# $\tau$-EXCEPTIONAL SEQUENCES

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*Dedicated to the memory of Ragnar-Olaf Buchweitz*

ABSTRACT. We introduce the notions of $\tau$-exceptional and signed $\tau$-exceptional sequences for any finite dimensional algebra. We prove that for a fixed algebra of rank $n$, and for any positive integer $t \leq n$, there is a bijection between the set of such sequences of length $t$, and (basic) ordered support $\tau$-rigid objects with $t$ indecomposable direct summands. If the algebra is hereditary, our notions coincide with exceptional and signed exceptional sequences. The latter were recently introduced by Igusa and Todorov, who constructed a similar bijection in the hereditary setting.

## INTRODUCTION

Exceptional sequences are sequences of objects in an abelian or triangulated category satisfying certain orthogonality conditions involving the vanishing of Hom and Ext-groups. They were first introduced in an algebraic geometry setting [4, 13, 14] (see also [20]). This motivated their consideration in the context of the representation theory of finite dimensional hereditary algebras (such as path algebras of quivers) [9, 19]. Although the definition makes sense for arbitary abelian categories, work in the module case has mainly dealt with hereditary algebras. See however [17] for an example of the use of exceptional sequences in a more general setting.

Signed exceptional sequences for hereditary finite dimensional algebras $H$ were recently introduced by Igusa and Todorov [12]. In this case, the projective objects appearing in the sequence can be signed. Such sequences were needed in order for the authors to define the *cluster morphism category* of $H$, whose objects are the finitely generated wide subcategories of mod $H$. Signed exceptional sequences were needed to explain the composition and associativity of maps in the cluster morphism category. In particular it was shown that complete signed exceptional sequences are in bijection with ordered cluster-tilting objects in the cluster category [7] corresponding to $H$. These are known to be in bijection with ordered clusters in the corresponding (acyclic) cluster algebra [6, 8].

Recently, Adachi, Iyama and Reiten [1] introduced $\tau$-tilting theory for finite-dimensional algebras and in particular the notions of $\tau$-rigid modules and support $\tau$-tilting objects. Motivated by this, we introduce the notion of a (signed) $\tau$-exceptional sequence of modules over a finite dimensional algebra. In the hereditary case, this coincides with the notion of a (signed) exceptional sequence, but in general it is different. We show that the complete signed $\tau$-exceptional sequences are in bijection with ordered

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support $\tau$-tilting objects, generalizing the result of Igusa and Todorov in the hereditary case. Our approach is very different from that in [12]. In particular, an important ingredient in our proof is the correspondence [1] between $\tau$-rigid modules and rigid 2-term complexes in the derived category.

A finite dimensional algebra is called *$\tau$-tilting finite* [10] if there is only a finite number of isomorphism classes of basic $\tau$-tilting objects. Motivated by the cluster morphism categories mentioned above, in a forthcoming paper [5] we construct a natural category whose objects are the wide subcategories of the module category of a $\tau$-tilting finite algebra. The construction relies heavily on the bijection established in this paper.

The paper is organized as follows. In Section 1 we give some notation and background, and state the main result. In Section 2 we give some background and results concerning 2-term silting objects in the derived category, and their links to $\tau$-rigid objects in module categories. In Sections 3 and 4 we prepare for the proof of the main result, which is then completed in Section 5. We conclude by giving some examples in Section 6.

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## 1. Notation and main result

Let $\Lambda$ be a basic finite dimensional algebra over a field $k$, and let $\operatorname{mod}\Lambda$ denote the category of finite dimensional left $\Lambda$-modules. Let $\mathcal{P}(\Lambda)$ denote the full subcategory of projective objects in $\operatorname{mod}\Lambda$. Similarly, if $\mathcal{X}$ is a subcategory of $\operatorname{mod}\Lambda$, let $\mathcal{P}(\mathcal{X})$ denote the full subcategory of $\mathcal{X}$ consisting of the Ext-projective objects in $\mathcal{X}$, i.e. the objects $P$ in $\mathcal{X}$ such that $\operatorname{Ext}^1(P, X) = 0$ for all $X \in \mathcal{X}$.

For an additive category $C$ and an object $X$ in $C$, we denote by $\operatorname{add} X$ the additive subcategory of $C$ generated by $X$, i.e. the full subcategory of $C$ whose objects are all direct summands of direct sums of copies of $X$. For a subcategory $\mathcal{X} \subseteq C$, we define $\mathcal{X}^{\perp} = \{Y \in C \mid \operatorname{Hom}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\}$, and define ${}^{\perp}\mathcal{X}$ similarly.

If $C$ is (skeletally small) and Krull-Schmidt, we denote by $\operatorname{ind}(C)$ the set of isomorphism classes of indecomposable objects in $C$. For an object $X$, we write $\operatorname{ind} X$ for $\operatorname{ind}(\operatorname{add}(X))$. For any basic object $X$ in $C$, let $\delta(X)$ denote the number of indecomposable direct summands of $X$. We denote $\delta(\Lambda)$ by $n$.

If $C$ is abelian and $X$ is an object of $C$, we denote by $\operatorname{Gen} X$ the full subcategory of $C$ consisting of all objects which are factors of objects in $\operatorname{add} X$.

In general, all subcategories considered are assumed to be full and closed under isomorphisms. All objects are taken to be basic where possible and considered up to isomorphism.

Let $\tau$ denote the Auslander-Reiten translate in $\operatorname{mod}\Lambda$. We now recall notation and definitions of from [1]. Note that our definitions are slightly different, but clearly equivalent to the corresponding definitions in [1].

An object $U$ in $\operatorname{mod}\Lambda$ is called *$\tau$-rigid* if $\operatorname{Hom}(U, \tau U) = 0$. Let $D^b(\operatorname{mod}\Lambda)$ denote the bounded derived category of $\operatorname{mod}\Lambda$, with shift functor denoted by [1]. We consider

$\operatorname{mod}\Lambda$ as a full subcategory of $D^b(\operatorname{mod}\Lambda)$ by regarding a module as a stalk complex concentrated in degree 0. Consider the full subcategory $C(\Lambda) = \operatorname{mod}\Lambda \amalg \operatorname{mod}\Lambda[1]$ of $D^b(\operatorname{mod}\Lambda)$.

**Definition 1.1.** *The object $M \amalg P[1]$ in $C(\Lambda)$ is called* support $\tau$-rigid *if*

(i) *The object $M$ lies in* $\operatorname{mod}\Lambda$ *and satisfies* $\operatorname{Hom}(M, \tau M) = 0$*, and*
(ii) *The object $P$ lies in $\mathcal{P}(\Lambda)$ and satisfies* $\operatorname{Hom}(P, M) = 0$.

*The object $M \amalg P[1]$ is called a* support $\tau$-tilting object *if $\delta(M \amalg P[1]) = n$. Moreover, $M$ is in this case called a* support $\tau$-tilting module *or just a $\tau$-tilting module if in addition $P = 0$.*

We want to consider all possible orderings of such objects, in the following sense.

**Definition 1.2.** *For a positive integer $t$, an ordered $t$-tuple of indecomposable objects $(\mathcal{T}_1, \ldots, \mathcal{T}_t)$ in $C(\Lambda)$ is called an* ordered support $\tau$-rigid object *if $\amalg_{i=1}^t \mathcal{T}_i$ is a support $\tau$-rigid object. If, in addition, $t = n$, then $(\mathcal{T}_1, \ldots, \mathcal{T}_t)$ is called an ordered support $\tau$-tilting object.*

For a full subcategory $\mathcal{Y}$ of $\operatorname{mod}\Lambda$, we shall denote by $C(\mathcal{Y})$ the full subcategory $\mathcal{Y} \amalg \mathcal{Y}[1]$ of $C(\Lambda)$. Let $U$ be a $\tau$-rigid $\Lambda$-module. Jasso [16] considered the category $J(U) = U^\perp \cap {}^\perp(\tau U)$, and in particular proved that if $U$ is indecomposable, the category $J(U)$ is equivalent to the module category of an algebra $\Lambda'$ with $\delta(\Lambda') = \delta(\Lambda) - 1$ (see Proposition 4.2 for more details). For a projective object $P$, we let $J(P[1]) = J(P) = P^\perp$.

This allows us to define signed $\tau$-exceptional sequences recursively as follows:

**Definition 1.3.** *For a positive integer $t$, an ordered $t$-tuple of indecomposable objects $(\mathcal{U}_1, \ldots, \mathcal{U}_{t-1}, \mathcal{U}_t)$ in $C(\Lambda)$ is called a* signed $\tau$-exceptional sequence*, if $\mathcal{U}_t$ is a $\tau$-rigid object in $C(\Lambda)$ and $(\mathcal{U}_1, \ldots, \mathcal{U}_{t-1})$ is a $\tau$-exceptional sequence in $C(J(U_t))$.*

*If $t = n$, then $(\mathcal{U}_1, \ldots, \mathcal{U}_t)$ is called a* complete signed $\tau$-exceptional sequence.

We can now state our main result.

**Theorem 1.4.** *Let $\Lambda$ be a finite dimensional algebra. For each $t \in \{1, \ldots, n\}$ there is a bijection between the set of ordered support $\tau$-rigid objects of length $t$ in $C(\Lambda)$ and the set of signed $\tau$-exceptional sequences of length $t$ in $C(\Lambda)$.*

For $t = n$, we obtain the following.

**Corollary 1.5.** *Let $\Lambda$ be a finite dimensional algebra. Then there is a bijection between the set of ordered support $\tau$-tilting objects in $C(\Lambda)$ and the set of complete signed $\tau$-exceptional sequences in $C(\Lambda)$.*

The following crucial result of [1], provides each $\tau$-rigid module $U$ with two support $\tau$-tilting objects having $U$ as a direct summand.

**Theorem 1.6.** [1, Section 2] *Let $U$ be a $\tau$-rigid module.*

(a) *Up to isomorphism, there is a unique basic module $B[U]$ such that $B[U] \amalg U$ is a $\tau$-tilting module and* $\operatorname{add}(B[U] \amalg U) = \mathcal{P}({}^\perp\tau U)$*. Moreover, we have ${}^\perp\tau U = {}^\perp\tau(U \amalg B[U]) = \operatorname{Gen}(U \amalg B[U])$, and the first equality characterizes $B[U]$.*

(b) *Up to isomorphism, there is a unique basic module* $C[U]$ *and a basic projective module* $Q$ *such that* $C[U] \amalg U \amalg Q[1]$ *is a support* $\tau$*-tilting object and* $\mathrm{add}(C[U] \amalg U) = \mathcal{P}(\mathrm{Gen}\, U)$. *In particular, we have* $\mathrm{add}\, Q = \mathcal{P}(\Lambda) \cap {}^{\perp}U$.

The modules $B[U]$ and $C[U]$ in Theorem 1.6 are known as the Bongartz and co-Bongartz complements of $U$. An important step towards our main theorem the construction of explicit bijections between the indecomposable direct summands of $B[U]$ and the indecomposable direct summands of $C[U] \amalg Q[1]$.

## 2. 2-term rigid objects

In this section we discuss 2-term silting objects in the derived category and their links to $\tau$-rigid objects in the module category.

We denote by $H^n$ the functor from $D^b(\mathrm{mod}\,\Lambda)$ which maps a complex to its $n$th homology. We regard the bounded homotopy category of projectives $\mathcal{K} = K^b(\mathcal{P}(\Lambda))$ as a full subcategory of $D^b(\mathrm{mod}\,\Lambda)$. An object $\mathbb{U}$ in $\mathcal{K}$ is said to be *rigid* if $\mathrm{Hom}(\mathbb{U}, \mathbb{U}[i]) = 0$ for all $i > 0$, and *silting* if in addition it generates $\mathcal{K}$ as a triangulated category (i.e. $\mathbb{U}$ is not contained in any proper triangulated subcategory of $\mathcal{K}$).

An object of the form

$$\cdots \to 0 \to P^{-1} \to P^0 \to 0 \to \cdots$$

in $\mathcal{K}$ is called a 2-*term object*. A 2-term object in $\mathcal{K}$ which is rigid (respectively, silting) is called 2-*term rigid* (respectively, 2-*term silting*).

For a module $U$, we let $\mathbb{P}_U$ denote the minimal projective presentation of $U$, considered as a 2-term object in $\mathcal{K}$. The following two lemmas are well-known.

**Lemma 2.1.** (a) *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be 2-term objects in* $\mathcal{K}$. *Then* $H^0$ *induces an epimorphism* $\mathrm{Hom}_{\mathcal{K}}(\mathbb{X}, \mathbb{Y}) \to \mathrm{Hom}(H^0(\mathbb{X}), H^0(\mathbb{Y}))$ *with kernel consisting of the maps factoring through* $\mathrm{add}\,\Lambda[1]$.

(b) *Let* $X, Y, Z$ *be in* $\mathrm{mod}\,\Lambda$. *If any map in* $\mathcal{K}$ *from* $\mathbb{P}_X$ *to* $\mathbb{P}_Z$ *factors through* $\mathbb{P}_Y$, *then any map from* $X$ *to* $Z$ *factors through* $Y$.

*Proof.* Part (a) is straightforward, and part (b) is a direct consequence of part (a). □

**Lemma 2.2.** *Let* $X$ *be in* $\mathrm{mod}\,\Lambda$. *Then* $\mathbb{P}_X$ *is indecomposable in* $\mathcal{K}$ *if and only if* $X$ *is an indecomposable module.*

*Proof.* Straightforward. □

We also need the following facts from [1].

**Lemma 2.3.** [1, Section 3] *Let* $U, X$ *be in* $\mathrm{mod}\,\Lambda$.

(a) $\mathrm{Hom}(U, \tau X) = 0$ *if and only if* $\mathrm{Hom}_{\mathcal{K}}(\mathbb{P}_X, \mathbb{P}_U[1]) = 0$. *In particular, the module* $U$ *is* $\tau$*-rigid if and only if* $\mathbb{P}_U$ *is rigid.*

(b) *If* $U \amalg P[1]$ *is a support* $\tau$*-rigid object in* $\mathcal{C}(\Lambda)$, *then* $\mathbb{P}_U \amalg P[1]$ *is a* 2*-term rigid object in* $\mathcal{K}$, *and it is* 2*-term silting if and only if* $U \amalg P[1]$ *is support* $\tau$*-tilting.*

(c) *If* $\mathbb{U}$ *is* 2*-term rigid in* $\mathcal{K}$, *then* $H^0(\mathbb{U})$ *is* $\tau$*-rigid, and it is support* $\tau$*-tilting if* $\mathbb{U}$ *is* 2*-term silting.*

(d) *The constructions of (b) and (c) give a bijection between* 2*-term silting objects in* $\mathcal{K}$ *and support* $\tau$*-tilting objects in* $\mathcal{C}(\Lambda)$.

Note that by Lemma 2.3 any 2-term silting object in $\mathcal{K}$ is, up to homotopy, of the form $\mathbb{P}_X \amalg Q[1]$, where $\mathbb{P}_X$ is a minimal projective presentation for some $\Lambda$-module $X$. In particular, this implies the following.

**Lemma 2.4.** *Let $\mathbb{U}$ be a 2-term silting object in $\mathcal{K}$. Then $H^0(\mathbb{U})$ is $\tau$-tilting if and only if $H^{-1}(\mathbb{U})$ has no projective direct summand.*

Parts (b) to (e) in the following Lemma are essentially contained in [2, Section 2], but we include proofs for convenience.

**Lemma 2.5.** *Let $\mathbb{U}$ and $\mathbb{X}$ be 2-term objects in $\mathcal{K}$ satisfying*

(i) $\mathbb{U} \amalg \mathbb{X}$ *is rigid;*
(ii) $\operatorname{add}\mathbb{U} \cap \operatorname{add}\mathbb{X} = 0$.

*Let*

(1) $$\mathbb{Y} \xrightarrow{\beta} \mathbb{U}' \xrightarrow{\alpha} \mathbb{X} \to$$

*be a triangle in which $\alpha$ is a minimal right* $\operatorname{add}\mathbb{U}$*-approximation. Then the following hold.*

(a) *The object $\mathbb{Y}$ is a 2-term object if and only if the induced map $H^0(\alpha)\colon H^0(\mathbb{U}') \to H^0(\mathbb{X})$ is an epimorphism.*
(b) *The object $\mathbb{Y} \amalg \mathbb{U}$ is rigid.*
(c) $\operatorname{add}\mathbb{U} \cap \operatorname{add}\mathbb{Y} = 0$.
(d) *The morphism $\mathbb{Y} \xrightarrow{\beta} \mathbb{U}'$ is a minimal left* $\operatorname{add}\mathbb{U}$*-approximation.*
(e) *The object $\mathbb{Y} \amalg \mathbb{U}$ is silting if and only if $\mathbb{X} \amalg \mathbb{U}$ is silting.*
(f) *If $\mathbb{X} \amalg \mathbb{U}$ is 2-term silting and $H^0(\alpha)$ is an epimorphism, then $\mathbb{Y} \amalg \mathbb{U}$ is 2-term silting.*

*Proof.* Note that we can assume that any differential appearing in an object in $\mathcal{K}$ is radical, i.e. that its image is contained in the radical of its target. Let $\operatorname{cone}(\alpha)$ be the mapping cone of $\alpha$, which has the form:

$$\cdots \to 0 \to U^{-1} \to U^0 \amalg X^{-1} \to X^0 \to 0 \to \cdots$$

Then $\mathbb{Y} \simeq \operatorname{cone}(\alpha)[-1]$. Clearly $\operatorname{cone}(\alpha)[-1]$ is 2-term if and only if the map $U^0 \amalg X^{-1} \to X^0$ is an epimorphism, which is equivalent to $U^0 \to X^0$ being a (split) epimorphism, since the map $X^{-1} \to X^0$ is radical. This holds if and only if the induced map $H^0(\alpha)\colon H^0(\mathbb{U}') \to H^0(\mathbb{X})$ is an epimorphism, giving (a).

We now prove (b). We apply $\operatorname{Hom}(\mathbb{U}, \ )$ to (1), obtaining the exact sequences

$$\operatorname{Hom}(\mathbb{U}, \mathbb{U}'[i]) \to \operatorname{Hom}(\mathbb{U}, \mathbb{X}[i]) \to \operatorname{Hom}(\mathbb{U}, \mathbb{Y}[i+1]) \to \operatorname{Hom}(\mathbb{U}, \mathbb{X}[i+1]),$$

for all $i$. Since $\alpha$ is a right $\operatorname{add}\mathbb{U}$-approximation, the map $\operatorname{Hom}(\mathbb{U}, \mathbb{U}') \to \operatorname{Hom}(\mathbb{U}, \mathbb{X})$ is an epimorphism. Hence $\operatorname{Hom}(\mathbb{U}, \mathbb{Y}[i]) = 0$ for all $i > 0$.

Applying $\operatorname{Hom}(\mathbb{X}, \ )$ to (1), and considering the exact sequence

$$\operatorname{Hom}(\mathbb{X}, \mathbb{X}[i-1]) \to \operatorname{Hom}(\mathbb{X}, \mathbb{Y}[i]) \to \operatorname{Hom}(\mathbb{X}, \mathbb{U}[i])$$

gives $\operatorname{Hom}(\mathbb{X}, \mathbb{Y}[i]) = 0$ for all $i > 1$.

Applying $\operatorname{Hom}( \ , \mathbb{Y})$ to (1) gives an exact sequence

$$\operatorname{Hom}(\mathbb{U}', \mathbb{Y}[i]) \to \operatorname{Hom}(\mathbb{Y}, \mathbb{Y}[i]) \to \operatorname{Hom}(\mathbb{X}, \mathbb{Y}[i+1]),$$

and hence $\operatorname{Hom}(\mathbb{Y}, \mathbb{Y}[i]) =$ for all $i > 0$.

Finally, applying $\operatorname{Hom}( \ , \mathbb{U})$ to (1) gives an exact sequence

$$\operatorname{Hom}(\mathbb{U}', \mathbb{U}[i]) \to \operatorname{Hom}(\mathbb{Y}, \mathbb{U}[i]) \to \operatorname{Hom}(\mathbb{X}, \mathbb{U}[i+1]),$$

and hence $\operatorname{Hom}(\mathbb{Y}, \mathbb{U}[i]) =$ for all $i > 0$. This finishes the proof of (b).

Part (c) follows directly from the minimality of $\alpha$.

For part (d) consider the exact sequence

$$\operatorname{Hom}(\mathbb{U}', \mathbb{U}) \to \operatorname{Hom}(\mathbb{Y}, \mathbb{U}) \to \operatorname{Hom}(\mathbb{X}, \mathbb{U}[1]),$$

which is part of the long exact sequence obtained by applying $\operatorname{Hom}( \ , \mathbb{U})$ to (1). Since the last term vanishes, the first map is an epimorphism, and hence the map $\mathbb{Y} \xrightarrow{\beta} \mathbb{U}'$ is a left $\operatorname{add} \mathbb{U}$-approximation. It is minimal since $\operatorname{add} \mathbb{U} \cap \operatorname{add} \mathbb{X} = 0$.

Part (e) follows from part (b) and the existence of the triangle (1), and part (f) is a direct consequence of parts (a) and (e). □

## 3. Exchange

Let $U$ be a $\tau$-rigid $\Lambda$-module. Recall from Theorem 1.6 the notation $B[U]$ for the Bongartz complement of $U$ and $C[U] \amalg Q[1]$ for the co-Bongartz complement of $U$ in $\mathcal{C}(\Lambda)$. We will denote $C[U]$ by $C$ in the sequel. The aim of this section is to give an explicit bijection between the indecomposable direct summands in these two complements of $U$.

**Remark 3.1.** *Let $\mathbb{C}_Q = \mathbb{P}_C \amalg Q[1]$ be the 2-term rigid object in $\mathcal{K}$ corresponding to the support $\tau$-rigid object $C \amalg Q[1]$ in $\mathcal{C}(\Lambda)$. By Lemma 2.3, we have that $\mathbb{C}_Q \amalg \mathbb{P}_U$ is a 2-term silting object.*

**Lemma 3.2.** *Let $U$ be a $\tau$-rigid module, and consider the support $\tau$-tilting object $C \amalg U \amalg Q[1]$ in $\mathcal{C}(\Lambda)$, where $C \amalg Q[1]$ is the co-Bongartz complement of $U$. Let $\mathbb{C}_Q = \mathbb{P}_C \amalg Q[1]$ be the corresponding 2-term rigid object in $\mathcal{K}$. Let $\alpha\colon \mathbb{P}'_U \to \mathbb{C}_Q$ be a minimal right* $\operatorname{add} \mathbb{P}_U$*-approximation of $\mathbb{C}_Q$, and complete it to a triangle:*

$$\mathbb{Y} \xrightarrow{\beta} \mathbb{P}'_U \xrightarrow{\alpha} \mathbb{C}_Q \to$$

*in $\mathcal{K}$. Then the following hold.*

(a) *The map $H^0(\alpha)$ is a minimal right* $\operatorname{add} U$*-approximation of $H^0(\mathbb{C}_Q) = C$.*
(b) *The object $\mathbb{Y}$ is 2-term.*
(c) *The object $\mathbb{Y} \amalg \mathbb{P}_U$ is 2-term silting.*
(d) *Let $B = H^0(\mathbb{Y})$. Then $B \amalg U$ is $\tau$-tilting, and $\mathbb{Y} = \mathbb{P}_B$.*
(e) *The map $\beta\colon \mathbb{Y} \to \mathbb{P}'_U$ is a minimal left* $\operatorname{add} \mathbb{P}_U$*-approximation.*

*Proof.* To prove (a), consider a map $\beta\colon U'' \to C$, with $U''$ in $\operatorname{add} U$. Let $\widetilde{\beta}\colon \mathbb{P}_{U''} \to \mathbb{P}_C$ be a map such that $H^0\left(\widetilde{\beta}\right) = \beta$. Consider the map $\gamma = \begin{pmatrix} \widetilde{\beta} \\ 0 \end{pmatrix} : \mathbb{P}_{U''} \to \mathbb{P}_C \amalg Q[1] = \mathbb{C}_Q$. Since $\alpha$ is a right $\operatorname{add} \mathbb{P}_U$-approximation, the map $\gamma$ factors through $\alpha$. But then $\beta = H^0(\gamma)$ factors through $H^0(\alpha)$. Clearly, the minimality of $\alpha$ implies that $H^0(\alpha)$ is minimal, giving part (a).

Since $C$ lies in $\operatorname{Gen} U$, the map $H^0(\alpha)$ is an epimorphism, and part (b) then follows from Lemma 2.5(a).

Since $\mathbb{P}_U \amalg \mathbb{C}_Q$ is 2-term silting by Remark 3.1, part (c) now follows from Lemma 2.5(f).

We now prove part (d). To show that $B \amalg U$ is a $\tau$-tilting module, by Lemma 2.4 it is enough to prove that $H^{-1}(\mathbb{Y})$ has no projective direct summands. We have that $\mathbb{Y}[1]$ is homotopic to the mapping cone of $\alpha$. This mapping cone is

$$\cdots \to 0 \to P^{-1}_{U'} \to P^0_{U'} \amalg P^{-1}_C \amalg Q \to P^0_C \to 0 \to \cdots$$

where $U' = H^0(\mathbb{P}'_U)$ is in $\operatorname{add} U$. By minimality of the map $P^{-1}_{U'} \to P^0_{U'}$, it is clear that $H^{-1}(\mathbb{Y})$ has no projective direct summand. This proves that $B \amalg U$ is $\tau$-tilting and also that $\mathbb{Y} = \mathbb{P}_B$.

Part (e) follows directly from Lemma 2.5(d). □

Remark on the proof of part (d): Since the induced map $U' \to C$ is an epimorphism, the map between the projective covers $P^0_{U'} \to P^0_C$ is a (split) epimorphism. Hence the mapping cone of $\alpha$ is actually homotopic to a complex

$$\cdots \to 0 \to P^{-1}_{U'} \to W \amalg P^{-1}_C \amalg Q \to 0 \to \cdots$$

where $W \amalg P^0_C \simeq P^0_{U'}$.

By Lemma 3.2(d), we can write the triangle in the statement of the lemma as:

$$\mathbb{P}_B \xrightarrow{\beta} \mathbb{P}'_U \xrightarrow{\alpha} \mathbb{C}_Q \to \tag{2}$$

Let $\mathbb{C}_Q = \amalg_i \mathbb{X}_i$ be a decomposition of $\mathbb{C}_Q$ into indecomposable direct summands. For each $i$, consider a minimal right $\operatorname{add}\mathbb{P}_U$-approximation $(\mathbb{P}'_U)_i \xrightarrow{\alpha_i} \mathbb{X}_i$. It is easy to check that $\amalg_i \alpha_i$ is a minimal right $\operatorname{add}\mathbb{P}_U$-approximation of $\mathbb{C}_Q$, and hence we may assume that $\alpha = \amalg_i \alpha_i$. Hence, we obtain for each $i$ a triangle

$$\mathbb{P}_{B_i} \xrightarrow{\beta_i} (\mathbb{P}'_U)_i \xrightarrow{\alpha_i} \mathbb{X}_i \to \tag{3}$$

obtained by completing a minimal right $\operatorname{add}\mathbb{P}_U$-approximation $\alpha_i$ of $\mathbb{X}_i$ to a triangle. We now have $\beta = \amalg_i \beta_i$, $B = \amalg_i B_i$ and $\mathbb{P}_B = \amalg_i \mathbb{P}_{B_i}$.

**Lemma 3.3.** *With notation as above, the map $\mathbb{X}_i \mapsto \mathbb{P}_{B_i}$ is a bijection between the indecomposable direct summands of $\mathbb{C}_Q$ and the indecomposable direct summands of $\mathbb{P}_B$.*

*Proof.* By Lemma 3.2 and the above discussion, each map $\mathbb{P}_{B_i} \xrightarrow{\beta_i} (\mathbb{P}'_U)_i$ is a minimal left $\operatorname{add}\mathbb{P}_U$-approximation. □

In particular we now have that the $B_i$ are indecomposable by Lemma 2.2.

**Lemma 3.4.** *The $\Lambda$-module $B$ is the Bongartz complement $B[U]$ of $U$; in particular $B$ is basic.*

*Proof.* We first prove that $B$ is basic. Suppose that $B_i \simeq B_j$ for some $i \neq j$. Then also $\mathbb{P}_{B_i} \simeq \mathbb{P}_{B_j}$, and, by Lemma 3.3, we have $\mathbb{X}_i \simeq \mathbb{X}_j$. But $\mathbb{C}_Q$ is basic, since by construction both $C$ and $Q$ are basic (see Theorem 1.6(b) and the remark afterwards). Hence $i = j$, and therefore $B$ is basic.

By Theorem 1.6(a), the Bongartz complement $B[U]$ of $U$ is characterized by the property that ${}^\perp\tau(B[U] \amalg U) = {}^\perp\tau U$. It is therefore sufficient to prove that ${}^\perp\tau U \subset {}^\perp\tau B$. That is, we need to prove that $\mathrm{Hom}(X, \tau U) = 0$ implies that $\mathrm{Hom}(X, \tau B) = 0$. By Lemma 2.3, this is equivalent to proving that $\mathrm{Hom}_{\mathcal{K}}(\mathbb{P}_U, \mathbb{P}_X[1]) = 0$ implies that $\mathrm{Hom}_{\mathcal{K}}(\mathbb{P}_B, \mathbb{P}_X[1]) = 0$.

For this, consider part of the long exact sequence obtained by applying $\mathrm{Hom}_{\mathcal{K}}( \ , \mathbb{P}_X[1])$ to the triangle (2):

$$\mathrm{Hom}((\mathbb{P}'_U, \mathbb{P}_X[1]) \to \mathrm{Hom}(\mathbb{P}_B, \mathbb{P}_X[1]) \to \mathrm{Hom}(\mathbb{C}_Q, \mathbb{P}_X[2])$$

The last term vanishes since the complexes $\mathbb{C}_Q$ and $\mathbb{P}_X$ are both 2-term. Hence $\mathrm{Hom}_{\mathcal{K}}(\mathbb{P}_U, \mathbb{P}_X[1]) = 0$ implies that $\mathrm{Hom}_{\mathcal{K}}(\mathbb{P}_B, \mathbb{P}_X[1]) = 0$, as required.

This proves that ${}^{\perp}\tau U \subset {}^{\perp}\tau B$, and hence that ${}^{\perp}\tau(B \amalg U) = {}^{\perp}\tau U$, which implies that $B = B[U]$. □

Recall that $\mathbb{C}_Q = \mathbb{P}_C \amalg Q[1]$, where $C = C[U]$ is the co-Bongartz complement of $U$. We now focus on the indecomposable direct summands $Q'[1]$ of $\mathbb{C}_Q$, where $Q'$ is an indecomposable direct summand of $Q$.

**Lemma 3.5.** *Let $\mathbb{X}_i$ be an indecomposable direct summand of $\mathbb{C}_Q$ of the form $Q'[1]$, where $Q'$ is an indecomposable direct summand of $Q$. Consider triangle (3). The induced map $H^{-1}(\mathbb{X}_i) \to H^0(\mathbb{P}_{B_i})$, that is $Q' \to B_i$, is a minimal left ${}^{\perp}\tau U$-approximation, and hence also a left $\mathcal{P}({}^{\perp}\tau U)$-approximation.*

*Proof.* Let $M$ be in ${}^{\perp}\tau U$. Then, by Lemma 2.3, we have $\mathrm{Hom}_{\mathcal{K}}(\mathbb{P}_U, \mathbb{P}_M[1]) = 0$. Apply $\mathrm{Hom}_{\mathcal{K}}( \ , \mathbb{P}_M)$ to the triangle (3) and consider the exact sequence

$$\mathrm{Hom}(\mathbb{P}_{B_i}, \mathbb{P}_M) \to \mathrm{Hom}(\mathbb{X}_i, \mathbb{P}_M[1]) \to \mathrm{Hom}((\mathbb{P}'_U)_i, \mathbb{P}_M[1])$$

Since the last term vanishes, every map $\mathbb{X}_i[-1] \to \mathbb{P}_M$ factors through $\mathbb{X}_i[-1] \to \mathbb{P}_{B_i}$. This means that every map from $Q$ (regarded as a complex concentrated in degree 0) to $\mathbb{P}_M$ factors through $\mathbb{P}_{B_i}$. Hence, by Lemma 2.1, the map $Q' \to B_i$ is a left ${}^{\perp}\tau U$-approximation. This map is non-zero, since ${}^{\perp}\tau U$ is sincere by [1, Theorem 2.10]. It is therefore minimal, since $B_i$ is indecomposable. Since $B_i$ is in $\mathcal{P}({}^{\perp}\tau U)$ by Lemma 3.4, the last statement also follows. □

Taking homology, the triangle (2) induces an exact sequence

$$H^{-1}(\mathbb{C}_Q) \to H^0(\mathbb{P}_B) \to H^0(\mathbb{P}'_U) \to H^0(\mathbb{C}_Q) \to 0.$$

Note that $C = H^0(\mathbb{C}_Q)$ is the co-Bongartz complement of $U$ and $B = H^0(\mathbb{B})$ is the Bongartz complement of $U$. Let $U' = H^0(\mathbb{P}'_U)$ and $H^{-1}(\mathbb{C}_Q) = \widetilde{Q}$, where $Q$ is a direct summand of $\widetilde{Q}$. The sequence above becomes:

(4) $$\widetilde{Q} \xrightarrow{\delta} B \xrightarrow{\mu} U' \xrightarrow{\gamma} C \to 0$$

Since the triangles (3) sum to the triangle (2), we also have that the sequence (4) is a direct sum of $n$ exact sequences

(5) $$Q_i \xrightarrow{\delta_i} B_i \xrightarrow{\mu_i} U'_i \xrightarrow{\gamma_i} C_i \to 0$$

where for each $i$ either

Case (i) $Q_i$ is an indecomposable projective direct summand of $Q$, the map $\mu_i$ is an epimorphism and $C_i = 0$ (this happens when $\mathbb{X}_i$ is the shift of an indecomposable projective direct summand of $Q$), or

Case (ii) $C_i$ is non-zero and is an indecomposable direct summand of $C$ (this happens when $\mathbb{X}_i$ is the minimal projective presentation of a summand $C_i$ of $C$).

**Lemma 3.6.** *With notation as above, consider the exact sequence (5). Then we have the following:*

(a) *The map $\mu_i$ is a minimal left* add $U$*-approximation.*
(b) *In case (i), the map $\gamma_i$ is the zero map and $\delta_i$ is a minimal left $\mathcal{P}({}^{\perp}\tau U)$-approximation, while in case (ii), the map $\gamma_i$ is a minimal right* add $U$*-approximation.*

*Proof.* Using the fact that each $\beta_i$ in (3) is a minimal left add $\mathbb{P}_U$-approximation, in combination with Lemma 2.1(b), it follows that each $\mu_i$ is a left add $U$-approximation. Minimality follows from the fact that add $U \cap$ add $C = 0$. This proves (a).

For (b), note that in case (i), the map $\gamma_i$ must be zero as $C_i = 0$. The fact that $\delta_i$ is a minimal left $\mathcal{P}({}^{\perp}\tau U)$-approximation follows from Lemma 3.5. In case (ii), the fact that $\gamma_i$ is a minimal right add $U$-approximation follows from Lemma 3.2. □

Let us summarize our findings.

**Proposition 3.7.** *Let $U$ be a $\tau$-rigid module. Let $B$ be the Bongartz complement of $U$ and $C$ the co-Bongartz complement of $U$, with corresponding support $\tau$-tilting object $C \amalg U \amalg Q[1]$ such that* $\mathrm{add}(C \amalg U) = \mathcal{P}(\mathrm{Gen}\, U)$*, as in Theorem 1.6.*

*Then there is a triangle*

$$\mathbb{P}_B \xrightarrow{\beta} \mathbb{P}'_U \xrightarrow{\alpha} \mathbb{C}_Q \to, \tag{6}$$

*where $\mathbb{C}_Q = \mathbb{P}_C \amalg Q[1]$ and $\beta$ (respectively, $\alpha$) is a minimal left (respectively, right)* add $\mathbb{P}_U$*-approximation. This triangle is the direct sum of $n$ triangles*

$$\mathbb{P}_{B_i} \xrightarrow{\beta_i} (\mathbb{P}'_U)_i \xrightarrow{\alpha_i} \mathbb{X}_i \to, \tag{7}$$

*where $B = \amalg_i B_i$ is a decomposition of $B$ into a direct sum of indecomposable modules and the $\mathbb{X}_i$ are the indecomposable direct summands of $\mathbb{C}_Q$.*

*Let $B = B' \amalg B''$, where $B'$ is the direct sum of all indecomposable direct summands of $B$ with the property that the minimal left* add $U$*-approximation $B_i \to U_i$ is not an epimorphism, and $B''$ is the complement of $B'$ in $B$.*

(a) *For each indecomposable direct summand $B_i$ of $B$ which is a summand of $B'$, there is an exact sequence*

$$B_i \xrightarrow{\mu_i} U'_i \xrightarrow{\gamma_i} C_i \to 0,$$

*where $\mu_i$ (respectively, $\gamma_i$) is a minimal left (respectively, right)* add $U$*-approximation and $C_i$ is an indecomposable direct summand of $C$. This arises from part of the long exact sequence associated to* (7)*:*

$$H^0(\mathbb{P}_{B_i}) \to H^0((\mathbb{P}'_U)_i) \to H^0(\mathbb{X}_i) \to H^1(\mathbb{P}_{B_i}) = 0,$$

*where $\mathbb{X}_i = \mathbb{P}_{C_i}$. The map $B_i \mapsto C_i$ is a correspondence between the indecomposable direct summands of $B'$ and the indecomposable direct summands of $C$.*

(b) *For each indecomposable direct summand $B_i$ of $B$ which is a summand of $B''$, there is an exact sequence*

$$Q_i \xrightarrow{\delta_i} B_i \xrightarrow{\mu_i} U'_i \to 0,$$

*where $Q_i$ is an indecomposable direct summand of $Q$, with $U'_i \in \operatorname{add} U$ and with $\delta_i\colon Q_i \to B_i$ a minimal left $\mathcal{P}({}^{\perp}\tau U) = \operatorname{add} B$-approximation. This arises from part of the long exact sequence associated to* (7)*:*

$$H^{-1}(\mathbb{X}_i) \to H^0(\mathbb{P}_{B_i}) \to H^0((\mathbb{P}'_U)_i) \to H^0(\mathbb{X}_i) = 0,$$

*where $\mathbb{X}_i = Q_i[1]$. The map $B_i \mapsto Q_i$ is a bijection between the indecomposable direct summands of $B''$ and the indecomposable direct summands of $Q$.*

(c) *There is a bijection between the indecomposable direct summands of $B$ and the indecomposable direct summands of the support $\tau$-rigid object $C \amalg Q[1]$ in $\mathcal{C}(\Lambda)$.*

*Proof.* Parts (a) and (b) follow from Lemmas 3.3, 3.4, 3.6 and the above discussion. Part (c) is a direct consequence of parts (a) and (b). □

We recall the following version of Wakamatsu's lemma from [1].

**Lemma 3.8.** [1, Lemma 2.6] *Let $U$ be a $\tau$-rigid module and let $\alpha\colon U' \to X$ a right* $\operatorname{add} U$*-approximation. Then* $\ker\alpha$ *lies in ${}^{\perp}(\tau U)$.*

Later we will need the following stronger version of Lemma 3.6(a), which is due to [1].

**Lemma 3.9.** [1, Lemma 2.20] *The map $\mu_i$ of Lemma 3.6 is a minimal left* $\operatorname{Gen} U$*-approximation.*

*Proof.* The proof is essentially contained in the proof of Lemma 2.20 of [1], but we give the details for convenience. For an object $V$ in $\operatorname{Gen} U$ there is a short exact sequence

$$0 \to Z \to U' \to V \to 0, \tag{8}$$

where $U' \to V$ is a right $\operatorname{add} U$-approximation, and $Z$ lies in ${}^{\perp}\tau U$ by Lemma 3.8. Since ${}^{\perp}\tau U \subseteq {}^{\perp}\tau B$, we have $\operatorname{Hom}(Z, \tau B) = 0$, and hence $\operatorname{Ext}^1(B, Z) = 0$, by the Auslander-Reiten formula. Applying $\operatorname{Hom}(B_i, \ )$ to the exact sequence (8) we get an exact sequence

$$\operatorname{Hom}(B_i, U') \to \operatorname{Hom}(B_i, V) \to \operatorname{Ext}^1(B_i, Z).$$

Since the last term vanishes, the first map is an epimorphism.

Consider an arbitrary map $B_i \to V$. By the above, it factors $B_i \to U' \to V$. But the map $\mu_i\colon B_i \to U_i$ is a left $\operatorname{add} U$-approximation by Lemma 3.6(a), and so $B_i \to U'$ factors $B_i \to U_i \to U'$, and hence $B_i \to V$ also factors through $\mu_i$. This concludes the proof. □

## 4. Reduction

We fix a $\tau$-rigid $\Lambda$-module $U$ throughout this section. Recall that a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\operatorname{mod}\Lambda$ is called a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. For a given torsion pair $(\mathcal{T}, \mathcal{F})$ and an arbitrary module $X$, there is a (unique up to isomorphism) exact sequence

$$0 \to t(X) \to X \to f(X) \to 0$$

with the property that $t(X)$ is in $\mathcal{T}$ and $f(X)$ is in $\mathcal{F}$. It is known as the *canonical sequence* for $X$.

**Lemma 4.1.** [3, Theorem 5.8] *The pair* $(\operatorname{Gen} U, U^{\perp})$ *is a torsion pair in* $\operatorname{mod} \Lambda$.

From now on we only consider this torsion pair, and use the notation $t$ and $f$ relative to this pair.

We next recall some results and notions, mostly from [16]. Let $J(U) = U^{\perp} \cap {}^{\perp}\tau U$. The following summarizes some important facts about $J(U)$.

**Proposition 4.2.** *Let $B$ be the Bongartz complement of $U$. Then we have:*

(a) [16, Theorem 3.8] *The subcategory $J(U)$ is equivalent to* $\operatorname{mod} \operatorname{End}(B \amalg U)/I$*, where $I$ is the ideal generated by all maps factoring through $U$.*
(b) [11, Theorem 3.28] *The subcategory $J(U)$ is an exact abelian (wide) subcategory of* $\operatorname{mod} \Lambda$.
(c) [16, Theorem 3.8] *If $U$ is indecomposable, then $J(U)$ has $n-1$ simple modules up to isomorphism.*

The following result of Auslander and Smalø is very useful.

**Lemma 4.3.** [3, Theorem 5.10] *For $X, Y$ in* $\operatorname{mod} \Lambda$ *we have* $\operatorname{Hom}(X, \tau Y) = 0$ *if and only if* $\operatorname{Ext}^1(Y, \operatorname{Gen} X) = 0$

**Lemma 4.4.** (a) *If $X$ is in* $\operatorname{Gen} U$*, then $X$ is in* $\mathcal{P}(\operatorname{Gen} U)$ *if and only if $X \amalg U$ is $\tau$-rigid.*
(b) *If $X \amalg U$ is $\tau$-rigid, then* $\operatorname{Ext}^1(X, t(Z)) = 0$ *for any module $Z$.*

*Proof.* We have (see Theorem 1.6) that $\mathcal{P}(\operatorname{Gen} U) = \operatorname{add}(C \amalg U)$, where $C \amalg U$ is $\tau$-rigid. Hence, if $X$ is in $\mathcal{P}(\operatorname{Gen} U)$, then $X \amalg U$ is $\tau$-rigid.

Conversely, assume $X \amalg U$ is $\tau$-rigid. Then it follows from Lemma 4.3 that $X$ lies in $\mathcal{P}(\operatorname{Gen} U)$. This proves (a), and (b) is a direct consequence of (a). □

Our next step towards the main result, is the following bijection.

**Proposition 4.5.** *For a $\tau$-rigid module $U$, and with notation as before, the map $f$ induces a bijection:*

$$\text{Objects } X \text{ in } \operatorname{ind} \operatorname{mod} \Lambda \text{ such that } X \amalg U \text{ is } \tau\text{-rigid and } X \text{ is not in } \operatorname{Gen} U$$
$$\updownarrow$$
$$\text{Objects in } \operatorname{ind} J(U) \text{ which are } \tau\text{-rigid in } J(U)$$

In order the prove Proposition 4.5, we will need the following lemmas.

**Lemma 4.6.** *If $X$ is an indecomposable $\Lambda$-module such that $X \amalg U$ is $\tau$-rigid, then either $f(X)$ is indecomposable or $f(X) = 0$. We have $f(X) = 0$ if and only if $X$ is in* $\operatorname{Gen} U$.

*Proof.* Note that $f(X) = 0$ if and only if $X$ in in $\operatorname{Gen} U$, by the definition of $f$. For the rest of the statement, it is sufficient to prove that there is a surjective ring map $\operatorname{End}(X) \to \operatorname{End}(f(X))$. The (well-known) functoriality of $f$ gives a map from $\operatorname{End}(X)$ to $\operatorname{End}(f(X))$; we recall the construction now.

Let $\phi$ be in $\operatorname{End}(X)$ and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t(X) & \xrightarrow{\eta_X} & X & \xrightarrow{\nu_X} & f(X) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \phi} & & & & \\ 0 & \longrightarrow & t(X) & \xrightarrow{\eta_X} & X & \xrightarrow{\nu_X} & f(X) & \longrightarrow & 0 \end{array}$$

Since $\operatorname{Hom}(t(X), f(X)) = 0$, there is a map $\theta\colon f(X) \to f(X)$ such that $u_X\phi = \theta u_X$. Since $\nu_X$ is an epimorphism, there is a unique map $\theta$ with this property, so this gives a well defined map $\operatorname{End}(X) \to \operatorname{End}(f(X))$. It is then easy to check that this map is a ring map. We claim that it is surjective.

Consider part of the long exact sequence

$$\operatorname{Hom}(X, X) \to \operatorname{Hom}(X, f(X)) \to \operatorname{Ext}^1(X, t(X))$$

obtained by applying $\operatorname{Hom}(X, \ )$ to the canonical sequence of $X$. We have that $\operatorname{Ext}^1(X, t(X)) = 0$ by Lemma 4.4. Hence the map $\operatorname{Hom}(X, X) \to \operatorname{Hom}(X, f(X))$ is surjective. Furthermore, applying $\operatorname{Hom}(\ , f(X))$ to the canonical sequence gives that $\operatorname{Hom}(f(X), f(X)) \simeq \operatorname{Hom}(X, f(X))$, since $\operatorname{Hom}(t(X), f(X)) = 0$. The claim follows. □

**Lemma 4.7.** *Let $X, Y$ be indecomposable modules not in* $\operatorname{Gen} U$*, and such that both $U \amalg X$ and $U \amalg Y$ are $\tau$-rigid. Then $f(X) \simeq f(Y)$ implies that $X \simeq Y$.*

*Proof.* Fix an isomorphism $\phi\colon f(X) \to f(Y)$, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t(X) & \longrightarrow & X & \xrightarrow{u_X} & f(X) & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & t(Y) & \longrightarrow & Y & \xrightarrow{u_Y} & f(Y) & \longrightarrow & 0 \end{array}$$

where the rows are the canonical sequences for $X$ and $Y$. Consider part of the long exact sequence:

$$\operatorname{Hom}(X, Y) \xrightarrow{(X, u_Y)} \operatorname{Hom}(X, f(Y)) \to \operatorname{Ext}^1(X, t(Y)).$$

We have that $\operatorname{Ext}^1(X, t(Y)) = 0$ by Lemma 4.4, and hence the map

$$\operatorname{Hom}(X, Y) \xrightarrow{\operatorname{Hom}(X, u_Y)} \operatorname{Hom}(X, f(Y))$$

is surjective. Now choose a map $\psi\colon X \to Y$ satisfying $u_Y\psi = \phi u_X$. By a symmetric argument, we can also choose a map $\psi'\colon Y \to X$ such that $u_X\psi' = \phi^{-1}u_Y$.

Now consider the composition $\psi'\psi$ in $\operatorname{End}(X)$. By Fitting's Lemma, any endomorphism of an indecomposable finite length module is either invertible or nilpotent. For any positive integer $n$ we have $u_X(\psi'\psi)^n = (\phi^{-1}\phi)^n u_X = u_X \neq 0$. Note that $u_X$ is non-zero, since $X$ is not in $\operatorname{Gen} U$. Hence $\psi'\psi$ is not nilpotent and thus an automorphism. Therefore $\psi$ is an isomorphism, and this finishes the proof. □

**Lemma 4.8.** *Assume $U \amalg X$ is $\tau$-rigid in* $\operatorname{mod} \Lambda$*. Then either*

(i) *$X$ is in $\mathcal{P}(\operatorname{Gen} U)$ and $f(X) = 0$, or*
(ii) *$f(X)$ is $\tau$-rigid in $J(U)$.*

*Proof.* Clearly $f(X) = 0$ if and only if $X$ is in $\operatorname{Gen} U$.

If $X$ is in $\operatorname{Gen} U$ and $U \amalg X$ is $\tau$-rigid, then $X$ is in $\mathcal{P}(\operatorname{Gen} U)$ by Lemma 4.4.

So assume $f(X) \neq 0$. Consider the following exact sequence, obtained by applying $\operatorname{Hom}( \ , \operatorname{Gen} X \cap J(U))$ to the canonical sequence for $X$:

$$\operatorname{Hom}(t_U(X), \operatorname{Gen} X \cap J(U)) \to \operatorname{Ext}^1(f_U(X), \operatorname{Gen} X \cap J(U)) \to \operatorname{Ext}^1(X, \operatorname{Gen} X \cap J(U))$$

Since $\operatorname{Hom}(X, \tau X) = 0$ we have $\operatorname{Ext}^1(X, \operatorname{Gen} X) = 0$ by Lemma 4.3, so in particular $\operatorname{Ext}^1(X, \operatorname{Gen} X \cap J(U)) = 0$. We have $\operatorname{Hom}(t_U(X), \operatorname{Gen} X \cap J(U)) = 0$ since $t_U(X)$ is in $\operatorname{Gen} U$, and $\operatorname{Hom}(\operatorname{Gen} U, J(U)) = 0$, since $J(U) \subseteq U^{\perp}$. Therefore also $\operatorname{Ext}^1(f_U(X), \operatorname{Gen} X \cap J(U)) = 0$. Using Lemma 4.3 again, it is sufficient to prove that $\operatorname{Ext}^1(f_U(X), \operatorname{Gen}_{J(U)} f_U(X)) = 0$. We have $\operatorname{Gen} f_U(X) \subseteq \operatorname{Gen} X$, and hence $\operatorname{Gen}_{J(U)} f_U(X) \subseteq \operatorname{Gen} X \cap J(U)$. It follows that $\operatorname{Ext}^1(f_U(X), \operatorname{Gen} X \cap J(U)) = 0$. □

Note that this could also be seen by completing $U \amalg X$ to a $\tau$-tilting module (using [1], and then applying [16, Theorem 3.15]).

We can now complete the proof of the Proposition.

*Proof of Proposition 4.5.* It follows from Lemma 4.8 that $f$ sends objects $X$ in $\operatorname{ind} \operatorname{mod} \Lambda$ such that $X \amalg U$ is $\tau$-rigid and not in $\operatorname{Gen} U$ to objects in $\operatorname{ind} J(U)$ which are $\tau$-rigid in $J(U)$. By Lemma 4.6 $f$ sends indecomposable modules to indecomposable modules and it follows from Lemma 4.7 that $f$ induces an injective map.

It is a direct consequence of Theorem 3.15 in [16] that $f$ induces a surjective map. □

**Lemma 4.9.** *The map $X \mapsto f(X)$ induces a bijection between* $\operatorname{ind} \mathcal{P}({}^{\perp}\tau U) \setminus \operatorname{ind} U$ *and* $\operatorname{ind} \mathcal{P}(J(U))$.

*Proof.* By Lemma 4.6, indecomposables are preserved by $f$. The result follows from this fact and a special case of [16, Prop. 3.14]. □

**Definition 4.10.** *Consider the map* $\rho \colon \{\operatorname{ind} \mathcal{P}(\operatorname{Gen} U) \setminus \operatorname{ind}(U)\} \cup \{\operatorname{ind}(\mathcal{P}(\Lambda) \cap {}^{\perp}U)\} \to \operatorname{ind} \mathcal{P}(J(U))$ *defined as follows.*

*If $X$ is in* $\{\operatorname{ind} \mathcal{P}(\operatorname{Gen} U) \setminus \operatorname{ind}(U)\}$*, consider the triangle $\mathbb{R}_X \to \mathbb{P}_{U_X} \to \mathbb{P}_X \to$, where the right map is a minimal right* $\operatorname{add} \mathbb{P}_U$*-approximation (so $U_X$ lies in* $\operatorname{add} U$*), and let $\rho(X) = f(H^0(\mathbb{R}_X))$. Since $X$ is a direct summand of the co-Bongartz complement of $U$, it follows from Lemmas 3.2 and 3.4 that $\mathbb{R}_X = \mathbb{P}_{B_X}$ for an indecomposable direct summand $B_X$ of the Bongartz complement of $U$. By Proposition 3.7(a), taking $C_i = X$, there is an exact sequence*

$$B_X \xrightarrow{a_X} U_X \xrightarrow{b_X} X \to 0,$$

*where $B_X = H^0(\mathbb{R}_X)$, the map $a_X$ is a minimal left* $\operatorname{add} U$*-approximation and $b_X$ is a minimal right* $\operatorname{add} U$*-approximation.*

*If $X$ lies in* $\operatorname{ind}(\mathcal{P}(\Lambda) \cap {}^{\perp}U)$*, let $\rho(X) = f(B_X)$, where $a_X \colon X \to B_X$ is a minimal left $\mathcal{P}({}^{\perp}\tau U)$-approximation of $X$ and $B_X$ is an indecomposable direct summand of the Bongartz complement of $U$. Since $X$ is a direct summand of $Q$, where* $Q[1]$ *is as in Theorem 1.6, it follows from Proposition 3.7(b), taking $Q_i = X$, that there is an exact sequence*

$$X \xrightarrow{c_X} B_X \xrightarrow{d_X} U_X \to 0,$$

*where $d_X$ is a minimal left* $\operatorname{add} U$*-approximation.*

Now, combining Proposition 3.7 and Lemma 4.9, we obtain the following.

**Proposition 4.11.** *The map $\rho$ defines a bijection*

$$\{\operatorname{ind}\mathcal{P}(\operatorname{Gen} U) \setminus \operatorname{ind}(U)\} \cup \{\operatorname{ind}\mathcal{P}(\Lambda) \cap {}^{\perp}U\}$$
$$\updownarrow$$
$$\operatorname{ind}\mathcal{P}(J(U))$$

We end this section with a lemma which will be useful for the proof of the main result.

Recall that a module $Y_0$ in a full subcategory $\mathbb{Y}$ of $\operatorname{mod}\Lambda$ is called *split projective* in $\mathbb{Y}$, if each epimorphism $Y \to Y_0$ with $Y$ in $\mathbb{Y}$ is split.

**Lemma 4.12.** *Let $B$ be the Bongartz complement of $U$. Then each direct summand $B_i$ in $B$ is split projective in ${}^{\perp}\tau U$.*

*Proof.* Note that by assumption $\operatorname{Gen} T = {}^{\perp}\tau T$ and $\operatorname{add} T = \mathcal{P}({}^{\perp}\tau T)$. Assume $B_i$ is not split projective. Then there is an epimorphism $Q \to B_i$, with $Q$ in $\operatorname{add} B/B_i \amalg U$. Let $B_i'$ be the kernel of this epimorphism. Then, by Lemma 3.8, we have that $B_i'$ is in ${}^{\perp}(\tau(B/B_i \amalg U)) = {}^{\perp}\tau U$. But then the sequence $0 \to B_i' \to Q \to B_i \to 0$ splits, since $B_i$ is Ext-projective in ${}^{\perp}\tau U$. This is a contradiction. □

## 5. Main Theorem

In this section we complete the proof of Theorem 1.4, using the reduction technique of Section 4 and the bijection of Section 3.

Note that for a indecomposable projective module $P$, there is primitive idempotent $e$ in $\Lambda$, such that $P \simeq \Lambda e$, and we have that $J(P) = P^{\perp}$ is equivalent to $\operatorname{mod}(\Lambda/\Lambda e\Lambda)$.

**Lemma 5.1.** *If $(\mathcal{U}_1, \dots, \mathcal{U}_{t-1}, \mathcal{U}_t)$ in $\mathcal{C}(\Lambda)$ is a signed $\tau$-exceptional sequence, then $t \leq n$.*

*Proof.* This is clear when $n = 1$. The statement follows by induction on $n$, since, by Lemma 4.2, we have that $\delta(\Lambda') = \delta(\Lambda) - 1$, when $\operatorname{mod}\Lambda'$ is equivalent to $J(\mathcal{U}_t)$. □

For an object $\mathbb{X}$ in $\mathcal{C}(\Lambda)$, we set

$$|\mathcal{X}| = \begin{cases} X, & \text{if } \mathcal{X} = X \text{ is in } \operatorname{mod}\Lambda; \\ Y, & \text{if } \mathcal{X} = Y[1] \text{ is in } (\operatorname{mod}\Lambda)[1]. \end{cases}$$

Recall that a module $M$ is called *exceptional* if $\operatorname{Ext}^1(M, M) = 0$. Warning: apart from $|\mathcal{U}_t|$, the modules $|\mathcal{U}_i|$ arising from a signed $\tau$-exceptional sequence $(\mathcal{U}_1, \dots, \mathcal{U}_t)$ are not necessarily $\tau$-rigid in $\operatorname{mod}\Lambda$. However, we have the following.

**Lemma 5.2.** *Let $U$ be a $\tau$-rigid object in* $\operatorname{mod}\Lambda$ *and suppose that $Y$ is $\tau$-rigid in $J(U)$. Then $Y$ is exceptional in* $\operatorname{mod}\Lambda$.

*Proof.* It follows from the Auslander-Reiten formula that $Y$ is exceptional in $J(U)$. But $J(U)$ is an exact abelian subcategory of $\operatorname{mod}\Lambda$, and hence $M$ is also exceptional in $\operatorname{mod}\Lambda$. □

**Corollary 5.3.** *Let $(\mathcal{U}_1, \dots, \mathcal{U}_t)$ in $\mathcal{C}(\Lambda)$ be a signed $\tau$-exceptional sequence. Then each $|\mathcal{U}_i|$ is an exceptional module in* $\operatorname{mod}\Lambda$.

*Proof.* Firstly, $|\mathcal{U}_t|$ is $\tau$-rigid in $\operatorname{mod}\Lambda$, giving the result for $i = t$ using the Auslander-Reiten formula. The module $|\mathcal{U}_{t-1}|$ is $\tau$-rigid in $J(\mathcal{U}_t)$, so the result for $i = t - 1$ follows from Lemma 5.2. The result for all $i$ follows from an inductive argument. □

We now restate our main result.

**Theorem 5.4.** *Let $\Lambda$ be a finite dimensional algebra. For each $t \in \{1, \dots, n\}$ there is a bijection between the set of ordered support $\tau$-rigid objects of length $t$ in $\mathcal{C}(\Lambda)$ and the set of signed $\tau$-exceptional sequences of length $t$ in $\mathcal{C}(\Lambda)$.*

For $t = n$, we obtain.

**Corollary 5.5.** *Let $\Lambda$ be a finite dimensional algebra. There is a bijection between the set of ordered support $\tau$-tilting objects in $\mathcal{C}(\Lambda)$ and the set of complete signed $\tau$-exceptional sequences in $\mathcal{C}(\Lambda)$.*

The remainder of this section is devoted to proving Theorem 5.4. The main idea of the proof is to work by induction on $t$, making use of Propositions 4.5 and 4.11, which we for convenience now reformulate as follows.

**Proposition 5.6.** *Let $U$ be a $\tau$-rigid module in* $\operatorname{mod}\Lambda$. *Then there is a bijection $\mathcal{E}_U$ between the sets*

$$\{X \in \operatorname{ind}\operatorname{mod}\Lambda \setminus \operatorname{ind} U \mid X \amalg U \textit{ is } \tau\textit{-rigid}\} \cup \{\operatorname{ind}(\mathcal{P}(\Lambda) \cap {}^{\perp}U)[1]\}$$

*and*

$$\{X \in \operatorname{ind} J(U) \mid X\ \tau\textit{-rigid in } J(U)\} \cup \{\operatorname{ind}(\mathcal{P}(J(U))[1]\}$$

*given by*

$$\mathcal{E}_U(X) = \begin{cases} f(X) & \textit{if } X \in \{\operatorname{ind}\operatorname{mod}\Lambda \setminus \operatorname{add} U \mid X \amalg U \textit{ is } \tau\textit{-rigid and } X \notin \operatorname{Gen} U\} \\ \rho(X) & \textit{if } X \in \{\operatorname{ind}\operatorname{mod}\Lambda \setminus \operatorname{add} U \mid X \amalg U \textit{ is } \tau\textit{-rigid and } X \in \operatorname{Gen} U\} \cup \\ & \operatorname{ind}(\mathcal{P}(\Lambda) \cap {}^{\perp}U)[1] \end{cases}$$

We extend the domain of $\mathcal{E}_U$ to

$$\operatorname{add}(\{X \in \operatorname{ind}\operatorname{mod}\Lambda \setminus \operatorname{add} U \mid X \amalg U \text{ is } \tau\text{-rigid}\} \cup \{\operatorname{ind}(\mathcal{P}(\Lambda) \cap {}^{\perp}U)[1]\})$$

by setting $\mathcal{E}_U(\amalg_{i=1}^s X_i) = \amalg_{i=1}^s \mathcal{E}_U(X_i)$ for objects $X_1, X_2, \dots X_s$ in

$$\{X \in \operatorname{ind}\operatorname{mod}\Lambda \setminus \operatorname{add} U \mid X \amalg U \text{ is } \tau\text{-rigid}\} \cup \{\operatorname{ind}(\mathcal{P}(\Lambda) \cap {}^{\perp}U)[1]\}.$$

**Proposition 5.7.** *Let $U$ be a $\tau$-rigid module in* $\operatorname{mod}\Lambda$ *with $\delta(U) = t'$. For any positive integer $t \leq n - t'$, the map $\mathcal{E}_U$ induces a bijection between the set of support $\tau$-rigid objects $X$ in $\mathcal{C}(\Lambda)$ such that $\delta(X) = t$, $X \amalg U$ is support $\tau$-rigid and* $\operatorname{add} X \cap \operatorname{add} U = 0$, *and the set of support $\tau$-rigid objects in $\mathcal{C}(J(U))$.*

*Proof.* We first need to prove that for any $X$ in $\mathcal{C}(\Lambda)$, if $X \amalg U$ is support $\tau$-rigid with $\operatorname{add} X \cap \operatorname{add} U = 0$, then $\mathcal{E}_U(X) = \widetilde{X}$ is support $\tau$-rigid in $\mathcal{C}(J(U))$. For this, let $X_i$ and $X_j$ be two indecomposable direct summands in $X$, and consider the following cases.

**Case I:** Let $X_i, X_j$ both be in $\operatorname{mod}\Lambda$ and not in $\operatorname{Gen} U$. Then by Lemma 4.8, we have that

$$\widetilde{X}_i \amalg \widetilde{X}_j = f(X_i) \amalg f(X_j) = f(X_i \amalg X_j)$$

is $\tau$-rigid in $J(U)$.

**Case II:** Let $X_i$ be in $\operatorname{Gen} U$, and assume $X_j$ is in $\operatorname{mod}\Lambda$ but not in $\operatorname{Gen} U$.

Then $\widetilde{X}_i[-1]$ is in $\mathcal{P}(J(U))$, and we need to prove that $\operatorname{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = 0$. Recall that $\widetilde{X}_i = f(B_{X_i})[1]$ (see Definition 4.10), where there is an exact sequence

$$B_{X_i} \xrightarrow{a_{X_i}} U_{X_i} \to X_i \to 0$$

with $a_{X_i}$ a minimal left $\operatorname{add} U$-approximation.

Moreover $\widetilde{X}_j = f(X_j)$, so we need to prove that $\operatorname{Hom}(f(B_{X_i}), f(X_j)) = 0$. We have that $B_{X_i}$ is in $\mathcal{P}({}^{\perp}\tau U)$ by Lemma 3.4, and $f(B_{X_i})$ is in $\mathcal{P}(J(U))$ by Lemma 4.9.

We have that $0 \to \operatorname{Hom}(f(B_{X_i}), f(X_j)) \to \operatorname{Hom}(B_{X_i}, f(X_j))$ is exact, so it suffices to show that $\operatorname{Hom}(B_{X_i}, f(X_j)) = 0$. For this, apply $\operatorname{Hom}(B_{X_i}, \ )$ to the exact sequence

$$0 \to t(X_j) \to X_j \to f(X_j) \to 0$$

and consider the long exact sequence

$$0 \to \operatorname{Hom}(B_{X_i}, t(X_j)) \to \operatorname{Hom}(B_{X_i}, X_j) \to \operatorname{Hom}(B_{X_i}, f(X_j)) \to \operatorname{Ext}^1(B_{X_i}, t(X_j)).$$

We have that $t(X_j)$ is in $\operatorname{Gen} U \subseteq {}^{\perp}\tau U$, and $B_{X_i}$ is in $\mathcal{P}({}^{\perp}\tau U)$, so $\operatorname{Ext}^1(B_{X_i}, t(X_j)) = 0$. Therefore it is sufficient to show that $\operatorname{Hom}(B_{X_i}, t(X_j)) \to \operatorname{Hom}(B_{X_i}, X_j)$ is an epimorphism.

So consider an arbitrary map $g\colon B_{X_i} \to X_j$. We first claim that this map factors through an object in $\operatorname{add} U$. Recall from Definition 4.10 that there is a triangle

$$\mathbb{P}_{B_{X_i}} \to \mathbb{P}_{U_{X_i}} \to \mathbb{P}_{X_i} \to$$

with $U_{X_i}$ in $\operatorname{add} U$. Now apply $\operatorname{Hom}_{\mathcal{K}}( \ , \mathbb{P}_{X_j})$ to this triangle, and consider the exact sequence

$$\operatorname{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}) \to \operatorname{Hom}(\mathbb{P}_{B_{X_i}}, \mathbb{P}_{X_j}) \to \operatorname{Hom}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1])$$

By assumption we have $\operatorname{Hom}(X_j, \tau X_i) = 0$, and this implies that the last term $\operatorname{Hom}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1])$ vanishes, by Lemma 2.3. Hence the map $\operatorname{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}) \to \operatorname{Hom}(\mathbb{P}_{B_{X_i}}, \mathbb{P}_{X_j})$ is an epimorphism, and so any map $\mathbb{P}_{B_{X_i}} \to \mathbb{P}_{X_j}$ factors through $\mathbb{P}_{U_{X_i}}$. By Lemma 2.1, this means that the map $g$ factors through $U_{X_i}$ in $\operatorname{add} U$. Assume $B_{X_i} \xrightarrow{g} X_j = B_{X_i} \xrightarrow{h} U_{X_i} \xrightarrow{p} X_j$. Then $U_{X_i} \xrightarrow{p} X_j$ factors through $t(X_j) \to X_j$. Hence $g$ also factors through $t(X_j) \to X_j$, and we have proved the claim that $\operatorname{Hom}(B_{X_i}, t(X_j)) \to \operatorname{Hom}(B_{X_i}, X_j)$ is an epimorphism. It then follows that $\operatorname{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = \operatorname{Hom}(B_{X_i}, f(X_j)) = 0$.

**Case III:** Now assume $X_i = P[1]$ for $P$ in $\mathcal{P}(\Lambda) \cap {}^{\perp}U$, and $X_j$ lies in $\operatorname{mod}\Lambda$ and not in $\operatorname{Gen} U$. Then $\operatorname{Hom}(P, X_j) = 0$. We need to prove that $\operatorname{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = 0$. By Definition 4.10, there is an exact sequence:

$$P \xrightarrow{c_P} B_P \xrightarrow{d_P} U_P \to 0 \tag{9}$$

By Definition 4.10, we have $\widetilde{X}_i = f(B_P)[1]$ and $\widetilde{X}_j = f(X_j)$.

Note that $\operatorname{Hom}(P, X_j) = 0$ implies that $\operatorname{Hom}(P, \widetilde{X}_j) = 0$, since $P$ is projective. We also have $\operatorname{Hom}(U, \widetilde{X}_j) = \operatorname{Hom}(U, f(X_j)) = 0$, by definition of $f$.

Applying $\operatorname{Hom}( \ , \widetilde{X}_j)$ to (9) gives an exact sequence

$$0 \to \operatorname{Hom}(U_P, \widetilde{X}_j) \to \operatorname{Hom}(B_P, \widetilde{X}_j) \to \operatorname{Hom}(P, \widetilde{X}_j)$$

The end terms vanish, and hence we obtain that $\operatorname{Hom}(B_P, \widetilde{X_j}) = 0$. Clearly then also $\operatorname{Hom}(\widetilde{X_i}[-1], \widetilde{X_j}) = \operatorname{Hom}(f(B_P), \widetilde{X_j}) = 0$. This finishes the proof of Case III.

Now combining the Cases I,II and III, the claim that $\mathcal{E}_U(X)$ is a support $\tau$-rigid object in $J(U)$ follows.

Now, let $M = \amalg_{i=1}^t M_i$ be a support $\tau$-rigid object in $\mathcal{C}(J(U))$ with $\delta(M) = t$, where each $M_i$ is indecomposable. There are, by Proposition 5.6, unique indecomposable modules $X_i$ such that $\mathcal{E}_U(X_i) = M_i$, the object $U \amalg X_i$ is support $\tau$-rigid, and $X_i \notin \operatorname{add} U$. We need to prove that $X = \amalg_{i=1}^t X_i$ is support $\tau$-rigid as well. For this we consider two arbitrary summands $X_i, X_j$ in $X$ and the following cases.

**Case I:** Assume $X_i = P[1]$ with $P$ in $\mathcal{P}(\Lambda) \cap {}^\perp U$, and assume that $X_j$ is in $\operatorname{mod}\Lambda$ is such that $X_j \amalg U$ is $\tau$-rigid, and $\operatorname{Hom}(\widetilde{X_i}[-1], \widetilde{X_j}) = 0$. We need to prove that $\operatorname{Hom}(P, X_j) = 0$.

Note that by Definition 4.10, we have $\widetilde{X_i}[-1] = f(B_P)$, where $P \to B_P$ is a minimal $\mathcal{P}({}^\perp\tau U)$-approximation of $P$. Moreover $\widetilde{X_j} = f(X_j)$.

We have that $\operatorname{Hom}(f(B_P), f(X_j)) = \operatorname{Hom}(\widetilde{X_i}[-1], \widetilde{X_j}) = 0$. Apply $\operatorname{Hom}( \ , f(X_j))$ to the exact sequence $0 \to t(B_P) \to B_P \to f(B_P) \to 0$ and consider the exact sequence

$$0 \to \operatorname{Hom}(f(B_P), f(X_j)) \to \operatorname{Hom}(B_P, f(X_j)) \to \operatorname{Hom}(t(B_P), f(X_j))$$

The last term vanishes since $t(B_P)$ is in $\operatorname{Gen} U$ and $f(X_j)$ is in $U^\perp$. We then obtain that also $\operatorname{Hom}(B_P, f(X_j)) = 0$.

Since by assumption $X$ is in ${}^\perp\tau U$, we also have that $f(X)$ is in ${}^\perp\tau U$. Hence any map $P \to f(X_j)$ factors through $B_P \to f(X_j)$, since $P \to B_P$ is a left ${}^\perp\tau U$-approximation. Since $\operatorname{Hom}(B_P, f(X_j)) = 0$, we have $\operatorname{Hom}(P, f(X_j)) = 0$.

We also have that $\operatorname{Hom}(P, U) = 0$ implies that $\operatorname{Hom}(P, \operatorname{Gen} U) = 0$, so in particular $\operatorname{Hom}(P, t(X_j)) = 0$.

Since $\operatorname{Hom}(P, f(X_j)) = 0 = \operatorname{Hom}(P, t(X_j))$, we indeed also have $\operatorname{Hom}(P, X_j) = 0$, and this finishes Case I.

**Case II:** If $X_i, X_j$ are both in $\operatorname{mod}\Lambda$ and not in $\operatorname{Gen} U$, then $\widetilde{X_i} = f(X_i), \widetilde{X_j} = f(X_j)$ is such that $\widetilde{X_i} \amalg \widetilde{X_j} = \widetilde{X_i \amalg X_j}$ is $\tau$-rigid in $J(U)$, then $X_i \amalg X_j$ is $\tau$-rigid according to [16, Cor. 3.18].

**Case III:** Now assume $X_i$ is in $\mathcal{P}(\operatorname{Gen} U)$, and hence $\widetilde{X_i}$ in $\mathcal{P}(J(U))[1]$, while $X_j$ is in $\operatorname{mod}\Lambda$ but not in $\operatorname{Gen} U$.

By assumption $\operatorname{Hom}(\widetilde{X_i}[-1], \widetilde{X_j}) = 0$, and we need to prove that $\operatorname{Hom}(X_i, \tau X_j) = 0 = \operatorname{Hom}(X_j, \tau X_i)$

Since $X_i$ is in $\operatorname{Gen} U$, and $\operatorname{Hom}(U, \tau X_j) = 0$, we have that also $\operatorname{Hom}(X_i, \tau X_j) = 0$.

By Lemma 2.3 we have that in order to prove that $\operatorname{Hom}(X_j, \tau X_i) = 0$, it is sufficient to prove that $\operatorname{Hom}_{\mathcal{K}}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1]) = 0$.

We apply $\operatorname{Hom}_{\mathcal{K}}( \ , \mathbb{P}_{X_j})$ to the triangle

$$\mathbb{P}_{B_{X_i}} \to \mathbb{P}_{U_{X_i}} \to \mathbb{P}_{X_i} \to$$

(see Definition 4.10) and consider the exact sequence

$$\operatorname{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}) \to \operatorname{Hom}(\mathbb{P}_{B_{X_i}}, \mathbb{P}_{X_j}) \to \operatorname{Hom}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1]) \to \operatorname{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}[1]) \tag{10}$$

We first note that $\operatorname{Hom}(X_j, \tau U) = 0$ implies that the last term $\operatorname{Hom}_{\mathcal{K}}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}[1])$ vanishes. It is therefore sufficient to prove that the first map $\operatorname{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}) \to \operatorname{Hom}(\mathbb{P}_{B_{X_i}}, \mathbb{P}_{X_j})$ is an epimorphism, that is: we claim that any map $\mathbb{P}_{B_{X_i}} \to \mathbb{P}_{X_j}$ factors through $\mathbb{P}_{B_{X_i}} \to \mathbb{P}_{U_{X_i}}$. For this, it is sufficient that any map $B_{X_i} \to X_j$ factors through $B_{X_i} \to U_{X_i}$. Consider the exact sequence

$$\operatorname{Hom}(B_{X_i}, t(X_j)) \to \operatorname{Hom}(B_{X_i}, X_j) \to \operatorname{Hom}(B_{X_i}, f(X_j)) \tag{11}$$

obtained by applying $\operatorname{Hom}(Y_i, \ )$ to the canonical sequence for $X_j$. We claim that the last term vanishes. For this consider the exact sequence

$$\operatorname{Hom}(f(B_{X_i}), f(X_j)) \to \operatorname{Hom}(B_{X_i}, f(X_j)) \to \operatorname{Hom}(t(B_{X_i}), f(X_j)) \tag{12}$$

obtained by applying $\operatorname{Hom}( \ , f(X_j))$ to the canonical sequence for $B_{X_i}$. In (12) the first term vanishes by assumption, and the last term vanishes since $t(B_{X_i})$ is in $\operatorname{Gen} U$ and $f(X_j)$ is in $U^{\perp}$. Hence $\operatorname{Hom}(B_{X_i}, f(X_j))$, which is the last term of sequence (11) also vanishes. This means that any map $B_{X_i} \to X_j$ factors $B_{X_i} \to t(X_j) \to X_j$.

By Lemma 3.9, the map $B_{X_i} \to U_{X_i}$ is a $\operatorname{Gen} U$-approximation. So the map $B_{X_i} \to t(X_j)$ factors $B_{X_i} \to U_{X_i} \to t(X_j)$, and hence our original map $B_{X_i} \to X_j$ factors $B_{X_i} \to U_{X_i} \to t(X_j) \to X_j$. We have now proved that any map $B_{X_i} \to X_j$ factors through $B_{X_i} \to U_{X_i}$, and hence any map $\mathbb{P}_{B_{X_i}} \to \mathbb{P}_{X_j}$ factors through $\mathbb{P}_{B_{X_i}} \to \mathbb{P}_{U_{X_i}}$. Therefore $\operatorname{Hom}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1]) = 0$, which implies $\operatorname{Hom}(X_j, \tau X_i) = 0$, and the claim is proved. This finishes the proof of case III.

Combining Cases I, II and III proves the claim that $X = \amalg X_i$ is support $\tau$-rigid, and this concludes the proof of the proposition. □

In particular, we then have the following.

**Corollary 5.8.** *Let $t > 1$. Let $U$ be an indecomposable $\tau$-rigid module. The map $\mathcal{E}_U$ induces a bijection between ordered support $\tau$-rigid objects in $\mathcal{C}(\Lambda)$ with last term $U$ and length $t$, and ordered support $\tau$-rigid objects of length $t-1$ in $\mathcal{C}(J(U))$.*

We need also to deal with the case where the last term in an ordered support $\tau$-rigid objects in $\mathcal{C}(\Lambda)$ is of the form $P[1]$. For this, we first observe the following.

**Lemma 5.9.** *Let $P$ be a projective $\Lambda$-module, and consider $J(P[1]) = P^{\perp}$.*

(a) *The $\tau$-rigid modules in $J(P[1])$ are exactly the $\tau$-rigid modules $X$ in* $\operatorname{mod} \Lambda$ *with* $\operatorname{Hom}(P, X) = 0$.
(b) *The map $Q \mapsto f_P(Q)$ gives a bijection between the indecomposables in $\mathcal{P}(\Lambda) \setminus$* $\operatorname{add} P$ *and the indecomposables in $\mathcal{P}(P^{\perp})$.*

*Proof.* Part (a) is a direct consequence of [1, Lemma 2.1]. Part (b) follows from Lemma 4.9. □

For a projective module $P$ in $\mathcal{P}(\Lambda)$, consider the map $\mathcal{E}_{P[1]}$ from

$$\operatorname{add}\{X \in \operatorname{ind} \operatorname{mod} \Lambda \mid X \ \tau\text{-rigid, and } \operatorname{Hom}(P, X) = 0\} \cup (\operatorname{ind} \mathcal{P}(\Lambda) \setminus \operatorname{ind} P)[1]$$

to

$$\operatorname{add}\{X \in \operatorname{ind} J(P[1])) \mid X \ \tau\text{-rigid}\} \cup \operatorname{ind} \mathcal{P}(J(P[1]))[1]$$

defined as follows. For $X$ indecomposable, we set

$$\mathcal{E}_{P[1]}(X) = \begin{cases} X & \text{if } X\ \tau\text{-rigid, and } \operatorname{Hom}(P, X) = 0 \\ f_P(X)[1] & \text{if } X \in \mathcal{P}(\Lambda) \setminus \operatorname{ind} P \end{cases}$$

For $X = \amalg_{i=1}^{t} X_i$, with each $X_i$ in

$$\{X \in \operatorname{ind} \operatorname{mod} \Lambda \mid X\ \tau\text{-rigid, and } \operatorname{Hom}(P, X) = 0\} \cup (\operatorname{ind} \mathcal{P}(\Lambda) \setminus \operatorname{ind} P)[1],$$

we set $\mathcal{E}_{P[1]}(X) = \amalg_{i=1}^{t} \mathcal{E}_{P[1]}(X_i)$.

**Proposition 5.10.** *Let $P$ be in $\mathcal{P}(\Lambda)$ with $\delta(P) = t'$.*

(a) *The map $\mathcal{E}_{P[1]}$ is a bijection between the sets*

$$\{X \in \operatorname{ind} \operatorname{mod} \Lambda \mid X \textit{ is } \tau\textit{-rigid and } \operatorname{Hom}(P, X) = 0\} \cup \{\operatorname{ind}(\mathcal{P}(\Lambda)) \setminus \operatorname{ind} P)[1]\}$$

*and*

$$\{X \in \operatorname{ind} J(P[1]) \mid X \textit{ is } \tau\textit{-rigid}\} \cup \{\operatorname{ind} \mathcal{P}(J(P[1]))[1]\}.$$

(b) *For any positive integer $t \leq n - t'$, the map $\mathcal{E}_{P[1]}$ induces a bijection between the set of support $\tau$-rigid objects $X$ in $\mathcal{C}(\Lambda)$ such that $\delta(X) = t$, the object $X \amalg P[1]$ is support $\tau$-rigid and* $\operatorname{add} X \cap \operatorname{add} P[1]$*=0, and the set of support $\tau$-rigid objects in $\mathcal{C}(P^{\perp})$ with $t$ indecomposable direct summands.*

*Proof.* Part (a) follows directly from Lemma 5.9.

Part (b): Let $Q \notin \operatorname{add} P$ be an indecomposable module in $\mathcal{P}(\Lambda)$, and let $X$ be in $P^{\perp}$. Apply $\operatorname{Hom}(\ , X)$ to the canonical sequence

$$0 \to t_P(Q) \to Q \to f_P(Q) \to 0.$$

Since $\operatorname{Hom}(P, X) = 0$, we have that $\operatorname{Hom}(\operatorname{Gen} P, X) = 0$ and thus that $\operatorname{Hom}(t_P(Q), X) = 0$. It follows that $\operatorname{Hom}(Q, X) \simeq \operatorname{Hom}(f_p(Q), X)$. The claim follows from combining this with part (a). □

**Corollary 5.11.** *Let* $t > 1$. *Let $P$ be an indecomposable projective module. Then the map $\mathcal{E}_{P[1]}$ induces a bijection between ordered support $\tau$-rigid objects in $\mathcal{C}(\Lambda)$ with last term $P[1]$ and length $t$, and ordered support $\tau$-rigid objects of length $t - 1$ in $\mathcal{C}(J(P[1])) = \mathcal{C}(P^{\perp})$.*

We can now prove Theorem 5.4.

*Proof of Theorem 5.4.* We prove the result by induction on $\delta(\Lambda) = n$. The statement for $t = 1$ is clear. In particular, this deals with the case $n = 1$. So we assume the result to be true for algebras with a smaller number of indecomposable projective modules up to isomorphism. Let $U$ be an indecomposable $\tau$-rigid object in $\operatorname{mod} \Lambda$. By Corollary 5.8, there is a bijection between the ordered support $\tau$-rigid objects in $\mathcal{C}(\Lambda)$ ending in $U$ and the ordered support $\tau$-rigid objects in $\mathcal{C}(J(U))$. The ordered support $\tau$-rigid objects in $\mathcal{C}(J(U))$ are by the induction hypothesis in bijection with the signed $\tau$-exceptional sequences in $\mathcal{C}(J(U))$. And by definition a sequence $(U_1, \ldots, U_{t-1})$ is a signed $\tau$-exceptional sequence in $\mathcal{C}(J(U))$ if and only if $(\mathcal{U}_1, \ldots, \mathcal{U}_{t-1}, U)$ is a signed $\tau$-exceptional sequence in $\mathcal{C}(\Lambda)$.

Let $P$ be an indecomposable module in $\mathcal{P}(\Lambda)$. In a similar way to the above, there is a bijection between ordered support $\tau$-rigid objects in $C(\Lambda)$ ending in $P[1]$ and signed $\tau$-exceptional sequences in $C(J(P[1]))$, using the induction hypothesis and Corollary 5.11. □

**Remark 5.12.** *We now give a more explicit description of the bijection constructed in Theorem 5.4, and the inverse of this bijection. Let* $\mathsf{W}$ *denote a wide subcategory of* $\operatorname{mod}\Lambda$ *which is equivalent to a module category, and let* $n_{\mathsf{W}}$ *denote its rank. Let* $\mathcal{U}$ *denote an indecomposable* $\tau$*-rigid object in* $C(\mathsf{W})$*, that is either* $\mathcal{U} = U$ *for an indecomposable* $\tau$*-rigid module in* $\mathsf{W}$*, or* $\mathcal{U} = P[1]$*, where* $P$ *is indecomposable projective in* $\mathsf{W}$*. Recall, in particular, that by Proposition 4.2, we have that* $J_{\mathsf{W}}(\mathcal{U})$ *is a wide subcategory of* $\mathsf{W}$*, and hence of* $\operatorname{mod}\Lambda$*, equivalent to a module category.*

*Now, consider the bijections obtained by combining Propositions 5.6 and 5.10.*

$$\{\mathcal{X} \in \operatorname{ind} C(\mathsf{W}) \setminus \mathcal{U} \mid \mathcal{X} \amalg \mathcal{U} \text{ is } \tau\text{-rigid}\}$$
$$\mathcal{E}_{\mathcal{U}}^{\mathsf{W}} \downarrow \quad \uparrow \mathcal{F}_{\mathcal{U}}^{\mathsf{W}}$$
$$\{\mathcal{X} \in \operatorname{ind} C(J(\mathcal{U})) \mid \mathcal{X} \ \tau\text{-rigid in } J(\mathcal{U})\}$$

*Consider, for each* $t = 1, \ldots, n_{\mathsf{W}}$*, the bijections*

$$\{\text{ordered } \tau\text{-rigid objects in } \mathsf{W} \text{ with } t \text{ indecomposable direct summands}\}$$
$$\Psi_t^{\mathsf{W}} \downarrow \quad \uparrow \Phi_t^{\mathsf{W}}$$
$$\{\tau\text{-exceptional sequences in } \mathsf{W} \text{ of length } t\}$$

*where* $\Psi_t^{\mathsf{W}}$ *is the bijection constructed in Theorem 5.4, and* $\Phi_t^{\mathsf{W}}$ *is its inverse. Then we have*

$$\Psi_t^{\mathsf{W}}(\mathcal{T}_1, \ldots, \mathcal{T}_t) = (\Psi_{t-1}^{J_{\mathsf{W}}(\mathcal{T}_t)}(\mathcal{E}_{\mathcal{T}_t}^{\mathsf{W}}(\mathcal{T}_1), \ldots, \mathcal{E}_{\mathcal{T}_t}^{\mathsf{W}}(\mathcal{T}_{t-1})), \mathcal{T}_t)$$

*Now let*

$$\begin{aligned} \mathsf{W}_t &= \mathsf{W} & \mathcal{U}_t &= \mathcal{T}_t \\ \mathsf{W}_{t-1} &= J(\mathcal{U}_t) & \mathcal{U}_{t-1} &= \mathcal{E}_{\mathcal{U}_t}(\mathcal{T}_{t-1}) \\ &\vdots & &\vdots \\ \mathsf{W}_i &= J_{\mathsf{W}_{i+1}}(\mathcal{U}_{i+1}) & \mathcal{U}_i &= \mathcal{E}_{\mathcal{U}_{i+1}}^{\mathsf{W}_{i+1}} \ldots \mathcal{E}_{\mathcal{U}_{t-1}}^{\mathsf{W}_{t-1}} \mathcal{E}_{\mathcal{U}_t}^{\mathsf{W}_t}(\mathcal{T}_i) \\ &\vdots & &\vdots \\ \mathsf{W}_1 &= J_{\mathsf{W}_2}(\mathcal{U}_1) & \mathcal{U}_1 &= \mathcal{E}_{\mathcal{U}_2}^{\mathsf{W}_2} \ldots \mathcal{E}_{\mathcal{U}_{t-1}}^{\mathsf{W}_{t-1}} \mathcal{E}_{\mathcal{U}_t}^{\mathsf{W}_t}(\mathcal{T}_1) \end{aligned}$$

*It is then straightforward to verify that*

$$\Psi_t^{\mathsf{W}}(\mathcal{T}_1, \ldots, \mathcal{T}_t) = (\mathcal{U}_1, \ldots, \mathcal{U}_t).$$

*and that the inverse bijection is given by*

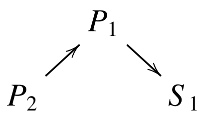
$$\Phi_t^{\mathsf{W}}(\mathcal{U}_1, \ldots, \mathcal{U}_t) = (\mathcal{F}_{\mathcal{U}_t}^{\mathsf{W}_t} \cdots \mathcal{F}_{\mathcal{U}_2}^{\mathsf{W}_2}(\mathcal{U}_1), \mathcal{F}_{\mathcal{U}_t}^{\mathsf{W}_t} \cdots \mathcal{F}_{\mathcal{U}_3}^{\mathsf{W}_3}(\mathcal{U}_2), \cdots, \mathcal{U}_t)$$

*where* $\mathsf{W}_t = \mathsf{W}$ *and* $\mathsf{W}_i = J_{\mathsf{W}_{i+1}}(\mathcal{U}_{i+1})$ *for all* $i$*, as above.*

## 6. Examples

Each example is given as the path algebra of a quiver modulo an admissible ideal of relations generated by paths. For each vertex $i$ of the quiver, we denote by $P_i, I_i$, $S_i$ the corresponding indecomposable projective (respectively, indecomposable injective, simple) module.

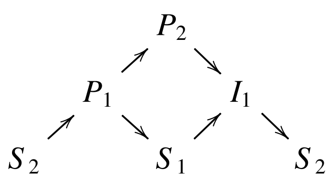
6.1. **Example 1:** Let $Q$ be the quiver $1 \longrightarrow 2$, and let $\Lambda = kQ$. There are three indecomposable modules, $P_1, P_2 = S_2, S_1$, and the AR-quiver of $\operatorname{mod}\Lambda$ is:




There are 5 support $\tau$-tilting modules (= support tilting modules, since $\Lambda$ is hereditary), and hence 10 ordered support $\tau$-tilting modules. We list these in the table below, together with the corresponding complete signed $\tau$-exceptional sequences.

| Ordered support $\tau$-tilting object | Signed $\tau$-exc. sequence | Ordered support $\tau$-tilting object | Signed $\tau$-exc. sequence |
|---|---|---|---|
| $(P_2, P_1)$ | $(P_2, P_1)$ | $(P_2, P_1[1])$ | $(P_2, P_1[1])$ |
| $(S_1, P_1)$ | $(P_2[1], P_1)$ | $(P_2[1], P_1[1])$ | $(P_2[1], P_1[1])$ |
| $(P_1, P_2)$ | $(S_1, P_2)$ | $(S_1, P_2[1])$ | $(S_1, P_2[1])$ |
| $(P_1[1], P_2)$ | $(S_1[1], P_2)$ | $(P_1[1], P_2[1])$ | $(S_1[1], P_2[1])$ |
| $(P_1, S_1)$ | $(P_1, S_1)$ | | |
| $(P_2[1], S_1)$ | $(P_1[1], S_1)$ | | |

6.2. **Example 2:** Let $Q'$ be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$, and let $\Lambda' = kQ'/I$, where $I$ is the ideal generated by the path $\beta\alpha$. There are 5 indecomposable modules, and the AR-quiver is:

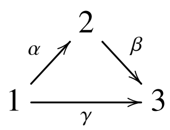



Note that the module $I_1$ is *not* $\tau$-rigid in $\operatorname{mod}\Lambda$, while the other four indecomposable modules are $\tau$-rigid.

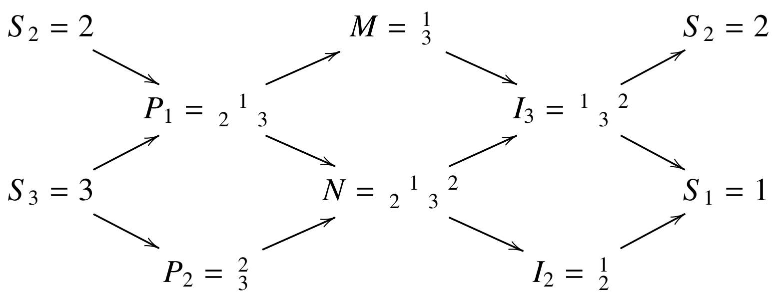
There are 6 support $\tau$-tilting modules, and hence 12 ordered support $\tau$-tilting modules We list these in the table below, together with the corresponding complete signed $\tau$-exceptional sequences.

| Ordered support $\tau$-tilting object | Signed $\tau$-exc. sequence | Ordered support $\tau$-tilting object | Signed $\tau$-exc. sequence |
|---|---|---|---|
| $(P_1, P_2)$ | $(S_1, P_2)$ | $(P_2[1]S_1)$ | $(P_1[1], S_1)$ |
| $(S_2, P_2)$ | $(S_1[1], P_2)$ | $(P_1, S_1)$ | $(P_1, S_1)$ |
| $(P_2, P_1)$ | $(S_2, P_1)$ | $(S_2, P_1[1])$ | $(S_2, P_1[1])$ |
| $(S_1, P_1)$ | $(S_2[1], P_1)$ | $(P_2[1], P_1[1])$ | $(S_2[1], P_1[1])$ |
| $(P_2, S_2)$ | $(I_1, S_2)$ | $(S_1, P_2[1])$ | $(S_1, P_2[1])$ |
| $(P_1[1], S_2)$ | $(I_1[1], S_2)$ | $(P_1[1], P_2[1])$ | $(S_1[1], P_2[1])$ |

6.3. **Example 3:** Let $Q''$ be the quiver




and let $\Lambda'' = kQ''/I$ where $I$ is the ideal generated by the path $\beta\alpha$. There are 9 indecomposable modules, and the AR-quiver is




where the notation indicates which simple modules occur in the radical layers of the module, so $N = {}_2{}^1{}_3{}^2$ is a module of length 4, of radical length 2, and with $N/\operatorname{rad} N \simeq S_1 \amalg S_2$.

The following table gives, for each $\tau$-rigid indecomposable $U$, a list of the indecomposable modules in $J(U)$ and an algebra $\Gamma_U$ such that $J(U) \simeq \Gamma - \text{mod}$.

| $U$ | $J(U)$ | $\Gamma_U$ |
|---|---|---|
| $S_2 = 2$ | $\{\begin{smallmatrix}2\\3\end{smallmatrix}, {}^{1}{}_{3}{}^{2}, 1\}$ | $k(\bullet \longrightarrow \bullet)$ |
| $S_3 = 3$ | $\{2, \begin{smallmatrix}1\\2\end{smallmatrix}, 1\}$ | $k(\bullet \longrightarrow \bullet)$ |
| $P_1 = {}_{2}{}^{1}{}_{3}$ | $\{3, \begin{smallmatrix}2\\3\end{smallmatrix}, 2\}$ | $k(\bullet \longrightarrow \bullet)$ |
| $P_2 = \begin{smallmatrix}2\\3\end{smallmatrix}$ | $\{3, \begin{smallmatrix}1\\3\end{smallmatrix}, 1\}$ | $k(\bullet \longrightarrow \bullet)$ |
| $M = \begin{smallmatrix}1\\3\end{smallmatrix}$ | $\{3, {}_{2}{}^{1}{}_{3}, \begin{smallmatrix}1\\2\end{smallmatrix}\}$ | $k(\bullet \longrightarrow \bullet)$ |
| $N = {}_{2}{}^{1}{}_{3}{}^{2}$ | $\{\begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}\}$ | $k(\bullet \quad \bullet)$ |
| $I_2 = \begin{smallmatrix}1\\2\end{smallmatrix}$ | $\{{}_{2}{}^{1}{}_{3}, \begin{smallmatrix}1\\3\end{smallmatrix}, {}_{2}{}^{1}{}_{3}{}^{2}, {}^{1}{}_{3}{}^{2}, 2\}$ | $k(\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet)/(\beta\alpha)$ |
| $S_1$ | $\{\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}\}$ | $k(\bullet \quad \bullet)$ |

We calculate the total number of signed $\tau$-exceptional sequences as follows. For

$$U \in \{S_2, S_3 = P_3, P_2, P_1, M, P_3[1], P_2[1], P_1[1]\}$$

there are (by Example 1) 10 signed $\tau$-exceptional sequences of the form $(-,-,U)$. For $U \in \{N, S_1\}$ there are 4 signed $\tau$-exceptional sequences of the form $(-,-,U)$, while there are (by Example 2) 12 signed $\tau$-exceptional sequences of the form $(-,-,I_2)$. Hence, in total there are 100 signed $\tau$-exceptional sequences for this algebra.

We conclude with examples illustrating how we compute which signed $\tau$-exceptional sequence is the image of a given support $\tau$-tilting object under our bijection. Consider the ordered support $\tau$-rigid object $(M, I_2, P_1)$. To compute $\mathcal{E}_{P_1}(M)$ we first note that $M$ is in Gen $P_1$, so $\mathcal{E}_{P_1}(M) = \rho(M)$. Furthermore we have that $\mathbb{P}_M$ is given by $P_2 \to P_1$ and so we have the triangle (7) in this case is:

$$P_2 \to P_1 \to \mathbb{P}_M$$

and hence $\rho(M) = (f_{P_1}(P_2))[1] = P_2[1]$. Similarly, we have that $\mathcal{E}_{P_1}(I_2) = \rho(I_2) = S_3[1]$. The ordered support $\tau$-rigid object $(P_2[1], S_3[1])$ in $J(P_1)$ corresponds according to the table of Example 1 to the signed $\tau$-exceptional sequence $(S_2[1], S_3[1])$. Hence our bijection maps the ordered support $\tau$-rigid object $(M, I_2, P_1)$ to the signed $\tau$-exceptional sequence $(S_2[1], S_3[1], P_1)$.

Consider the ordered support $\tau$-rigid object $(M, P_1, I_2)$. Note that $\mathrm{Hom}(I_2, M) = 0 = \mathrm{Hom}(I_2, P_1)$, so that $\mathcal{E}_{I_2}(M) = M$ and $\mathcal{E}_{I_2}(P_1) = P_1$. The ordered support $\tau$-rigid object $(M, P_1)$ in $J(I_2)$ corresponds, according to the table of Example 2, to the signed $\tau$-exceptional sequence $(S_2[1], P_1)$ in $J(I_2)$ (note that $S_2$ is projective in $J_{J(I_2)}(P_1)$). Hence our bijection maps the ordered support $\tau$-rigid object $(M, P_1, I_2)$ to the signed $\tau$-exceptional sequence $(S_2[1], P_1, I_2)$.

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