} [\lambda_{j+1} + 1] e_{\mathbf{i}_j^-} & \text{if } e_{\mathbf{i}_j^-} \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

(ii)

$$K_j.e_{\mathbf{i}} = v^{\lambda_j - \lambda_{j+1}} e_{\mathbf{i}},$$

(iii)

$$E_j.e_{\mathbf{i}} = \begin{cases} [\lambda_j + 1]e_{\mathbf{i}_j^+} & \textit{if } e_{\mathbf{i}_j^+} \textit{ exists,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: The proof of (ii) is easy and follows from the definition and basic properties of $e_{\mathbf{i}}$, and the action of K_i under the comultiplication Δ.

The proof of (iii) is analogous to the proof of (i), so we only prove (i).

By consideration of the action of F_j on tensor space, we find that each basis element of $V^{\otimes r}$ occurring with nonzero coefficient in $F_j.e_{\mathbf{i}}$ is of form $e_{k_1} \otimes \cdots \otimes e_{k_r}$, where the sequence $(k_1, \dots, k_r)$ is a permutation of $\mathbf{i}_j^-$. (If $\mathbf{i}_j^-$ is not defined, then F_i must act as zero as required. We may therefore assume $\lambda_j > 0$.) Lemma 2.3 now implies that $F_j.e_{\mathbf{i}}$ is some scalar multiple of $e_{\mathbf{i}_j^-}$, so it remains to prove that the scalar is as claimed.

Lemma 2.3 (ii) shows that it is enough to show that the coefficient of the leading term $l_k := e_{k_r} \otimes \cdots \otimes e_{k_1}$ in $F_j.e_{\mathbf{i}}$ is equal to $[\lambda_{j+1} + 1]$.

Denote the leading term of $e_{\mathbf{i}}$ by $l_{\mathbf{i}} := e_{i_r} \otimes \cdots \otimes e_{i_1}$. Consider the set of basis elements occurring in $e_{\mathbf{i}}$ with nonzero coefficients which yield a nonzero coefficient of $l_{\mathbf{i}}$ after the action of F_i. This set consists of $l_{\mathbf{i}}$ together with certain elements $l_{\mathbf{i},p}$, as p ranges from 1 to λ_{j+1} inclusive. (If $\lambda_{j+1} = 0$ then this set consists only of $l_{\mathbf{i}}$.) The elements $l_{\mathbf{i},p}$ are permutations of $l_{\mathbf{i}}$, identical except that the leftmost occurrence of e_j (note that there must be one since we are assuming $\lambda_j > 0$) is shifted p places to the left. Note also that $l_{\mathbf{i}} = l_{\mathbf{i},0}$. We illustrate this with an example.

Suppose $n = 2, r = 5$, $l_{\mathbf{i}} = e_2 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1$ and $j = 1$. Then $l_k = e_2 \otimes e_2 \otimes e_2 \otimes e_2 \otimes e_1$, $\lambda_1 = 2$, $\lambda_2 = 3$, and we have

$$\begin{aligned} l_{\mathbf{i},0} &= e_2 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1, \\ l_{\mathbf{i},1} &= e_2 \otimes e_2 \otimes e_1 \otimes e_2 \otimes e_1, \\ l_{\mathbf{i},2} &= e_2 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1, \\ l_{\mathbf{i},3} &= e_1 \otimes e_2 \otimes e_2 \otimes e_2 \otimes e_1. \end{aligned}$$

One can see from the proof of Lemma 2.3 that the coefficient of $l_{\mathbf{i},p}$ in $e_{\mathbf{i}}$ is v^{-p}. By consideration of the action of F_j under the comultiplication, we see that l_k appears in $F_j.l_{\mathbf{i},p}$ with coefficient $v^{\lambda_{j+1}}.v^{-p}$.

This means that the overall coefficient of l_k in $F_j.e_{\mathbf{i}}$ is

$$\sum_{p=0}^{\lambda_{j+1}} v^{\lambda_{j+1}} v^{-2p},$$

i.e. $[\lambda_{j+1}+1]$. The result now follows. □

We now show how $S_q^r(V)$ may be generated as a U-module by a single vector. Denote by y_1 the vector $[e_1 \vee \cdots \vee e_1] = e_1 \otimes \cdots \otimes e_1$.

Lemma 2.5 *Let $x = [e_{i_r} \vee \cdots \vee e_{i_1}]$ satisfy $i_1 \leq \cdots \leq i_r$. For each integer j satisfying $1 \leq j \leq n$, let λ_j be the multiplicity with which e_j appears in x. For each k satisfying $1 \leq k < n$, let $c_k = \sum_{j=k+1}^{n} \lambda_j$. Then*

$$F_{n-1}^{(c_{n-1})} \cdots F_1^{(c_1)}.y_1 = x.$$

Proof: We proceed by induction on $t = \sum_{j=1}^{n-1} c_j$. The case $t = 0$ is trivial. For the general case, let l be the largest index such that c_l is nonzero. (We may assume $l \geq 1$, otherwise we are in the case $t = 0$ which has been dealt with.) This implies that e_k does not occur in x unless $k \leq l+1$. Let x' be the basis element of $S_q^r(V)$ obtained from x by replacing all the occurrences of e_{l+1} by e_l. Then by the inductive hypothesis we have

$$F_{l-1}^{(c_{l-1})} \cdots F_1^{(c_1)}.y_1 = x'.$$

We need to show that $F_l^{(c_l)}.x' = x$. It is enough to prove that

$$F_l^{c_l}.x' = [c_l]!\, x,$$

and this follows from repeated application of Lemma 2.4 (i). □

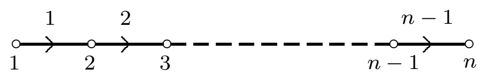
We now find that the quantized symmetric power is a highest weight U-module.

Corollary 2.6 *The space $S_q^r(V)$ is a highest weight U-module of highest weight $(r, 0, \ldots, 0)$ and with highest weight vector y_1.*

Proof: Lemma 2.5 shows that $S_q^r(V) = U^-.y_1$. It is easy to check that $K_i.y_1 = v^{\delta_{i,1}r}.y_1$. So y_1 has weight $(r, 0, \ldots, 0)$. We see from Lemma 2.4 (iii) that $E_i.y_1 = 0$ for all i. This completes the proof. □

3 The Canonical Basis

Let $\mathcal{L}$ be the R-submodule of V generated by $e_1, e_2, \dots, e_n$, and let $B = \{b_1, b_2, \dots, b_n\}$ be the set of images of these elements under the natural projection from $\mathcal{L}$ to $\mathcal{L}/v^{-1}\mathcal{L}$. Then $(\mathcal{L}, B)$ is a crystal basis for V. The crystal graph is:



The edge labels are written above the edges, and vertex j corresponds to the crystal basis element b_j (see [6, §3.2]). We know from §1 that for $r \in \mathbb{N}, r \geq 1$, there is an irreducible finite-dimensional highest weight module V^r for U with highest weight $(r, 0, 0, \dots, 0)$, and highest weight vector x_1. According to Kashiwara and Nakashima (see [6, 3.4.2]), a crystal basis for V^r is $(\mathcal{L}^r, B^r)$, where $\mathcal{L}^r = \mathcal{L}^{\otimes r}$ and

$$B^r = \{b_{i_r} \otimes b_{i_{r-1}} \otimes \cdots \otimes b_{i_1} \ : \ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n\}.$$

Note that, if $k \in [1, n-1]$ then in B, b_k is of type u_+ for k, b_{k+1} is of type u_- for k, and all other b_p's are of type u_0 for k. For $1 \leq j \leq n$, suppose that j appears with multiplicity λ_j in the sequence $i_1, i_2, \dots, i_r$. Define $c_k = \sum_{j=k+1}^{n} \lambda_j$, for $k = 1, 2, \dots n-1$. Put $\widehat{x_1} = b_1 \otimes b_1 \otimes \cdots \otimes b_1$ in B^r. Then $\widehat{x_1}$ is the vector of highest weight in B^r. It is clear from Lemma 1.6 that:

$$\widetilde{F}_1^{c_1} \widehat{x_1} = b_2 \otimes b_2 \otimes \cdots \otimes b_2 \otimes b_1 \otimes b_1 \otimes \cdots b_1 \tag{1}$$

(where there are c_1 b_2's in the expression). Since λ_1 is equal to the number of 1's in $i_1, i_2, \dots i_r$, and $c_1 = r - \lambda_1$, it can be seen that in the expression on the right hand side in equation (1) there are the same number of b_1's as in the expression $b_{i_r} \otimes b_{i_{r-1}} \otimes \cdots \otimes b_{i_1}$. If we now apply $\widetilde{F}_2^{c_2}$ to both sides of (1) we get in a similar way an expression with the same number of b_1's and b_2's as $b_{i_r} \otimes b_{i_{r-1}} \otimes \cdots \otimes b_{i_1}$. Repeating this argument, we see that:

$$b_{i_r} \otimes b_{i_{r-1}} \otimes \cdots \otimes b_{i_1} = \widetilde{F}_{n-1}^{c_{n-1}} \widetilde{F}_{n-2}^{c_{n-2}} \cdots \widetilde{F}_1^{c_1} \widehat{x_1}.$$

Note that here we must have $0 \leq c_{n-1} \leq c_{n-2} \leq \cdots \leq c_1 \leq r$, and also that if $c'_1, c'_2, \dots c'_{n-1}$ are any integers satisfying $0 \leq c'_{n-1} \leq c'_{n-2} \leq \cdots \leq c'_1 \leq r$, then there exist integers $i_1, i_2, \dots, i_r$ satisfying $1 \leq i_1 \leq i_2 \cdots \leq i_r \leq n$ such that if we define the c's the same way as above, then $c_j = c'_j$ for $j = 1, 2, \dots, n-1$. So, we have

$$B^r = \{\widetilde{F}_{n-1}^{c_{n-1}} \widetilde{F}_{n-2}^{c_{n-2}} \cdots \widetilde{F}_1^{c_1} \widehat{x_1} \ : \ 0 \leq c_{n-1} \leq c_{n-2} \leq \cdots \leq c_1 \leq r\} \tag{2}$$

Suppose that $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$, and $c_k = \sum_{j=k+1}^{n} \lambda_j$, for $k = 1, 2, \dots, n$, where λ_j is the number of occurrences of j in $i_1, i_2, \dots, i_r$. Fix $i \in [1, n-1]$. Then since E_i commutes with F_j if $i \neq j$, we have that in V^r, $E_i(F_{i-1}^{(c_{i-1})} F_{i-2}^{(c_{i-2})} \cdots F_1^{(c_1)} x_1) = 0$. Also, if $i = 1$

then $K_i F_{i-1}^{(c_{i-1})} F_{i-2}^{(c_{i-2})} \cdots F_1^{(c_1)} x_1 = K_1 x_1 = v^r x_1$, where $r \geq c_1$, and if $i \neq 1$ then we have that $K_i F_{i-1}^{(c_{i-1})} F_{i-2}^{(c_{i-2})} \cdots F_1^{(c_1)} x_1 = v^{c_{i-1}} F_{i-1}^{(c_{i-1})} F_{i-2}^{(c_{i-2})} \cdots F_1^{(c_1)} x_1$, and $c_{i-1} \geq c_i$. (We use the relations of U and the definition of the module V^r). So in either case, the Kashiwara decomposition with respect to i (see Definition 1.1) of $F_{i-1}^{(c_{i-1})} F_{i-2}^{(c_{i-2})} \cdots F_1^{(c_1)} x_1$ is just $F_i^{(0)} x_{0,c}$, where $c \geq c_i$. It follows that $\widetilde{F}_i^{c_i} F_{i-1}^{(c_{i-1})} F_{i-2}^{(c_{i-2})} \cdots F_1^{(c_1)} x_1 = F_i^{(c_i)} F_{i-1}^{(c_{i-1})} F_{i-2}^{(c_{i-2})} \cdots F_1^{(c_1)} x_1$. Therefore,

$$\widetilde{F}_{n-1}^{c_{n-1}} \widetilde{F}_{n-2}^{c_{n-2}} \cdots \widetilde{F}_1^{c_1} x_1 = F_{n-1}^{(c_{n-1})} F_{n-2}^{(c_{n-2})} \cdots F_1^{(c_1)} x_1.$$

Thus these elements are all fixed under $\bar{\ }$, so by their definition, (2) and Theorem 1.7 form the canonical basis for V^r. We have proved the following theorem:

Theorem 3.1 *Suppose $r \in \mathbb{N}$, $r \geq 1$. Then the canonical basis for V^r, the irreducible finite-dimensional highest weight module of highest weight $(r, 0, 0, \ldots, 0)$ is given by the following set:*

$$\{F_{n-1}^{(c_{n-1})} F_{n-2}^{(c_{n-2})} \cdots F_1^{(c_1)} x_1 \ : \ 0 \leq c_{n-1} \leq c_{n-2} \leq \cdots \leq c_1 \leq r\}.\square$$

We are now ready to formalize the link between the quantized symmetric power and the canonical basis discussed in §3.

Lemma 3.2 *The quantized symmetric power $S_q^r(V)$ is isomorphic as a U-module to the irreducible finite-dimensional highest weight U-module V^r of highest weight $(r, 0, \ldots, 0)$. The isomorphism can be chosen to send $[e_1 \vee \cdots \vee e_1]$ to x_1.*

Proof: We know from Corollary 2.6 that $S_q^r(V)$ is a highest weight U-module with highest weight $(r, 0, \ldots, 0)$. We deduce from the representation theory of quantized enveloping algebras in characteristic zero that it has as a quotient the irreducible U-module of highest weight $(r, 0, \ldots, 0)$. The basis described in Lemma 2.5 shows that the dimensions of the two modules are the same, so the modules are isomorphic. Since they are both highest weight modules, the isomorphism must send highest weight vectors to highest weight vectors, and thus we can choose the isomorphism as stated. $\square$

We can now state the actions of the generators of U on the canonical basis.

Theorem 3.3 *Let $x = F_{n-1}^{(c_{n-1})} \cdots F_1^{(c_1)} x_1$ be an element of the canonical basis described in §3. Define $c_0 = 0$ and $c_n = r$. Then the actions of E_i, F_i and K_i on x are given as follows:*

(i)

$$F_i.x = \begin{cases} [c_i - c_{i+1} + 1]\, F_{n-1}^{(c_{n-1})} \cdots F_{i+1}^{(c_{i+1})} F_i^{(c_i+1)} F_{i-1}^{(c_{i-1})} \cdots F_1^{(c_1)}.x_1 & \textit{if } c_i < c_{i-1}, \\ 0 & \textit{otherwise,} \end{cases}$$

(ii)

$$K_i.x = v^{c_{i-1} - 2c_i + c_{i+1}} x,$$

(iii)

$$E_i.x = \begin{cases} [c_{i-1} - c_i + 1]\, F_{n-1}^{(c_{n-1})} \cdots F_{i+1}^{(c_{i+1})} F_i^{(c_i-1)} F_{i-1}^{(c_{i-1})} \cdots F_1^{(c_1)}.x_1 & \textit{if } c_i > c_{i+1}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: Since x is an element of the canonical basis, we must have $r \geq c_1 \geq \cdots \geq c_{n-1} \geq 0$. Let $\lambda_k := c_{k-1} - c_k$, for each $1 \leq k \leq n$.

Using Lemma 2.5, we can identify x with the element of $S_q^r(V)$ given by $x' := [e_{i_r} \vee \cdots \vee e_{i_1}]$, where $i_1 \leq \cdots \leq i_r$ and e_j occurs in x' with multiplicity λ_j. Also note that replacing an occurrence of e_i in x' by e_{i+1} corresponds to increasing c_i by one; this is possible if and only if $c_i < c_{i-1}$. Similarly, replacing an occurrence of e_{i+1} by e_i corresponds to decreasing c_i by one; this is possible if and only if $c_i > c_{i+1}$. The assertions now follow from Lemma 2.4. □

Remark 3.4 It is worth noting that E_i and F_i take canonical basis elements to $\mathcal{A}$-multiples of other canonical basis elements, or to zero, so the matrices representing them with respect to the canonical basis have elements in $\mathcal{A}$. The fact that the matrices can be written over $\mathcal{A}$ is also a consequence of [7, Theorems 14.4.11(a) & 14.4.12].

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Richard Green,
Mathematical Institute,
24-29 St. Giles',
Oxford,
OX1 3LB,
United Kingdom.
greenr@maths.ox.ac.uk

Robert Marsh,
Department of Mathematics,
University of Glasgow,
University Gardens,
Glasgow, G12 8QW,
United Kingdom.
rjm@maths.gla.ac.uk