

TORSION PAIRS AND RIGID OBJECTS IN TUBES

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ABSTRACT. We classify the torsion pairs in a tube category and show that they are in bijection with maximal rigid objects in the extension of the tube category containing the Prüfer and adic modules. We show that the annulus geometric model for the tube category can be extended to the larger category and interpret torsion pairs, maximal rigid objects and the bijection between them geometrically. We also give a similar geometric description in the case of the linear orientation of a Dynkin quiver of type A.

INTRODUCTION

Torsion pairs in abelian categories were introduced by Dickson [12] (see the introduction to [5] for further details). They play an important role in the study of localisation (see e.g. [5]) and in tilting theory (see e.g. [2, 5]). Also, torsion pairs in the case of triangulated categories, as defined in [17, 2.2], have been considered recently by a number of authors, e.g. [16, 17, 19, 21, 22, 28].

The main object of study of this article is the collection of torsion pairs in a tube category, which is a hereditary abelian category. Such categories arise as full subcategories of module categories over tame hereditary algebras, and are so-called because their Auslander-Reiten quivers have the shape of a tube.

A geometric model for tube categories has been given in [3, 27] (see also [7]). The indecomposable objects are parametrized by a collection of oriented arcs in an annulus with n marked points on one of its boundary components. The dimension of the Ext-group between two indecomposable objects coincides with the negative intersection number of the corresponding pair of arcs.

In this article, we classify the torsion pairs in a tube category \mathbf{T} . We build on results in [8] giving a bijection between torsion pairs in a tube category and cotilting objects in the category obtained by taking arbitrary direct limits of modules in the tube. We show that all torsion pairs in the tube category arise in this way or via a dual construction. Thus they are in bijection with maximal rigid objects in the category $\overline{\mathbf{T}}$ obtained from the tube by taking arbitrary direct or inverse limits of objects in the tube. We give an explicit description of this bijection and its inverse.

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We further show that the geometric model referred to above can be extended to the indecomposable objects in $\overline{\mathbf{T}}$, i.e. to include Prüfer and adic modules associated to the tube. These extra objects are represented by certain infinite arcs in the annulus which spiral in towards the inner boundary component. The result concerning the dimensions of Ext-groups extends to this case also. This enables us to give a characterisation of maximal rigid objects in \mathbf{T} in the geometric model. We also give a characterisation of torsion pairs in the geometric model in terms of certain closure properties corresponding to closure properties of the subcategories in a torsion pair. In particular, the collection of arcs corresponding to a subcategory in a torsion pair must form an *oriented Ptolemy diagram*, an oriented version in the annulus case of the Ptolemy diagrams appearing in [16, 22], as well as satisfying additional criteria. We also give a geometric description of the above bijection and its inverse.

In order to give the geometric interpretation, we first give a similar model for the linearly oriented quiver in type A (note that M. Warkentin [27] also suggests such a model) and show how to interpret tilting modules and torsion pairs in this model.

We remark that, in independent work, T. Holm, P. Jørgensen and M. Rubey [15] have classified the torsion pairs in the cluster category associated to a tube (as opposed to the tube itself, which we study here). The torsion pairs in the cluster case are different, although we note that unoriented Ptolemy diagrams play a role in the cluster tube case, while oriented Ptolemy diagrams appear here.

In Section 1, we recall the definition and some of the properties of torsion pairs in abelian categories. In Section 2, we discuss the type A case. In Section 3, we recall the geometric model of the tube and show how it can be extended to include Prüfer and adic modules. We also discuss a certain reflection map (from [8]) on the indecomposable objects of the tube, and its properties. In Section 4 we classify the torsion pairs in the tube and prove that they are in bijection with maximal rigid objects in $\overline{\mathbf{T}}$, giving an explicit description of the bijection and its inverse. In Section 5, we give a geometric interpretation of maximal rigid objects in \mathbf{T} and torsion pairs in \mathbf{T} and the bijection between them.

1. PRELIMINARIES

We shall adopt the convention throughout that all subcategories considered are strictly full (i.e. full and closed under isomorphism). We shall also consider modules up to isomorphism. Let \mathcal{A} be an abelian category. If \mathcal{X} is a subcategory of \mathcal{A} , we define ${}^{\perp_H}\mathcal{X}$ (respectively, ${}^{\perp_E}\mathcal{X}$) to be the additive subcategory of \mathcal{A} consisting of the objects Y satisfying $\mathrm{Hom}_{\mathcal{A}}(Y, X) = 0$ (respectively, $\mathrm{Ext}_{\mathcal{A}}^1(Y, X) = 0$) for all X in \mathcal{X} . For an object X of \mathcal{A} we define ${}^{\perp_H}X = {}^{\perp_H}\mathrm{add} X$, and ${}^{\perp_E}X = {}^{\perp_E}\mathrm{add} X$, where $\mathrm{add} X$ denotes the additive subcategory of \mathcal{A} whose objects are finite direct sums of direct summands of X . We similarly define \mathcal{X}^{\perp_H} , \mathcal{X}^{\perp_E} , X^{\perp_H} and X^{\perp_E} . For additive subcategories \mathcal{X}_i , $i = 1, 2, \dots, m$, of \mathcal{A} , we write $\coprod_{i=1}^m \mathcal{X}_i$ for the smallest additive subcategory of \mathcal{A} containing the \mathcal{X}_i .

If \mathcal{X} is an additive subcategory of \mathcal{A} , we write $\mathrm{Gen} \mathcal{X}$ for the subcategory consisting of objects which are quotients of objects of \mathcal{X} and, for an object X , $\mathrm{Gen} X = \mathrm{Gen}(\mathrm{add} X)$. Similarly, we write $\mathrm{Cogen} \mathcal{X}$ for the subcategory consisting of objects

which are subobjects of objects of \mathcal{X} and $\text{Cogen } X = \text{Cogen}(\text{add } X)$. We write $\text{ind } \mathcal{X}$ for the subcategory of \mathcal{X} whose objects are the indecomposable objects in \mathcal{X} .

We next recall some of the theory of torsion pairs. Recall that a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of \mathcal{A} is said to be a *torsion pair* [12, Sect. 1] if $\text{Hom}(T, F) = 0$ for all objects T in \mathcal{T} and F in \mathcal{F} and for every object A in \mathcal{A} there is a short exact sequence

$$(1) \quad 0 \rightarrow T \rightarrow A \rightarrow F \rightarrow 0$$

with T in \mathcal{T} and F in \mathcal{F} . We say that \mathcal{T} is the *torsion* part and \mathcal{F} is the *torsion-free* part of the pair.

Recall that an abelian category \mathcal{A} is said to be *finite length* if it is skeletally small and every object in it is artinian and noetherian. Equivalently, it is skeletally small and every object has a finite length composition series.

Thus, for example, the category $\text{mod } A$ of finite dimensional modules over a finite-dimensional algebra is a finite length category. Parts (a), (b) of the following can be proved using arguments as in [12, Thm. 2.1] (see also [8, Lemma 1.7]); part (c) is a well-known corollary.

Lemma 1.1. *Suppose that \mathcal{A} is a finite length abelian category.*

- (a) *Let \mathcal{T} be a subcategory of \mathcal{A} and let $\mathcal{F} = \mathcal{T}^{\perp_H}$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in \mathcal{A} if and only if \mathcal{T} is closed under quotients and extensions.*
- (b) *Let \mathcal{F} be a subcategory of \mathcal{A} and let $\mathcal{T} = {}^{\perp_H} \mathcal{F}$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in \mathcal{A} if and only if \mathcal{F} is closed under subobjects and extensions.*
- (c) *A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of \mathcal{A} is a torsion pair if and only if $\mathcal{T}^{\perp_H} = \mathcal{F}$ and ${}^{\perp_H} \mathcal{F} = \mathcal{T}$.*

Recall that a finite length abelian category is said to be *serial* if every object is a direct sum of indecomposable uniserial objects. We recall the following well-known result.

Lemma 1.2. *Let \mathcal{A} be a serial finite length abelian category. Then*

- (a) *A morphism from an indecomposable object of \mathcal{A} to a direct sum of indecomposable objects is a monomorphism if and only if at least one of its components is a monomorphism.*
- (b) *Let \mathcal{X} be a subcategory of $\text{ind } \mathcal{A}$. Then we have $\text{ind Gen}(\text{add } \mathcal{X}) = \text{ind Gen } \mathcal{X}$ and $\text{ind Cogen}(\text{add } \mathcal{X}) = \text{ind Cogen } \mathcal{X}$.*

Note that the dual of Lemma 1.2(a) also holds. We fix an algebraically-closed field K and denote by D the vector space duality $\text{Hom}_K(-, K)$. Next, consider a finite dimensional K -algebra Λ . We denote by $\text{Mod } \Lambda$ the category of all left Λ -modules, and by $\text{mod } \Lambda$ the subcategory of all finitely generated Λ -modules. Recall that a module U in $\text{mod } \Lambda$ is called *tilting* if

- (a) The projective dimension of U is at most 1;
- (b) $\text{Ext}^1(U, U) = 0$;
- (c) There is a short exact sequence

$$0 \rightarrow \Lambda \rightarrow U_0 \rightarrow U_1 \rightarrow 0$$

with U_0 and U_1 in $\text{add } U$.

Cotilting modules in $\text{mod } \Lambda$ are defined dually. We only consider basic tilting or cotilting modules, i.e. we assume that $U = \coprod U_i$, with U_i indecomposable and $U_i \not\cong U_j$ for $i \neq j$.

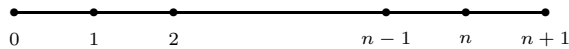
2. DYNKIN TYPE A

We consider a linearly oriented quiver Q of type A_n :

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n-1 \longleftarrow n$$

We write S_i , $i = 1, \dots, n$, for the simple KQ -modules. Let M_{ij} denote the indecomposable KQ -module with composition factors S_{i+1}, \dots, S_{j-1} (starting from the socle). If i lies in $\{j-1, j\}$ we regard M_{ij} as zero. Note that $\text{mod } KQ$ is serial.

2.1. Geometric model. We now describe a geometric model for the indecomposable KQ -modules. Note that a similar such model is also suggested in [27, Remark 4.28]. Here we will indicate how this model can incorporate torsion theories in $\text{mod } KQ$ — later we will indicate how this can be done for tubes. We also give some more explicit information as preparation for this. We consider a line segment ℓ_n with marked points $0, 1, \dots, n+1$:



and associate the module M_{ij} with the arc $[i, j]$ above ℓ_n from i to j oriented towards j , $0 \leq i < j-1 \leq n$. This gives a bijection between indecomposable KQ -modules and the set $\mathcal{A}(\ell_n)$ of arcs up to isotopy between marked points of ℓ_n , above ℓ_n , which are not isotopic to boundary arcs.

It is easy to check that, for $[i, j], [i', j'] \in \mathcal{A}(\ell_n)$, we have $\text{Ext}^1(M_{ij}, M_{i'j'}) \cong K$ if there is a negative crossing between $[i, j]$ and $[i', j']$ (see Figure 1) and is zero otherwise. In the former case, the non-split extension takes the form:

$$0 \rightarrow M_{i'j'} \rightarrow \begin{array}{c} M_{i'j} \\ \amalg \\ M_{ij'} \end{array} \rightarrow M_{ij} \rightarrow 0$$

up to equivalence, if $j' > i+1$, while if $j' = i+1$, it takes the form

$$0 \rightarrow M_{i'j'} \rightarrow M_{i'j} \rightarrow M_{ij} \rightarrow 0,$$

This is interpreted geometrically in Figure 2 where the indecomposable summands of the middle term of the short exact sequence are indicated by dotted lines.



FIGURE 1. A negative crossing between $[i, j]$ and $[i', j']$.

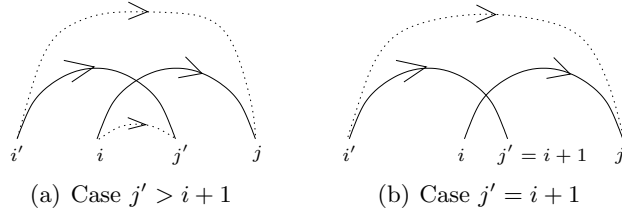


FIGURE 2. Non-split extension

The AR-translation moves an arc one step to the left (or gives zero if this is not defined). Furthermore, the indecomposable quotients of a module correspond to moving the starting point weakly to the right, while submodules correspond to moving the ending point weakly to the left. We call the former arcs *left-shortenings* of $[i, j]$ and the latter arcs *right-shortenings* of $[i, j]$.

By [6], a KQ -module is tilting if and only if it is a maximal rigid KQ -module. Since $M_{0,n+1}$ is projective-injective, it is a summand of every tilting module. It follows that in the above model, tilting modules are in bijection with triangulations of a polygon with $n + 2$ sides.

Motivated by [22, 16] and the above description of extensions, we call a collection of arcs in $\mathcal{A}(\ell_n)$ an *oriented Ptolemy diagram* if, whenever $[i, j]$ and $[i', j']$ lie in the collection, with $i' < i < j' < j$, we have that the arcs $[i, j']$ (provided $j' > i + 1$) and $[i', j]$ also lie in the collection.

2.2. Torsion pairs. By Lemma 1.1 and Lemma 1.2, a collection of arcs in $\mathcal{A}(\ell_n)$ corresponds to the torsion (respectively, torsion-free) part of a torsion pair in $\text{mod } KQ$ if and only if it forms an oriented Ptolemy diagram and is closed under left-shortening (respectively, right-shortening).

Given a tilting KQ -module U , the pair $(\text{Gen } U, \text{Cogen } \tau U) = (U^{\perp_E}, U^{\perp_H})$ is known to form a torsion pair (see [2, VI.2]). A torsion pair arises in this way if and only if \mathcal{T} contains all the indecomposable injective modules, if and only if \mathcal{F} contains no non-zero injective module (see [2, VI.6] for the first equivalence; the second is easy to check). The first equivalence holds for an arbitrary finite dimensional hereditary algebra of finite representation type, and the equivalence of the second two statements holds for any finite dimensional hereditary algebra.

Noting that a KQ -module is tilting if and only if it is cotilting, we have:

Corollary 2.1. *The map:*

$$U \mapsto (\text{Gen } U, \text{Cogen } \tau U) = (U^{\perp_E}, U^{\perp_H})$$

gives a bijection between tilting KQ -modules and torsion pairs $(\mathcal{T}, \mathcal{F})$ for which \mathcal{T} contains all the indecomposable injective modules. The map:

$$U \mapsto (\text{Gen } \tau^{-1}U, \text{Cogen } U) = ({}^{\perp_H}U, {}^{\perp_E}U)$$

gives a bijection between tilting KQ -modules and torsion pairs $(\mathcal{T}, \mathcal{F})$ for which \mathcal{F} contains all the indecomposable projective modules.

Using Lemma 1.2, the first map in Corollary 2.1 can be interpreted in the geometric model: $\text{Gen } U$ is obtained from U by closure under left shortening and $\text{Cogen } \tau U$ is obtained from U by shifting to the left one step (deleting arcs starting at 0) and then closing under right shortening (in both cases we then take the additive closure).

Conversely, if $(\mathcal{T}, \mathcal{F})$ is a torsion pair of the kind considered in Corollary 2.1, then U can be recovered as the direct sum of the indecomposable Ext-projectives in \mathcal{T} , i.e. the objects

$$(2) \quad \{X \in \text{ind}(\mathcal{T}) : \text{Ext}^1(X, T) = 0 \text{ for all } T \in \mathcal{T}\},$$

by [2, VI.2.5]. Geometrically, this means taking all of the arcs X in $\text{ind } \mathcal{T}$ which do not have a negative crossing with an arc in $\text{ind } \mathcal{T}$.

There is also a geometric description of the second map in Corollary 2.1 and its inverse. We leave the details to the reader.

3. TUBES

3.1. Categorical description. Fix an integer $n \geq 1$. Consider the quiver Q

$$(3) \quad \begin{array}{ccccccc} & & 2 & \longleftarrow & 3 & \longleftarrow & \cdots & \longleftarrow & n & & \\ & \swarrow & & & & & & & & \searrow & \\ 1 & & & \longleftarrow & & & & & & & n+1 \end{array}$$

of Euclidean type $\tilde{\mathbb{A}}_{1,n}$. The path algebra $\Lambda = KQ$ is tame hereditary, and the module category $\text{mod } KQ$ has an extension closed subcategory \mathbf{T}_n , which can be realized as the extension closure of the modules L, S_2, \dots, S_n , where S_i denotes the simple corresponding to vertex i , and L denotes the unique indecomposable module with composition factors S_1 and S_{n+1} . The category \mathbf{T}_n is called a *tube* of rank n . Note that the indecomposables of \mathbf{T} form a standard component (see e.g. [2]) of the AR-quiver of KQ .

Actually \mathbf{T}_n is a hereditary finite length abelian category with n simple objects, and equivalent categories appear in various settings in representation theory and algebraic geometry. For each pair of objects X, Y in \mathbf{T}_n , the spaces $\text{Hom}(X, Y)$ and $\text{Ext}^1(X, Y)$ have finite K -dimension, and there is an autoequivalence τ on \mathbf{T}_n , induced by the Auslander-Reiten translate on $\text{mod } \Lambda$, with the property that $\text{Hom}(Y, \tau X) \simeq D \text{Ext}^1(X, Y)$. Let $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$ be the map taking i to $i+n$. From now on we denote the simples in $\mathbf{T} = \mathbf{T}_n$ by $M_{i, i+2}$ for $i = 0, \dots, n-1$, in such a way that $\tau M_{i, i+2} = M_{i-1, i+1}$, where we regard $M_{\sigma^k(i), \sigma^k(j)}$ as equal to $M_{i, j}$ for any integer k . The category \mathbf{T}_n is serial; thus the indecomposable objects in \mathbf{T}_n are uniserial and uniquely determined by their simple socle and length (in \mathbf{T}_n). We denote by $M_{i, i+l+1}$ an indecomposable with socle $M_{i, i+2}$ and length l . Then $\tau M_{i, i+l+1} = M_{i-1, i+l}$, and the AR-quiver of the tube \mathbf{T}_n is as in Figure 3 (with the columns on the left- and right-hand sides identified). Note that each indecomposable object has a unique name M_{ij} (with $j - i \geq 2$) if we insist that i lies in $\{0, 1, \dots, n-1\}$.

An additive subcategory of \mathbf{T} is said to be of *finite type* if it contains only finitely many indecomposable objects. Otherwise it is said to be of *infinite type*. Some particular subcategories of \mathbf{T} are important for this paper. For each fixed i in

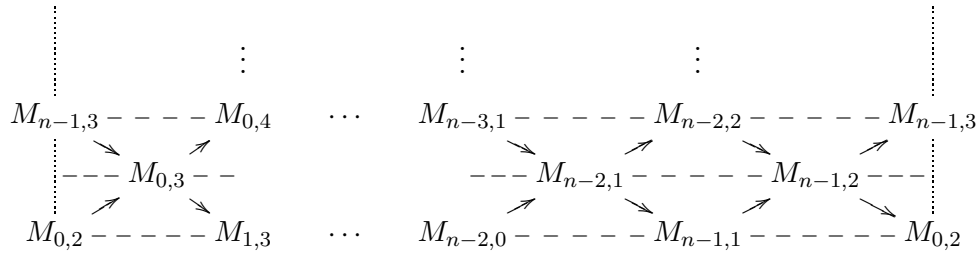


FIGURE 3. The AR-quiver of \mathbf{T}_n

$\{0, \dots, n-1\}$, we consider the additive subcategory whose objects are the indecomposable objects $M_{i,i+t}$, for all $t > 1$. This is called a *ray*, and is denoted \mathcal{R}_i . Dually, for each i in $\{0, \dots, n-1\}$, we consider the additive subcategory whose indecomposable objects are all indecomposable objects $M_{i-u,i}$, for all $u > 1$. This is called a *coray*, and denoted \mathcal{C}_i .

For each i in $\{0, \dots, n-1\}$, and each $t > 1$, the *wing* $\mathcal{W}_{i,i+t}$ is the additive subcategory of \mathbf{T} whose indecomposable objects are the $M_{j,j+u}$ with $u \geq 2$, $i \leq j \leq i+t-2$ and $j+u \leq i+t$. It contains a unique indecomposable object $M_{i,i+t}$ of maximal length. The objects $M_{i,i+2}$ and $M_{i+t-2,i+t}$ lie at the bottom left and bottom right corners of the wing, respectively, in the collection of vertices corresponding to the indecomposable objects of the wing in the AR-quiver of \mathbf{T} . For $t \leq 1$ we let $\mathcal{W}_{i,i+t}$ be the zero subcategory. We also denote the wing of an indecomposable object X by \mathcal{W}_X .

Due to the following well-known fact, we can apply results from the previous section in our analysis of \mathbf{T} .

Lemma 3.1. *For $u \leq n+1$ the wing $\mathcal{W}_{i,i+u}$ in \mathbf{T}_n is equivalent to the module category $\text{mod } KQ$, where Q is a linearly oriented quiver of Dynkin type A_{u-1} . If $u \leq n$ the equivalence is exact.*

Note that if $u = n+1$, the object corresponding to the projective-injective indecomposable module in $\text{mod } KQ$ under the above equivalence is not Ext-projective in the wing.

We denote by $\text{Mod } \Lambda$ the category of all left Λ -modules. Let $\varinjlim \mathbf{T}$ be the subcategory of $\text{Mod } \Lambda$ whose objects are direct limits of filtered direct systems of objects in \mathbf{T} . Note that $\varinjlim \mathbf{T}$ contains the *Prüfer modules*, i.e. the modules $M_{i,\infty}$, $i = 0, 2, \dots, n-1$ obtained as direct limits of the indecomposable objects in the rays, i.e.

$$M_{i,\infty} = \varinjlim (M_{i,i+2} \rightarrow M_{i,i+3} \rightarrow M_{i,i+4} \rightarrow \dots).$$

Let $\varprojlim \mathbf{T}$ be the subcategory of $\text{Mod } \Lambda$ whose objects are inverse limits of filtered inverse systems of objects in \mathbf{T} . This category contains the *adic modules*, which are obtained as inverse limits along a coray:

$$M_{-\infty,i} = \varprojlim (\dots \rightarrow M_{i-4,i} \rightarrow M_{i-3,i} \rightarrow M_{i-2,i}).$$

Let $\overline{\mathbf{T}}$ be the subcategory of $\text{Mod } \Lambda$ whose objects are all filtered direct limits or filtered inverse limits of objects in \mathbf{T} . This category clearly contains $\varinjlim \mathbf{T}$ and $\varprojlim \mathbf{T}$. We extend the definition of σ to Prüfer and adic modules with the convention that $\sigma(\pm\infty) = \pm\infty$: note that any Prüfer module has a unique name $M_{i,\infty}$ if we take i in $\{0, 1, \dots, n-1\}$, and similarly for adic modules.

Recall that a Λ -module M is *pure-injective* if the canonical map $M \rightarrow DDM$ is a split monomorphism. For background on pure-injective modules, and other definitions, see e.g. [18] or [24]. It can be shown (see [8]) that the category $\overline{\mathbf{T}}$ has the following properties:

- All objects are pure-injective as Λ -modules.
- Any object is determined by its indecomposable direct summands.
- The indecomposables in $\overline{\mathbf{T}}$ are exactly the indecomposables M_{ij} in \mathbf{T} , the Prüfer modules $M_{i,\infty}$ and the adic modules $M_{-\infty,i}$.

A module M in $\overline{\mathbf{T}}$ is called *rigid* if $\text{Ext}_{\Lambda}^1(M, M) = 0$. Note that since all objects in $\overline{\mathbf{T}}$ are pure-injective, this definition is equivalent to having $\text{Ext}_{\Lambda}^1(M', M'') = 0$ for all indecomposable direct summands M', M'' of M . Now, a rigid module M in $\overline{\mathbf{T}}$ is called *maximal rigid*, if $\text{Ext}_{\Lambda}^1(M \amalg X, M \amalg X) = 0$ for an indecomposable X in $\overline{\mathbf{T}}$ implies that X is isomorphic to a direct summand in M .

3.2. Geometric model. We now give a geometric model for $\overline{\mathbf{T}}$. This extends the known geometric model for \mathbf{T} [3, 27]. Consider an annulus $\mathbb{A}(n)$ with n marked points on the outer boundary. The points are labelled $0, 1, \dots, n-1$, and arranged anticlockwise (see Figure 4).

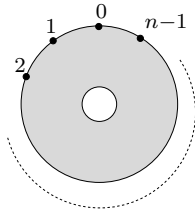


FIGURE 4. An annulus with n marked points on its outer boundary.

Let $\mathbb{U}(n)$ denote the universal cover of $\mathbb{A}(n)$, with marked points corresponding to \mathbb{Z} (and with $0, 1, \dots, n-1$ lying in a fundamental domain). See Figure 5.

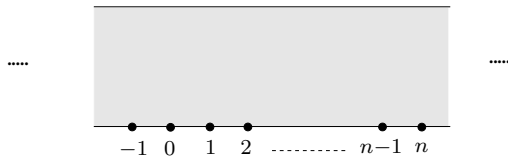


FIGURE 5. The universal cover of the annulus in Figure 4.

For integers i, j with $i + 2 \leq j$, let $[i, j]$ denote the arc in $\mathbb{U}(n)$ with starting point i and ending point j , oriented from i to j . We also allow arcs which have



FIGURE 6. Infinite arcs in $\mathbb{U}(n)$.

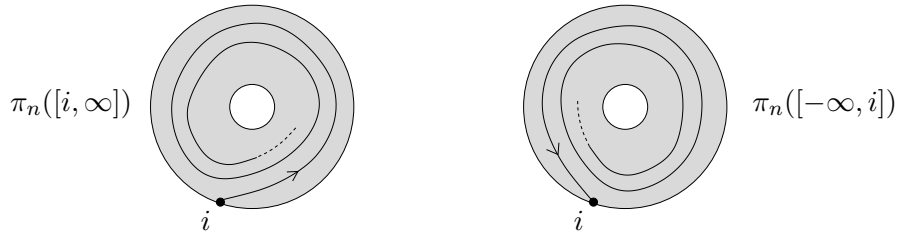


FIGURE 7. Infinite arcs in $\mathbb{A}(n)$.

only one end-point: arcs of the form $[i, \infty]$ (respectively, $[-\infty, j]$ which start at i (respectively, end at j) and are oriented in the positive x direction. See Figure 6.

Let $\pi_n([i, j])$ denote the corresponding arc in $\mathbb{A}(n)$ and let $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}(\mathbb{A}(n))$ denote the set of (isotopy classes of) such arcs. It contains the set $\mathcal{A} = \mathcal{A}(\mathbb{A}(n))$ of arcs of the form $\pi_n([i, j])$ with i, j finite. The map $\psi : \tilde{\mathcal{A}} \rightarrow \text{ind } \overline{\mathbf{T}}$ sending $\pi_n([i, j])$ to M_{ij} is a bijection. The infinite arcs are displayed in Figure 7.

Define a quiver with vertices given by the elements in \mathcal{A} and arrows:

$$\pi_n([i, j]) \rightarrow \pi_n([i, j + 1])$$

and

$$\pi_n([i, j]) \rightarrow \pi_n([i + 1, j]) \text{ (if } j \neq i + 2)$$

Defining a translate using the formula $\tau(\pi_n([i, j])) = \pi_n([i - 1, j - 1])$, this becomes a translation quiver. We call this the *(translation) quiver of $\mathcal{A}(\mathbb{A}(n))$* .

By [3, Lemma 2.5], [27, 4.18] (or, using unoriented arcs, [7, §3.4], [13]), we have:

Proposition 3.2. *The restriction of ψ to \mathcal{A} gives an isomorphism between the translation quiver of $\mathcal{A}(\mathbb{A}(n))$ and the AR-quiver of \mathbf{T}_n .*

Note that the convention that $M_{\sigma^k(i), \sigma^k(j)} = M_{ij}$ corresponds exactly to the fact $\pi_n([\sigma^k(i), \sigma^k(j)]) = \pi_n([i, j])$ for any integer k .

For arcs α, β in $\mathcal{A}(\mathbb{A}(n))$, let $I(\alpha, \beta)$ be the minimum number of intersections between arcs in the isotopy classes α and β , not allowing non-transverse or multiple intersections. Similarly we let $I^+(\alpha, \beta)$ (resp. $I^-(\alpha, \beta)$) denote the number of positive (resp. negative) crossings between α and β (see Figure 1 for an example of a negative crossing). We will now prove the following result:

Theorem 3.3. *Given indecomposable objects M_{ij} and $M_{i'j'}$ in $\overline{\mathbf{T}}$. Then:*

$$\mathrm{Ext}^1(M_{ij}, M_{i'j'}) \cong \prod_{I^-(\pi_n([i,j]), \pi_n([i',j']))} K.$$

In the case where i, i', j, j' are all finite, the result is proved in [3, Thm. 3.7], [27, Thm. 4.23]. See also [7] for further results in this direction. We first recall some results we will need:

Lemma 3.4. *Let X, Y be arbitrary Λ -modules, $(X_j)_j$ an arbitrary filtered direct system of modules and $(Y_j)_j$ an arbitrary filtered inverse system of modules.*

- (a) $\mathrm{Hom}(\varinjlim X_j, Y) \simeq \varprojlim \mathrm{Hom}(X_j, Y)$.
- (b) *If X is finitely generated, then $\mathrm{Hom}(X, \varinjlim Y_j) \simeq \varinjlim \mathrm{Hom}(X, Y_j)$*
- (c) *If the Y_j are finitely generated, then $\varinjlim Y_j \simeq D \varinjlim DY_j$.*
- (d) *If Y is pure-injective, then $\mathrm{Ext}^1(\varinjlim X_j, Y) \simeq \varprojlim \mathrm{Ext}^1(X_j, Y)$.*
- (e) *If the Y_j are finitely generated, then $\mathrm{Ext}^1(X, \varinjlim Y_j) \simeq \varprojlim \mathrm{Ext}^1(X, Y_j)$.*

Proof. For (a) see, for example, [26]. For (b), see [20, Lemma 1.6] or [11, Sect. 1.5]. For (c), we have, using part (a):

$$\begin{aligned} D \varinjlim DY_j &= \mathrm{Hom}(\varinjlim \mathrm{Hom}(Y_j, K), K) \\ &\simeq \varprojlim \mathrm{Hom}(\mathrm{Hom}(Y_j, K), K) \simeq \varprojlim DDY_j \simeq \varprojlim Y_j, \end{aligned}$$

as required. Part (d) is proved in [1, Prop. I.10.1]. For (e), we recall that $\mathrm{Ext}^1(X, DY) \simeq \mathrm{Ext}^1(Y, DX)$ for all modules X and Y . Using parts (c) and (d) and the fact [24, Prop. 4.3.29] that DX is pure-injective for any module X , we have:

$$\begin{aligned} \mathrm{Ext}^1(X, \varinjlim Y_j) &\simeq \mathrm{Ext}^1(X, D(\varinjlim DY_j)) \simeq \mathrm{Ext}^1(\varinjlim DY_j, DX) \\ &\simeq \varprojlim \mathrm{Ext}^1(DY_j, DX) \simeq \varprojlim \mathrm{Ext}^1(X, DDY_j) \\ &\simeq \varprojlim \mathrm{Ext}^1(X, Y_j), \end{aligned}$$

and (e) is shown. □

We also need the following (see e.g. [11, Sect. 3.1]).

Lemma 3.5. *For modules X and Y with X finitely generated, we have $D \mathrm{Ext}^1(X, Y) \cong \mathrm{Hom}(Y, \tau X)$ and $\mathrm{Ext}^1(Y, X) \cong D \mathrm{Hom}(\tau^{-1} X, Y)$.*

Note that if X, Y are finitely generated, then the first formula can also be written $\mathrm{Ext}^1(X, Y) \cong D \mathrm{Hom}(Y, \tau X)$. We recall the following (see, for example, [23, p46]).

Lemma 3.6. (a) *If X is a Prüfer module and Y is a finitely generated module then $\mathrm{Hom}(X, Y) = 0$.*

- (b) *If X is a finitely generated module and Y is an adic module then $\mathrm{Hom}(X, Y) = 0$.*

With arguments as in [3], the crossing numbers can now be computed as follows. Recall that $\sigma: \mathbb{Z} \rightarrow \mathbb{Z}$ is the function $i \mapsto i + n$.

Proposition 3.7. *We have the following:*

- (a) $I^-(\pi_n([i, \infty]), \pi_n([a, b])) = |\{m \in \mathbb{Z} : a < \sigma^m(i) < b\}|$;
- (b) $I^-(\pi_n([a, b]), \pi_n([i, \infty])) = I^+(\pi_n([i, \infty]), \pi_n([a, b])) = 0$;
- (c) $I^-(\pi_n([-\infty, j]), \pi_n([a, b])) = 0$;
- (d) $I^-(\pi_n([a, b]), \pi_n([-\infty, j])) = I^+(\pi_n([-\infty, j]), \pi_n([a, b])) = |\{m \in \mathbb{Z} : a < \sigma^m(i) < b\}|$;
- (e) $I^-(\pi_n([i, \infty]), \pi_n([-\infty, i'])) = \aleph_0$, for all i, i' in $\{0, \dots, n-1\}$;
- (f) $I^-(\pi_n([i, \infty]), \pi_n([i', \infty])) = 0$ for all i, i' in $\{0, \dots, n-1\}$;
- (g) $I^-(\pi_n([-\infty, i]), \pi_n([i', \infty])) = 0$ for all i, i' in $\{0, \dots, n-1\}$;
- (h) $I^-(\pi_n([-\infty, i]), \pi_n([-\infty, i'])) = 0$ for all i, i' in $\{0, \dots, n-1\}$.

Proof of Theorem 3.3. We need to compute $\text{Ext}^1(X, Y)$ for all pairs of indecomposables X, Y in $\overline{\mathbf{T}}$, and compare these with the crossing-numbers from Proposition 3.7. We first determine $\text{Ext}^1(M_{i, \infty}, M_{ab})$. By Lemma 3.5,

$$\text{Ext}^1(M_{i, \infty}, M_{ab}) \cong D \text{Hom}(M_{a+1, b+1}, M_{i, \infty}).$$

Any morphism $f : M_{a+1, b+1} \rightarrow M_{i, \infty}$ must factor through the unique submodule of $M_{i, \infty}$ of length $b-a-1$, which is isomorphic to $M_{i, i+b-a}$. Hence $\text{Hom}(M_{a+1, b+1}, M_{i, \infty}) \cong \text{Hom}(M_{a+1, b+1}, M_{i, i+b-a})$. Thus $\text{Ext}^1(M_{i, \infty}, M_{ab})$ equals the dimension of this last space, i.e. the number of times the simple top $M_{b-1, b+1}$ of $M_{a+1, b+1}$ appears as a composition factor in $M_{i, i+b-a}$, which, using arguments as in [3], is given by

$$|\{n \in \mathbb{Z} : a < \sigma^n(i) < b\}|.$$

Using a dual argument, we obtain:

$$\dim \text{Ext}^1(M_{i, -\infty}, M_{ab}) = |\{n \in \mathbb{Z} : a < \sigma^n(i) < b\}|.$$

By Lemma 3.5,

$$D \text{Ext}^1(M_{ab}, M_{i, \infty}) \simeq \text{Hom}(M_{i, \infty}, M_{a-1, b-1}).$$

We see that $\text{Ext}^1(M_{ab}, M_{i, \infty}) = 0$ by Lemma 3.6(a).

Similarly, by Lemma 3.5, we have that

$$\text{Ext}^1(M_{-\infty, i}, M_{ab}) \simeq D \text{Hom}(\tau^{-1} M_{ab}, M_{-\infty, i}) = 0,$$

using Lemma 3.6(b).

We next compute $\text{Ext}^1(M_{i, \infty}, M_{-\infty, i'})$. Using Lemma 3.4(e), we have that

$$\text{Ext}^1(M_{i, \infty}, M_{-\infty, i'}) = \text{Ext}^1(M_{i, \infty}, \varprojlim M_{j, i'}) \simeq \varprojlim \text{Ext}^1(M_{i, \infty}, M_{j, i'}),$$

where j is the dummy variable in the limit. The maps

$$\text{Ext}^1(M_{i, \infty}, M_{j, i'}) \rightarrow \text{Ext}^1(M_{i, \infty}, M_{j-1, i'})$$

are surjective. As j tends to $-\infty$, the dimension of $\text{Ext}^1(M_{i, \infty}, M_{j, i'})$ is unbounded (by the above formula for it). Therefore, this limit evaluates to $\prod_{\aleph_0} K$.

It is shown in [8, Lemma 2.7], that, for all i, i' in $\{0, \dots, n-1\}$, all of $\text{Ext}^1(M_{-\infty, i}, M_{i', \infty})$, $\text{Ext}^1(M_{i, \infty}, M_{i', \infty})$ and $\text{Ext}^1(M_{-\infty, i}, M_{-\infty, i'})$ vanish.

Comparing the crossing numbers with the dimensions of the corresponding Ext-groups concludes the proof of the theorem. \square

3.3. Reflection. Recall (see [8]) that there is a map $M \mapsto M^\vee$, which gives a bijection on the indecomposable objects in $\overline{\mathbf{T}}$. With our notation, the map is given by $M_{ij} \mapsto M_{-j, -i}$ for i, j in $\mathbb{Z} \cup \{\pm\infty\}$.

For a subcategory \mathcal{X} of $\overline{\mathbf{T}}$, we let \mathcal{X}^\vee denote the subcategory $\{X^\vee \mid X \in \mathcal{X}\}$. The map has the following properties.

- Lemma 3.8.**
- (a) $\dim \operatorname{Hom}(X, Y) = \dim \operatorname{Hom}(Y^\vee, X^\vee)$ for X, Y in $\overline{\mathbf{T}}$;
 - (b) $\operatorname{Ext}^1(X, Y) = 0$ if and only if $\dim \operatorname{Ext}^1(Y^\vee, X^\vee) = 0$ for X, Y in $\overline{\mathbf{T}}$;
 - (c) $(\tau X)^\vee = \tau^{-1} X^\vee$ for all indecomposables X in \mathbf{T} .
 - (d) For each object X in $\overline{\mathbf{T}}$, we have $(X^{\perp_H})^\vee = {}^{\perp_H}(X^\vee)$ and $(X^\vee)^{\perp_E} = ({}^{\perp_E} X)^\vee$.

We will also need the following.

Lemma 3.9. For a maximal rigid object U in $\overline{\mathbf{T}}$, we have

- (a) $(\operatorname{Gen} U \cap \mathbf{T})^\vee = \operatorname{Cogen}(U^\vee) \cap \mathbf{T}$;
- (b) $(\operatorname{Cogen} U \cap \mathbf{T})^\vee = \operatorname{Gen}(U^\vee) \cap \mathbf{T}$.

Proof. This follows directly from the definition of the map using Lemma 3.6 and the fact that an indecomposable X in \mathbf{T} is generated by a maximal rigid object U if and only if it is generated by an indecomposable direct summand in U (see also [9]). \square

4. TORSION PAIRS AND MAXIMAL RIGID OBJECTS

In this section we give an improvement of a result from [8], and discuss a combinatorial interpretation. The idea is to link maximal rigid objects in $\overline{\mathbf{T}}$ with torsion pairs in \mathbf{T} .

4.1. A bijection. A rigid object in $\overline{\mathbf{T}}$ is said to be of *Prüfer type* if it has a Prüfer module as a direct summand, and it is said to be of *adic type* if it has an adic module as a direct summand.

An object U in $\varinjlim \mathbf{T}$ is maximal rigid if and only if it is cotilting in the category $\varinjlim \mathbf{T}$, in the sense of Colpi [10]. This is proved in [9] (see [9, 1.10] and note that, by [9, Lemma 1.2], a finitely presented $\tilde{\Lambda}_n$ -module is maximal rigid if and only if it is a tilting module, where $\tilde{\Lambda}_n$ denotes the completion of the path algebra of an oriented n -cycle).

We consider equivalence classes of maximal rigid objects in $\overline{\mathbf{T}}$, where two maximal rigid objects are considered to be equivalent if they have the same indecomposable direct summands. The following is proved in [8].

Proposition 4.1. Let U be a rigid object in $\overline{\mathbf{T}}$.

- (a) U is maximal rigid in $\overline{\mathbf{T}}$ if and only if it has n pairwise nonisomorphic indecomposable direct summands.
- (b) If U is maximal rigid, it is either of Prüfer type or of adic type, but not both.
- (c) A rigid object of Prüfer type lies in the subcategory $\varinjlim \mathbf{T} \subset \overline{\mathbf{T}}$.

We will give a parallel result concerning torsion pairs in \mathbf{T} . A torsion pair $(\mathcal{T}, \mathcal{F})$ in \mathbf{T} is said to be of *ray type* if \mathcal{F} contains at least one ray of \mathbf{T} . It is said to be of *coray type* if \mathcal{T} contains at least one coray of \mathbf{T} . A class of objects (or subcategory)

\mathcal{X} of a category \mathcal{C} is said to be *generating*, if for each map f in \mathcal{C} , there is an object X in \mathcal{X} with $\text{Hom}(X, f) \neq 0$. Dually, one can define *cogenerating* classes of objects, i.e. \mathcal{X} is cogenerating if, for each map f in \mathcal{C} , there is an object X in \mathcal{X} with $\text{Hom}(f, X) \neq 0$.

Lemma 4.2. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in \mathbf{T} . Then it is not the case that both \mathcal{T} and \mathcal{F} are of finite type.*

Proof. Suppose that \mathcal{T} and \mathcal{F} are both of finite type. By the definition of a torsion pair, every object M in \mathbf{T} must be the middle term of an exact sequence

$$0 \rightarrow T \rightarrow M \rightarrow F \rightarrow 0,$$

where $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Since \mathbf{T} is serial, if M is indecomposable then so are T and F . It follows that \mathbf{T} itself is of finite type, a contradiction. Hence, we cannot have that both \mathcal{T} and \mathcal{F} are of finite type. \square

We next prove that the torsion pairs with \mathcal{T} of infinite type are exactly those of coray type. For an indecomposable object X in \mathbf{T} we write \mathcal{C}_X for the coray containing X and \mathcal{R}_X for the ray containing X .

Lemma 4.3. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in \mathbf{T} , where \mathbf{T} has rank n . Assume \mathcal{T} is of infinite type. Then the following hold:*

- (i) \mathcal{T} contains an indecomposable object X , with $l(X) = n$.
- (ii) \mathcal{F} is of finite type.
- (iii) \mathcal{T} contains the coray \mathcal{C}_X .
- (iv) \mathcal{T} cogenerates \mathbf{T} .

Proof. First note that since \mathcal{T} is of infinite type, there is no limit on the length of the indecomposable objects in \mathcal{T} . Since \mathcal{T} is closed under factor objects, it must therefore contain an indecomposable object with $l(X) = n$. Hence (i) holds. Since \mathcal{T} is closed under factor objects, the indecomposable objects in the coray \mathcal{C}_X below X are contained in \mathcal{T} ; more precisely $\text{ind}(\mathcal{C}_X) \cap \{Y \mid l(Y) \leq n\} \subset \mathcal{T}$. Assume first that $n > 1$. Let X' be the (uniquely defined) indecomposable object in \mathbf{T} such that there is an irreducible monomorphism $X' \rightarrow X$. Since $l(X) = n$, we have that $\text{Hom}(X, Y) = 0$ for an indecomposable object Y in \mathbf{T} if and only if Y is in the wing $\mathcal{W}_{X'}$. By the definition of a torsion pair, we have that \mathcal{F} is contained in $\mathcal{W}_{X'}$ so is of finite type. If $n = 1$, we have $\text{Hom}(X, Y) \neq 0$ for any indecomposable object Y in \mathbf{T} , so \mathcal{F} is the zero subcategory and (ii) holds. By Lemma 1.1, (iii) holds, while (iv) is a direct consequence of (iii). \square

We state the dual version of Lemma 4.3.

Lemma 4.4. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in \mathbf{T} , where \mathbf{T} has rank n . Assume \mathcal{F} is of infinite type. Then the following hold:*

- (i) \mathcal{F} contains an indecomposable object X , with $l(X) = n$.
- (ii) \mathcal{T} is of finite type.
- (iii) \mathcal{F} contains the ray \mathcal{R}_X .
- (iv) \mathcal{F} generates \mathbf{T} .

Combining Lemma 4.2, 4.3 and 4.4, we obtain the following direct consequence.

Corollary 4.5. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in \mathbf{T} .*

- (a) *The following are equivalent*
- (i) *\mathcal{T} is of infinite type;*
 - (ii) *\mathcal{T} contains a coray;*
 - (iii) *\mathcal{T} cogenerates \mathbf{T} ;*
 - (iv) *\mathcal{F} is of finite type.*
- (b) *$(\mathcal{T}, \mathcal{F})$ is either of ray or of coray type (and not both).*

Moreover, by Lemma 1.1 and a direct application of Lemma 3.8, we obtain the following.

Lemma 4.6. *The map $M \rightarrow M^\vee$, maps a torsion pair $(\mathcal{T}, \mathcal{F})$ to a torsion pair $(\mathcal{F}^\vee, \mathcal{T}^\vee)$. Moreover if $(\mathcal{T}, \mathcal{F})$ is of ray-type, then $(\mathcal{F}^\vee, \mathcal{T}^\vee)$ is of coray-type and vice-versa.*

For a maximal rigid object U of Prüfer type, consider the subcategory $\mathcal{F}_U = {}^{\perp_E}U \cap \mathbf{T}$. We set $\mathcal{T}_U = {}^{\perp_H}(\mathcal{F}_U) \cap \mathbf{T}$. For a maximal rigid object U of adic type, we define $\mathcal{T}_U = U^{\perp_E} \cap \mathbf{T}$ and $\mathcal{F}_U = (\mathcal{T}_U)^{\perp_H} \cap \mathbf{T}$. We have the following reformulation of a result of [9]:

Theorem 4.7. *The map $U \mapsto (\mathcal{T}_U, \mathcal{F}_U)$ gives a one-to-one correspondence between equivalence classes of maximal rigid objects in $\varinjlim \mathbf{T}$ and torsion pairs in \mathbf{T} with the property that \mathcal{F} generates \mathbf{T} .*

Now, using Lemma 3.8, we obtain a commutative square (*):

$$\begin{array}{ccc} \{\text{maximal rigid objects of Prüfer type in } \overline{\mathbf{T}}\} & \longrightarrow & \{\text{torsion pairs of ray-type in } \mathbf{T}\} \\ \downarrow & & \downarrow \\ \{\text{maximal rigid objects of adic type in } \overline{\mathbf{T}}\} & \longrightarrow & \{\text{torsion pairs of coray-type in } \mathbf{T}\} \end{array}$$

where the horizontal maps are given by $U \mapsto (\mathcal{T}_U, \mathcal{F}_U)$ and the vertical maps are induced by $M \mapsto M^\vee$.

As a direct consequence of Lemma 3.8 (as also observed in [8]), we have that the left vertical map in (*) is a bijection. We have already observed that the right vertical map is a bijection. The upper horizontal map is bijective by Theorem 4.7, combining with Proposition 4.1 and Lemma 4.4, and it follows that the lower horizontal map is bijective.

Combining the commutative diagram of bijections (*) with Corollary 4.5 we obtain the following improvement of Theorem 4.7.

Theorem 4.8. *The map $U \mapsto (\mathcal{T}_U, \mathcal{F}_U)$ gives a bijection between*

- *Equivalence classes of maximal rigid objects in $\overline{\mathbf{T}}$*
- *Torsion pairs $(\mathcal{T}, \mathcal{F})$ in \mathbf{T} , and*

Corollary 4.9. *The number of torsion pairs in \mathbf{T} is $2^{\binom{2n-1}{n-1}}$.*

Proof. By [9, 2.4, 5.2 and B.1], the number of cotilting objects in $\varinjlim \mathbf{T}$ is $\binom{2n-1}{n-1}$. The result then follows from Proposition 4.1, the above diagram of maps, and Theorem 4.8. \square

Note that the above results hold in the case $n = 1$: in this case there are two maximal rigid objects: the unique Prüfer and adic modules, and two corresponding torsion pairs, $(0, \mathbf{T})$ and $(\mathbf{T}, 0)$ respectively.

4.2. Alternative and explicit descriptions of \mathcal{T}_U and \mathcal{F}_U . We give alternative and more explicit descriptions for the subcategories \mathcal{T}_U and \mathcal{F}_U corresponding to a maximal rigid object U in $\overline{\mathbf{T}}$.

The following observation is useful.

Lemma 4.10. *Let \mathbf{T} be a tube of rank n . Then we have:*

- (a) ${}^{\perp E}M_{i,\infty} \cap \mathbf{T} = \mathbf{T}$;
- (b) $M_{i,\infty}^{\perp E} \cap \mathbf{T} = \mathcal{W}_{i,i+n}$

We now give an explicit description of the torsion pair $(\mathcal{T}_U, \mathcal{F}_U)$ corresponding to a maximal rigid module U . We first of all give a combinatorial lemma concerning wings, which is easy to check.

Lemma 4.11. *Let $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n - 1$ be integers. Then*

(i) *We have:*

$$\bigcap_{r=0}^{k-1} \mathcal{W}_{i_r, i_r+n} = \prod_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}}$$

(where the subscripts are interpreted modulo k and, for $r = k-1$, we interpret $i_{r+1} = i_0$ as $i_0 + n$).

- (ii) *The wing $\mathcal{W}_{i_r, i_{r+1}}$ is zero if and only if $i_{r+1} - i_r = 1$.*
- (iii) *The wings $\mathcal{W}_{i_r, i_{r+1}}$ do not overlap, i.e. each indecomposable object in \mathbf{T} belongs to at most one $\mathcal{W}_{i_r, i_{r+1}}$.*
- (iv) *If X and Y lie in different wings among the $\mathcal{W}_{i_r, i_{r+1}}$, then $\text{Ext}^1(X, Y) = 0 = \text{Ext}^1(Y, X)$.*

For an example illustrating Lemma 4.11(i), with $n = 10$, $k = 4$, $i_0 = 0$, $i_1 = 4$, $i_2 = 7$ and $i_3 = 8$, see Figure 8.

In the following proposition and the sequel, we adopt the same convention for the wings as in Lemma 4.11(i).

Proposition 4.12. *Let U be maximal rigid in $\overline{\mathbf{T}}$.*

(a) *If U is of Prüfer type, then*

$$(\mathcal{T}_U, \mathcal{F}_U) = ({}^{\perp H}U \cap \mathbf{T}, {}^{\perp E}U \cap \mathbf{T}) = (\tau^{-1}(\text{Gen } U \cap \mathbf{T}), \text{Cogen } U \cap \mathbf{T}).$$

(b) *Assume U is of Prüfer type with Prüfer summands $M_{i_r, \infty}$ for $r = 0, \dots, k-1$ where $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n - 1$. Then*

$$(\mathcal{T}_U, \mathcal{F}_U) = \left(\prod_{r=0}^{k-1} \mathcal{T}_r, \left(\prod_{r=0}^{k-1} \mathcal{F}_r \right) \amalg \mathcal{F}_{\infty} \right),$$

where $(\mathcal{T}_r, \mathcal{F}_r)$ is a torsion pair in $\mathcal{W}_{i_r, i_{r+1}+1}$ with \mathcal{F}_r containing all of the projective objects in $\mathcal{W}_{i_r, i_{r+1}+1}$ and $\mathcal{F}_\infty = \coprod_{r=0}^{k-1} \mathcal{R}_{i_r}$.

(c) If U is of adic type, then

$$(\mathcal{T}_U, \mathcal{F}_U) = (U^{\perp E} \cap \mathbf{T}, U^{\perp H} \cap \mathbf{T}) = (\text{Gen } U \cap \mathbf{T}, \tau(\text{Cogen } U \cap \mathbf{T})).$$

(d) Assume U is of adic type with adic summands $M_{i_r, \infty}$ for $r = 0, \dots, k-1$, where $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n-1$. Then

$$(\mathcal{T}_U, \mathcal{F}_U) = \left(\left(\coprod_{r=0}^{k-1} \mathcal{T}_r \right) \amalg \mathcal{T}_\infty, \coprod_{r=0}^{k-1} \mathcal{F}_r \right),$$

where $(\mathcal{T}_r, \mathcal{F}_r)$ is a torsion pair in $\mathcal{W}_{i_r, i_{r+1}+1}$ with \mathcal{T}_r containing all of the injective objects in $\mathcal{W}_{i_r, i_{r+1}+1}$ and $\mathcal{T}_\infty = \coprod_{r=0}^{k-1} \mathcal{C}_{i_{r+1}+1}$.

Proof. We give the details for (a) and (b), while statements (c) and (d) can be proved similarly (or using Lemmas 3.8 and 3.9). Let U be maximal rigid of Prüfer type in $\overline{\mathbf{T}}$. Our aim is to compute $\mathcal{F}_U = {}^{\perp E}U \cap \mathbf{T}$ and then $\mathcal{T}_U = {}^{\perp H}\mathcal{F}_U \cap \mathbf{T}$. We use the Prüfer direct summands of U in order to compute \mathcal{F}_U more precisely in terms of a set of wings in \mathbf{T} . We then use this to compute \mathcal{T}_U using the theory of torsion pairs in type A (see Section 2).

Let $U_{\mathbf{T}}$ be the direct sum of all indecomposable direct summands of U which are finitely generated. Let $M_{i_0, \infty}, \dots, M_{i_{k-1}, \infty}$ with $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n-1$ be the indecomposable Prüfer summands of U .

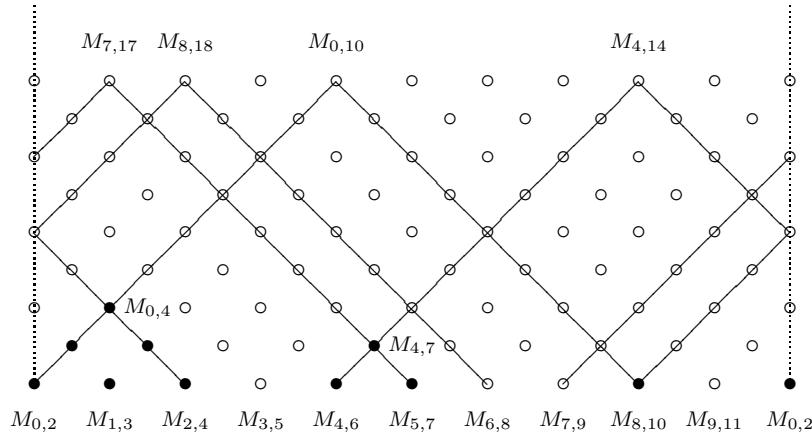


FIGURE 8. The indecomposable objects in the intersection of the four wings $\mathcal{W}_{0,10}, \mathcal{W}_{4,14}, \mathcal{W}_{7,17}$ and $\mathcal{W}_{8,18}$ in a tube of rank 10 corresponding to a maximal rigid object of Prüfer type. The objects lying in the intersection are indicated by filled-in circles. The intersection coincides with $\mathcal{W}_{0,4} \amalg \mathcal{W}_{4,7} \amalg \mathcal{W}_{8,10}$ (note that the wing $\mathcal{W}_{i_3, i_4} = \mathcal{W}_{7,8}$ is zero).

By Lemma 4.10, we then have that the finite part $U_{\mathbf{T}}$ lies in $\cap_{r=0}^{k-1} \mathcal{W}_{i_r, i_r+n}$, which coincides with $\prod_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}}$ by Lemma 4.11(i). We draw attention to the fact that the wings in the statement of the proposition are slightly larger than these — this will become clearer later in the proof.

Hence, we consider the decomposition $U_{\mathbf{T}} = \prod_{r=0}^{k-1} U_r$, where each U_r is rigid in $\mathcal{W}_{i_r, i_{r+1}}$ (note that some of the U_r then might be 0, i.e. in the case when $i_{r+1} - i_r = 1$). Then it follows easily from Lemma 4.11(iv) that U_r is maximal rigid in $\mathcal{W}_{i_r, i_{r+1}}$. Since U_r is maximal rigid, its restriction to the abelian category $\mathcal{W}_{i_r, i_{r+1}}$ is also cotilting. Our aim is to use this decomposition of U to compute \mathcal{F}_U .

If $\mathcal{W}_{i_r, i_{r+1}}$ is nontrivial, $M_{i_r, i_{r+1}}$ is necessarily an indecomposable direct summand of U_r . It is then straightforward to check that

$$\perp^E (\prod_{r=0}^{k-1} M_{i_r, i_{r+1}} \cap \mathbf{T}) = \left(\prod_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}} \right) \amalg \prod_{r=0}^{k-1} \mathcal{R}_{i_r},$$

and from this it follows that $\mathcal{F} \subset \left(\prod_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}} \right) \amalg \left(\prod_{r=0}^{k-1} \mathcal{R}_{i_r} \right)$.

We also have that $\perp^E U_{\mathbf{T}} \cap \mathcal{W}_{i_r, i_{r+1}} = \perp^E U_r \cap \mathcal{W}_{i_r, i_{r+1}}$, which gives us the following description of \mathcal{F} :

$$\mathcal{F} = \left(\prod_{r=0}^{k-1} (\perp^E U_r) \cap \mathbf{T} \right) \amalg \left(\prod_{r=0}^{k-1} \mathcal{R}_{i_r} \right).$$

Since U_r is cotilting in $\mathcal{W}_{i_r, i_{r+1}}$, we have that $\perp^E U_r \cap \mathcal{W}_{i_r, i_{r+1}} = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}}} U_r$. From this it follows that

$$\mathcal{F} = \left(\prod_{r=0}^{k-1} \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}}} U_r \right) \amalg \prod_{r=0}^{k-1} \mathcal{R}_{i_r}.$$

Noting that $\text{Cogen } M_{i, \infty} \cap \mathbf{T} = \mathcal{R}_i$, we see that $\mathcal{F} = \text{Cogen } U \cap \mathbf{T}$ as claimed in (a).

Next, we consider slightly larger wings, $\mathcal{W}_{i_r, i_{r+1}+1}$ in order to obtain the description of \mathcal{F} in (b). For each r , define $\tilde{U}_r = U_r \amalg M_{i_r, i_{r+1}+1}$. Then \tilde{U}_r is a cotilting module in $\mathcal{W}_{i_r, i_{r+1}+1}$. Note that if U has exactly one Prüfer summand, we need to regard $\mathcal{W}_{i_r, i_{r+1}+1} = \mathcal{W}_{i_0, i_0+n+1}$ as being a type A category non-exactly embedded in the tube, since the image of the projective-injective object is not rigid.

Note that

$$\text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{U}_r = (\text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}}} U_r) \amalg \text{add } M_{i_r, i_{r+1}+1}.$$

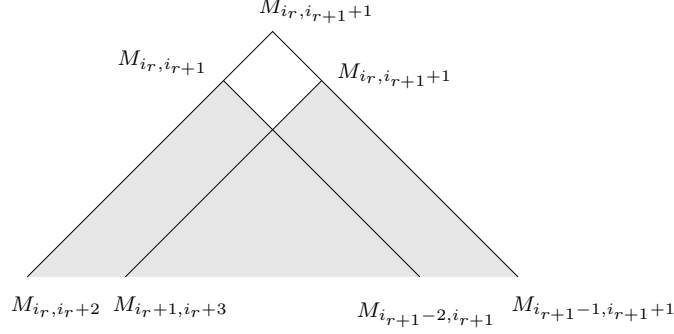
We can thus rewrite \mathcal{F} as follows:

$$\mathcal{F} = \left(\prod_{r=0}^{k-1} \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{U}_r \right) \amalg \left(\prod_{r=0}^{k-1} \mathcal{R}_{i_r} \right).$$

So, setting $\mathcal{F}_r = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{U}_r$, we obtain a description of \mathcal{F} as claimed in (b).

Next, we compute \mathcal{T} . By definition, we have that $\mathcal{T} = \perp^H \mathcal{F} \cap \mathbf{T}$. Furthermore,

$$\perp^H \left(\prod_{r=0}^{k-1} M_{i_r, i_{r+1}} \right) \cap \mathbf{T} = \left(\prod_{r=0}^{k-1} \tau^{-1} \mathcal{W}_{i_r, i_{r+1}} \right) \amalg \left(\prod_{r_0}^{k-1} \mathcal{C}_{i_{r_0+1}} \right).$$

FIGURE 9. A wing $\mathcal{W}_{i_r, i_{r+1}+1}$.

Noting that $\text{Hom}(M_{l, i_{r+1}+1}, M_{i_r, i_{r+1}+1}) \neq 0$ for $l \leq i_r$, we see that

$$\mathcal{T} \subset \prod_{r=0}^{k-1} \tau^{-1}(\mathcal{W}_{i_r, i_{r+1}}) \subset \prod_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}+1}.$$

From this, it follows that $\mathcal{T} = \prod_{r=0}^{k-1} ({}^{\perp H} \mathcal{F}_r \cap \mathcal{W}_{i_r, i_{r+1}+1})$, noting that

$$\mathcal{F}_r = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{U}_r.$$

Using the fact that \tilde{U}_r is cotilting in $\mathcal{W}_{i_r, i_{r+1}+1}$, combined with Corollary 2.1, we see that $\mathcal{T} = \prod_{r=0}^{k-1} \mathcal{T}_r$, where

$$\mathcal{T}_r = {}^{\perp H} \tilde{U}_r \cap \mathcal{W}_{i_r, i_{r+1}+1} = \text{Gen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tau_{\mathcal{W}_{i_r, i_{r+1}+1}}^{-1} \tilde{U}_r$$

is the torsion part of the torsion pair in $\mathcal{W}_{i_r, i_{r+1}+1}$ with torsion-free part \mathcal{F}_r . We have that ${}^{\perp H}(\prod_{r=0}^{k-1} M_{i_r, \infty}) = \prod_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}}$, and therefore $\mathcal{T} = {}^{\perp H} U \cap \mathbf{T}$.

Since $\text{Gen } M_{i, \infty} \cap \mathbf{T}$ is zero for all i and since we have $\tau_{\mathcal{W}_{i_r, i_{r+1}+1}}^{-1} \tilde{U}_r = \tau_{\mathcal{W}_{i_r, i_{r+1}+1}}^{-1} U_r = \tau^{-1} U_r$, we see that $\mathcal{T} = \tau^{-1} \text{Gen } U \cap \mathbf{T}$.

See Figure 9 for an illustration of one of the wings $\mathcal{W}_{i_r, i_{r+1}+1}$. We have that $\mathcal{F}_r = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{U}_r$. The indecomposable objects in \mathcal{F}_r apart from $M_{i_r, i_{r+1}+1}$ lie in the left hand shaded triangle. Also, $\mathcal{T}_r = \text{Gen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tau_{\mathcal{W}_{i_r, i_{r+1}+1}}^{-1} \tilde{U}_r$, and the indecomposable objects in \mathcal{T}_r lie in the right hand shaded triangle. \square

4.3. The maximal rigid module corresponding to a torsion pair. In this section we give an explicit description of the inverse of the bijection $U \mapsto (\mathcal{T}_U, \mathcal{F}_U)$ in Theorem 4.8 between maximal rigid objects in $\overline{\mathbf{T}}$ and torsion pairs in \mathbf{T} .

Lemma 4.13. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in \mathbf{T} . Then an indecomposable object in $\varinjlim(\text{ind } \mathcal{F})$ either lies in $\text{ind } \mathcal{F}$ or it is a Prüfer module which is the direct limit along a ray in $\text{ind } \mathcal{F}$.*

Proof. It is clear that any indecomposable in \mathcal{F} or Prüfer module which is the direct limit along a ray in $\text{ind } \mathcal{F}$ lies in $\text{ind}(\varinjlim \mathcal{F})$. So suppose that $X = \varinjlim X_j$ is an indecomposable in $\overline{\mathbf{T}}$, where each X_j is in $\text{ind } \mathcal{F}$. Firstly note that X cannot be an

adic module, since the adic modules do not lie in $\varinjlim \mathbf{T}$. Next, for any object T in \mathcal{T} , we have

$$\mathrm{Hom}(T, X) = \mathrm{Hom}(T, \varinjlim X_j) \simeq \varinjlim \mathrm{Hom}(T, X_j) = 0,$$

using Lemma 3.4(b), since each X_j lies in \mathcal{F} . It follows that if X is in $\mathrm{ind} \mathbf{T}$, we have that X lies in $\mathrm{ind} \mathcal{F}$. The only other possibility is if X is a Prüfer module. If Y is any indecomposable object in the corresponding ray in \mathbf{T} , we have an embedding $Y \rightarrow X$. Hence $\mathrm{Hom}(T, Y) = 0$ for any object T in \mathcal{T} , so Y lies in \mathcal{F} . The result follows. \square

Proposition 4.14. (a) *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair of ray type. Then $(\mathcal{T}, \mathcal{F})$ can be written in the form $(\mathcal{T}_U, \mathcal{F}_U)$, where U is the direct sum of the indecomposable objects in*

$$(4) \quad \varinjlim(\mathrm{ind} \mathcal{F}) \cap (\varinjlim(\mathrm{ind} \mathcal{F}))^{\perp E}.$$

Compare [8, Thm. 1.5].

(b) *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair of coray type. Then $(\mathcal{T}, \mathcal{F})$ can be written in the form $(\mathcal{T}_U, \mathcal{F}_U)$, where U is the direct sum of the indecomposable objects in*

$$(5) \quad \varprojlim(\mathrm{ind} \mathcal{T}) \cap {}^{\perp E}(\varprojlim(\mathrm{ind} \mathcal{T})).$$

Proof. We only consider (a); the proof of (b) is similar. By Theorem 4.8 and Proposition 4.12(b) and its proof, there is a maximal rigid object U in $\overline{\mathbf{T}}$ of Prüfer type such that $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}_U, \mathcal{F}_U)$ and $\mathcal{T}_U, \mathcal{F}_U$ have the following description. Let the Prüfer summands of U be $M_{i_r, \infty}$ for $r = 0, \dots, k-1$ where $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n-1$. Then

$$(\mathcal{T}_U, \mathcal{F}_U) = \left(\prod_{r=0}^{k-1} \mathcal{T}_r, \left(\prod_{r=0}^{k-1} \mathcal{F}'_r \right) \amalg \mathcal{F}_\infty \right),$$

where $\mathcal{T}_r, \mathcal{F}_\infty$ are as in Proposition 4.12(b) and $\mathcal{F}'_r = \mathrm{Cogen}_{\mathcal{W}_{i_r, i_{r+1}}} U_r$, where $U|_{\mathbf{T}} = \prod_{r=0}^{k-1} U_r$ with U_r in $\mathcal{W}_{i_r, i_{r+1}}$.

By (2) in Section 2,

$$\begin{aligned} \mathrm{ind} U_r &= \{X \in \mathrm{ind} \mathcal{F}'_r : \mathrm{Ext}^1(Y, X) = 0 \text{ for all } Y \in \mathcal{F}'_r\} \\ &= \{X \in \mathrm{ind} \mathcal{F}'_r : \mathrm{Ext}^1(Y, X) = 0 \text{ for all } Y \in \mathrm{ind} \mathcal{F}\}. \end{aligned}$$

Note that $\mathcal{F}_\infty = \coprod_{r=0}^{k-1} \mathcal{R}_{i_r} \subset \mathrm{ind} \mathcal{F}$. It follows that any indecomposable object X in \mathbf{T} which satisfies $\mathrm{Ext}^1(Y, X) = 0$ for all Y in $\mathrm{ind} \mathcal{F}$ must lie inside some wing $\mathcal{W}_{i_r, i_{r+1}}$, and hence in some \mathcal{F}'_r . Therefore

$$\begin{aligned} \mathrm{ind} U|_{\mathbf{T}} &= \mathrm{ind} \left(\prod_{r=0}^{k-1} U_r \right) \\ &= \{X \in \mathrm{ind} \mathcal{F} : \mathrm{Ext}^1(Y, X) = 0 \text{ for all } Y \in \mathrm{ind} \mathcal{F}\}. \end{aligned}$$

Next, by Lemma 4.13, an indecomposable object in $\varinjlim(\mathrm{ind} \mathcal{F})$ either lies in $\mathrm{ind} \mathcal{F}$ or is the direct limit along a ray in \mathcal{F} , i.e. it is a Prüfer summand $M_{i_r, \infty}$ of U . If Y is one of the latter summands and X lies in $\mathrm{ind} \mathcal{F}$, we have $D \mathrm{Ext}^1(Y, X) \simeq \mathrm{Hom}(X, \tau Y) = 0$, by Lemmas 3.5 and 3.6. Since the Prüfer summands themselves satisfy $\mathrm{Ext}^1(M_{i_r, \infty}, M_{i_{r'}, \infty}) = 0$ for all r, r' , the result follows. \square

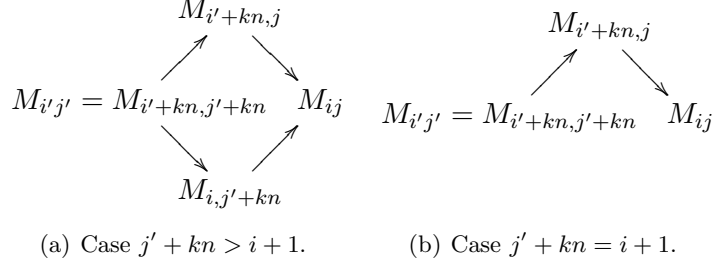


FIGURE 10. Objects and paths in the Auslander-Reiten quiver of \mathbf{T} corresponding to an intersection in $\mathbb{U}(n)$ between arcs $[i, j]$ and $[i' + kn, j' + kn]$.

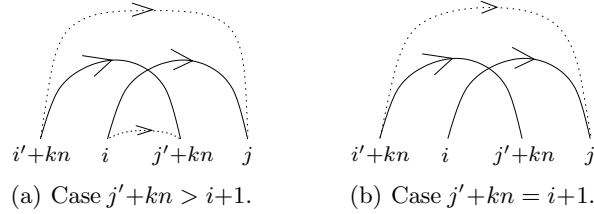


FIGURE 11. Non-split extensions in \mathbf{T} represented geometrically.

5. GEOMETRIC INTERPRETATION IN THE TUBE CASE

5.1. Short exact sequences. Let $[i, j], [i', j']$ be arcs in $\mathcal{A}(\mathbb{U}(n))$, giving rise to arcs $\pi_n([i, j]), \pi_n([i', j'])$ in $\mathcal{A}(\mathbb{A}(n))$ with corresponding indecomposable objects $M_{ij}, M_{i'j'}$ in U . Suppose that $I_{\mathbb{U}(n)}^-([i, j], [i' + kn, j' + kn])$ (i.e. the intersection number between arcs $[i, j]$ and $[i' + kn, j' + kn]$ in $\mathbb{U}(n)$) is equal to 1 for some k in \mathbb{Z} . Then, if $j' + kn > i + 1$, we have four objects in the Auslander-Reiten quiver of \mathbf{T} as shown in Figure 10(a); if $j' + kn = i + 1$, we have three objects as shown in Figure 10(b).

In the case $j' + kn > i + 1$, this corresponds to a non-split short exact sequence:

$$(6) \quad 0 \rightarrow M_{i',j'} \xrightarrow{f} M_{i'+kn,j} \amalg M_{i,j'+kn} \xrightarrow{g} M_{ij} \rightarrow 0,$$

and in the case $j' + kn = i + 1$, it corresponds to a short exact sequence:

$$(7) \quad 0 \rightarrow M_{i',j'} \xrightarrow{f} M_{i'+kn,j} \xrightarrow{g} M_{ij} \rightarrow 0$$

in \mathbf{T} . As in [27, Remark 4.25], these can be interpreted geometrically in $\mathbb{U}(n)$: see Figure 11.

If $j' + kn > i + 1$, write $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = (g_1 \ g_2)$. Then f_1 and g_2 are monomorphisms and f_2 and g_1 are epimorphisms. The f_i and g_i are uniquely determined up to a choice of scalars. If $j' + kn = i + 1$, f is a monomorphism and g is an epimorphism, again uniquely determined up to a choice of scalars.

Lemma 5.1. *Any non-split short exact sequence with first term $M_{i'j'}$ and last term M_{ij} has the same form as (6) or (7).*

Proof. Let

$$(8) \quad 0 \rightarrow M_{i'j'} \xrightarrow{u} E \xrightarrow{v} M_{ij} \rightarrow 0$$

be an arbitrary non-split short exact sequence with first term $M_{i'j'}$ and last term M_{ij} . By Lemma 1.2 we may write $E = E_1 \amalg E_2$ where E_1 is indecomposable and such that we have that, decomposing $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = (v_1, v_2)$, we have that u_1 a monomorphism. Since the sequence is not split, u_1 is not an isomorphism, so, denoting the length of an object M in \mathbf{T} by $\ell(M)$, we have

$$\ell(E_2) = \ell(M_{ij}) + \ell(M_{i'j'}) - \ell(E_1) < \ell(M_{ij}).$$

Since v_2 is not an epimorphism, v_1 must be an epimorphism, again using Lemma 1.2.

Case (i): Suppose first that $v_1 u_1 \neq 0$. If an integer k is such that $i' + kn \leq i$ and $i + 2 \leq j' + kn \leq j$, there is a homomorphism in \mathbf{T} from $M_{i'j'}$ to M_{ij} obtained by composing an epimorphism from $M_{i'j'}$ to $M_{i,j'+kn}$ with a monomorphism from $M_{i,j'+kn}$ to M_{ij} (i.e. the two maps in the lower edges of the diamond in Figure 10).

It is easy to check that the homomorphisms of this kind (allowing k to vary) form a basis of $\text{Hom}(M_{i'j'}, M_{ij})$. Since $vu = 0$ and $v_1 u_1 \neq 0$ and $v_1 u_1$ is a scalar multiple of such a basis element, there must be an indecomposable summand X of E_2 such that $v_X u_X$ is a scalar multiple of $v_1 u_1$ (where u_X, v_X are the corresponding components of u_2, v_2). But

$$(9) \quad \ell(X) \leq \ell(M_{ij}) + \ell(M_{i'j'}) - \ell(E_1),$$

and there is a unique path from $M_{i'j'}$ to M_{ij} through such an X giving rise to $g_1 f_1$ (i.e. with $X = M_{i,j'+kn}$) from which it follows that we have equality in (9) and thus that $E_2 = X$ is indecomposable and u_2 is an epimorphism and v_2 is a monomorphism. It follows that the short exact sequence (8) is of the form (6) up to a choice of scalars.

Case (ii): Now assume that $v_1 u_1 = 0$. This implies that $\ell(E_1) \geq \ell(M_{ij}) + \ell(M_{i'j'})$, but we also have $\ell(E_1) \leq \ell(E_1 \amalg E_2) = \ell(M_{ij}) + \ell(M_{i'j'})$, so we must have equality and $E = E_1$ is indecomposable. It follows that (8) is of the form (7) up to a choice of scalars. The proof is complete. \square

Definition 5.2. *We call a collection \mathcal{S} of arcs in $\mathcal{A}(\mathbb{A}(n))$ an oriented Ptolemy diagram (in $\mathbb{A}(n)$) if, whenever $\pi_n([i, j])$ and $\pi_n([i', j'])$ lie in \mathcal{S} with $i' < i < j' < j$ then $\pi_n([i, j'])$ (when $j' > i + 1$) and $\pi_n([i', j])$ also lie in \mathcal{S} (see related definitions in Section 2.1 and [22, 16]).*

We note that the additive closure of a collection of indecomposable objects in \mathbf{T} is closed under extensions if and only if the corresponding collection of arcs is an oriented Ptolemy diagram in $\mathbb{A}(n)$. It is also easy to check that if $\pi_n([i, j])$ is an arc in $\mathcal{A}(\mathbb{A}(n))$, then the indecomposable quotients of M_{ij} are the $M_{i'j}$ where $i \leq i' \leq j - 2$, i.e. arcs in $\mathbb{A}(n)$ corresponding to arcs in $\mathbb{U}(n)$ with the same ending

point and with starting point weakly to the right of i . Call these the *left-shortenings* of $\pi_n([i, j])$. Similarly, submodules are given by right-shortenings.

We make the following remark, which follows from Proposition 3.7. Recall that $\tilde{\mathcal{A}}(\mathbb{A}(n))$ denotes $\mathcal{A} = \mathcal{A}(\mathbb{A}(n))$ extended to include the homotopy classes of the arcs $\pi_n([i, \infty])$ and $\pi_n([-\infty, i])$.

Remark 5.3. *The bijection $\psi : \pi_n([i, j]) \mapsto M_{ij}$ between $\tilde{\mathcal{A}}(\mathbb{A}(n))$ and $\text{ind}(\overline{\mathbf{T}})$ induces a bijection between maximal noncrossing collections of arcs in $\mathbb{A}(n)$ (including the infinite arcs) and maximal rigid objects in $\overline{\mathbf{T}}$.*

We can now describe the conditions on collections of arcs appearing in torsion pairs in \mathbf{T} , using the above and Lemmas 1.1 and 1.2.

Proposition 5.4. (a) *A collection \mathcal{S} of arcs in $\mathcal{A}(\mathbb{A}(n))$ corresponds to the torsion part of a torsion pair in \mathbf{T} if and only if \mathcal{S} is an oriented Ptolemy diagram in $\mathbb{A}(n)$ and \mathcal{S} is closed under left-shortening.*
 (b) *A collection \mathcal{S} of arcs in $\mathcal{A}(\mathbb{A}(n))$ corresponds to the torsion-free part of a torsion pair in \mathbf{T} if and only if \mathcal{S} is an oriented Ptolemy diagram in $\mathbb{A}(n)$ and \mathcal{S} is closed under right-shortening.*

We remark that, if U is of Prüfer type, by Proposition 4.12, $\psi(\text{ind } \mathcal{T}_U)$ can be obtained by taking the closure of the set of arcs corresponding to finitely generated indecomposable summands of U under left shortening and rotating all resulting arcs one step to the right. We obtain $\psi(\text{ind } \mathcal{F}_U)$ by taking the closure of the set of arcs corresponding to *all* of the summands of U under right shortening.

Similarly, by Proposition 4.14, if $(\mathcal{T}, \mathcal{F})$ is a torsion pair where \mathcal{F} generates \mathbf{T} (i.e. of ray type), then $\psi(\varinjlim(\text{ind } \mathcal{F}))$ can be obtained from $\psi(\text{ind } \mathcal{F})$ by first adding any infinite arc $\pi_n([i, \infty])$ for which all arcs $\pi_n([i, j])$ for $j \geq a$ for some a lie in $\psi(\text{ind } \mathcal{F})$. Then U is the direct sum of the indecomposable objects corresponding to the arcs α in $\psi(\varinjlim(\text{ind } \mathcal{F}))$ such that the pair (β, α) of arcs has no negative intersections for all β in $\psi(\varinjlim(\text{ind } \mathcal{F}))$.

Similar descriptions can be given in the adic/coray type case.

Finally, we give an example in a tube of rank $n = 14$ to illustrate Proposition 4.12 and the results in this section. The arcs corresponding to the indecomposable direct summands of U are displayed in Figure 12 (only the beginnings of the infinite arcs are shown). Note that the Prüfer modules which are indecomposable summands of U are $M_{0,\infty}, M_{6,\infty}, M_{10,\infty}$ and $M_{13,\infty}$, so $i_0 = 0, i_1 = 6, i_2 = 10$ and $i_3 = 13$. The arcs corresponding to the indecomposable objects in \mathcal{T}_U are displayed in Figure 13.

The arcs corresponding to the indecomposable objects in the $\mathcal{F}_U \cap \mathcal{W}_{i_r, i_{r+1}+1}$ are displayed in Figure 14 (with dotted arcs indicating the indecomposable summands of U which are not in \mathbf{T} (or $\text{ind } \mathcal{F}$)). Note that there are infinitely many additional arcs not displayed, corresponding to indecomposables in \mathcal{F}_∞ but not in any of the \mathcal{F}_r . The missing arcs are $\pi_{14}([0, j])$ for $j \geq 8$, $\pi_{14}([6, j])$ for $j \geq 12$, $\pi_{14}([10, j])$ for $j \geq 15$ and $\pi_{14}([13, j])$ for $j \geq 16$. Note that, as indicated by Proposition 4.12, the intersections of \mathcal{T}_U and \mathcal{F}_U with the wings $\mathcal{W}_{i_r, i_{r+1}+1}$ (which are $\mathcal{W}_{0,7}, \mathcal{W}_{6,11}, \mathcal{W}_{10,14}$ and $\mathcal{W}_{13,15}$) are torsion pairs.

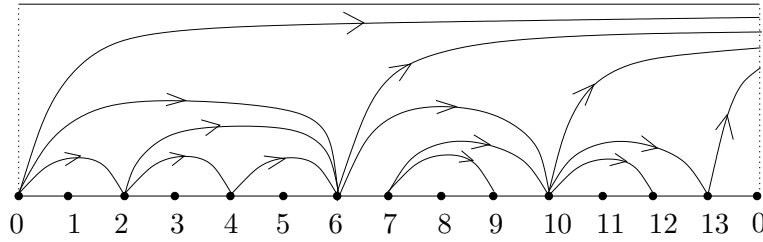


FIGURE 12. The maximal rigid object U

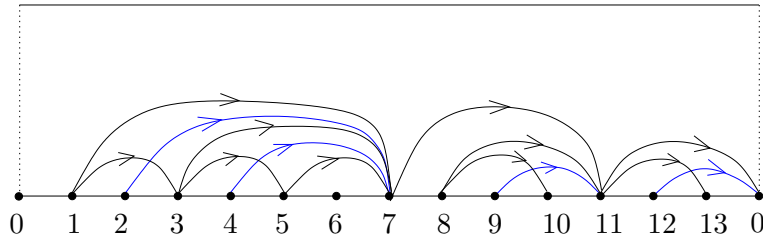


FIGURE 13. The torsion part, $\mathcal{T}_U = \tau^{-1}(\text{Gen } U \cap \mathbf{T})$, of the torsion pair corresponding to U . The arcs corresponding to indecomposable objects not in $\tau^{-1}(\text{add } U \cap \mathbf{T})$ are drawn in blue.

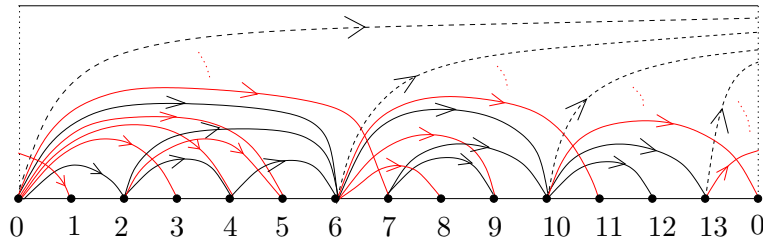


FIGURE 14. The torsion-free part, $\mathcal{F}_U = \text{Cogen } U \cap \mathbf{T}$, of the torsion pair corresponding to U . The arcs not in $\text{add } U$ are drawn in red. The dashed arcs are the indecomposable summands of U which are not of finite length (and thus not in \mathcal{F}_U). The arcs $\pi_{14}([0, j])$ for $j \geq 8$, $\pi_{14}([6, j])$ for $j \geq 12$, $\pi_{14}([10, j])$ for $j \geq 15$ and $\pi_{14}([13, j])$ for $j \geq 16$ have been omitted for clarity.

In Figure 15, we show the indecomposable summands of \mathbf{T} and the indecomposable objects in \mathcal{T}_U and \mathcal{F}_U in the AR-quiver of the tube.

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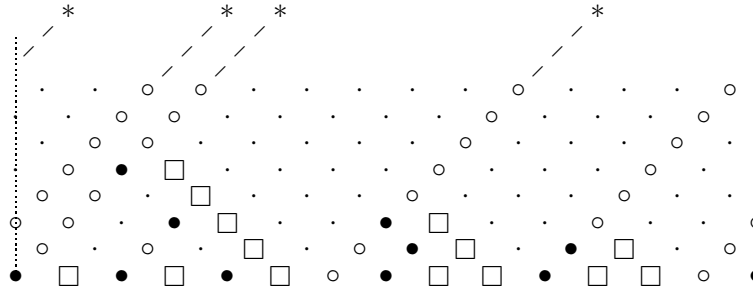


FIGURE 15. The AR-quiver of the tube, showing the indecomposable summands of U (\bullet or $*$), $\text{ind}(\mathcal{T}_U)$ (\square) and $\text{ind}(\mathcal{F}_U)$ (\circ or \bullet). The Prüfer direct summands of U are shown (symbolically) at the top of the diagram as asterisks.

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REFERENCES

- [1] M. Auslander, *Functors and morphisms determined by objects*, Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976), pp. 1244. Lecture Notes in Pure Appl. Math., Vol. 37, Dekker, New York, 1978.
- [2] I. Assem, D. Simson, A. Skowronski, *Elements of the Representation Theory of Associative Algebras. 1: Techniques of Representation Theory*, LMS Student Texts 65, 2006.
- [3] K. Baur, R.J. Marsh, *A geometric model of tube categories*, J. Algebra 362 (2012), 178–191.
- [4] A.A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers* (French) [Perverse sheaves], Analysis and topology on singular spaces, I (Luminy, 1981), 5-171, Astérisque, 100, Soc. Math. France, Paris, 1982.
- [5] A. Beligiannis, I. Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. 188 (2007), no. 883.
- [6] K. Bongartz, *Tilted algebras*, in: M. Auslander, E. Lluís (Eds.), Representations of algebras (Puebla, 1980), Lecture Notes in Math., vol. 903, Springer, Berlin, New York, 1981, 26-38.
- [7] T. Brüstle, J. Zhang, *On the cluster category of a marked surface without punctures*, Algebra Number Theory 5 (2011), no. 4, 529566.
- [8] A.B. Buan, H. Krause, *Cotilting modules over tame hereditary algebras*, Pacific J. Math. 211 (2003), no. 1, 41-59
- [9] A.B. Buan, H. Krause, *Tilting and cotilting for quivers of type \tilde{A}_n* , J. Pure Appl. Algebra 190 (2004), no. 1-3, 1-21
- [10] R. Colpi, *Tilting in Grothendieck categories*, Forum Math. 11 (1999), no. 6, 735759.
- [11] W. Crawley-Boevey, *Infinite-dimensional modules in the representation theory of finite-dimensional algebras*, Canadian Math. Soc. Conf. Proc., 23 (1998), 29-54.
- [12] E.S. Dickson, *A torsion theory for Abelian categories*, Trans. Amer. Math. Soc. 121 (1966) 223-235.
- [13] B. Geigig, *Geometric Realizations of Cluster Categories*, Master's thesis, Winter 2009/2010, <http://www.uni-graz.at/~baurk/thesis-Geigig.pdf>
- [14] D. Happel, L. Unger, *Almost complete tilting modules*, Proc. Amer. Math. Soc. 107, no. 3, 603-610, 1989.

- [15] T. Holm, P. Jørgensen and M. Rubey, *Torsion pairs in cluster tubes*, Preprint arXiv:1207.3206v1 [math.RT], 2012.
T. Holm, P. Jørgensen, Private communication, 2011.
- [16] T. Holm, P. Jørgensen, M. Rubey, *Ptolemy diagrams and torsion pairs in the cluster category of Dynkin type A_n* , J. Algebraic Combin. 34 (2011), 507-523
- [17] O. Iyama, Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. 172 (2008), no. 1, 117-168.
- [18] C.U. Jensen, H. Lenzing, *Model-theoretic algebra with particular emphasis on fields, rings, modules*, Algebra, Logic and Applications, 2. Gordon and Breach Science Publishers, New York, 1989.
- [19] S. Koenig and B. Zhu, *From triangulated categories to abelian categories: cluster tilting in a general framework*. Math. Z. 258 (2008), no. 1, 143-160.
- [20] H. Krause, Ø. Solberg, *Applications of torsion pairs*, J. London Math. Soc. (2) 68 (2003), 631-650.
- [21] H. Nakaoka, *General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories*, Appl. Categ. Structures 19, no. 6, 879-899, 2011.
- [22] P. Ng, *A characterization of torsion theories in the cluster category of Dynkin type A_∞* , Preprint, arXiv:1005.4364v1 [math.RT], 2010.
- [23] C.M. Ringel, *Infinite length modules. Some examples as introduction*. Infinite length modules (Bielefeld, 1998), 1-73, Trends Math., Birkhäuser, Basel, 2000.
- [24] M. Prest, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications, 121. Cambridge University Press, Cambridge, 2009.
- [25] R. Colpi, J. Trlifaj, *Tilting modules and tilting torsion theories*. J. Algebra 178 (1995), no. 2, 614-634.
- [26] J. Trlifaj, *Ext and inverse limits*, Illinois J. Math. Volume 47, Number 1-2 (2003), 529-538.
- [27] M. Warkentin, *Fadenmoduln über \widetilde{A}_n und Cluster-Kombinatorik (String modules over \widetilde{A}_n and cluster combinatorics)*, Diploma Thesis, University of Bonn, December 2008. Available from <http://nbn-resolving.de/urn:nbn:de:bsz:ch1-qucosa-94793>.
- [28] Y. Zhou, B. Zhu, *Mutation of torsion pairs in triangulated categories and its geometric realization*, Preprint arXiv:1105.3521v1 [math.RT], 2011.

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