

Dimer models and cluster categories of Grassmannians

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ABSTRACT

We associate a dimer algebra A to a Postnikov diagram D (in a disk) corresponding to a cluster of minors in the cluster structure of the Grassmannian $Gr(k, n)$. We show that A is isomorphic to the endomorphism algebra of a corresponding Cohen-Macaulay module T over the algebra B used to categorify the cluster structure of $Gr(k, n)$ by Jensen-King-Su. It follows that B can be realised as the boundary algebra of A , that is, the subalgebra eAe for an idempotent e corresponding to the boundary of the disk. The construction and proof uses an interpretation of the diagram D , with its associated plabic graph and dual quiver (with faces), as a dimer model with boundary. We also discuss the general surface case, in particular computing boundary algebras associated to the annulus.

Introduction

Postnikov diagrams (also known as alternating strand diagrams) are collections of curves in a disk satisfying certain axioms. They were introduced by Postnikov in his study of total positivity of the Grassmannian $Gr(k, n)$ of k -planes in \mathbb{C}^n [26]. A class of Postnikov diagrams was used by Scott [27] to show that the homogeneous coordinate ring of $Gr(k, n)$ is a cluster algebra, in which each such diagram corresponds to a seed whose (extended) cluster consists of minors (i.e., Plücker coordinates). The combinatorics of the diagram gives both the quiver of the cluster and the minors: the k -subsets of $\{1, 2, \dots, n\}$ corresponding to the minors appear as labels of alternating regions and the quiver can be read off geometrically. By a more recent result of Oh-Postnikov-Speyer [25], every cluster consisting of minors arises in this way (see also related results in [6]).

An additive categorification of this cluster algebra structure has been given by Geiss-Leclerc-Schröer [14] in terms of a subcategory of the category of finite dimensional modules over the preprojective algebra of type A_{n-1} . However, there is a single cluster coefficient, the minor corresponding to the k -subset $\{1, 2, \dots, k\}$ of $\{1, 2, \dots, n\}$, which is not realised in the category. Thus the categorification in [14] is strictly only of the coordinate ring of the affine open cell in the Grassmannian given by the non-vanishing of this minor. The cluster structure is then lifted to the homogeneous coordinate ring in an explicit and natural way.

Recently Jensen-King-Su [19] have given a full and direct categorification of the cluster structure on the homogeneous coordinate ring, using the category of (maximal) Cohen-Macaulay modules over the completion of an algebra B , which is a quotient of the preprojective algebra of type \hat{A}_{n-1} . In particular, a rank one Cohen-Macaulay B -module \mathbb{M}_I is associated to every k -subset I of $\{1, 2, \dots, n\}$.

Given a Postnikov diagram D , let

$$T_D = \bigoplus \mathbb{M}_I,$$

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where the direct sum is over the I labelling the alternating regions of D . As noted in [19], the completion of T_D is a cluster-tilting module and the work of Buan-Iyama-Reiten-Smith [4] would lead one to ask whether the completion of the endomorphism algebra $\text{End}_B(T_D)$ is a frozen Jacobian algebra ([4, Def. 1.1]). We will see that this is indeed the case, in a very particular way.

We associate to D a quiver with faces $Q(D)$. The subgraph containing the arrows incident with internal vertices of $Q(D)$ corresponds to the skew-symmetric matrix associated to D in Scott [27, §5], but there are additional arrows between the boundary vertices. The faces of $Q(D)$ correspond to the oriented regions of D . From the construction, $Q(D)$ may be embedded in the disk in which D is drawn and it is natural to interpret it as a dimer model with boundary, as a generalisation of dimer models on a torus or a more general closed surface (see [2], [3], [7], [12]). Such a generalisation has also been introduced recently by Franco [11].

To formalise this, we give an abstract definition of a dimer model with boundary as a quiver with faces satisfying certain axioms; in particular, the arrows are divided into internal arrows and boundary arrows. Such a dimer model has a natural embedding into a compact surface with boundary in which each component of the boundary is identified with an unoriented cycle of boundary arrows in Q . In the case without boundary, this corresponds closely to the definition of a dimer model given by Bocklandt [2, 2.2]. We associate to any dimer model Q with boundary a *dimer algebra* A_Q which coincides with the usual dimer algebra, defined by a superpotential or commutation relations, in the case where the boundary is empty. In that case, the completion with respect to the arrow ideal coincides with the usual Jacobian algebra of the quiver with potential (Q, W) (as in [9, §3]). If the boundary is nonempty, the dimer algebra can still be defined via a potential, but the relations do not include the derivatives of the potential with respect to boundary arrows. This is a slightly different convention to [4, Def. 1.1], because it is convenient to allow dimer model quivers to have 2-cycles.

Our main result is that $\text{End}_B(T_D)$ is isomorphic to the dimer algebra $A_D = A_{Q(D)}$ associated to the dimer model $Q(D)$. There is a natural grading on A_D by subsets of $\{1, 2, \dots, n\}$, which has a simple definition in terms of D , and there is also a similar grading on $\text{End}_B(T_D)$. We use these to show that there is a well-defined (graded) homomorphism

$$g: A_D \rightarrow \text{End}_B(T_D)$$

taking each arrow $I \rightarrow J$ in $Q(D)$ to the homomorphism $\mathbb{M}_I \rightarrow \mathbb{M}_J$ which generates $\text{Hom}_B(\mathbb{M}_I, \mathbb{M}_J)$ freely as a $\mathbb{C}[t]$ -module, where the polynomial ring $\mathbb{C}[t]$ is the centre of B . We also note that the centre of A_D is $\mathbb{C}[u]$, where u is the sum of all minimal loops in $Q(D)$, i.e. the loops around the faces, and that $g(u) = t$.

To see that g is surjective, we show that between any two vertices I, J of $Q(D)$ there is a path of minimal degree, which (as an element of A_D) must then map to the generator of $\text{Hom}_B(\mathbb{M}_I, \mathbb{M}_J)$. Such a path is constructed inductively, with the induction step depending on a careful analysis of the local behaviour of strands in D near a vertex or face of $Q(D)$. As an aside, we note that this local analysis implies that $Q(D)$ can be isoradially embedded into a planar disk and we relate this to the embedding of $Q(D)$ constructed as a ‘plabic tiling’ in [25, §9].

To see that g is injective, we observe that, since the dimer model $Q(D)$ is ‘consistent’ in an appropriate sense (cf. [25, Rk. 6.4]), we can adapt arguments from [2, §5] to show that any path between two vertices in $Q(D)$ is equal (in A_D) to a path of minimal degree multiplied by a power of u . Thus g is an isomorphism and we also prove that g induces an isomorphism between the corresponding completed algebras.

Let $e \in A_D$ be the sum of the primitive idempotents corresponding to the boundary vertices. We call the algebra eA_De the *boundary algebra* of $Q(D)$ and it is a notable fact

that this algebra is independent of the choice of Postnikov diagram D , once k and n are fixed. This follows immediately from the isomorphism $A_D \cong \text{End}_B(T_D)$, because the B -modules corresponding to the idempotents in e are the indecomposable projectives and so

$$eA_De \cong \text{End}_B(B) \cong B^{\text{opp}}.$$

However, we also give a direct proof of the independence by showing that the boundary algebra is invariant under the untwisting, twisting and geometric exchange moves [26, §14] (see also [27, §3]) for Postnikov diagrams.

Finally, for any integer $k \geq 1$, we consider a notion of Postnikov diagram of degree k on a marked surface with boundary in which all of the marked points lie on the boundary, generalizing the usual notion of an Postnikov diagram which can be regarded as the disk case. We say that such a diagram is a *weak Postnikov diagram* if it is not required to satisfy the global ‘consistency’ axioms (conditions (b1) and (b2) in Definition 2.1). Adapting a construction of [27, §3], we associate a weak Postnikov diagram of degree 2 to a triangulation of any such marked surface. We compute the corresponding boundary algebra in the case of an annulus with at least one marked point on each of its boundary components and show that it is independent of the choice of triangulation.

The structure of the paper is as follows. In Section 1 we set up some of the notation. In Section 2, we recall the definition of a Postnikov diagram [26, §14] and its corresponding plabic graph, as well as the corresponding quiver [27, §5]. In Section 3, we give the definition of a dimer model with boundary and its corresponding dimer algebra, noting that the quiver associated to a Postnikov diagram can be given such a structure.

In Section 4 we define a weighting on the arrows in $Q(D)$, computing the weight of the boundary of a face of $Q(D)$ and the sum of the weights of the arrows incident with a vertex of $Q(D)$. In Section 5 we show how these results can be used to embed $Q(D)$ isoradially into a disk.

In Section 6, we show how the results in Section 4 can be used to construct the first arrow in the minimal path mentioned above. In Section 7 we recall the algebra B (and the completed version \widehat{B}) from [19] and define the B -module T_D . In Section 8, we construct a minimal path. In Section 9, we show that there is a unique element of A_D which can be written as such a path, and that any path in $Q(D)$ is equal in A_D to this element multiplied by a power of a minimal loop. In Section 10, we prove that A_D is isomorphic to $\text{End}_B(T_D)$ and show that eA_De is isomorphic to B . In Section 11, we give the completed version of these results. In Section 12, we give a proof in terms of Postnikov diagrams that the algebra eA_De is independent of the choice of Postnikov diagram, and in Section 13 we consider the surface case.

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1. Set-up and notation

Fix a positive integer n and an integer k with $1 \leq k \leq n - 1$. We will write $\mathbb{Z}_n = \{1, 2, \dots, n\}$. We consider a circular graph C with vertices $C_0 = \mathbb{Z}_n$ clockwise around a circle and edges C_1 also labelled by \mathbb{Z}_n , with edge i joining vertices $i - 1$ and i ; see Figure 1 for the case $n = 7$. For integers $a, b \in \{1, 2, \dots, n\}$, we denote by $[a, b]$ the closed cyclic interval consisting of the elements of the set $\{a, a + 1, \dots, b\}$ reduced mod n . We similarly have the open interval (a, b) .

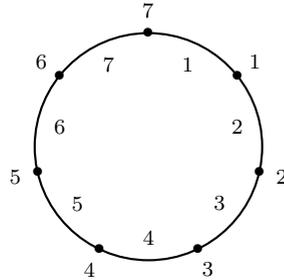


FIGURE 1. The graph C

For a subset S of C_1 , define S_0 to be the set of vertices incident with an edge in S . So, in particular, $(a, b)_0 = [a, b - 1]$, the set of vertices incident with an edge in the set (a, b) .

In general for sets S, S' we write $S - S'$ for the set of elements in S but not in S' . For $s \in S$, we use the shorthand $S - s$ for $S - \{s\}$ and for any z we use the shorthand $S + z$ for $S \cup \{z\}$.

2. Postnikov diagrams

We recall a special case of the definition of a *Postnikov diagram* (alternating strand diagram) [26, §14].

DEFINITION 2.1. A (k, n) -Postnikov diagram D consists of n directed curves, called *strands*, in a disk with n marked vertices on its boundary, labelled by the elements of C_1 (in clockwise order). The strands are also labelled by the elements of C_1 , with strand i starting at vertex i and ending at vertex $i + k$. The following axioms must be satisfied.

Local axioms:

- (a1) Only two strands can cross at a given point and all crossings are transverse.
- (a2) There are finitely many crossing points.
- (a3) Proceeding along a given strand, the other strands crossing it alternate between crossing it left to right and right to left.

Global axioms:

- (b1) A strand cannot intersect itself.
- (b2) If two strands intersect at distinct points U and V , then one strand is oriented from U to V and the other is oriented from V to U .

Note: for axiom (a3), strands $i - k$ and i are regarded as crossing at the boundary vertex i in the obvious way. Note also that because the disk is compact, condition (a2) is effectively local.

We shall often refer to (k, n) -Postnikov diagrams as simply Postnikov diagrams when k and n are clear from the context. A Postnikov diagram is defined up to isotopies fixing the boundary. Two Postnikov diagrams are said to be *equivalent* if one can be obtained from the other using the untwisting and twisting moves illustrated in Figure 2 (or the opposite versions, obtained from these diagrams by reflection in a horizontal line). Note that these moves are local: there must be a disk containing the initial configuration, and no other strands are involved in the move. We call an untwisting or twisting move at the boundary a *boundary untwisting* or *twisting* move (the lower diagram in the figure).

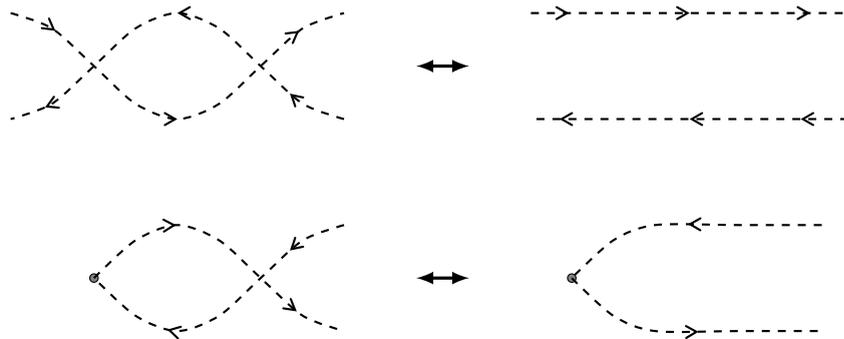


FIGURE 2. Untwisting and twisting moves in a Postnikov diagram. The moves obtained by reflecting these diagrams in a horizontal line are also allowed

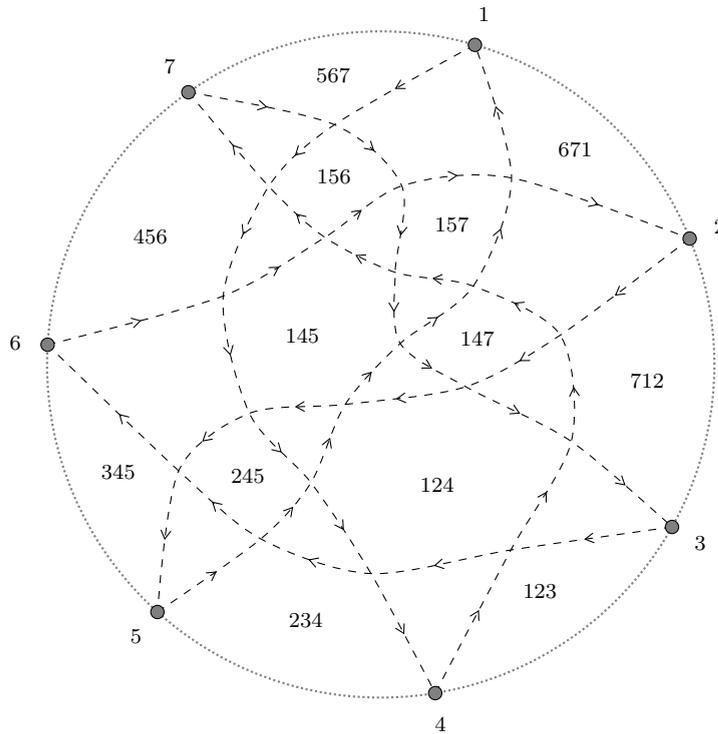


FIGURE 3. A $(3, 7)$ -Postnikov diagram

DEFINITION 2.2. We shall say that a Postnikov diagram is of *reduced type* if no untwisting move or boundary untwisting move (i.e. going from left to right in Figure 2) can be applied to it.

Note that in a Postnikov diagram of reduced type, while the first crossing of strand i is with strand $i - k$, as is required, the second crossing must be with a different strand. Figure 3 shows an example of a $(3, 7)$ -Postnikov diagram which is of reduced type.

A Postnikov diagram divides the interior of the disk into bounded regions, the connected components of the complement of the strands in the diagram. A region not adjacent to the

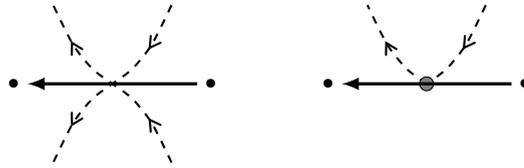


FIGURE 4. Orientation convention for the quiver $Q(D)$

boundary of the disk is called *internal* and the other regions are referred to as a *boundary* region. A region is said to be *alternating* if the strands incident with it alternate in orientation going around the boundary (ignoring the boundary of the disk). It is said to be *oriented* if the strands around its boundary are all oriented clockwise (or all anticlockwise). It is easy to check that every region of a Postnikov diagram must be alternating or oriented.

Each strand divides the disk into two parts, consisting of those regions on the left hand side of the strand (when flowing along it) and those on the right hand side. Each alternating region is labelled with the k -subset I of C_1 consisting of the numbers of those strands which have the region on their left hand side. The labels of the alternating regions are all distinct. We denote the set of labels of D by $\mathcal{C}(D)$. The alternating regions have been labelled in Figure 3, using the convention that a subset $\{i_1, i_2, \dots, i_s\}$ of C_1 is displayed as $i_1 i_2 \dots i_s$.

REMARK 2.3. For $i \in C_0$, let $E_i = [i - k + 1, i] \subset C_1$, i.e. the set of labels of the vertices between edges $i - k$ and i . Then the labels of the boundary alternating regions are precisely the k -subsets E_1, E_2, \dots, E_n (see [27, §3]).

Recall that a quiver Q is a directed graph encoded by a tuple $Q = (Q_0, Q_1, h, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows and $h, t: Q_1 \rightarrow Q_0$, so that each $\alpha \in Q_1$ is an arrow $t\alpha \rightarrow h\alpha$. We will write $Q = (Q_0, Q_1)$, with the remaining data implicit, and we will also regard it as an oriented 1-dimensional CW-complex.

DEFINITION 2.4. The *quiver* $Q(D)$ of a Postnikov diagram D has vertices $Q_0(D) = \mathcal{C}(D)$ given by the labels of the alternating regions of D . The arrows $Q_1(D)$ correspond to intersection points of two alternating regions, with orientation as in Figure 4. The diagram on the right indicates the boundary case, which can also occur in the opposite sense. We refer to the arrows between boundary vertices as *boundary arrows*.

REMARK 2.5. We can embed $Q(D)$ into the disk, with each vertex plotted at some point in the interior of the alternating region it corresponds to, except for boundary regions, in which case we plot the point on the boundary of the disk. Each arrow is drawn within the two regions corresponding to its end-points and passing through the corresponding crossing in D . Boundary arrows are drawn along the boundary.

For example, the quiver of the Postnikov diagram in Figure 3 is shown in Figure 5, embedded as in Remark 2.5.

DEFINITION 2.6. A *plabic graph* [26, §11] is a planar graph embedded into a disk with n vertices on the boundary, each of degree 1, and a colouring of the internal vertices with two colours (which we take to be black and white). In this paper, we will additionally assume

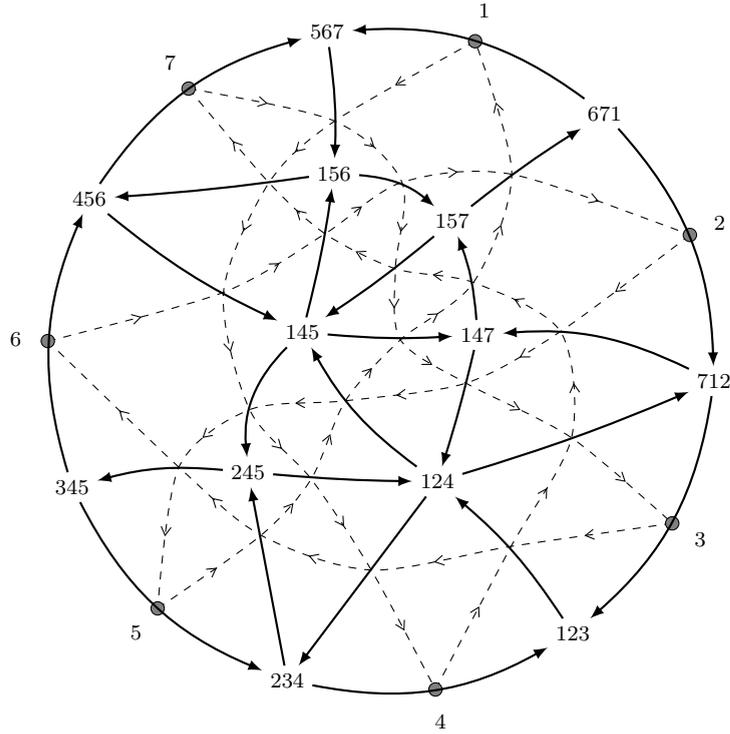


FIGURE 5. The quiver of the Postnikov diagram in Figure 3 (see Remark 3.4)

that the graph is bipartite, i.e. the end points of internal edges have different colours, and that no internal vertex has degree 1. Note that the boundary vertices may best be considered as the mid-points of half-edges, which we also call boundary edges.

Postnikov [26, §14] makes the following definition (see also [16, 2.1]).

DEFINITION 2.7. To any Postnikov diagram D , there is an associated plabic graph $G(D)$, defined as follows. The boundary vertices are those of D , while the internal vertices correspond to the oriented regions of D and are coloured black or white when the boundary of the region is oriented anticlockwise or clockwise, respectively. The internal edges of $G(D)$ correspond to the points of intersection of pairs of oriented regions. For each oriented boundary region of D , there is a boundary edge between the vertex corresponding to that region and the boundary point that it touches.

The graph $G(D)$ can be embedded in the disk, with each internal vertex mapped to a point inside its corresponding oriented region and internal edges drawn as arcs passing through the two oriented regions and their point of intersection. A boundary edge corresponding to a boundary oriented region is drawn as an arc inside this region joining the corresponding internal vertex to the boundary vertex. Thus $G(D)$ is indeed a plabic graph; Figure 6 shows an embedded graph for the Postnikov diagram in Figure 3.

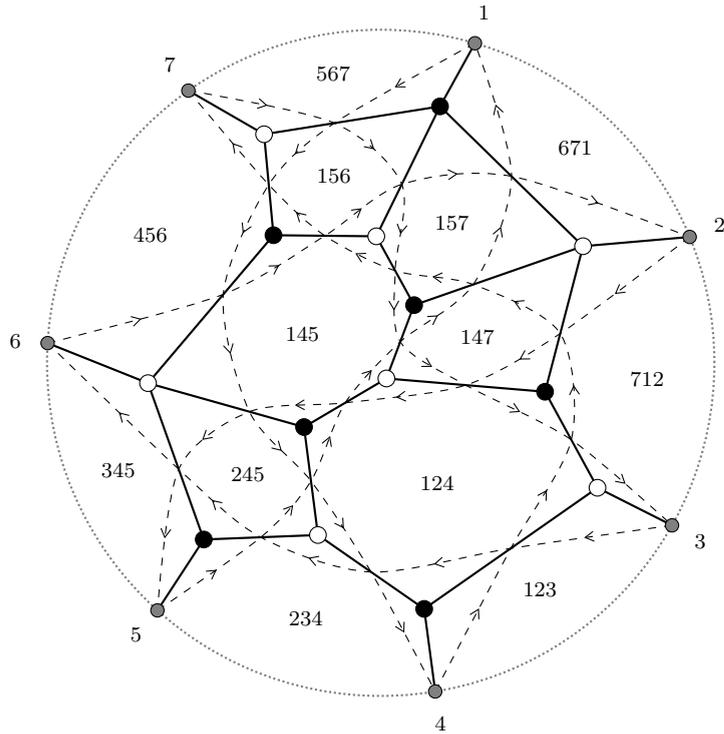


FIGURE 6. The plabic graph corresponding to the Postnikov diagram in Figure 3

REMARK 2.8. Figures 5 and 6 may be considered as pictures of a dimer model, or bipartite field theory (in the sense of Franco [11]), in a disk. We will make this more precise in the next section, adapting the more quiver focussed formalism of Bocklandt [1, 2], Davison [7] and Broomhead [3] to the boundary case.

3. Dimer models with boundary

In this section, we formalise the notion of a dimer model with boundary and show how the quiver of a Postnikov diagram can be interpreted as a dimer model in a disk. Given a quiver Q , we write Q_{cyc} for the set of oriented cycles in Q (up to cyclic equivalence). We start with a more general definition.

DEFINITION 3.1. A quiver with faces is a quiver $Q = (Q_0, Q_1)$, together with a set Q_2 of faces and a map $\partial: Q_2 \rightarrow Q_{\text{cyc}}$, which assigns to each $F \in Q_2$ its boundary $\partial F \in Q_{\text{cyc}}$.

We shall often denote a quiver with faces by the same letter Q , regarded now as the triple (Q_0, Q_1, Q_2) . We say that Q is finite if Q_0, Q_1 and Q_2 are all finite sets. The number of times an arrow $\alpha \in Q_1$ appears in the boundaries of the faces in Q_2 will be called the face multiplicity of α . The (unoriented) incidence graph of Q , at a vertex $i \in Q_0$, has vertices given by the arrows incident with i . The edges between two arrows α, β correspond to the

paths of the form

$$\xrightarrow{\alpha} i \xrightarrow{\beta}$$

occurring in a cycle bounding a face.

DEFINITION 3.2. A (finite, oriented) *dimer model with boundary* is given by a finite quiver with faces $Q = (Q_0, Q_1, Q_2)$, where Q_2 is written as disjoint union $Q_2 = Q_2^+ \cup Q_2^-$, satisfying the following properties:

- (a) the quiver Q has no loops, i.e. no 1-cycles, but 2-cycles are allowed,
- (b) all arrows in Q_1 have face multiplicity 1 (*boundary arrows*) or 2 (*internal arrows*),
- (c) each internal arrow lies in a cycle bounding a face in Q_2^+ and in a cycle bounding a face in Q_2^- ,
- (d) the incidence graph of Q at each vertex is connected.

Note that, by (b), each incidence graph in (d) must be either a line (at a *boundary* vertex) or an unoriented cycle (at an *internal* vertex).

REMARK 3.3. We will only encounter oriented dimer models in this paper, but it is possible to consider unoriented ones by not writing Q_2 as a disjoint union and dropping condition (c). We will also only encounter finite dimer models, but infinite dimer models can also be considered, e.g. the universal cover of any finite dimer model on a torus. One should then add to (d) the condition that each incidence graph is finite, so Q is ‘locally finite’. We choose not to require that the quiver Q is connected. However, note that, if it is, then it is actually strongly connected, because every arrow is contained in a face, whose boundary also includes a path going in the opposite direction (cf. [1, Def. 6.1]). Condition (a) is included to avoid unpleasant degeneracies in Definition 3.5.

If we realise each face F of a dimer model Q as a polygon, whose edges are labelled (cyclically) by the arrows in ∂F , then we may, in the usual way, form a topological space $|Q|$ by gluing together the edges of the polygons labelled by the same arrows, in the manner indicated by the directions of the arrows. Then, arguing as in [1, Lemma 6.4], we see that conditions (b) and (d) ensure that $|Q|$ is a surface with boundary, while (c) means that it can be oriented by declaring the boundary cycles of faces in Q_2^+ to be oriented positive (or anticlockwise) and those of faces in Q_2^- to be negative (or clockwise). Note also that each component of the boundary of $|Q|$ is (identified with) an unoriented cycle of boundary arrows in Q . If Q is a dimer model with boundary, for which $|Q|$ is homeomorphic to a disk, then we will call Q a *dimer model in a disk*.

On the other hand, suppose that we are given an embedding of a finite quiver $Q = (Q_0, Q_1)$ into a compact (oriented) surface Σ with boundary, such that the complement of Q in Σ is a disjoint union of disks, each of which is bounded by a cycle in Q . Then we may make Q into an (oriented) dimer model in the above sense, for which $|Q| \cong \Sigma$, by setting Q_2 to be the set of connected components of the complement of Q in Σ , which can be separated into Q_2^+ and Q_2^- when Σ is oriented.

REMARK 3.4. By Remark 2.5, we have precisely such an embedding of the quiver $Q(D)$, associated to a Postnikov diagram D in a disk (see Figure 5). Thus $Q(D)$ can be considered to be not just a quiver, but actually a dimer model in a disk in the above sense. As is well-known, the Postnikov diagram can be reconstructed by drawing strand segments inside each face of $Q(D)$ (in an embedding into a disk) from the mid-point of each arrow to the

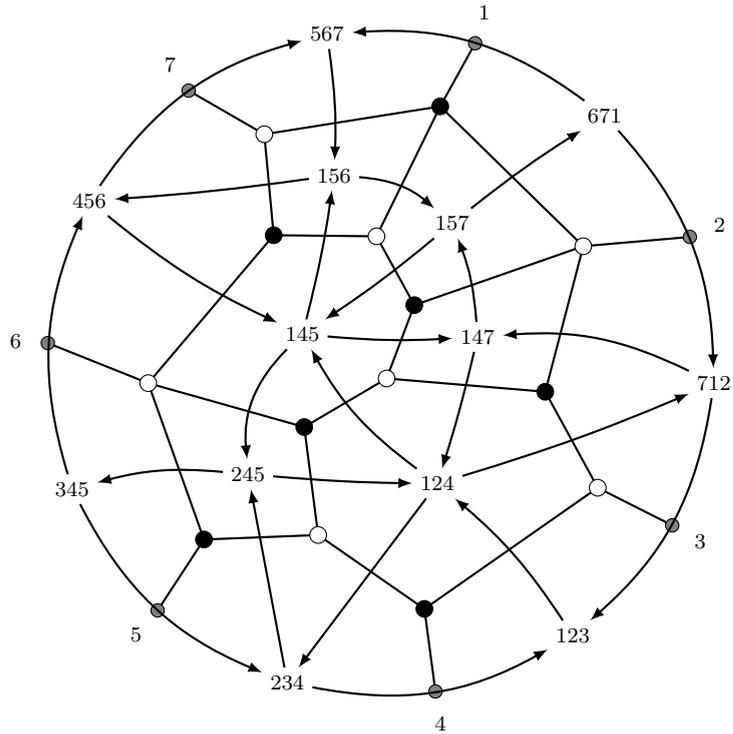


FIGURE 7. The quiver and plabic graph associated to the Postnikov diagram in Figure 3

mid-point of the next arrow in the cycle which is the boundary of the face, oriented in the same direction. (In fact, the strands correspond to the zig-zag paths in the disk; see [2, §5], [16, §§1.7, 2.1]). Note that carrying out this procedure for an arbitrary dimer model in a disk will give a diagram satisfying the local axioms (a1) – (a3) of Definition 2.1, but not necessarily satisfying the global axioms (b1) and (b2).

We may also describe $Q(D)$, as a quiver with faces, directly and more combinatorially as the dual of the plabic graph $G(D)$, as in [11, §2.1] for a general bipartite field theory. In other words, $Q_0(D)$ is in bijection with the set of faces of $G(D)$ and $Q_1(D)$ with the set of edges, with boundary arrows corresponding to boundary edges. An arrow joins the two faces in $G(D)$ that share the corresponding edge and is oriented so that the black vertex is on the left and/or the white vertex is on the right. The faces (plaquettes in [11]) $F \in Q_2^+(D)$ correspond to the internal black vertices, while those in $Q_2^-(D)$ correspond to the white vertices. The boundary ∂F is given by the arrows corresponding to the edges incident with the internal vertex of $G(D)$ corresponding to F , ordered anticlockwise round black vertices and clockwise round white ones. This duality is illustrated in Figure 7, for $Q(D)$ as in Figure 5 and $G(D)$ as in Figure 6.

DEFINITION 3.5. Given a dimer model with boundary Q , we define the *dimer algebra* A_Q as follows. For each internal arrow $\alpha \in Q_1$, there are (unique) faces $F^+ \in Q_2^+$ and $F^- \in Q_2^-$ such that $\partial F^\pm = \alpha p_\alpha^\pm$, for paths p_α^+ and p_α^- from $h\alpha$ to $t\alpha$. Then the dimer algebra A_Q is

the quotient of the path algebra $\mathbb{C}Q$ by the relations

$$p_\alpha^+ = p_\alpha^-, \tag{3.1}$$

for every internal arrow $\alpha \in Q_1$.

REMARK 3.6. Note that the orientation is not strictly necessary to define A_Q ; we only need to know that F^\pm are the two faces that contain the internal arrow α in their boundaries, but not which is which. On the other hand, given the orientation, we may also define a (super)potential W_Q by the usual formula (e.g. [12, §2])

$$W_Q = \sum_{F \in Q_2^+} \partial F - \sum_{F \in Q_2^-} \partial F,$$

defined up to cyclic equivalence. Then A_Q may also be described as the quotient of the path algebra $\mathbb{C}Q$ by the so-called ‘F-term’ relations

$$\partial_\alpha(W_Q) = 0,$$

for each *internal* arrow α in Q , where ∂_α is the usual cyclic derivative (e.g. [15, §1.3] or [1, §3]). Thus, in the absence of boundary arrows in Q , the algebra A_Q is the usual Jacobi (or superpotential) algebra (e.g. [1, §3], [3, §2.1.3]).

In the boundary case, the idea of only considering F-term relations for internal arrows has arisen independently in work of Franco [11, §6.1] and Buan-Iyama-Reiten-Smith [4, Defn. 1.1]. In the latter case, a slightly different approach is used, whereby any arrow joining two boundary (or frozen) vertices is considered to be frozen and hence does not contribute an F-term relation, while in our case, we may have internal arrows with both end-points being boundary vertices (see Figures 23 and 28). Buan-Iyama-Reiten-Smith [4] give a description [4, Thm. 6.6] of the endomorphism algebras of some cluster-tilting objects over preprojective algebras as frozen Jacobian algebras in the sense of [4, Defn 1.1]. Demonet-Luo [8] give a 2-Calabi-Yau categorification \mathcal{C} of the Grassmannian $Gr(2, n)$ using frozen Jacobian algebras in the sense of [4, Defn. 1.1]; these algebras are the endomorphism algebras of cluster-tilting objects in \mathcal{C} (see [8, Thm. 1.3]).

DEFINITION 3.7. We write A_D for the dimer algebra $A_{Q(D)}$ associated to the dimer model $Q(D)$. It follows from the defining relations that, for any vertex $I \in Q_0(D)$, the product in A_D of the arrows in any cycle that starts at I and bounds a face is the same. We denote this element by u_I , and write

$$u = \sum_{I \in Q_0(D)} u_I. \tag{3.2}$$

It similarly follows from the relations that u commutes with every arrow and hence is in the centre of A_D .

REMARK 3.8. A dimer algebra is a special case of an algebra defined by a quiver Q with commutation relations, that is, it is a quotient $\mathbb{C}Q/I$, where the ideal I is generated by $\{p_i - q_i : i \in \mathcal{I}\}$ for paths p_i and q_i in Q with the same start and end points. Any such algebra has a couple of elementary properties, which we note down for future reference and provide proofs for the convenience of the reader, although these properties seem well-known and ‘obvious’. Firstly,

(a) every path in Q gives a non-zero element of $\mathbb{C}Q/I$.

This is an immediate corollary of a stronger property, that requires one to first observe that commutation relations define a natural equivalence relation \sim on the set of paths in Q ,

generated by requiring that $p \sim q$ if p has a subpath p_i and q is obtained from p by replacing p_i with q_i , for some $i \in \mathcal{I}$. Then, secondly,

(b) the equivalence classes of \sim form a basis of $\mathbb{C}Q/I$.

Note that any equivalence class \bar{p} of paths does determine a well-defined element $p + I$ of $\mathbb{C}Q/I$ and these elements evidently span. To see that they are independent, observe that there is a well-defined algebra $\mathbb{C}(Q/\sim)$ with basis given by the set of equivalence classes of \sim , with multiplication given by concatenation, where possible, and zero otherwise, extended linearly. The natural map

$$\pi: \mathbb{C}Q \rightarrow \mathbb{C}(Q/\sim): p \mapsto \bar{p}$$

has each $p_i - q_i$, for $i \in \mathcal{I}$, in its kernel and so induces a map $\bar{\pi}: \mathbb{C}Q/I \rightarrow \mathbb{C}(Q/\sim)$, which is the inverse of the map $\bar{p} \mapsto p + I$.

Alternatively, (a) has a direct proof as follows. Let M be the algebra of square matrices over \mathbb{C} of size $|Q_0|$. Then (numbering of the vertices Q_0) there is a morphism of algebras $\theta: \mathbb{C}Q \rightarrow M$ taking the idempotent e_i to the elementary matrix E_{ii} and any arrow from i to j to the matrix E_{ji} . The kernel of θ contains every possible commutation relation, and hence contains I , so θ induces an algebra morphism $\bar{\theta}: \mathbb{C}Q/I \rightarrow M$. However, $\bar{\theta}$ sends every path in Q to a non-zero element of M .

4. Weights

In this section, we introduce a weighting on the arrows in the quiver $Q = Q(D)$ of a Postnikov diagram D , via elements of $\mathbb{N}C_0$. Our main aim is to compute the weight of the boundary of a face of $Q(D)$ and the sum of the weights of the arrows incident with a vertex of $Q(D)$.

These results will be then used in Section 6 in order to find the first step in a path between any pair of vertices in Q_0 whose weight does not include every element of C_0 ; we call such a path a *minimal path*. Such a minimal path itself will be constructed in Section 8, and is a key component in the proof of surjectivity of the morphism we shall construct in Section 10 from the total algebra to the endomorphism algebra.

DEFINITION 4.1. For any arrow $\alpha: I \rightarrow J$ in $Q_1(D)$, let $c \in C_1$ be the number of the strand crossing α from right to left and $d \in C_1$ the number of the strand crossing α from left to right. In other words, $J = I - c + d$. Identifying a subset of C_0 with the sum of its elements in $\mathbb{N}C_0$, we give α the *weight*

$$w_\alpha = (c, d)_0;$$

see Figure 8. The weight of a path in $Q(D)$ is then defined to be the sum of the weights of the arrows in the path.

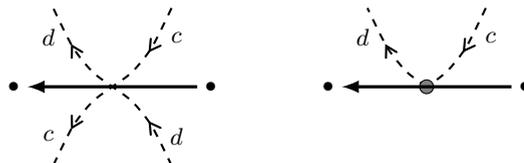


FIGURE 8. Internal and boundary arrows of weight $(c, d)_0$

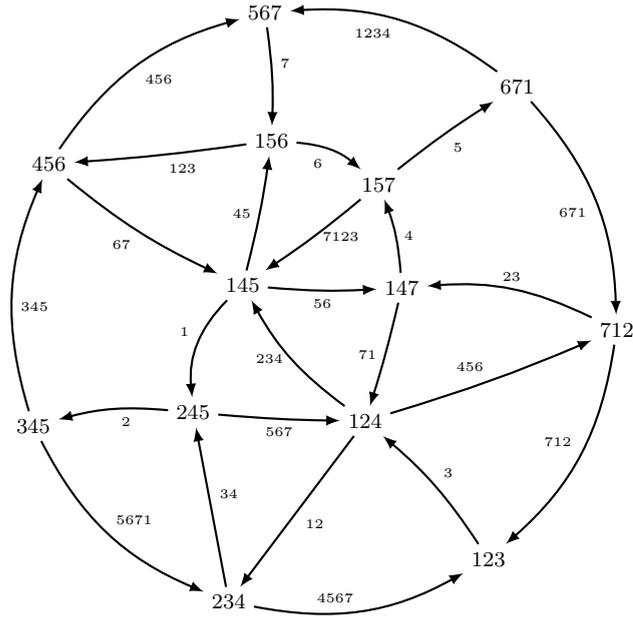


FIGURE 9. Weights on the quiver of the Postnikov diagram in Figure 5

REMARK 4.2. Note that by definition the weight of an arrow cannot be the whole of C_0 . It also cannot be zero (i.e. the empty set) because crossing strands are distinct by Definition 2.1(b1). The weights of the arrows in the quiver in Figure 5 are shown in Figure 9.

LEMMA 4.3. Let F be a face in $Q_2(D)$. The ordering of the arrows in ∂F induces a cyclic ordering on the strands in D entering F on its boundary. Then the starting points of these strands appear on the boundary of the disk in the same order. The same result holds for the end points of the strands exiting F .

Proof. We argue as in the proof of necessity in [2, Thm. 6.6]. For an arrow α in ∂F , let R_α denote the part of the strand entering F at α from its starting point until its crossing point with α (we refer to this as a *backward ray*). Note that if α is a boundary arrow then R_α is a single point.

Suppose that α and β are arrows in ∂F and that R_α and R_β cross outside F . Let $\pi(\alpha, \beta)$ be the path in ∂F strictly between α and β which is homotopic to the composition of the path from the intersection of α with R_α , backwards along R_α to its last crossing point with R_β , and the path from this crossing point to the intersection of β and R_β .

We show by induction on l that there are no crossings outside F between R_α and R_β for which the length of $\pi(\alpha, \beta)$ is l . If R_α and R_β cross outside F and the length of $\pi(\alpha, \beta)$ is zero, then we have a contradiction to Definition 2.1(b2). So fix $l \geq 1$ and suppose the result is true for smaller l . If R_α and R_β cross and $\pi(\alpha, \beta)$ has length l , then, for any arrow γ in $\pi(\alpha, \beta)$, R_γ must cross R_α or R_β . Since $\pi(\alpha, \beta) = \pi(\gamma, \beta) \circ \gamma \circ \pi(\alpha, \gamma)$, we see that both $\pi(\alpha, \gamma)$ and $\pi(\gamma, \beta)$ have length less than l , so we have a contradiction and R_α and R_β cannot cross.

By induction, none of the backward rays R_α cross, and the result follows. □

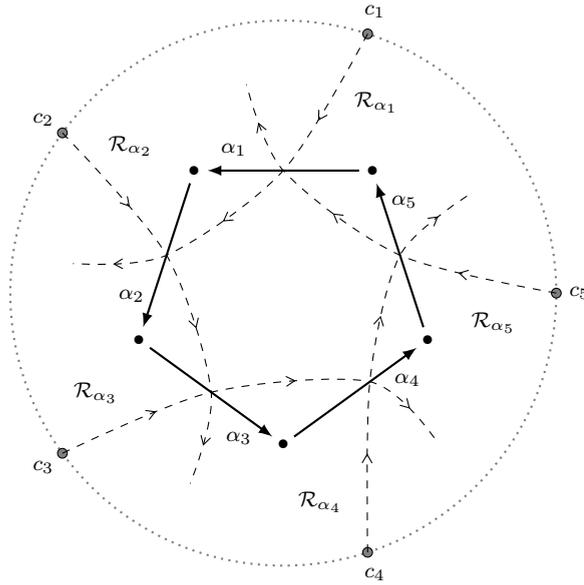


FIGURE 10. The ordering of strands around a face

COROLLARY 4.4. For every $F \in Q_2(D)$, the weight of ∂F is C_0 .

Proof. Suppose first that $F \in Q_2^+(D)$. Let $\alpha_1, \dots, \alpha_r$ be the arrows in $\partial(F)$, taken in anticlockwise order. Let c_i be the number of strand \mathcal{R}_{α_i} , for $i = 1, \dots, r$. Then the weight of α_i is $(c_i, c_{i-1})_0$ (with subscripts modulo r), and the result follows from Lemma 4.3. A similar argument applies to faces in $Q_2^-(D)$. \square

REMARK 4.5. It follows from Corollary 4.4 that the weighting of arrows $\alpha \mapsto w_\alpha$ given in Definition 4.1 induces an $\mathbb{N}C_0$ -grading on A_D (see Definition 3.5).

REMARK 4.6. For $i \in C_0$, let $\mathcal{P}_i = \{\alpha \in Q_1(D) : i \in w_\alpha\}$. By Corollary 4.4, the cycle bounding each face in $Q_2(D)$ contains exactly one arrow from \mathcal{P}_i . Thus each such \mathcal{P}_i may be considered as a perfect matching on $Q(D)$ or, equivalently, on the dual $G(D)$ (see [11, §2.2]).

Next, we consider the strands around a vertex in a similar way. We fix an internal vertex I in $Q_0(D)$, and suppose that there are $2r$ arrows incident with I in $Q(D)$. We label them α_i , $i = 1, 2, \dots, 2r$, in order anticlockwise around I , where the α_i for i odd are outgoing arrows and the others are incoming, and treat the subscripts modulo $2r$. Let I_i be the end-point of α_i not equal to I . For i odd, we suppose that strand f_i crosses α_i from right to left (looking along the arrow), and f_{i+1} crosses α_i from left to right, so that α_i has weight $(f_i, f_{i+1})_0$ (again treating subscripts modulo $2r$). Note that if i is even, then α_i also has weight $(f_i, f_{i+1})_0$. See Figure 11 for the case $r = 3$. Note that we have:

$$I_i = \begin{cases} I - f_i + f_{i+1}, & i \text{ odd;} \\ I - f_{i+1} + f_i, & i \text{ even.} \end{cases} \tag{4.1}$$

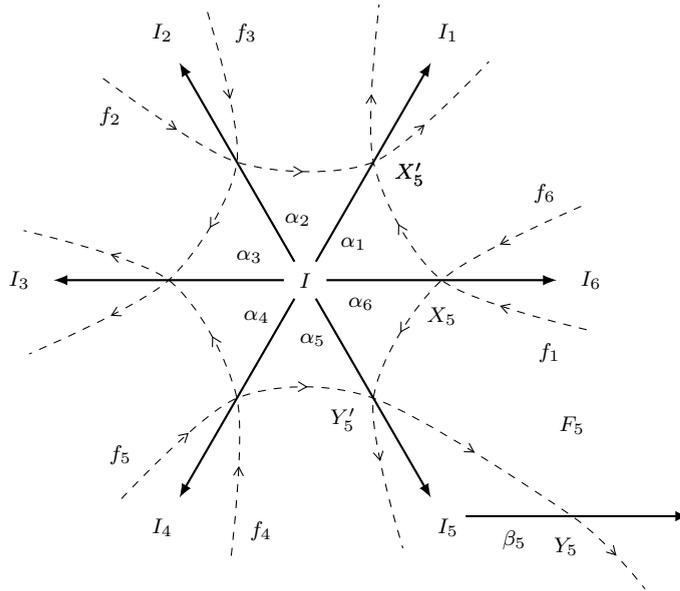


FIGURE 11. The arrows incident with an internal vertex I

- LEMMA 4.7. Let I be an internal vertex of $Q(D)$, with notation as defined above. Then
- (a) For each i , the strand labels f_i, f_{i+1}, f_{i+2} occur in clockwise order in C .
 - (b) The strand labels $f_1, f_3, \dots, f_{2r-1}$ appear in anticlockwise order in C , and
 - (c) The strand labels f_2, f_4, \dots, f_{2r} appear in anticlockwise order in C .

Proof. We argue as in the proof of necessity in [2, Thm. 6.6]. Fix $i \in [1, 2r]$, odd, and let F_i be the face whose boundary includes $\alpha_i \alpha_{i+1}$. Let β_i be the arrow following α_i in ∂F_i . The strands leaving F_i at α_i, β_i , and α_{i+1} (ordered clockwise around ∂F_i) are f_i, f_{i+1} and f_{i+2} respectively. Part (a) for i odd then follows from an application of Lemma 4.3.

Let X_i be the point on α_{i+1} where strand f_{i+2} leaves F_i and let Y_i be the point on β_i where strand f_i leaves F_i (note that Y_i could coincide with X_i). By Lemma 4.3, strands f_{i+2} and f_i do not cross after they leave F_i .

Let X'_i be the next crossing point of strand f_{i+2} with an arrow after X_i (i.e. on arrow α_{i+2}), and let Y'_i be the previous crossing point of strand f_i with an arrow before Y_i (i.e. on arrow α_i). Since the part of strand f_i between Y'_i and Y_i lies inside F_i and the part of strand f_{i+2} between X_i and X'_i lies outside F_i , it follows that strand f_{i+2} , after X'_i , and strand f_i , after Y'_i , do not cross. See Figure 11 for an illustration in the case $i = 5$. Part (b) then follows, via an argument similar to that used in Lemma 4.3.

The proofs of part (a) for i even and part (c) are similar to the above. □

Properties (b) and (c) of Lemma 4.7 are similar to the notion of ‘proper ordering’ introduced by Gullota [17, §3] for dimer models on a torus T^2 . In this case, each strand (or zig-zag path) has a two component winding number, i.e. its homology class in $H_1(T^2) \cong \mathbb{Z}^2$. The dimer model is said to be properly ordered if the circular order of the strands around any vertex of the bipartite graph is the same as the circular order of the directions of their winding numbers.

DEFINITION 4.8. Let I be an internal vertex in $Q_0(D)$ and let the f_i be defined as above. Then consecutive intervals $(f_i, f_{i+1})_0$ of vertices follow on from each other, and do not overlap. Thus gluing these intervals together creates a path on C which we denote $\xi(I)$.

We have:

PROPOSITION 4.9. Fix an internal vertex I of $Q(D)$ incident with $2r$ arrows. Then the path $\xi(I)$ wraps around C exactly $r - 1$ times.

Proof. By Lemma 4.7(b), we have:

$$\sum_{i=1}^r (f_{2i+1}, f_{2i-1})_0 = C_0, \quad (4.2)$$

By Lemma 4.7(a), we have

$$(f_{2i-1}, f_{2i+1})_0 = (f_{2i-1}, f_{2i})_0 + (f_{2i}, f_{2i+1})_0,$$

So

$$(f_{2i+1}, f_{2i-1})_0 = C_0 - (f_{2i-1}, f_{2i})_0 - (f_{2i}, f_{2i+1})_0.$$

Hence, by (4.2),

$$\sum_{i=1}^r (C_0 - (f_{2i-1}, f_{2i})_0 - (f_{2i}, f_{2i+1})_0) = C_0.$$

So

$$\sum_{i=1}^r ((f_{2i-1}, f_{2i})_0 + (f_{2i}, f_{2i+1})_0) = (r - 1)C_0. \quad (4.3)$$

As mentioned above, consecutive intervals $(f_i, f_{i+1})_0$ of vertices follow on from each other and the result follows. \square

We next consider the case of a boundary vertex E_j in $Q(D)$. If there is an arrow from E_j to E_{j+1} (respectively, E_{j-1}) we take α_{out}^+ (respectively, α_{out}^-) to be this arrow; otherwise we take it to be the arrow outgoing from E_j immediately clockwise (respectively, anticlockwise) of this arrow. Let $W_{\text{out}}(E_j)$ denote the set of all arrows (whether outgoing or ingoing) incident with E_j in the wedge clockwise of α_{out}^+ and anticlockwise of α_{out}^- (including both α_{out}^+ and α_{out}^-). Let r_{out} be the number of arrows in $W_{\text{out}}(E_j)$ starting at E_j , so we have that $|W_{\text{out}}| = 2r_{\text{out}} - 1$. See Figure 12 for an illustration.

If there is an arrow from E_{j+1} (respectively, E_{j-1}) to E_j we take α_{in}^+ (respectively, α_{in}^-) to be the this arrow; otherwise we take it to be the arrow incoming to E_j immediately clockwise (respectively, anticlockwise) of this arrow. Let $W_{\text{in}}(E_j)$ denote the set of all arrows (whether outgoing or ingoing) incident with E_j in the wedge clockwise of α_{in}^+ and anticlockwise of α_{in}^- (including both α_{in}^+ and α_{in}^-). Let r_{in} be the number of arrows in $W_{\text{in}}(E_j)$ ending at E_j , so we have that $|W_{\text{in}}| = 2r_{\text{in}} - 1$.

We write

$$W_{\text{out}}(E_j) = \{\alpha_1, \alpha_2, \dots, \alpha_{2r_{\text{out}}-1}\},$$

numbering the arrows anticlockwise around E_j . Note that the α_i for i odd are outgoing from E_j and the others are incoming to E_j .

There are two possibilities for arrows incident with E_j not lying in $W_{\text{out}}(E_j)$. If the arrow between E_j and E_{j-1} (respectively, E_{j+1}) points towards E_j (and so does not lie in

$W_{\text{out}}(E_j)$), label it α_0 (respectively, α_{2r}), where $r = r_{\text{out}}$. Thus the arrows incident with I are $\alpha_0, \alpha_1, \dots, \alpha_{2r-1}, \alpha_{2r}$, in anticlockwise order around i , with the first and last arrows in the list appearing only when they are defined.

As before, we set I_i be the end-point of α_i not equal to I . And for i odd, we suppose that strand f_i crosses α_i from right to left (looking along the arrow), and f_{i+1} crosses α_i from left to right, so that α_i has weight $(f_i, f_{i+1})_0$. Note that if i is even, then α_i also has weight $(f_i, f_{i+1})_0$. Note that statement (4.1) holds in this case also. See Figure 12 for an illustration of the possible cases.

Since $E_j = E_{j-1} \cup \{j\} \setminus \{j-k\}$ and strand f_1 crosses the arrow between E_{j-1} and E_j (whichever direction it is in), with E_j on its left, we always have that $f_1 = j$. Similarly, we always have that $f_{2r} = j+1$.

We make similar definitions for the wedge $W_{\text{in}}(E_j)$.

DEFINITION 4.10. Let E_j be an external vertex in $Q_0(D)$ and let the f_i be defined as above. Then consider the intervals $(f_i, f_{i+1})_0$ of vertices, for $i = 1, 2, \dots, 2r_{\text{out}} - 1$. Consecutive intervals in this set follow on from each other exactly and do not overlap. We denote the path obtained by gluing these intervals together by $\xi_{\text{out}}(E_j)$. Similarly, we have a path $\xi_{\text{in}}(E_j)$.

PROPOSITION 4.11. *Let E_j be a boundary vertex of Q . Then we have the following.*

- (a) *The path $\xi_{\text{out}}(E_j)$ starts at vertex j , wraps $r_{\text{out}} - 1$ times around C_0 times, then ends by revisiting the vertex j .*
- (b) *The path $\xi_{\text{in}}(E_j)$ starts at vertex $j - k$, wraps $r_{\text{in}} - 1$ times around C_0 times, then ends by revisiting the vertex $j - k$.*

Proof. We prove part (a) only; the proof of (b) is similar. We set $r = r_{\text{out}}$. The result will follow if we can show that:

$$\sum_{\alpha \in W_{\text{out}}(E_j)} w_\alpha = (r - 1)C_0 + j, \tag{4.4}$$

since the weights of the α_i are intervals forming the path $\xi_{\text{out}}(E_j)$. Arguing as in Lemma 4.7, we see that:

- (a) $f_1, f_3, \dots, f_{2r-1}$ appear in anticlockwise order in C_0 , and
- (b) for $1 \leq i \leq 2r - 3$, f_i, f_{i+1}, f_{i+2} occur in clockwise order in C_0 .

Hence

$$\sum_{i=1}^{r-1} (f_{2i+1}, f_{2i-1})_0 = C_0 - (f_1, f_{2r-1})_0,$$

and, for $1 \leq i \leq r - 1$,

$$(f_{2i-1}, f_{2i+1})_0 = (f_{2i-1}, f_{2i})_0 + (f_{2i}, f_{2i+1})_0 = w_{\alpha_{2i-1}} + w_{\alpha_{2i}}.$$

So, for $1 \leq i \leq r - 1$,

$$(f_{2i+1}, f_{2i-1})_0 = C_0 - w_{\alpha_{2i-1}} - w_{\alpha_{2i}},$$

and we have:

$$\sum_{i=1}^{r-1} (C_0 - w_{\alpha_{2i-1}} - w_{\alpha_{2i}}) = C_0 - (f_1, f_{2r-1})_0. \tag{4.5}$$

If the arrow between E_j and E_{j+1} points towards E_{j+1} then it is labelled α_{2r-1} . Since $E_j = E_{j+1} \cup \{j-k+1\} \setminus \{j+1\}$ and strand f_{2r-1} cuts this arrow with E_j on its left, we

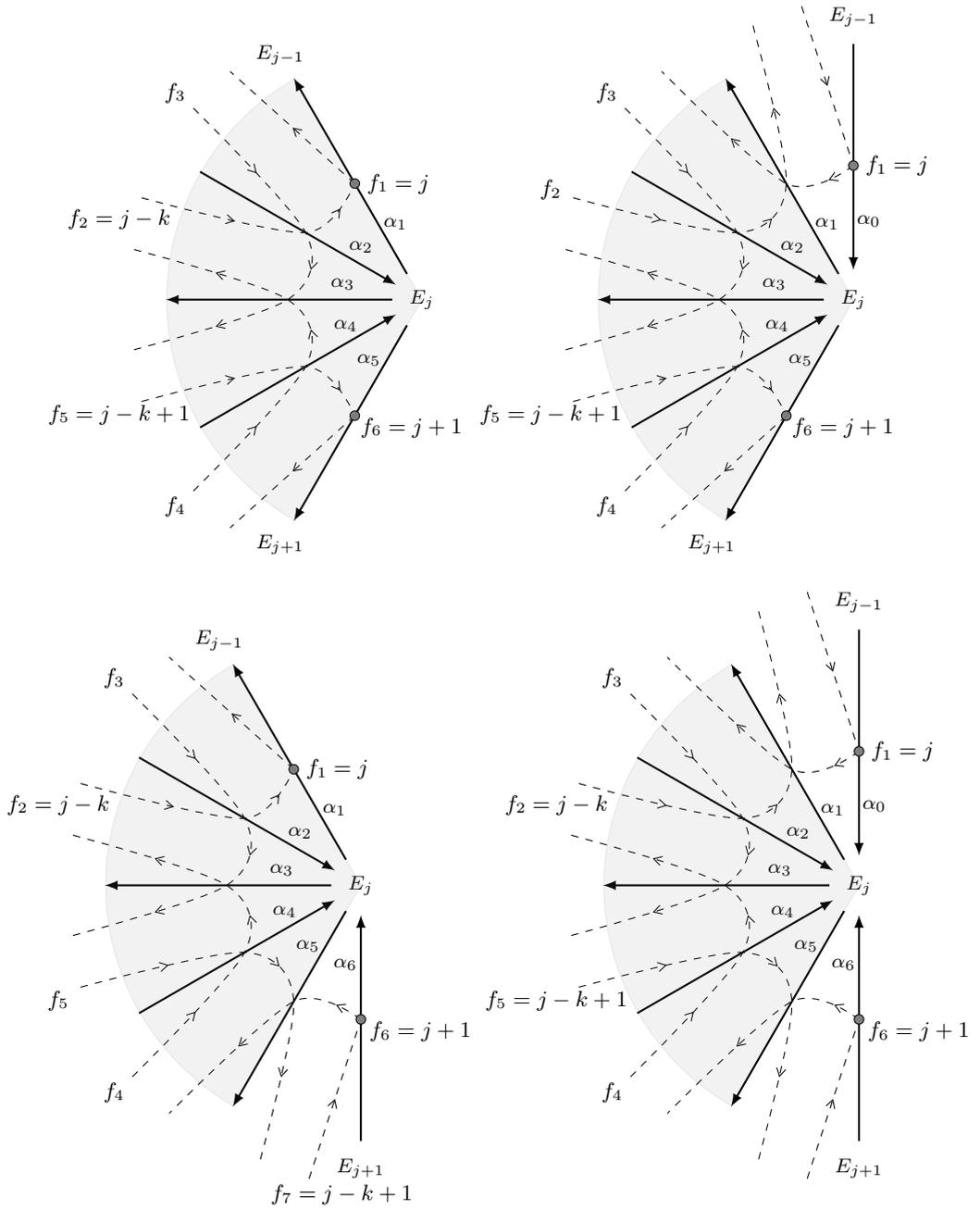


FIGURE 12. The arrows incident with a boundary vertex E_j . The shaded region indicates the wedge W_{out}

have that $f_{2r-1} = j - k + 1$. The other strand crossing this arrow is $f_{2r} = j + 1$. Recall also that $f_1 = j$. Hence, in this case, (4.5) can be rewritten as

$$\sum_{i=1}^{2r-2} w_{\alpha_i} = (r-2)C_0 + (j, j-k+1)_0$$

We then have, using the fact that $w_{\alpha_{2r-1}} = (f_{2r-1}, f_{2r})_0 = (j-k+1, j+1)_0$, that

$$\sum_{i=1}^{2r-1} w_{\alpha_i} = (r-2)C_0 + (j, j-k+1)_0 + (j-k+1, j+1)_0,$$

implying (4.4) as required.

If the arrow between E_j and E_{j+1} points towards E_j then it is labelled α_{2r} . Then strand $f_{2r} = j + 1$ crosses this arrow from right to left and $f_{2r+1} = j - k + 1$ crosses it from left to right. Arguing as above, we obtain the following variation of (4.5):

$$\sum_{i=1}^r (C_0 - w_{\alpha_{2i-1}} - w_{\alpha_{2i}}) = C_0 - (f_1, f_{2r+1})_0,$$

which can be rewritten as:

$$\sum_{i=1}^{2r} w_{\alpha_i} = (r-1)C_0 + (j, j-k+1)_0.$$

We then have, using the fact that $w_{\alpha_{2r}} = (j+1, j-k+1)_0$, that

$$\sum_{i=1}^{2r-1} w_{\alpha_i} = (r-1)C_0 + (j, j-k+1)_0 - (j+1, j-k+1)_0,$$

implying (4.4) as required. □

5. Angles and isoradial embedding

Fix a Postnikov diagram D . In this section we investigate the implications of the results in the previous section for embedding $Q(D)$ isoradially into a planar polygon. This leads us to a notion of consistent boundary R -charge. We compare this embedding with results of [25]. The results in this section are not needed for the main result of the paper, but seem to be interesting from the point of view of understanding $Q(D)$ as a dimer model.

For each $i \in C_0$, let $\theta_i \in (0, 2\pi)$ be an angle, with the property that $\sum_{i \in C_0} \theta_i = 2\pi$. For each arrow α in $Q_1(D)$, let

$$\theta_\alpha = \sum_{i \in w_\alpha} \theta_i.$$

We then have the following result. It has a geometric interpretation which will be explained in Corollary 5.4.

LEMMA 5.1. *The θ_α satisfy the following conditions:*

(a) *For all $F \in Q_2(D)$,*

$$\sum_{\alpha \in \partial F} \theta_\alpha = 2\pi.$$

(b) *Let $I \in Q_0(D)$ be an internal vertex. Then we have:*

$$\sum_{\alpha, \beta} \pi - \frac{1}{2}(\theta_\alpha + \theta_\beta) = 2\pi, \tag{5.1}$$

where the sum is over all pairs of arrows α, β incident with I such that α, β are consecutive arrows in a face of $Q(D)$. Equivalently,

$$\sum_{\alpha} (\pi - \theta_{\alpha}) = 2\pi, \quad (5.2)$$

where the sum is over all arrows α incident with I .

(c) Let $I = E_j \in Q_0(D)$ be an external vertex. Then we have:

$$\sum_{\alpha, \beta} \pi - \frac{1}{2}(\theta_{\alpha} + \theta_{\beta}) = \pi - \frac{1}{2}(\theta_j + \theta_{j-k}). \quad (5.3)$$

where the sum is over all pairs of arrows α, β incident with I such that α, β are consecutive arrows in a face of $Q(D)$. Equivalently,

$$\sum_{\alpha \in W_{\text{out}}(E_j)} \pi - \theta_{\alpha} = \pi - \theta_j. \quad (5.4)$$

Proof. Part (a) is equivalent to Corollary 4.4. In parts (b) and (c), we use the same notation as above for the arrows incident with I in $Q(D)$. For part (b), note that $\sum_{i=1}^{2r} \theta_{\alpha_i} = 2(r-1)\pi$, by equation (4.3) in the proof of Proposition 4.11. This is easily seen to be equivalent to the two formulas in (b).

We now consider part (c). Let γ^- be the arrow between E_j and E_{j-1} , and let γ^+ be the arrow between E_j and E_{j+1} . Then we have

$$w_{\gamma^-} = \begin{cases} [j, j-k-1] & \text{if } \gamma^- \text{ points away from } E_j; \\ C_0 - [j, j-k-1] & \text{if } \gamma^- \text{ points towards } E_j; \end{cases} \quad (5.5)$$

$$w_{\gamma^+} = \begin{cases} [j-k+1, j] & \text{if } \gamma^+ \text{ points away from } E_j; \\ C_0 - [j-k+1, j] & \text{if } \gamma^+ \text{ points towards } E_j. \end{cases}$$

The sum in part (c) can be rewritten as follows:

$$\begin{aligned} \sum_{\alpha, \beta} \pi - \frac{1}{2}(\theta_{\alpha} + \theta_{\beta}) &= (\pi - \frac{1}{2}(\theta_{\alpha_0} + \theta_{\alpha_1})) + (\pi - \frac{1}{2}(\theta_{\alpha_{2r-1}} + \theta_{\alpha_{2r}})) + \sum_{i=1}^{2r-2} \pi - \frac{1}{2}(\theta_{\alpha_i} + \theta_{\alpha_{i+1}}) \\ &= (\pi - \frac{1}{2}(\theta_{\alpha_0} + \theta_{\alpha_1})) + (\pi - \frac{1}{2}(\theta_{\alpha_{2r-1}} + \theta_{\alpha_{2r}})) + \frac{1}{2}(\theta_{\alpha_1} + \theta_{\alpha_{2r-1}}) - \pi + \sum_{i=1}^{2r-1} (\pi - \theta_{\alpha_i}). \end{aligned}$$

where the first (respectively, second) term on the right hand side appears if the arrow between E_j and E_{j-1} (respectively, E_{j+1}) points towards E_j . Hence,

$$\sum_{\alpha, \beta} \pi - \frac{1}{2}(\theta_{\alpha} + \theta_{\beta}) - \sum_{i=1}^{2r-1} (\pi - \theta_{\alpha_i}) + \pi - \theta_j = \frac{1}{2}(\psi_1 + \psi_2 - 2\theta_j), \quad (5.6)$$

where

$$\psi_1 = \begin{cases} \theta_{\alpha_1} & \text{if } \gamma^- \text{ points away from } E_j; \\ 2\pi - \theta_{\alpha_0} & \text{if } \gamma^- \text{ points towards } E_j; \end{cases}$$

and

$$\psi_2 = \begin{cases} \theta_{\alpha_{2r-1}} & \text{if } \gamma^+ \text{ points away from } E_j; \\ 2\pi - \theta_{\alpha_{2r}} & \text{if } \gamma^+ \text{ points towards } E_j. \end{cases}$$

Note that $\gamma^- = \alpha_1$ if γ^- points away from E_j and $\gamma^- = \alpha_0$ otherwise; see Figure 11. Similarly, $\gamma^+ = \alpha_{2r-1}$ if γ^+ points away from E_j and $\gamma^+ = \alpha_{2r}$ otherwise. Applying (5.5),

we see that the right hand side of (5.6) simplifies to

$$\frac{1}{2} \left(-2\theta_j + \sum_{i \in [j, j-k-1]} \theta_i + \sum_{i \in [j-k+1, j]} \theta_i \right) = \pi - \frac{1}{2}(\theta_j + \theta_{j-k}).$$

Therefore, recalling that $W_{\text{out}}(E_j) = \{\alpha_1, \dots, \alpha_{2r-1}\}$, equation (5.4) is equivalent to equation (5.3), which holds by equation (4.4) in the proof of Proposition 4.11. \square

REMARK 5.2. For each arrow $\alpha \in Q_1(D)$, let $R_\alpha = \theta_\alpha/\pi$. Then Lemma 5.1(a) states that for all $F \in Q_2(D)$,

$$\sum_{\alpha \in \partial F} R_\alpha = 2. \tag{5.7}$$

Equation (5.2) becomes:

$$\sum_{\alpha} (1 - R_\alpha) = 2, \tag{5.8}$$

for every internal vertex $I \in Q_0(D)$, where the sum is over all arrows α incident with I . Equation (5.4) becomes:

$$\sum_{\alpha \in W_{\text{out}}(E_j)} 1 - R_\alpha = 1 - \theta_j/\pi. \tag{5.9}$$

We can regard equations (5.7)–(5.9) as the definition of a *consistent boundary R-charge*.

Following [2, §6], we make the following definition (noting that Bocklandt uses the term ‘embedding with isoradial cycles’).

DEFINITION 5.3. Let Q be a quiver with faces. We will say that a map $v: Q_0 \rightarrow \mathbb{R}^2$ is an *isoradial embedding* of Q if the following hold:

- (i) The map v induces an embedding of (Q_0, Q_1) into \mathbb{R}^2 , taking an arrow to the line segment between the images of its endpoints.
- (ii) For each $F \in Q_2$ with

$$\partial F = I_1 \rightarrow I_2 \rightarrow \dots \rightarrow I_r \rightarrow I_1,$$

the points $v(I_1), v(I_2), \dots, v(I_r)$, taken in order, form a polygon \mathbb{T}_F inscribed on a unit circle.

- (iii) Two polygons of form \mathbb{T}_F which have non-empty intersection can intersect only in a single common edge or point.

Note that in an isoradial embedding, two polygons \mathbb{T}_F intersect in an edge (respectively, a point, the empty set) if and only if the corresponding faces have boundaries containing a unique common arrow (respectively, a unique common vertex, no common vertex) in Q . The strictly convex polygons \mathbb{T}_F tile a subset of \mathbb{R}^2 , which we shall refer to as the *image* of Q under v .

COROLLARY 5.4. *If D is of reduced type (see Definition 2.2), then the quiver with faces $Q(D)$ can be isoradially embedded into an n -sided polygon with vertices E_j , $j = 1, \dots, n$, and internal angle $\pi - \frac{1}{2}(\theta_j + \theta_{j-k})$ at vertex E_j .*

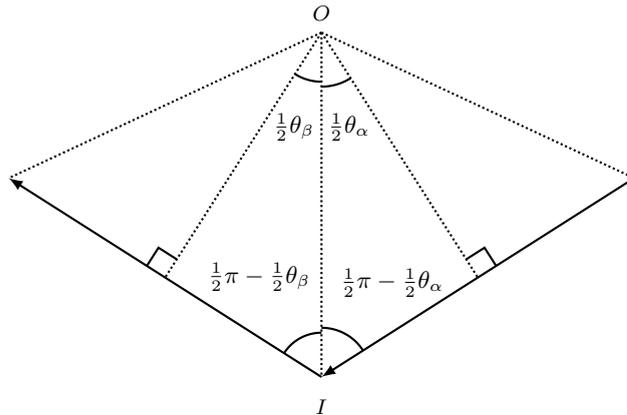


FIGURE 13. The angle between successive arrows incident with a vertex is $\pi - \frac{1}{2}(\theta_\alpha + \theta_\beta)$

Proof. We modify the discussion after [2, Rk 6.2] for the boundary case (see also [18, §3] for the dual case). Part (a) of Lemma 5.1 means that each cycle in $Q_2(D)$ can be embedded into the plane as a polygon inscribed on a unit circle such that each arrow α stands on an arc of angle θ_α . The angle between successive arrows α, β in a cycle in $Q_2(D)$ is $\pi - \frac{1}{2}(\theta_\alpha + \theta_\beta)$ (see Figure 13), so it follows from Lemma 5.1(b) and (c) that these polygons (which all have at least three sides since D is of reduced type) tile the polygon mentioned above and hence give an isoradial embedding. \square

Note that if $\theta_i = 2\pi/n$, the polygon in Corollary 5.4 is regular. If D is not of reduced type, we still obtain a tiling of the same polygon using the above procedure, if we allow degenerate 2-sided tiles. But is not an isoradial embedding since the arrows in a two-cycle map onto the same line in the plane.

We remark that an isoradial embedding of $Q(D)$ in the above sense has been constructed explicitly in [25] in the case where D is of reduced type. We recall this construction here in order to compare with the above.

Recall that a pair I, J of k -subsets of C_1 is said to be *noncrossing*, or *weakly separated* [22] (see also [27, Defn. 3]) if there are no elements a, b, c, d , cyclically ordered around C_1 , such that $a, c \in I - J$ and $b, d \in J - I$. Note that a pair I, I is always noncrossing. A collection of k -subsets is said to be noncrossing if it is pairwise noncrossing.

By [27, Cor. 1], the set $\mathcal{C} = \mathcal{C}(D)$ of k -subsets of C_1 labelling a Postnikov diagram D is a maximal noncrossing collection. By [25, Thm. 7.1], every such collection arises in this way.

DEFINITION 5.5. Let \mathcal{C} be any maximal noncrossing collection of k -subsets of C_1 . We can define a quiver with faces $\Gamma = \Gamma(\mathcal{C})$, with vertex set $\Gamma_0 = \mathcal{C}$. The arrows Γ_1 and faces Γ_2 are determined as follows, mimicking the definition of the CW-complex $\Sigma(\mathcal{C})$ in [25, §9].

If K is any $(k - 1)$ -subset of C_1 , then the *white clique* $\mathcal{W}(K)$ of K is the set

$$\{I \in \mathcal{C} : K \subseteq I\},$$

which is given a cyclic order

$$K + a_1, K + a_2, \dots, K + a_r, K + a_1,$$

where $a_1, a_2, \dots, a_r, a_1$ are cyclically ordered clockwise in C_1 .

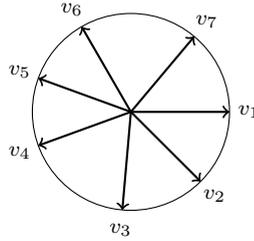


FIGURE 14. Typical points v_1, \dots, v_n in the case $n = 7$

If L is any $(k + 1)$ -subset of C_1 , then the *black clique* $\mathcal{B}(L)$ of L is the set

$$\{I \in \mathcal{C} : I \subseteq L\},$$

which is given a cyclic order

$$L - b_s, L - b_{s-1}, \dots, L - b_1, L - b_s,$$

where $b_1, b_2, \dots, b_s, b_1$ are cyclically ordered clockwise in C_1 .

A clique is said to be *nontrivial* when it has at least 3 elements. We let Γ_2 be the set of nontrivial cliques, or more precisely, we set

$$\Gamma_2^- = \{K : |\mathcal{W}(K)| \geq 3\}, \quad \Gamma_2^+ = \{L : |\mathcal{B}(L)| \geq 3\}.$$

For $I, J \in \mathcal{C}$, there is an arrow $\alpha: I \rightarrow J$ in Γ_1 if J follows I in the cyclic ordering of some nontrivial clique. Note: even if this occurs in more than one clique, there is only one arrow. Then we may also define $\partial: \Gamma_2 \rightarrow \Gamma_{\text{cyc}}$ in the obvious way.

Note that an arrow $\alpha: I \rightarrow J$ in $\Gamma(\mathcal{C})$ can occur in the boundary of at most 2 cliques, namely $\mathcal{W}(I \cap J)$ and $\mathcal{B}(I \cup J)$, and that it will occur with the same orientation in both. Indeed, if both these cliques are nontrivial, then $\alpha: I \rightarrow J$ must be an arrow in both boundaries, by [25, Lemma 9.2].

In [25, §9], an isoradial embedding of $\Gamma(\mathcal{C})$ is constructed as follows. For all $i \in C_1$, choose unit vectors v_i in \mathbb{R}^2 , in the same clockwise order around the unit circle as in C_1 (see Figure 14 for an example the case $n = 7$). Note that these points form a strictly convex polygon, which is the condition required by [25] (see [25, Lemma 9.3]). Here we assume the stronger property that they form a polygon inscribed in a unit circle.

This is extended to a map v on all subsets J of C_1 by setting

$$v(J) = \sum_{i \in J} v_i.$$

Denote the point with position vector $v(J)$ by J . Any $i \in C_0$ is common to the two adjacent edges $i, i + 1 \in C_1$. Thus it makes sense to set θ_i to be the angle between v_i and v_{i+1} for $i \in C_0$.

LEMMA 5.6. *Let $\alpha: I \rightarrow J$ be an arrow in Q . Set $K = I \cap J$ and $L = I \cup J$. Then*

$$\angle JKI = \theta_\alpha = \angle ILJ,$$

Proof. Firstly, note that θ_α cannot be equal to 0 or 2π , by the remarks immediately following Definition 4.1. Now we know that

$$\overrightarrow{KI} = v_I - v_K = v_{I-J}, \quad \overrightarrow{KJ} = v_J - v_K = v_{J-I}.$$

The formula for the angle $\angle JKI$ then follows from the definition of the θ_i and the fact that $w_\alpha = (c, d)_0$, where $I - J = \{c\}$ and $J - I = \{d\}$. The formula for the angle $\angle ILJ$ is proved similarly. \square

Suppose Q is a quiver with faces such that the boundary of every face is a cycle of length at least 3 and that Q has an isoradial embedding v whose image is a subset of the plane bounded by a polygon, so the map v induces an embedding of the quiver (Q_0, Q_1) into a disk. Then the given structure of quiver with faces on (Q_0, Q_1) coincides with the structure inherited from the embedding and it follows that Q is a dimer model in a disk.

THEOREM 5.7. *Let D be a (k, n) -Postnikov diagram of reduced type and $\mathcal{C} = \mathcal{C}(D)$ the set of labels of the alternating regions. Let $\Gamma = \Gamma(\mathcal{C})$ be the associated quiver with faces, as in Definition 5.5, and let $v_{\mathcal{C}} : \Gamma_0 \rightarrow \mathbb{R}^2$ be the restriction of v to $\Gamma_0 = \mathcal{C}$.*

- (a) *The map $v_{\mathcal{C}}$ is an isoradial embedding of Γ . The image of Γ under $v_{\mathcal{C}}$ is the subset of the plane bounded by a convex polygon with vertices $v(E_1), v(E_2), \dots, v(E_n)$ arranged clockwise around the boundary.*
- (b) *As quivers with faces, we have $\Gamma(\mathcal{C}) \cong Q(D)$.*

Proof. We first prove part (a). To check part (ii) of Definition 5.3, notice that, if $K \in \Gamma_2^-$, then $v(K + a) - v(K) = v_a$, for any $a \in C_1$. Hence, the points

$$v(K + a_1), v(K + a_2), \dots, v(K + a_r)$$

lie in order clockwise around a unit circle centred at $v(K)$. A similar statement holds for $L \in \Gamma_2^+$. For parts (i) and (iii) of Definition 5.3 we use [25, Prop. 9.4], noting that by [25, Thm. 9.12] there cannot be a pair of vertices $I, J \in \mathcal{C}$ with $\mathcal{W}(I \cap J) = \mathcal{B}(I \cup J) = \{I, J\}$, so $\Gamma(\mathcal{C})$ corresponds to $\Sigma(\mathcal{C})$ in [25, §9]. The claim concerning the image follows from [25, Prop. 9.8, Rk. 9.9 and Thm. 9.12].

For part (b), we note that by [25, Thm. 9.12], $\Gamma(\mathcal{C})$ is the dual of $G(D)$. By Remark 3.4, the dual of $G(D)$ is $Q(D)$ and the result follows. \square

In particular, we see, using Lemma 5.6 and Theorem 5.7, that $v_{\mathcal{C}}$ has the property mentioned in the proof of Corollary 5.4, i.e. that each arrow α stands on an arc of angle θ_α .

We colour a tile \mathbb{T}_F in an isoradial embedding black (respectively, white) if the anticlockwise (respectively, clockwise) ordering of its vertices corresponds to the ordering of the cycle ∂F . Note that the tiling is bipartite in the sense that two tiles sharing an edge must be of opposite colour. The bipartite tiling corresponding to the isoradial embedding in Theorem 5.7(a) is referred to as the *plabic tiling* corresponding to \mathcal{C} in [25]. For the Postnikov diagram D in Figure 3 and the vectors in Figure 14, the plabic tiling and the image of $\Gamma(\mathcal{C}(D))$ under the isoradial embedding in Theorem 5.7(a) are shown in Figure 15. Note that the quiver is the same as the one in Figure 5, but a little straightened.

6. Construction of a minimal path: first arrow

Our goal in this section is to show that, given any distinct pair of vertices $I, J \in Q_0(D)$, there is an arrow, starting at I , whose weight is constrained in a way that makes it a

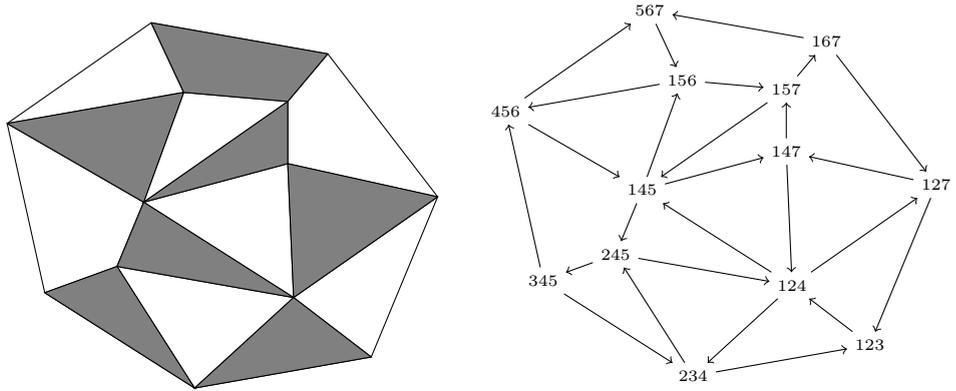


FIGURE 15. The plabic tiling and the image of $Q(D)$ under $v_{C(D)}$ for the Postnikov diagram D in Figure 3 and v_i in Figure 14

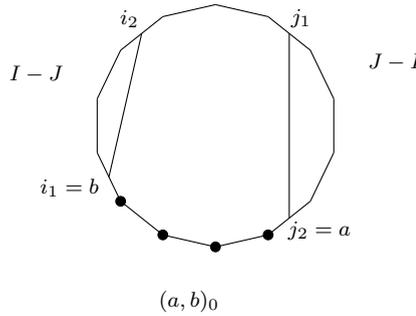


FIGURE 16. A noncrossing pair

candidate for the first arrow in a minimal path from I to J . We will then construct this path in Section 8.

DEFINITION 6.1. Suppose that I, J are distinct noncrossing k -subsets of C_1 . Then $I - J$ and $J - I$ are nonempty and contained in non-overlapping cyclic intervals in C_1 . Let $[i_1, i_2]$ and $[j_1, j_2]$ be the smallest cyclic intervals containing $I - J$ and $J - I$, respectively. We define two elements of C_1 associated to the ordered pair (I, J) by setting

$$a = a(I, J) = j_2 \in J - I, \quad b = b(I, J) = i_1 \in I - J.$$

Observe that, since $a \neq b$, the interval $(a, b)_0$ in C_0 is nonempty. See Figure 16 for a picture; the vertices in $(a, b)_0$ are indicated by black dots.

Note that, if I, J are distinct k -subsets labelling regions of the same Postnikov diagram D , then they are noncrossing, by [27, Cor. 1]. Our goal may now be stated more precisely as follows.

PROPOSITION 6.2. Let D be a Postnikov diagram and let I, J be distinct vertices in $Q_0(D)$. Let $a = a(I, J)$ and $b = b(I, J)$. Then there is an arrow α , starting at I , whose weight w_α (as a subset of C_0) is disjoint from $(a, b)_0$.

Proof. We divide the proof into two cases, when I is internal or external. The required statements are then proved as Lemmas 6.4 and 6.6 below. \square

Thus, we start by assuming that I is an internal vertex of $Q(D)$ and J is an arbitrary vertex. We use the notation introduced after Remark 4.6.

LEMMA 6.3. *Fix $1 \leq i \leq r$. Then $[f_{2i}, f_{2i+1}] \not\subseteq (a, b)$.*

Proof. Suppose, for a contradiction, that $[f_{2i}, f_{2i+1}] \subseteq (a, b)$. Then a, f_{2i}, f_{2i+1}, b appear in cyclic order clockwise in C_1 . Since $f_{2i} \notin I$ and $f_{2i} \in (a, b)$, we must have $f_{2i} \notin J$, so $f_{2i} \in I_{2i} - J$ by (4.1). Similarly, $f_{2i+1} \in J - I_{2i}$. We also have that $a \notin I_{2i}$ since $a \notin I$ and $a \neq f_{2i}$, so $a \in J - I_{2i}$. Similarly, $b \in I_{2i} - J$. This implies that I_{2i} and J are crossing, which is the required contradiction, as $I_{2i}, J \in Q_0(D)$. \square

Let $\xi(I)$ be the path associated to I in Definition 4.8. By Proposition 4.9, it wraps around C exactly $r - 1$ times. Let $C(r - 1)$ be a connected $(r - 1)$ -fold cover of the graph C . By choosing appropriate consecutive lifts $(\tilde{f}_i, \tilde{f}_{i+1})_0$, we can lift ξ to $C(r - 1)$, obtaining a path $\tilde{\xi}$ which encircles $C(r - 1)$ exactly once.

LEMMA 6.4. *There is an $i \in \{1, 2, \dots, r\}$ such that the weight $(f_{2i-1}, f_{2i})_0$ of the arrow $\alpha_{2i-1}: I \rightarrow I_{2i-1}$ is disjoint from $(a, b)_0$.*

Proof. The preimage of $(a, b)_0$ in $C(r - 1)$ consists of a disjoint union of $r - 1$ lifts of $(a, b)_0$. If one of these, say $(\tilde{a}, \tilde{b})_0$, had a nonempty intersection with more than one of the lifts $(\tilde{f}_{2i-1}, \tilde{f}_{2i})_0$ in $\tilde{\xi}$ it would have non empty intersection with two consecutive odd intervals $(\tilde{f}_{2i-1}, \tilde{f}_{2i})_0$ and $(\tilde{f}_{2i+1}, \tilde{f}_{2i+2})_0$. Then we'd have $[\tilde{f}_{2i}, \tilde{f}_{2i+1}] \subseteq (\tilde{a}, \tilde{b})$, and thus that $[f_{2i}, f_{2i+1}] \subseteq (a, b)$, a contradiction to Lemma 6.3.

Hence each of the $r - 1$ lifts of $(a, b)_0$ can have nonempty intersection with at most one of the r intervals $(\tilde{f}_{2i-1}, \tilde{f}_{2i})_0$. Thus there must be one outgoing arrow α_{2i-1} whose weight $(f_{2i-1}, f_{2i})_0$ does not intersect $(a, b)_0$, as required. \square

We next consider the case where $I = E_j$ is a boundary vertex. We use the notation for the incident arrows etc., introduced just before Definition 4.10. The statement of Lemma 6.3 also still holds in this case, but we need an extra lemma too.

LEMMA 6.5. *We have that $j \notin (a, b)_0$.*

Proof. Since $E_j = [j - k + 1, j]$, we have $E_j - J \subseteq [j - k + 1, j]$, while $J - E_j \subseteq [j + 1, j - k]$. It follows from the definition of a and b that j, a, b, j appear in clockwise order around C , with $j \neq a$, $a \neq b$ (but possibly $b = j$). The result follows. \square

We consider the path $\xi_{\text{out}}(E_j)$ associated to E_j in Definition 4.10. By Proposition 4.11(a), $\xi_{\text{out}}(E_j)$ wraps around C exactly $r - 1$ times but the vertex j appears at the start and the end of $\xi_{\text{out}}(E_j)$ (so that there are two edges in the overlap of ξ at its start and end): we have $f_1 = j$ and $f_{2r} = j + 1$.

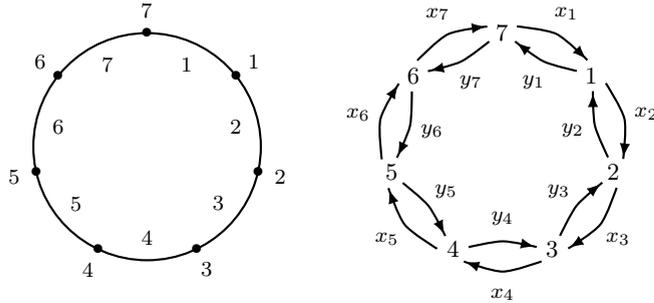


FIGURE 17. The graph C and corresponding quiver

By choosing appropriate consecutive lifts $(\tilde{f}_i, \tilde{f}_{i+1})_0$, we can lift $\xi_{\text{out}}(E_j)$ to $C(r-1)$, obtaining a path ξ which encircles $C(r-1)$ exactly once plus the single vertex (a lift of j) in the overlap at the start and the end.

LEMMA 6.6. *There is an $i \in \{1, 2, \dots, r\}$ such that the weight $(f_{2i-1}, f_{2i})_0$ of the outgoing arrow $\alpha_{2i-1}: E_j \rightarrow I_{2i-1}$ is disjoint from $(a, b)_0$.*

Proof. The interval $(a, b)_0$ has $r-1$ distinct lifts in $C(r-1)$. By Lemma 6.5, no lift can have nonempty intersection with both intervals $(\tilde{f}_1, \tilde{f}_2)_0$ and $(\tilde{f}_{2r-1}, \tilde{f}_{2r})_0$, since then it would contain a lift of j . Arguing as in the proof of Lemma 6.4, we obtain that a lift of $(a, b)_0$ can only have nonempty intersection with at most one of the intervals $(\tilde{f}_{2i-1}, \tilde{f}_{2i})_0$. The $r-1$ copies in total have nonempty intersection with at most $r-1$ such intervals. But there are r such intervals in total, so there is at least one such interval that has empty intersection with $(a, b)_0$, as required. \square

7. The rank one modules \mathbb{M}_I .

Consider the quiver with vertices C_0 and, for each edge $i \in C_1$, a pair of arrows $x_i: i-1 \rightarrow i$ and $y_i: i \rightarrow i-1$. Then let B be the quotient of the path algebra (over \mathbb{C}) of this quiver by the ideal generated by the $2n$ relations $xy = yx$ and $x^k = y^{n-k}$, interpreting x and y as arrows of the form x_i, y_i appropriately and starting at any vertex. See Figure 17 for an example when $n = 5$.

The completion \widehat{B} of B coincides with the quotient of the completed path algebra of the graph C , i.e. the doubled quiver as above, by the closure of the ideal generated by the relations above. The algebras B and \widehat{B} were introduced in [19, §3].

The centre Z of B is the polynomial ring $\mathbb{C}[t]$, where $t = \sum_{i=1}^n x_i y_i$. The (maximal) Cohen-Macaulay B -modules are precisely those which are free as Z -modules. Indeed, such a module M is given by a representation $\{M_i : i \in C_0\}$ of the quiver with each M_i a free Z -module of the same rank, which is the rank of M (cf. [19, §3]).

DEFINITION 7.1. For I any k -subset of C_1 , we define a rank one B -module

$$\mathbb{M}_I = (U_i, i \in C_0; x_i, y_i, i \in C_1)$$

as follows (cf. [19, §5]). For each vertex $i \in C_0$, set $U_i = \mathbb{C}[t]$ and, for each edge $i \in C_1$, set $x_i: U_{i-1} \rightarrow U_i$ to be multiplication by 1 if $i \in I$ and by t if $i \notin I$,

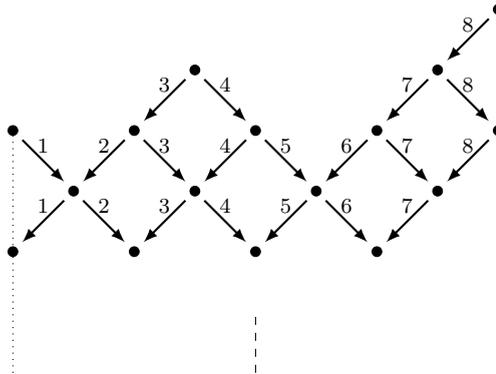


FIGURE 18. The module $\mathbb{M}_{\{1,4,5\}}$ in the case $k = 3, n = 8$

$y_i: U_i \rightarrow U_{i-1}$ to be multiplication by t if $i \in I$ and by 1 if $i \notin I$.

The module \mathbb{M}_I can be represented by a lattice diagram \mathcal{L}_I in which $U_0, U_1, U_2, \dots, U_n$ are represented by columns from left to right (with U_0 and U_n to be identified). The vertices in each column correspond to the natural monomial basis of $\mathbb{C}[t]$. The column corresponding to U_{i+1} is displaced half a step vertically downwards (respectively, upwards) in relation to U_i if $i \in I$ (respectively, $i \notin I$), and the actions of x_i and y_i are shown as diagonal arrows. Note that the k -subset I can then be read off as the set of labels on the arrows pointing down to the right which are exposed to the top of the diagram. For example, the lattice picture $\mathcal{L}_{\{1,4,5\}}$ in the case $k = 3, n = 8$, is shown in Figure 18.

REMARK 7.2. Fix $j \in C_0$, and consider the module \mathbb{M}_{E_j} . The corresponding lattice diagram is bounded at the top by paths going down from $j - k$ to j . It is easy to see that there is a bijection between the vertices of the diagram and a basis of Be_{j-k} , where e_{j-k} is the idempotent corresponding to the vertex $j - k$. We see that the module \mathbb{M}_{E_j} is the projective indecomposable B -module corresponding to the vertex $j - k$.

Every B -module has a canonical endomorphism t , that is, multiplication by $t \in Z$. For \mathbb{M}_J this corresponds to shifting \mathcal{L}_J one step downwards. Since Z is central, $\text{Hom}_B(M, N)$ is a Z -module for arbitrary B -modules M and N . If M, N are free Z -modules, then so is $\text{Hom}_B(M, N)$.

DEFINITION 7.3. A monomial morphism $\varphi: \mathbb{M}_I \rightarrow \mathbb{M}_J$ is one determined by an embedding of \mathcal{L}_I in \mathcal{L}_J . Since the cokernel of any such φ is clearly finite dimensional, we may define the degree of φ to be the element of NC_0 that counts the multiplicities of the simple modules in a composition series for its cokernel. As for the weights on arrows, we identify a subset of C_0 with the sum of its elements, and refer to a monomial morphism as being sincere if the support of its degree is C_0 , and insincere otherwise.

Note that the composition of monomial morphisms is monomial and that degree is additive under composition. The degree of the endomorphism t is precisely C_0 , without multiplicity.

Given k -subsets I, J of C_1 , let

$$g_{JI}: \mathbb{M}_I \rightarrow \mathbb{M}_J \tag{7.1}$$

denote the monomial morphism with minimal codimension. This can be represented by embedding \mathcal{L}_I into \mathcal{L}_J as high as possible, i.e. so that at least one vertex on the top boundary of \mathcal{L}_J lies in image of \mathcal{L}_I . Thus g_{JI} is insincere. Note that g_{II} is the identity map on \mathbb{M}_I . We have the following, which explains the terminology *monomial morphism*.

LEMMA 7.4. *Let I, J be k -subsets of C_1 . Then $\text{Hom}_B(\mathbb{M}_I, \mathbb{M}_J)$ is a free rank one $\mathbb{C}[t]$ -module generated by g_{JI} . Furthermore, $t^m g_{JI}$ may be characterised as the unique monomial morphism of its degree $\mathbb{M}_I \rightarrow \mathbb{M}_J$. In particular, g_{JI} is the unique insincere monomial morphism $\mathbb{M}_I \rightarrow \mathbb{M}_J$.*

Proof. For the first part, observe that there is a basis of $\text{Hom}_B(\mathbb{M}_I, \mathbb{M}_J)$ consisting of monomial morphisms, given by taking all embeddings of \mathcal{L}_I into \mathcal{L}_J , which are clearly just downward shifts of the highest possible one. In other words, they are the maps $t^m g_{JI}$, for all $m \geq 0$, as required. For the second part, note that the degree of $t^m g_{JI}$ is equal to the degree of g_{JI} plus mC_0 . The last part follows from this and the fact that g_{JI} is insincere, by the remarks immediately following Definition 7.3. \square

For a nonempty subset V of C_0 and k -subsets I, J of C_1 , we write $I \leq_V J$ if the support of the degree of g_{JI} does not contain any element of V .

LEMMA 7.5. *Let V be a subset of C_0 . Then the relation \leq_V is a partial order on the collection of all k -subsets of C_1 .*

Proof. Reflexivity is clear. If $I \leq_V J$ and $J \leq_V K$, then $\text{coker } g_{JI}$ and $\text{coker } g_{KJ}$ have composition series containing no simple S_i with $i \in V$. It follows that the composition of these two morphisms has the same property. It thus coincides with g_{JI} , and it follows that $I \leq_V J$, so \leq_V is transitive. Suppose that $I \leq_V J$ and $J \leq_V I$. Then the composition $g_{IJ}g_{JI}: \mathbb{M}_I \rightarrow \mathbb{M}_I$ has cokernel containing no simple S_i with $i \in V$, and thus must be the identity map and we have $I = J$ as required. \square

8. Existence of minimal path

In this section, we show that there is a minimal path between any pair of vertices in the quiver $Q(D)$ of a Postnikov diagram D .

PROPOSITION 8.1. *Let I, J be distinct noncrossing k -subsets of C_1 . Let $a = a(I, J)$ and $b = b(I, J)$, as in Definition 6.1. Let $I - J = \{b_1, \dots, b_s\}$, writing the elements in clockwise order starting at $b = b_1$. Let $J - I = \{a_1, \dots, a_s\}$, writing the elements in anticlockwise order starting at $a = a_1$. Then the degree of $g_{JI}: \mathbb{M}_I \rightarrow \mathbb{M}_J$ is equal to $\sum_{j=1}^s (b_j, a_j)_0$.*

To prove this, we first need the following technical lemma:

LEMMA 8.2. *Let I be a k -subset of C_1 , and suppose that $J = I - c + d$ for $c \neq d$ in C_1 . Then \mathcal{L}_J can be obtained from \mathcal{L}_I by adding one vertex to the top of each of the columns*

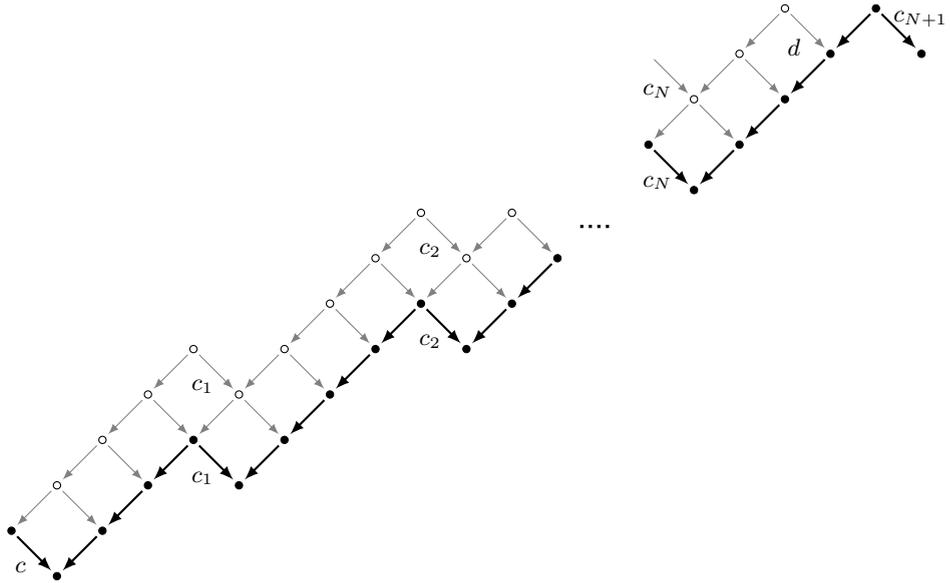


FIGURE 19. Proof of Lemma 8.2

corresponding to the vertices in $(c, d)_0$, together with the required additional arrows, i.e. arrows pointing downwards between any new vertex v and the vertices in adjacent columns.

Proof. Let c_1, c_2, \dots, c_N be the elements of I which are clockwise of c and anticlockwise of d . Let c_{N+1} be the first element of I clockwise past d (possibly equal to c). The added vertices and arrows cover the arrows c, c_1, c_2, \dots, c_N exposed to the top of the diagram. The new exposed arrows are c_1, c_2, \dots, c_N, d . Thus we see that the diagram constructed as indicated corresponds to the k -subset $I - c + d$ as required. This is illustrated in Figure 19, which shows the top part of \mathcal{L}_I between the columns of arrows c_1 and c_{N+1} (which may or not be identified). The new arrows are shown in gray. \square

Proof Proof of Proposition 8.1. We proceed by induction on s . The base case is $s = 1$. Then $I - J = \{b_1\}$ and $J = I = \{a_1\}$ for elements a_1, b_1 . We apply Lemma 8.2 to the pair I and $J = I - b + a = I - b_1 + a_1$, and see that \mathcal{L}_J can be obtained from \mathcal{L}_I by adding a layer of vertices to the columns $b_1, \dots, a_1 - 1$ to the lattice picture, giving the result in this case.

Suppose that $s > 1$ and the result holds for smaller s . Let $I_1 = I - b + a = I - b_1 + a_1$. Then, by again applying Lemma 8.2, \mathcal{L}_{I_1} can be obtained from \mathcal{L}_I by adding a layer of vertices to the columns $b_1, \dots, a_1 - 1$. We have $I_1 - J = (I - J) - b_1 = \{b_2, \dots, b_s\}$ and $J - I_1 = (J - I) - a_1 = \{a_2, \dots, a_s\}$. The result now follows from applying the inductive hypothesis to the pair I_1, J . \square

COROLLARY 8.3. Let $\alpha: I \rightarrow J$ be an arrow in $Q_1(D)$. Then the degree of g_{JI} is equal to the weight w_α (see Definition 4.1).

Proof. We apply Lemma 8.2 to the pair I, J , noting that $J = I - b + a$, where a, b are the strands crossing α , in such a way that the weight of α is $(b, a)_0$. \square

We obtain the following corollary:

COROLLARY 8.4. *Let I, J be distinct noncrossing k -subsets of C_1 . For any $c, d \in C_1$, we have $I \leq_{(c,d)_0} J$ if and only if $(c, d)_0 \subseteq (a(I, J), b(I, J))_0$.*

Proof. By Proposition 8.1, we have $I \leq_{(c,d)_0} J$ if and only if no element of $(c, d)_0$ lies in the support of $\sum_{j=1}^s (b_j, a_j)_0$. This support is $(b_1, a_1)_0 = (b(I, J), a(I, J))_0$, so $I \leq_{(c,d)_0} J$ holds if and only if $(c, d)_0 \subseteq (a(I, J), b(I, J))_0$, as required. \square

Note that Corollary 8.4 means, in particular, that $I \leq_{(a(I,J),b(I,J))_0} J$ for any pair of (distinct) vertices $I, J \in Q_0(D)$.

LEMMA 8.5. *Let $I \neq J \in Q_0(D)$ and set $a = a(I, J)$, $b = b(I, J)$. Suppose that there is an arrow $\alpha: I \rightarrow I_1$ in $Q_1(D)$ whose weight w_α is disjoint from $(a, b)_0$. Then $I_1 \leq_{(a,b)_0} J$.*

Proof. If $I_1 = I - c + d$, then, by assumption, the weight $w_\alpha = (c, d)_0$ is a subset of $(b, a)_0$. Applying Lemma 8.2 to the pair I, I_1 , we see that \mathcal{L}_{I_1} is obtained from \mathcal{L}_I by adding vertices at the top of the columns corresponding to entries in the set $(c, d)_0$.

This additional layer is part of the first layer added to \mathbb{M}_I in the proof of Proposition 8.1. Additional layers are added to this to eventually reach \mathbb{M}_J . Since $I \leq_{(a,b)_0} J$, all of these layers involve only columns corresponding to elements of $(b, a)_0$, and we see that $I_1 \leq_{(a,b)_0} J$, as required. \square

DEFINITION 8.6. We call a path p *sincere* if the support of its weight is equal to C_0 , and *insincere* otherwise.

PROPOSITION 8.7. *Let $I \neq J \in Q_0(D)$ and set $a = a(I, J)$, $b = b(I, J)$. Then there is an insincere path from I to J in $Q(D)$.*

Proof. By Corollary 8.4, we have $I \leq_{(a,b)_0} J$. By Proposition 6.2, there is a vertex $I_1 \in Q_0(D)$ and an arrow $\alpha: I \rightarrow I_1$ such that w_α does not contain any element of $(a, b)_0$. By Lemma 8.5 we have $I_1 \leq_{(a,b)_0} J$. Let $a_1 = a(I_1, J)$ and $b_1 = b(I_1, J)$. Then, by Corollary 8.4, we have $(a, b)_0 \subseteq (a_1, b_1)_0$. From the proof of Lemma 8.5 we also see that $\text{coker } g_{JI_1}$ has strictly smaller dimension than $\text{coker } g_{JI}$.

We can now repeat this argument for I_1 , since $I_1 \leq_{(a_1,b_1)_0} J$ again by Corollary 8.4. We get $I_2 \leq_{(a_2,b_2)_0} J$, where $a_2 = a(I_2, J)$ and $b_2 = b(I_2, J)$. Continuing in this way, we obtain a path

$$I \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

which satisfies

$$(a, b)_0 \subseteq (a_1, b_1)_0 \subseteq (a_2, b_2)_0 \subseteq \dots,$$

and thus that the weight of each arrow in the path avoids $(a, b)_0$. The path must eventually reach J because of the decreasing dimension of $\text{coker } g_{JI_i}$ as i increases, and we are done. \square

9. Description of paths

Before we can prove the main result, we need some more information about A_D . By Proposition 8.7, there is an insincere path between any two vertices I, J of $Q(D)$. In this section we will show that there is a unique element of A_D which can be written as such a path, and furthermore that any path from I to J is equal in A_D to a power of u multiplied by this path, adapting arguments from [2, §5]. We shall use the fact that $Q(D)$ is a dimer model in a disk (see Remark 3.4).

We recall the following from [2, §5]. Let p be a path in $Q(D)$. A subpath p of maximal length with respect to being contained in the boundary of a face in Q_2^+ (respectively, Q_2^-) is called a *positive arc* (respectively, *negative arc*) of p . Then p can be written as a concatenation of its positive arcs or a concatenation of its negative arcs. The path p is defined to be *positively reducible* (respectively, *negatively reducible*) if (at least) one of its positive (respectively, negative) arcs contains all of the arrows but one in lying in the boundary of a face in Q_2^+ (respectively, in Q_2^-).

If p is any path in $Q(D)$, we denote by p^{-1} the path in the underlying unoriented graph of $Q(D)$ obtained by reading p in reverse. We say that a cycle in the underlying unoriented graph *does not self-intersect* if it does not visit the same vertex twice (except the start and end points). Note that this includes the case of a cycle which goes along an arrow in $Q(D)$ and then returns along the same arrow.

Suppose that p and q are distinct paths in $Q(D)$ with the same start and end points, such that $q^{-1}p$ (regarded as path in the underlying unoriented graph) is a clockwise cycle which does not self-intersect. Then we say that the pair p, q is *reducible* if either p is negatively reducible or q is positively reducible. We make a similar definition if $q^{-1}p$ is an anticlockwise cycle.

PROPOSITION 9.1. *Let p, q be distinct paths in $Q(D)$, with the same start and end points, such that $q^{-1}p$ does not self-intersect. Then the pair p, q is reducible.*

Proof. Let $D_0 = R_{pq}$ be the interior of the region enclosed by the cycle $q^{-1}p$, which is a disk by the assumption on p and q . We first show that either p or q has a *backwards arrow* in the interior of R_{pq} , that is, an arrow starting at some vertex on the path which ends at an earlier vertex of the path (e.g. the thick curved arrows in Figure 20).

Consider the strand S_1 which enters D_0 through the last arrow a_0 of q . Then S_1 leaves D_0 through an arrow a_1 of p or q . We assume that a_1 lies in p , as illustrated in the left hand diagram in Figure 20; the argument when a_1 lies in q is similar and is illustrated in the right hand diagram in Figure 20. Since S_1 does not intersect itself (Definition 2.1(b1)), it divides D_0 into two disks (and also $a_1 \neq a_0$). Let D_1 be the disk on the right of S_1 and let S_2 be the strand that crosses S_1 immediately before a_1 . Note that $S_2 \neq S_1$, again by Definition 2.1(b1), and that S_2 will be leaving D_1 at this crossing, by the alternating crossing property (Definition 2.1(a3)).

By Definition 2.1(b2), S_2 cannot enter D_1 through the part of S_1 before its crossing point with S_2 , so it must enter at an arrow a_2 of p after or equal to a_1 . The strand S_2 divides D_1 into two disks and we take D_2 to be the disk on the right of S_2 .

In the case $a_2 = a_1$, we claim that no strand can cross S_2 between its two crossing points with S_1 . It then follows that the arrow corresponding to the crossing not at a_1 is a backwards arrow, with head/tail equal to the tail/head of a_1 , and we are done in this case. To prove the claim, consider the first strand crossing S_2 after a_1 . If it were not S_1 , then it would be entering D_2 . But it can not leave D_2 through S_2 , by Definition 2.1(b2), and it can not

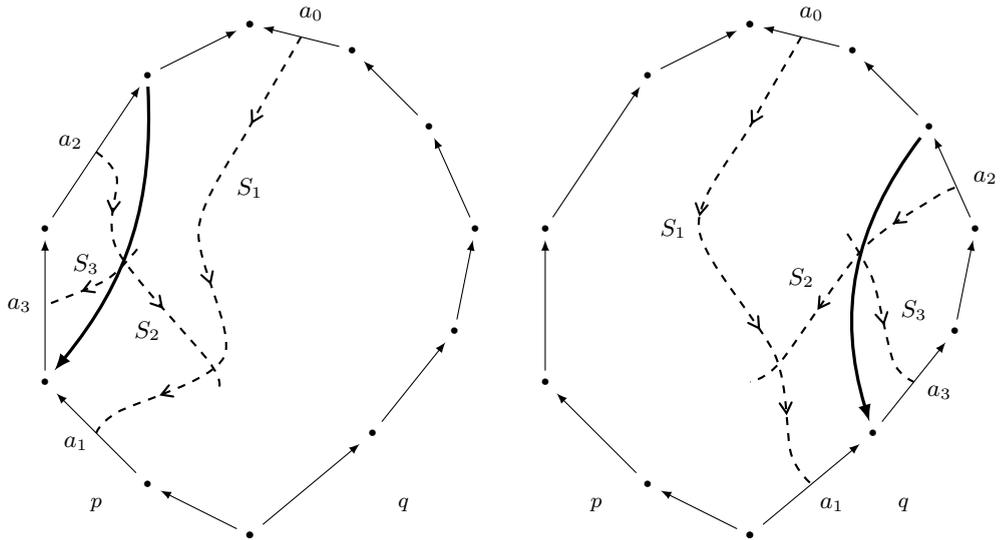


FIGURE 20. Proof of Proposition 9.1, first part

leave through S_1 , as the only two available crossing points are already taken by S_2 . This contradiction proves the claim.

In the case $a_2 \neq a_1$, let S_3 be the strand that crosses S_2 immediately after a_2 . If $S_3 = S_1$, then the arrow corresponding to the crossing of S_1 and S_2 has tail at the head of a_2 and head at the tail of a_1 , so is the required backwards arrow for p . If $S_3 \neq S_1$, then, as in the previous case, S_3 cannot leave D_2 by crossing S_2 or S_1 , so it must leave through an arrow a_3 of p lying strictly after a_1 and before or equal to a_2 . If $a_3 = a_2$, then we get a backwards arrow as in the case $a_2 = a_1$. Otherwise a_3 lies strictly between a_2 and a_1 .

We may continue in this way to define successive strands S_i entering or leaving disks D_{i-1} at arrows a_i until, by the finiteness of p , we necessarily reach a point when $S_i = S_{i-2}$ or $a_i = a_{i-1}$ and we obtain a backwards arrow as above.

Thus p or q has a backwards arrow. For the second part of the proof, we show that, if p has a backwards arrow β , then p is negatively reducible. A similar argument shows that if q has a backwards arrow, then q is positively reducible. As there are only finitely many backwards arrows for p , we may further assume that β is ‘minimal’ in the sense that there is no other backwards arrow in the disk bounded by β and the part p' of p between the endpoints of β . Let $q' = p'_\beta$ (as in Definition 3.5) and observe that our goal is achieved by showing that the paths p' and q' coincide.

So suppose, for contradiction, that p' and q' do not coincide. If p' and q' intersect, we replace them with subpaths which start and end at the same vertices but don’t intersect. As in the first part of the proof, let S_1 be the strand entering the region $R_{p'q'}$ at the last arrow a_0 of q' . If S_1 were to leave $R_{p'q'}$ at an arrow of q' , then, since q' is a part of p'_β , it must re-enter $R_{p'q'}$ at the following arrow of q' and, after doing this possibly several times (see Figure 21), it would have to cross itself (or be a loop), contradicting Definition 2.1(b1). On the other hand, if S_1 were to leave $R_{p'q'}$ at an arrow of p' , then we can argue as in the first part to show that p' has a backwards arrow in $R_{p'q'}$, contradicting the minimality of β and completing the proof. \square

Suppose that I and J are vertices in $Q_0(D)$, that p, q are paths from I to J in $Q(D)$ and that $q^{-1}p$ does not self-intersect (except at its endpoints). We will denote by $\overline{R_{pq}}$ the region

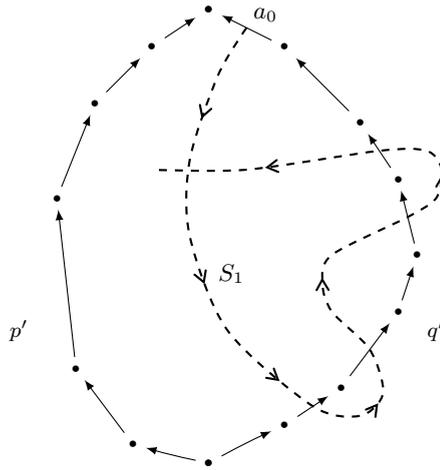


FIGURE 21. Proof of Proposition 9.1, second part

between p and q , including p and q . If $p = q = \gamma$ for some arrow γ then $\overline{R_{pq}} = \gamma$; otherwise it is a disk. We shall call a non-trivial path s in $\overline{R_{pq}}$ which intersects $q^{-1}p$ only at its end points a *chord* in $\overline{R_{pq}}$.

If, furthermore, the end points of s both lie on p and s is oriented in the same direction as (respectively, in the opposite direction to) p , we shall call s a forwards (respectively, backwards) p -chord. In particular, we allow an arrow of p to be a forwards p -chord. We make a similar definition for q .

We also define the *area* of a region bounded by a cycle in the underlying unoriented graph of $Q(D)$ which does not self-intersect to be the number of faces in $Q_2(D)$ contained in it.

LEMMA 9.2. *Let I, J be vertices in $Q_0(D)$ and p, q be paths from I to J in $Q(D)$ such that $q^{-1}p$ does not self-intersect. Then there is a path r in $\overline{R_{pq}}$ such that, in A_D , $p = u^{N_p}r$ and $q = u^{N_q}r$ for some nonnegative integers N_p and N_q .*

Proof. Note that we include the case where $p = q = \gamma$ for some arrow γ . We first observe that if the result holds in a case where $q = e_I$ is trivial, we must have $r = e_I$ and so $p = u^{N_p}e_I$; similarly if p is trivial. To prove the lemma, we argue by induction on the area of $\overline{R_{pq}}$. If this is zero, then $p = q = \gamma$ for some arrow γ , and the result is trivial. We suppose that $\overline{R_{pq}}$ has non-zero area, and assume that the result holds in the case where it has smaller area.

Without loss of generality, we may assume $q^{-1}p$ is a clockwise cycle. By Proposition 9.1, either p is negatively reducible or q is positively reducible. We suppose that p is negatively reducible (a similar argument holds in the case where q is positively reducible). Then p contains a subpath of the form p_{α}^{-} for some backwards arrow $\alpha : L \rightarrow K$ for p in $\overline{R_{pq}}$. We can apply the relation $p_{\alpha}^{-} = p_{\alpha}^{+}$ in A_D to p to produce a new path p' from I to J in $\overline{R_{pq}}$. Note that we have $p = p'$ in A_D .

Since $p_{\alpha}^{-}\alpha$ and $p_{\alpha}^{+}\alpha$ are faces of $Q(D)$, and $Q(D)$ is a dimer model, there cannot be any arrows inside the disks they bound. It follows that p_{α}^{+} (and hence also p') is contained in $\overline{R_{pq}}$. Note that p_{α}^{+} can be written uniquely as a composition of chords. We will apply the inductive hypothesis to the parts of $\overline{R_{pq}}$ between each chord and $q^{-1}p$, which we can do since they each have area smaller than that of $\overline{R_{pq}}$. Several types of chord can appear in the

composition and we need to deal with each type in a slightly different way, which we now detail.

Let $p[K, I]$ be the subpath of p from I to K and $p[J, L]$ the subpath of p from L to J , so that $p' = p[J, L]p_\alpha^+p[K, I]$. It is helpful to distinguish two cases for the path p_α^+ from K to L :

- (a) The path p_α^+ is of the form $p_5p_4p_3p_2p_1$, where p_1 and p_5 are compositions of backwards p -chords, p_2 is either an idempotent (if I lies on p_α^+) or a chord from a vertex on $p[K, I]$ to a vertex on q , p_3 is a composition of forwards q -chords and p_4 is either an idempotent (if J lies on p_α^+) or a chord from a vertex on q to a vertex on $p[J, L]$.
- (b) The path p_α^+ is of the form $p_3p_2p_1$, where p_1 and p_3 are compositions of backwards p -chords and p_2 is a forwards p -chord.

We also allow p_1, p_3 and p_5 in (a) and p_1 and p_3 in (b) to be idempotents, considered as empty compositions.

See Figure 22 for examples. We reiterate that it is possible for forwards q -chords in p_3 to be arrows in q , as in case (a) in Figure 22.

In case (a), each backwards p -chord in p_1 and p_5 corresponds to a loop in p' whose area is less than that of $\overline{R_{pq}}$: by the inductive hypothesis and the remark at the start of the proof, each such loop is equal to a power of u times an idempotent. We see that p' is the product of a power of u and $s_2p_4p_3p_2s_1$, where s_1 is an initial subpath of $p[K, I]$ and s_2 is a final subpath of $p[J, L]$. But each forwards q -chord in p_3 , together with the subpath of q with the same start and end vertices, form a pair of paths satisfying the assumptions in the lemma. The same applies to p_2s_1 together with an initial subpath of q , and to s_2p_4 together with a final subpath of q . Applying the inductive hypothesis to all of these pairs gives the result for p', q , and hence for p, q .

Similarly, in case (b), each backwards p -chord in p_1 and p_3 corresponds to a loop in p' which, by induction, is equal to a power of u times an idempotent. We see that p' is the product of a power of u and $s_2p_2s_1$, where s_1 is an initial subpath of $p[K, I]$ and s_2 is a final subpath of $p[J, L]$. We may then apply the inductive hypothesis to the pair $s_2p_2s_1, q$ to get the result for p, q . □

PROPOSITION 9.3. *Let p, q be arbitrary paths from I to J in $Q(D)$. Then there is a path r in $Q(D)$ such that, in A_D , $p = u^{N_p}r$ and $q = u^{N_q}r$ for some nonnegative integers N_p and N_q .*

Proof. We will prove the result by induction on the length of $q^{-1}p$. If the length of $q^{-1}p$ is zero, then $p = q = e_I$ and the result holds with $r = e_I$. We assume that the length of $q^{-1}p$ is positive and that the result holds when it has shorter length.

We write p as:

$$I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_i$$

and q as:

$$I_j \rightarrow I_{j-1} \rightarrow \cdots \rightarrow I_i,$$

where $j \geq i$. Thus, I_0, \dots, I_j are the vertices of $Q(D)$ visited by $q^{-1}p$, in order, and $I_j = I_0$. Let m be minimal such that $I_m = I_l$ for some $l < m$. Suppose first that $m \leq i$. Then, by Lemma 9.2, the part of p between I_l and I_m is equal to a power of u multiplied by e_{I_l} . The result follows in this case by applying the inductive hypothesis to the pair p, q with the part of p between I_l and I_m removed. A similar argument applies in the case where $l \geq i$.

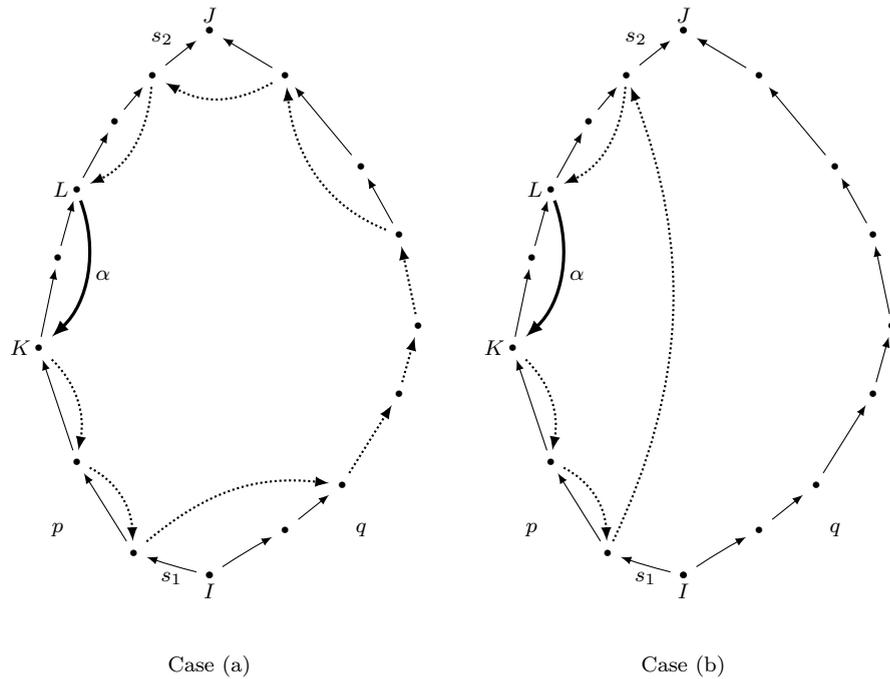


FIGURE 22. Examples for the proof of Lemma 9.2. Dotted arrows in $\overline{R_{pq}}$ represent chords and p_α^+ is the composition of all of them

The remaining possibility is that $l < i$ and $m > i$. But then we can write $p = p_2 p_1$ and $q = q_2 q_1$ as compositions of paths, such that $(q_2)^{-1} p_2$ is the part of $q^{-1} p$ between I_l and I_m and meets itself only at its starting and ending points. Note that the case $p_2 = q_2 = \gamma$ for some arrow γ may occur here. The result for (p, q) then follows by applying Lemma 9.2 to the pair (p_2, q_2) and the induction hypothesis to the pair (p_1, q_1) . \square

COROLLARY 9.4. *Let I, J be vertices in $Q(D)$. Then there is a unique element p_{JI} of A_D which can be written as an insincere path in $Q(D)$ from I to J . Furthermore, the elements*

$$\{u^N p_{JI} : N \geq 0\}$$

form a basis of $e_J A_D e_I$.

Proof. By Proposition 8.7, there is an insincere path p in $Q(D)$ from I to J (taking $p = e_I$ if $I = J$). Suppose that q is another such path and that $p = q$ in A_D . Since p, q are insincere and the support of u is C_0 , it follows from Proposition 9.3 that $p = q$ in A_D . We denote the common element of A_D arising from an insincere path in $Q(D)$ from I to J by p_{JI} .

If p is any path in $Q(D)$ from I to J , then by Proposition 9.3 we must have $p = u^N p_{JI}$ in A_D for some nonnegative integer N , since p_{JI} is insincere. Furthermore, the elements $u^N p_{JI}$ are non-zero in A_D by Remark 3.8(a) and independent (using Remark 4.5) since they have distinct weights. The result follows. \square

10. Isomorphism of algebras

Recall that B denotes the (polynomial case) algebra introduced in [19], as defined in Section 7, and that we have associated a B -module \mathbb{M}_I to each k -subset I of C_1 (see Definition 7.1). To any Postnikov diagram D , we may associate a B -module

$$T_D = \bigoplus_{I \in Q_0(D)} \mathbb{M}_I, \tag{10.1}$$

where $Q_0(D) = \mathcal{C}(D)$ is the set of labels on the alternating regions of D and thus the set of vertices of the dimer model $Q(D)$.

REMARK 10.1. Since the degree of a product of monomial morphisms is the sum of their degrees, the degree map induces an $\mathbb{N}C_0$ -grading on $\text{End}_B(T_D)$; see Remark 7.4.

Our goal in this section is to show that the dimer algebra $A_D = A_{Q(D)}$, as in Definition 3.5, is isomorphic to $\text{End}_B(T_D)$ as an $\mathbb{N}C_0$ -graded algebra, and, as a corollary, that the idempotent subalgebra of A_D corresponding to the boundary vertices of $Q(D)$ is isomorphic to B^{opp} . We start by defining a homomorphism from A_D to $\text{End}_B(T_D)$.

LEMMA 10.2. *There is a homomorphism $g: A_D \rightarrow \text{End}_B(T_D)$ of $\mathbb{N}C_0$ -graded algebras determined (uniquely) by the following properties:*

- (a) *If $I \in Q_0(D)$, then $g(e_I) = \text{id}_{\mathbb{M}_I}$.*
- (b) *If $\alpha: I \rightarrow J$ is an arrow in $Q_1(D)$, then $g(\alpha) = g_{JI}$.*

Proof. As A_D is a quotient of the path algebra of $Q(D)$, such a homomorphism is certainly uniquely determined by (a) and (b) and what we must check is that the morphisms $g(\alpha)$, for all $\alpha \in Q_1(D)$, satisfy the defining relations of A_D , namely (3.1). So, suppose that $\alpha: I \rightarrow J$ is an internal arrow. Then α lies in the boundary of a face $F^+ \in Q_2^+$. This boundary is a cycle

$$I \rightarrow J \rightarrow I_1 \rightarrow \dots \rightarrow I_m \rightarrow I.$$

Then $g(p_\alpha^+) = g_{I I_m} \cdots g_{I_1 J}$, which is insincere by Corollaries 8.3 and 4.4. Since this morphism is monomial, we must have $g(p_\alpha^+) = g_{IJ}$, by Lemma 7.4. The same argument applies to $g(p_\alpha^-)$ and so it is equal to $g(p_\alpha^+)$, as required. The fact that g is a homomorphism of $\mathbb{N}C_0$ -graded algebras follows from Corollary 8.3. □

We can now prove our main result.

THEOREM 10.3. *Let D be an arbitrary (k, n) -Postnikov diagram. Let A_D be the associated dimer algebra and T_D the associated B -module, as in (10.1). Then the map $g: A_D \rightarrow \text{End}_B(T_D)$, as in Lemma 10.2, is an isomorphism of graded algebras.*

Proof. Fix $I, J \in Q_0(D)$ and consider the minimal codimension map $g_{JI}: \mathbb{M}_I \rightarrow \mathbb{M}_J$ defined in (7.1). Let p_{JI} be the element of $e_J A_D e_I$ from Corollary 9.4. Its image under g is a monomial morphism, since the composition of monomial morphisms is monomial. As g is a morphism of graded algebras (Lemma 10.2), the degree of $g(u^m p_{JI})$ coincides with the degree of $t^m g_{JI}$. Hence, by Lemma 7.4, we have:

$$g(u^N p_{JI}) = t^N g_{JI} \tag{10.2}$$

for all $N \geq 0$.

By Corollary 9.4, the set $\{u^N p_{JI} : N \geq 0\}$ is a basis of $e_J A_D e_I$. On the other hand, by Lemma 7.4, the set $\{t^N g_{JI} : N \geq 0\}$ is a basis for

$$\mathrm{Hom}_B(\mathbb{M}_I, \mathbb{M}_J) = g(e_J) \mathrm{End}_B(T_D) g(e_I).$$

Hence, by (10.2), g maps a basis of A_D to a basis of $\mathrm{End}_B(T_D)$, so g is an isomorphism. \square

Let e be the sum, in A_D , of the idempotents e_I corresponding to the boundary vertices I of the Postnikov diagram D . We call the algebra $e A_D e$ the *boundary algebra* of D .

COROLLARY 10.4. *The boundary algebra $e A_D e$ is isomorphic to B^{opp} , i.e. the opposite of the algebra B in Section 7. In particular, it is independent of the choice of Postnikov diagram D , up to isomorphism.*

Proof. Using the isomorphism in Theorem 10.3, we see that

$$e A_D e \cong g(e) \mathrm{End}_B(T_D) g(e) = \mathrm{End}_B(P),$$

where P is the direct sum of the modules \mathbb{M}_I corresponding to the labels I of the boundary regions of D . These are exactly the projective B -modules (see Remarks 2.3 and 7.2), so $e A e$ is isomorphic to B^{opp} . \square

REMARK 10.5. If D, D' are any two (k, n) -Postnikov diagrams, then, by Corollary 10.4, we have isomorphisms $e A_D e \cong B^{\mathrm{opp}}$ and $B^{\mathrm{opp}} \cong e A_{D'} e$. It follows from the proof that the isomorphism obtained by composing these two isomorphisms takes $e_I \in e A_D e$ to $e_I \in e A_{D'} e$, for every boundary label I .

REMARK 10.6. Using the map from $\mathbb{N}C_0$ to \mathbb{N} mapping an element of $\mathbb{N}C_0$ to the sum of its coefficients, we get an \mathbb{N} -grading on A_D and hence on $e A_D e$. The algebra B (and hence also B^{opp}) has a natural grading in which x has degree $n - k$ and y has degree k .

Furthermore, since $g(p_{E_{j+1}E_j}) = g_{E_{j+1}E_j}$, the isomorphism in Corollary 10.4 takes $p_{E_{j+1}E_j}$ to y_{j+1-k} , regarded as an element of B^{opp} . If $p_{E_{j+1}E_j}$ is an arrow, it has weight E_j (since strand $j + 1$ starts and strand $j + 1 - k$ ends on this arrow). If it is a path, then there is an arrow from E_j to E_{j+1} of weight $C_0 \setminus E_j$ and it again follows that the weight of $p_{E_{j+1}E_j}$ is E_j . Hence in the \mathbb{N} -grading, $p_{E_{j+1}E_j}$ has degree $|E_j| = k$, which is the same as the degree of y_{j+1-k} in B^{opp} . A similar argument shows that $p_{E_j E_{j+1}}$ maps to x_{j+1-k} , both elements of degree $n - k$. We see that the isomorphism in Corollary 10.4 preserves the \mathbb{N} -grading.

REMARK 10.7. Consider a Postnikov diagram D with a face labelled I . Then we can regard $e_I A_D e$ as a right $e A_D e$ module and hence, by Corollary 10.4, as a left B -module. The isomorphism g in Theorem 10.3 induces an isomorphism between $e_I A_D e$ and $g(e_I) \mathrm{End}_B(T_D) g(e)$ as left B -modules. But

$$g(e_I) \mathrm{End}_B(T_D) g(e) = \mathrm{id}_{\mathbb{M}_I} \mathrm{End}_B(T_D) \mathrm{id}_B = \mathrm{Hom}_B(B, \mathbb{M}_I),$$

which is isomorphic to \mathbb{M}_I as a left B -module. We thus obtain that $e_I A_D e$ is isomorphic to \mathbb{M}_I as a left B -module. It follows that T_D is isomorphic to $A_D e$ as a left B -module.

REMARK 10.8. We can also obtain that the algebra A_D is a cancellation algebra. To see this, we must show that if $\alpha p = \alpha q$ (or $p\alpha = q\alpha$) in A_D for paths p, q in $Q(D)$ and an arrow

α in $Q_1(D)$, then $p = q$. If $\alpha p = \alpha q$ then let F be a face whose boundary contains α . Let r be a path be such that $\partial F = r\alpha$. We get $r\alpha p = r\alpha q$, so $up = uq$. Hence it is enough to show that $up = uq$ (or $pu = qu$) implies $p = q$. We focus on the case $up = uq$; the other case is similar.

Note that we can assume that p, q start at the same vertex, say I , and end at the same vertex, say J . By Corollary 9.4, there are nonnegative integers N_p, N_q such that $p = u^{N_p}p_{JI}$ and $q = u^{N_q}p_{JI}$. Since $up = uq$, we have $u^{N_p+1}p_{JI} = u^{N_q+1}p_{JI}$. The elements on each side of this equation are non-zero in A_D by Remark 3.8(a). Comparing their weights, we see that $N_p = N_q$ and hence $p = q = u^N p_{JI}$ as required.

11. Completion

The definition of the Jacobian algebra associated to a quiver with potential involves taking a quotient of the completed path algebra (completed with respect to the arrow ideal) by the closure of the ideal generated by the relations determined by the potential; see [9]. If the potential lies in the path algebra, this coincides with the completion of the quotient of the path algebra by the relations. Thus is is natural to consider the completion \widehat{A} of the total algebra with respect to the arrow ideal. We also want to relate our results to those in [19]. So, in this section, we obtain analogues of Theorem 10.3 and Corollary 10.4 for the completed total algebra.

LEMMA 11.1. *Let m_A, m_B denote the arrow ideals of A_D and B , respectively. Then there is a nonnegative integer N_A such that $m_A^{N_A} \subseteq (u) \subseteq m_A$ and a nonnegative integer N_B such that $m_B^{N_B} \subseteq (t) \subseteq m_B$.*

Proof. By Corollary 9.4, any insincere path p from I to J in $Q(D)$ must be equal (in A_D) to p_{JI} . Hence, in particular, its weight must be equal to the weight of p_{JI} . The length of p is less than or equal to the sum of the entries in its weight, so it is bounded. Allowing I and J to vary, we see that there is a positive integer N_A such that any insincere path in $Q(D)$ has length at most $N_A - 1$.

Hence any path p of length at least N_A must be sincere and so, by Corollary 9.4, we have $p = u^s p_{JI}$ in A_D for some positive integer s . This proves the first part, since clearly also $(u) \subseteq m_A$.

For the second part, note that, in B , $x^{k+1} = y^{n-k}x$ and $y^{n-k+1} = x^k y$, so any path in the quiver of B with at least $N_B := \max(k + 1, n - k + 1)$ steps is equal in B to an element of (t) . □

Let \widehat{A} and \widehat{B} be the completions of A and B with respect to (u) and (t) respectively. By Lemma 11.1, these completions are isomorphic to the completions with respect to m_A and m_B , respectively. Similarly, we denote the completion of a B -module M with respect to (t) by \widehat{M} ; this completion is isomorphic to the completion with respect to m_B .

Recall that, as we observed in the introduction, \widehat{T}_D is a cluster-tilting object in the category of Cohen-Macaulay \widehat{B} -modules, by [19, Rk. 5.5] and [27, Cor. 1]

THEOREM 11.2. *Let D be a Postnikov diagram. Then the isomorphism g in Theorem 10.3 induces an isomorphism $\widehat{g}: \widehat{A} \rightarrow \text{End}_{\widehat{B}}(\widehat{T}_D)$. Let e be the sum in \widehat{A}_D of the idempotents e_I for I a boundary vertex in D . Then $e\widehat{A}_D e \cong \widehat{B}^{opp}$.*

Proof. By Lemma 7.4 and Theorem 10.3, A is finitely generated as a $\mathbb{C}[u]$ -module and by [19, Cor. 3.4], B is finitely generated as a $\mathbb{C}[t]$ -module. Since polynomial rings are Noetherian, the natural maps induce morphisms of $\mathbb{C}[[u]]$ and $\mathbb{C}[[t]]$ -modules as follows:

$$\begin{aligned}\widehat{A} &\cong A \otimes_{\mathbb{C}[u]} \mathbb{C}[[u]]; \\ \widehat{B} &\cong B \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]; \\ \widehat{M}_I &\cong M_I \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]].\end{aligned}\tag{11.1}$$

Since u is central in A the completion of A as a $\mathbb{C}[u]$ -module with respect to the ideal (u) in $\mathbb{C}[u]$ coincides with the completion of A with respect to the ideal (u) in A , and the first isomorphism above is an isomorphism of algebras. Similarly for B .

Since $\mathbb{C}[[t]]$ is a flat $\mathbb{C}[t]$ -module we have, using a base change argument analogous to [24, Thm. 7.11], that for any finitely presented B -modules M, N ,

$$\mathrm{Hom}_B(M, N) \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]] = \mathrm{Hom}_{B \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]}(M \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]], N \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]).\tag{11.2}$$

As B is Noetherian, any finitely generated B -module is finitely presented. By (11.1) and (11.2), we obtain \widehat{g} as the composition:

$$\begin{aligned}\widehat{A} &\cong A \otimes_{\mathbb{C}[u]} \mathbb{C}[[u]] \\ &\cong \mathrm{End}_B(T) \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]] \\ &\cong \mathrm{End}_{\widehat{B}}(\widehat{T}),\end{aligned}$$

where the second isomorphism is induced by g . The proof of the second part then goes through in the same way as for Corollary 10.4. \square

12. Geometric exchange

By Corollary 10.4, the algebra $eA_D e$ does not depend on the choice of (k, n) -Postnikov diagram D . In this section, we give an alternative proof of this fact by showing directly that $eA_D e$ is invariant under the untwisting and twisting moves and the boundary untwisting and twisting moves (see Figure 2) and the geometric exchange move (see Figure 24).

The following lemma shows that A_D is invariant under equivalence of Postnikov diagrams.

LEMMA 12.1. *The algebra A_D is invariant (up to isomorphism) under the untwisting and twisting moves and the boundary untwisting and twisting moves (see Figure 2).*

Proof. It is enough to consider the untwisting moves in Figure 2. Invariance under the twisting moves follows, and the arguments for the moves obtained by reflecting those in Figure 2 in a horizontal line of symmetry are similar. Suppose first that D' is obtained from D by applying a untwisting move, as in the top diagram in Figure 2. Then, locally, the quivers $Q(D)$ and $Q(D')$ are as shown in the top diagram of Figure 23 (with part of D and $Q(D)$ shown on the left). In general we will denote the path in $Q(D')$ corresponding to a path π in $Q(D)$ by π' ; this is well defined for any path not passing along the arrows α or β .

Let F_1 be the face in $Q_2(D)$ on the left hand side of α , F_2 the face with boundary $\beta\alpha$ and F_3 the face on the right hand side of β . Then $\partial F_1 = q\alpha$ for some path q and $\partial F_2 = p\beta$ for some path p . The corresponding part of $Q(D')$ has only one face, F' , with $\partial F' = q'p'$. The other faces of $Q(D)$ and $Q(D')$ are in a natural one-to-one correspondence.

Recall that each arrow of $Q(D)$ determines a defining relation

$$p_\alpha^+ = p_\alpha^-,$$

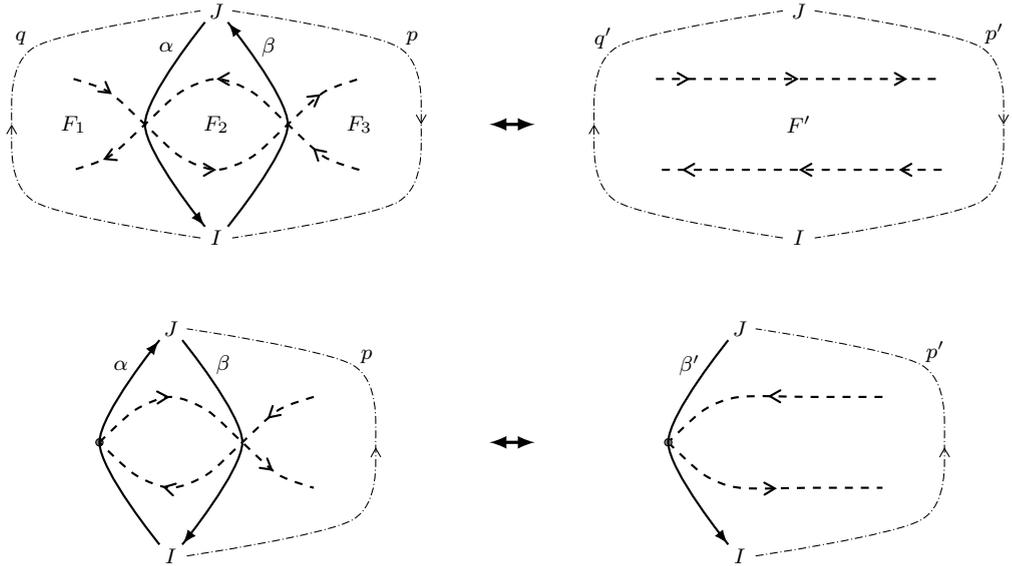


FIGURE 23. The effect of an untwisting move or boundary untwisting move on the quiver of a Postnikov diagram

of A_D (see equation (3.1)). The defining relation corresponding to the arrow α (respectively, β) is $q = \beta$ (respectively, $p = \alpha$). If γ is an arrow in the path p , so that $p = p_2\gamma p_1$ for some paths p_1 and p_2 , then $p_{\bar{\gamma}}$ is the path $p_1\beta p_2$. Similarly, if δ is an arrow in the path q , so that $q = q_2\delta q_1$ for some paths q_1, q_2 , then $q_{\bar{\delta}}$ is the path $q_1\alpha q_2$.

Making the substitutions $\alpha = p$ and $\beta = q$, we obtain $p_{\bar{\gamma}} = p_1qp_2$ and $q_{\bar{\delta}} = q_1pq_2$ in A_D . Since we have, in $\mathbb{C}Q(D')$, that $(p_{\bar{\gamma}})' = p'_1q'p'_2$ and $(q_{\bar{\delta}})' = q'_1p'q'_2$, it follows that A_D is isomorphic to $A_{D'}$, since the defining relations for $A_{D'}$ and A_D correspond precisely away from the local area affected by the untwisting move.

For the boundary untwisting move case (the lower pair of diagrams in Figure 2), the corresponding change in the quiver is displayed in the lower pair of diagrams in Figure 23. The defining relation of A_D corresponding to the arrow β is $p = \alpha$. If γ is an arrow in p , so that $p = p_2\gamma p_1$, then $p_{\bar{\gamma}} = p_1\beta p_2$, corresponding to $(p_{\bar{\gamma}})' = p'_1\beta'p'_2$ in $\mathbb{C}Q(D')$. Noting that β' is a boundary arrow in $Q(D')$ (so has no corresponding defining relation), we see that the substitution $p = \alpha$ gives an isomorphism between A_D and $A_{D'}$ in this case. \square

Note that, by Lemma 12.1, we see that A_D is isomorphic to $\text{End}_B(T_D)$ for any Postnikov diagram D ; see Theorem 10.3. Since the isomorphism in Lemma 12.1 sends the arrow ideal to the arrow ideal, it follows that $\widehat{A}_D \cong \text{End}_{\widehat{B}}(\widehat{T}_D)$ also.

There is a notion of *geometric exchange* on a Postnikov diagram, i.e. applying the local rule shown in Figure 24 [26, §14] (see also [27, §3]); we also illustrate the change in the quiver. The effect of this transformation on the plabic graph (respectively, quiver) is sometimes referred to as *urban renewal*. Urban renewal is discussed in [5, 20], whose authors refer to unpublished work of G. Kuperberg. See also the discussion in [16, 1.7], where it is referred to as a *spider move*. The effect on the quiver is *Seiberg duality* [12, §6]; see also Remark 12.3 below. Note that the k -subsets labelling the vertices remain unchanged under a geometric exchange except for the central region, which gets a new label.

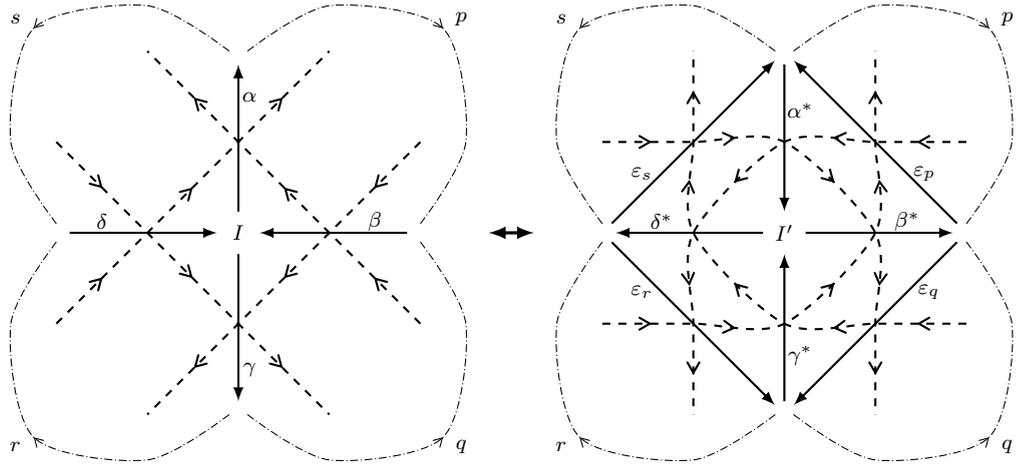


FIGURE 24. A geometric exchange and the corresponding change in the quiver

PROPOSITION 12.2. *Let D and D' be Postnikov diagrams and suppose that D' is obtained from D by applying the geometric exchange move at an internal vertex labelled by a k -subset I , as in Figure 24. Let I' be the label of the corresponding vertex in D' . Let*

$$e^I = \sum_{J \in D, J \neq I} e_J, \quad e^{I'} = \sum_{J \in D', J \neq I'} e_J.$$

Then $e^I A_D e^I \cong e^{I'} A_{D'} e^{I'}$.

Proof. We use the labelling of arrows as in Figure 24. Since the vertex I is internal, the arrows $\alpha, \beta, \gamma, \delta$ are not boundary arrows, so there is a face F whose boundary is $p\alpha\beta$ for some path p . In a similar way, we let q be the completion of $\gamma\beta$, r the completion of $\gamma\delta$ and s the completion of $\alpha\delta$ to face boundaries. We compute $e^I A_D e^I$ as a quiver with relations.

Let \mathcal{I}_D be the ideal of relations defining A_D . Then we have

$$e^I A_D e^I = e^I \left(\frac{\mathbb{C}Q(D)}{\mathcal{I}_D} \right) e^I = \frac{e^I \mathbb{C}Q(D) e^I}{e^I \mathcal{I}_D e^I}.$$

It is easy to check that $e^I \mathbb{C}Q(D) e^I$ is isomorphic to $\mathbb{C}\Gamma$, where Γ is obtained from $Q(D)$ by removing the vertex I and all incident arrows and adding new arrows $\varepsilon_p, \varepsilon_q, \varepsilon_r$ and ε_s , corresponding to the paths $\alpha\beta, \gamma\beta, \gamma\delta$ and $\alpha\delta$ respectively. Note that ε_p goes between the same vertices that p does, only in the opposite direction; similarly for $\varepsilon_q, \varepsilon_r$ and ε_s . We shall see that these arrows correspond to the arrows in $Q(D')$ with the same names.

The new relations generating $e^I \mathcal{I}_D e^I$ can be taken to be the old relations between vertices other than I together with new relations coming from the old relations between I and itself or other vertices, obtained by premultiplying or postmultiplying the old relations by an arrow. The relations we have to consider are $\beta p = \delta s, p\alpha = q\gamma, \beta q = \delta r$ and $s\alpha = r\gamma$, corresponding to the arrows α, β, γ and δ , respectively.

The relation $\beta p = \delta s$ gives $\alpha\beta p = \alpha\delta s$ and $\gamma\beta p = \gamma\delta s$, i.e. $\varepsilon_p p = \varepsilon_s s$ and $\varepsilon_q p = \varepsilon_r s$. The other relations give $p\varepsilon_p = q\varepsilon_q$ and $p\varepsilon_s = q\varepsilon_r, \varepsilon_q q = \varepsilon_r r$ and $\varepsilon_p q = \varepsilon_s r$, and $s\varepsilon_s = r\varepsilon_r$ and $s\varepsilon_p = r\varepsilon_q$ respectively.

We next do a similar computation for $e^{I'} A_{D'} e^{I'}$. Let p, q, r, s be the paths in $Q(D')$ corresponding to the paths with the same names in $Q(D)$. Let $\mathcal{I}_{D'}$ be the ideal of relations

defining $A_{D'}$. Then $e^{I'}\mathbb{C}Q(D')e^{I'}$ is isomorphic to $\mathbb{C}\Gamma'$, where Γ' is obtained from $Q(D')$ by removing the vertex I' and adding extra arrows $\zeta_p, \zeta_q, \zeta_r$ and ζ_s corresponding to the paths $\beta^*\alpha^*, \beta^*\gamma^*, \delta^*\gamma^*$ and $\delta^*\alpha^*$ respectively. We must consider the relations $\varepsilon_s\delta^* = \varepsilon_p\beta^*$, $\alpha^*\varepsilon_p = \gamma^*\varepsilon_q$, $\varepsilon_q\beta^* = \varepsilon_r\delta^*$ and $\gamma^*\varepsilon_r = \alpha^*\varepsilon_s$, corresponding to the arrows $\alpha^*, \beta^*, \gamma^*$ and δ^* , respectively.

The relation $\varepsilon_s\delta^* = \varepsilon_p\beta^*$ gives $\varepsilon_s\delta^*\alpha^* = \varepsilon_p\beta^*\alpha^*$ and $\varepsilon_s\delta^*\gamma^* = \varepsilon_p\beta^*\gamma^*$, i.e. $\varepsilon_s\zeta_s = \varepsilon_p\zeta_p$ and $\varepsilon_s\zeta_r = \varepsilon_p\zeta_q$. The other relations give $\zeta_p\varepsilon_p = \zeta_q\varepsilon_q$ and $\zeta_s\varepsilon_p = \zeta_r\varepsilon_q$, $\varepsilon_q\zeta_q = \varepsilon_r\zeta_r$ and $\varepsilon_q\zeta_p = \varepsilon_r\zeta_s$, $\zeta_r\varepsilon_r = \zeta_s\varepsilon_s$ and $\zeta_q\varepsilon_r = \zeta_p\varepsilon_s$, respectively.

Thus $e^{I'}A_{D'}e^{I'}$ is isomorphic to the quotient of $\mathbb{C}\Gamma'$ by the ideal generated by the above relations and the old relations in $A_{D'}$ between vertices not equal to I' .

This means that in $e^{I'}A_{D'}e^{I'}$, we also have the relations $\beta^*\alpha^* = p$, $\beta^*\gamma^* = q$, $\delta^*\gamma^* = r$ and $\delta^*\alpha^* = s$ coming from the arrows $\varepsilon_p, \varepsilon_q, \varepsilon_r$ and ε_s respectively. Thus, we have $\zeta_p = p$, $\zeta_q = q$, $\zeta_r = r$ and $\zeta_s = s$ in $e^{I'}A_{D'}e^{I'}$, so we can remove the arrows $\zeta_p, \zeta_q, \zeta_r$ and ζ_s from the quiver Γ' and replace them with p, q, r, s in the above relations. These relations become: $\varepsilon_s s = \varepsilon_p p$, $\varepsilon_s r = \varepsilon_p q$, $p\varepsilon_p = q\varepsilon_q$, $s\varepsilon_p = r\varepsilon_q$, $\varepsilon_q q = \varepsilon_r r$, $\varepsilon_q p = \varepsilon_r s$, $r\varepsilon_r = s\varepsilon_s$ and $q\varepsilon_r = p\varepsilon_s$, corresponding to the relations defining $e^I A_D e^I$ computed above. Since the other defining relations in $e^I A_D e^I$ and $e^{I'} A_{D'} e^{I'}$ are the same, we see that $e^I A_D e^I$ is isomorphic to $e^{I'} A_{D'} e^{I'}$ as required. \square

We remark that an alternative proof of Proposition 12.2 can be given by using Theorem 10.3 and noting that $e^I A_D e^I \cong \text{End}_{\mathcal{B}}(T_D/\widehat{\mathbb{M}}_I)$.

REMARK 12.3. The effect on $Q(D)$ of applying the geometric exchange is to carry out the first two steps of Fomin-Zelevinsky quiver mutation [13] at the vertex I , i.e.

- (a) For all paths of length two (with multiplicity) $J \rightarrow I \rightarrow K$, add an arrow $J \rightarrow K$.
- (b) Reverse all arrows incident with I .

The third step would usually be to cancel all two-cycles appearing after the first two steps. Instead, we carry out a slightly modified version of the third step, corresponding to applying Lemma 12.1, i.e.

- (c) Cancel all two-cycles consisting of non-boundary arrows.
- (d) For all two-cycles consisting of a boundary arrow and a non-boundary arrow, remove the boundary arrow and convert the non-boundary arrow into a boundary arrow.

The abstract rewriting system describing the individual moves in (c) and (d) is clearly terminating (as the number of arrows decreases with every step) and it is also easy to check that it is locally confluent. Hence, by the Diamond Lemma, it is confluent, and thus convergent. In other words, it does not matter in which order the individual steps in (c) and (d) are carried out; the resulting quiver will be independent of the order.

By Corollary 10.4, the boundary algebra $eA_D e$ is not dependent on the choice of D up to isomorphism. Lemmas 12.1 and 12.2 give an alternative proof of this fact.

COROLLARY 12.4. *Let D, D' be any two (k, n) -Postnikov diagrams. Then the corresponding boundary algebras $eA_D e$ and $e'A_{D'} e'$ are isomorphic.*

Proof. By Lemmas 12.1 and 12.2, this holds whenever D' can be obtained from D by a geometric exchange: if the exchange takes place at a vertex I , replacing it with I' , then we have $e'A_{D'} e' = e'e^I A_D e^I e'$ is isomorphic to $ee^I A_D e^I e$. In the general case, by [26, §14]

(see also [27]), there is a sequence of geometric exchanges and untwisting or twisting moves or boundary untwisting or twisting moves taking D to D' , and the result follows. \square

13. Surfaces

In this section, we generalize the context we are working in to surfaces with boundary. We note that dimer models (bipartite field theories) on surfaces with boundary have also been considered in independent work of S. Franco [11], and Postnikov diagrams on surfaces are also considered in [16, 23].

Let (X, M) be a marked oriented Riemann surface with nonempty boundary, where M is the set of marked points. We may assume that each boundary component is a circle. We also suppose that each marked point lies on a boundary component and that each boundary component has at least one marked point. This is sometimes referred to as the ‘unpunctured case’. We assume that (X, M) is not a disk with 1 or 2 marked points. Label the boundary components C_1, C_2, \dots, C_b ; suppose there are r_i marked points on boundary component C_i , for each i . We label the marked points around a boundary component C_i anticlockwise around the component as $p_{i1}, p_{i2}, \dots, p_{ir_i}$.

DEFINITION 13.1. We define a *weak Postnikov diagram* D on (X, M) to be a diagram consisting of directed curves embedded in (X, M) , one starting at each marked point and ending on the same boundary component on which it starts and exactly one strand ending and one strand starting at each marked point. The diagram must satisfy the local axioms (a1)–(a3) in Definition 2.1 and is considered up to isotopy. It need not necessarily be of reduced type. We say that D is a *Postnikov diagram* if, in addition, the global axioms (b1) and (b2) also hold. We say that a (weak) Postnikov diagram D has degree k if the strand starting at p_{ij} ends at $p_{i,j+k}$, where the second subscript is interpreted modulo r_i .

Thus, the (k, n) -Postnikov diagrams of Definition 2.1 are Postnikov diagrams of degree k on a disk with n marked points on its boundary, considered up to the untwisting and twisting moves (see Figure 2).

If D is a weak Postnikov diagram on a marked surface (X, M) , we define the corresponding dimer model $Q(D)$ as in the disk case, following Definition 2.4 and Remark 3.4. Let $A_D = A_{Q(D)}$ denote the corresponding dimer algebra, defined as in the disk case. Let e be the sum of the idempotents in A_D corresponding to the boundary vertices of $Q(D)$. Then, as in the disk case, we may define the *boundary algebra* of D to be $B_D = eA_De$. Note that it is not clear whether it is independent of the choice of D .

REMARK 13.2. Note that, as in the disk case, the boundary vertices of $Q(D)$ correspond to the alternating regions of D that have as one (unoriented) edge an interval on the boundary between two marked points. Thus the idempotents of the boundary algebra are naturally labelled by the *boundary intervals* of the marked surface, and not the marked (boundary) points. To draw the quiver $Q(D)$ on the surface, it is natural to locate each boundary vertex on the corresponding boundary interval.

We recall a construction of Scott [27, §3]. Given a triangulation \mathcal{T} of the disk with n marked points on its boundary, each triangle is replaced with a local configuration of strands, as in Figure 25, to produce a global configuration $D(\mathcal{T})$ of strands.

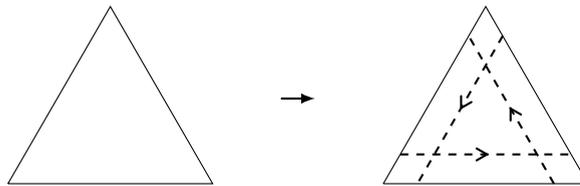


FIGURE 25. *Scott's construction*

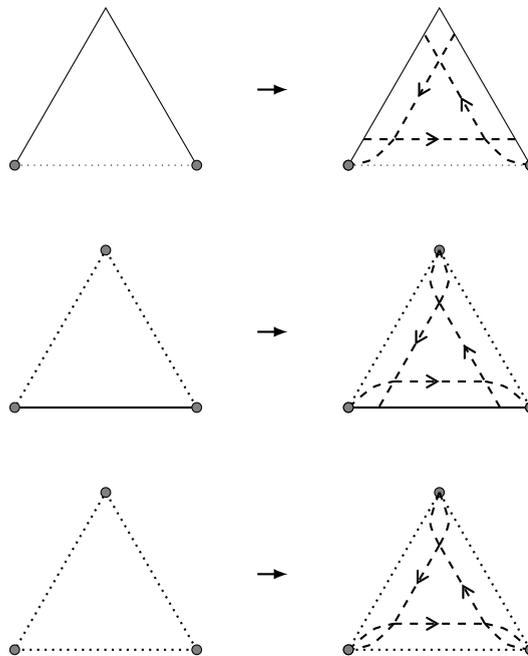


FIGURE 26. *Modified version of Scott's construction. The dotted lines indicate boundary edges*

We modify the conventions of Scott slightly, since we are following Postnikov [26]. We apply the above rule for an internal triangle of the triangulation, i.e. one all of whose edges are internal to X . For triangles with boundary edges, we apply the same rule except that the intersection of a strand with an edge of the triangle which is part of the boundary of X is slid along to the adjacent corner of the triangle. The upper diagram in Figure 26 illustrates this in the case where the horizontal edge in the figure is a boundary edge (and the others are internal); the middle diagram illustrates this in the case where the upper two edges are boundary edges, while the case where all three edges of the triangle are boundary edges is shown in the lower diagram. In each case the boundary edges are indicated by dotted lines, and internal edges by full lines.

By [27, Cor. 2], the map $\mathcal{T} \mapsto D(\mathcal{T})$ gives a bijection between triangulations of the disk with n marked points on its boundary and Postnikov diagrams of degree 2 on the disk (with the same marked points). The vertices of $Q(D(\mathcal{T}))$ correspond to the edges in \mathcal{T} (including boundary edges), and applying the geometric exchange at a vertex I corresponds to applying a quadrilateral flip at the corresponding edge.

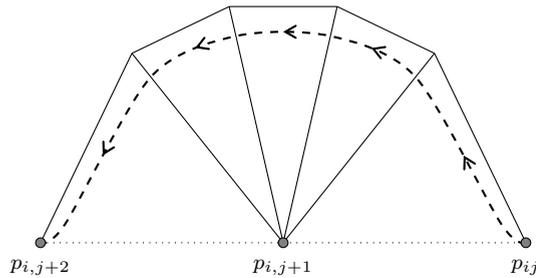


FIGURE 27. Path of a strand in $D(\mathcal{T})$ for \mathcal{T} a triangulation

We can generalize the map $\mathcal{T} \mapsto D(\mathcal{T})$ to a map from triangulations of (X, M) to weak Postnikov diagrams on (X, M) (from the construction, it is clear that axioms (a1)-(a3) in Definition 2.1 all hold for $D(\mathcal{T})$). We have:

LEMMA 13.3. *Let (X, M) be a marked surface with all marked points on the boundary, and let \mathcal{T} be a triangulation of (X, M) . Then $D(\mathcal{T})$ is a weak Postnikov diagram of degree 2.*

Proof. Axioms (a1) to (a3) in Definition 2.1 follow from the local construction of $D(\mathcal{T})$ (and the way in which two triangles are fitted together). So we just need to check that the weak Postnikov diagram constructed is of degree 2. Let p_{ij} be a marked point on boundary component \mathcal{C}_i of (X, M) . Anticlockwise of p_{ij} are the marked points $p_{i,j+1}$ and $p_{i,j+2}$ (allowing the possibility that one or both of them coincides with p_{ij}). Figure 27 shows the two triangles adjacent to the boundary arcs between p_{ij} and $p_{i,j+1}$ and between $p_{i,j+1}$ and $p_{i,j+2}$, together with all the arcs incident with $p_{i,j+1}$. Note that the two triangles may coincide. The dotted line at the base of the figure indicates the boundary of the surface (but note that some of the edges at the top of the figure may also be on the boundary). The figure also shows the strand which starts at p_{ij} ; we see that it ends at $p_{i,j+2}$ as required; note that its path is not affected by whether the edges at the top of the figure are boundary edges or not. \square

We may thus associate a dimer model $Q(D(\mathcal{T}))$ to a triangulation \mathcal{T} as in Lemma 13.3. As in Remark 3.6, we have an associated potential. Deleting the boundary arrows of the quiver, and all terms in the potential containing them, we obtain the quiver associated to \mathcal{T} in [10] with potential as in [21].

As an example, we consider the annulus. Fix positive integers n, m . Let $\Lambda_{n,m} = \mathbb{C}Q_{n,m}/\mathcal{I}$ be the algebra defined as follows. The quiver $Q_{n,m}$ is embedded into an annulus. The vertices are $1, 2, \dots, n$ clockwise on the outer boundary and $\bar{1}, \bar{2}, \dots, \bar{m}$ clockwise on the inner boundary. There are arrows $x_i: i-1 \rightarrow i$ and $y_i: i \rightarrow i-1$ on the outer boundary (end points taken mod n) and arrows $\bar{x}_i: \bar{i}-\bar{1} \rightarrow \bar{i}$ and $\bar{y}_i: \bar{i} \rightarrow \bar{i}-\bar{1}$ on the inner boundary (end points taken mod m), as well as arrows $r: 1 \rightarrow \bar{1}$ from the outer to inner boundary and $s: \bar{m} \rightarrow n$ from the inner to outer boundary. See Figure 28.

The relations are given by the following, where we omit the subscripts for x and y where they are determined by the starting vertex. Firstly, we have the relations:

$$xy = yx, \quad \bar{x}\bar{y} = \bar{y}\bar{x},$$

where the first relation (respectively, the second relation) starts at an arbitrary vertex on the outer boundary (respectively, the inner boundary). In addition, we have:

$$y^2 = x^{n-1-i} s \bar{x}^{m+1} r x^i \tag{13.1}$$

$$\bar{y}^2 = \bar{x}^{m-1-i} r x^{n+1} s \bar{x}^i \tag{13.2}$$

$$r = \bar{x}^m r x^n \tag{13.3}$$

$$s = x^n s \bar{x}^m \tag{13.4}$$

$$y x s = s \bar{x} \bar{y} \tag{13.5}$$

$$\bar{x} \bar{y} r = r y x \tag{13.6}$$

There is an instance of relation (13.1) for each vertex on the outer boundary; the exponent $i \geq 0$ is the minimum power of x such that x^i , when starting at that vertex, ends at vertex 1, the starting vertex of r . Similarly, there is an instance of relation (13.2) for each vertex on the inner boundary; the exponent $i \geq 0$ is the minimum power of \bar{x} such that \bar{x}^i , when starting at that vertex, ends at vertex \bar{m} , the starting vertex of s .

REMARK 13.4. The algebra $\Lambda_{n,m}$ is more symmetric than its presentation suggests. If we define $\Lambda'_{n,m}$ analogously to $\Lambda_{n,m}$, but with r replaced by an arrow R from 2 to $\bar{1}$ and s replaced by an arrow S from \bar{m} to 1, then the map $\Lambda_{n,m} \rightarrow \Lambda'_{n,m}$ taking r to $\bar{x}^m R x$ and s to $x^{n-1} S$ is an isomorphism; it has an inverse of a similar form. A corresponding isomorphism can be constructed that moves the endpoints of r and s on the inner boundary. Thus the algebra $\Lambda_{n,m}$ may be presented by any quiver ‘similar’ to Figure 28, i.e. with the same *labelled* boundary, but with the arrows r, s joining any pair of adjacent vertices on one boundary to any pair of adjacent vertices on the other boundary, and subject to essentially the same relations.

The existence of such similar presentations implies that $\Lambda_{n,m}$ has automorphisms that cycle the vertex idempotents on either boundary. In particular, it has non-trivial automorphisms that fix all the vertex idempotents, corresponding naturally to non-trivial elements of the (boundary fixing) mapping class group of the annulus.

In fact, the algebra $\Lambda_{n,m}$ also has other presentations in which the arrows r and s go between non-adjacent boundary vertices and the relations are adjusted a little. However, these will not be so relevant here.

PROPOSITION 13.5. *Let \mathcal{T} be any triangulation of the annulus, with $n > 0$ marked points on the outer boundary component and $m > 0$ marked points on the inner boundary component, as above, and let $D(\mathcal{T})$ be the corresponding weak Postnikov diagram of degree 2, as in Lemma 13.3. Then the boundary algebra $B_{D(\mathcal{T})}$ is isomorphic to $\Lambda_{n,m}$.*

Proof. Let \mathcal{T} be a triangulation of the annulus and set $A = A_{D(\mathcal{T})}$. Then \mathcal{T} must include an edge linking the two boundary components; we choose such an edge, and set e_0 to be the corresponding idempotent in A . Since the complement of this edge in the annulus is the interior of a disk, we can apply Corollary 10.4 (with some identifications) to get a description of $(e + e_0)A(e + e_0)$ as a quiver with relations, where e is the sum of the boundary idempotents in A . An explicit description of the boundary algebra eAe , of the above form, can then be obtained directly from this. □

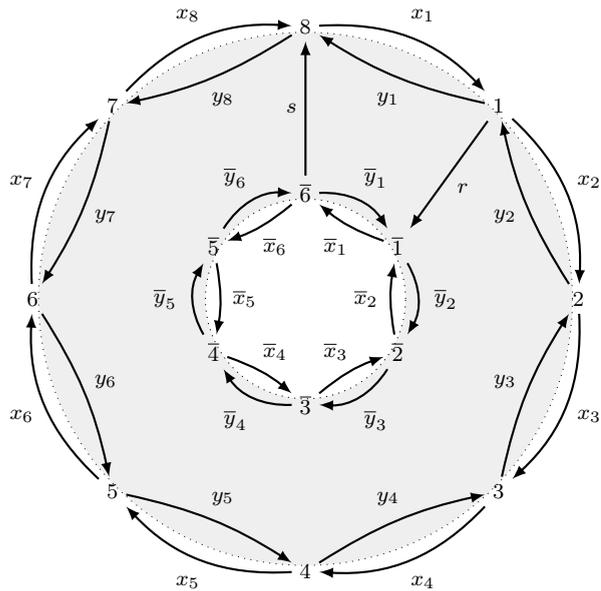


FIGURE 28. The boundary algebra of an annulus

By Remark 13.4, the isomorphism in Proposition 13.5 can (and indeed should) be chosen to be compatible with an *a priori* identification of the vertex idempotents of $\Lambda_{n,m}$ with the boundary vertices of $Q(D(\mathcal{T}))$, i.e. with the boundary intervals of (X, M) (cf. Remark 13.2). Thus, if we consider a second triangulation, then the two isomorphisms will yield an isomorphism of boundary algebras that preserves their (labelled) idempotents. In other words, the boundary algebra of a dimer model coming from a triangulated annulus does not depend on the triangulation (up to such an isomorphism).

Recall that, by Lemma 12.1 and Proposition 12.2, or by Corollary 10.4 and Remark 10.5, the boundary algebra of a Postnikov diagram of degree k on a disk also does not depend on the choice of diagram. Furthermore, it might be expected that Lemma 12.1 and Proposition 12.2 hold for the surface case. Thus it seems reasonable to make the following conjecture:

CONJECTURE 13.6. *Let (X, M) be a marked surface with nonempty boundary with marked points on its boundary only which is not a monogon or digon. Suppose that there is at least one marked point on each boundary component. Then the boundary algebra of a weak Postnikov diagram on (X, M) does not depend on the choice of diagram, up to isomorphism preserving the idempotents.*

References

1. R. BOCKLANDT, Calabi-Yau algebras and weighted quiver polyhedra, *Math. Z.* 273 (2013) 311–329.
2. R. BOCKLANDT, Consistency conditions for dimer models, *Glasg. Math. J.* 54 (2012) 429–447.
3. N. BROOMHEAD, Dimer models and Calabi-Yau algebras, *Mem. Amer. Math. Soc.* 215, no. 1011 (2012).
4. A.B. BUAN, O. IYAMA, I. REITEN, D. SMITH, Mutation of cluster-tilting objects and potentials, *Amer. J. Math.* 133 (2011) 835–887.
5. M. CIUCU, A complementation theorem for perfect matchings of graphs having a cellular completion, *J. Combin. Theory Ser. A* 81 (1998) 34–68.

6. V.I. DANILOV, A.V. KARZANOV, G.A. KOSHEVOY, On maximal weakly separated set-systems, *J. Algebraic Combin.* 32 (2010) 497–531.
7. B. DAVISON, Consistency conditions for brane tilings, *J. Algebra* 338 (2011) 1–23.
8. L. DEMONET, X. LUO, Ice quivers with potentials associated with triangulations and Cohen-Macaulay modules over orders, *Trans. Amer. Math. Soc.* 368 (2016) 4257–4293.
9. H. DERKSEN, J. WEYMAN, A. ZELEVINSKY, Quivers with potentials and their representations. I. Mutations., *Selecta Math. (N.S.)* 14 (2008) 59–119.
10. S. FOMIN, M. SHAPIRO, D. THURSTON, Cluster algebras and triangulated surfaces. I. Cluster complexes., *Acta Math.* 201 (2008) 83–146.
11. S. FRANCO, Bipartite Field Theories: from D-Brane Probes to Scattering Amplitudes, *JHEP* 11 (2012) 141.
12. S. FRANCO, A. HANANY, K.D. KENAWAY, D. VEGH, B. WECHT, Brane Dimers and Quiver Gauge Theories, *JHEP* 01 (2006) 096.
13. S. FOMIN, A. ZELEVINSKY, Cluster algebras. I. Foundations, *J. Amer. Math. Soc.* 15 (2002) 497–529.
14. C. GEISS, B. LECLERC, J. SCHRÖER, Partial flag varieties and preprojective algebras, *Ann. Inst. Fourier* 58 (2008) 825–876.
15. V. GINZBURG, Calabi-Yau algebras, arXiv:math/0612139v3 [math.AG] Jan 2007.
16. A.B. GONCHAROV, R. KENYON, Dimers and cluster integrable systems, *Ann. Sci. Éc. Norm. Sup. (4)* 46 (2013) 747–813.
17. D.R. GULOTTA, Properly ordered dimers, R-charges, and an efficient inverse algorithm, *JHEP* 10 (2008) 014.
18. A. HANANY, D. VEGH, Quivers, tilings, branes and rhombi, *JHEP* 10 (2007) 029.
19. B. JENSEN, A. KING, X. SU, A categorification of Grassmannian Cluster Algebras, arXiv:1309.7301v2 [math.RT] Dec 2014. To appear in *Proc. Lond. Math. Soc.*
20. R.W. KENYON, J.G. PROPP, D.B. WILSON, Trees and matchings, *Electron. J. Combin.* 7 (2000) R25.
21. D. LABARDINI-FRAGOSO, Quivers with potentials associated to triangulated surfaces, *Proc. Lond. Math. Soc. (3)* 98 (2009) 797–839.
22. B. LECLERC, A. ZELEVINSKY, Quasicommuting families of quantum Plücker coordinates, in: Kirillov’s seminar on representation theory, *Amer. Math. Soc. Transl. Ser. 2* 181 Amer. Math. Soc., Providence, RI, (1998) 85–108.
23. R.J. MARSH, J.S. SCOTT, Twists of Plücker coordinates as dimer partition functions, *Comm. Math. Phys.* 341 (2016) 821–884.
24. H. MATSUMURA, Commutative ring theory, Translated from the Japanese by M. Reid. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge (1986)
25. S. OH, A. POSTNIKOV, D.E. SPEYER, Weak Separation and Plabic Graphs, *Proc. Lond. Math. Soc. (3)* 110 (2015) 721–754.
26. A. POSTNIKOV, Total positivity, Grassmannians, and networks, arXiv:math/0609764v1 [math.CO] Sep 2006.
27. J.S. SCOTT, Grassmannians and cluster algebras, *Proc. London Math. Soc. (3)* 92 (2006) 345–380.

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