

TORSION PAIRS AND RIGID OBJECTS IN TUBES

KARIN BAUR, ASLAK BAKKE BUAN, AND ROBERT J. MARSH

ABSTRACT. We classify the torsion pairs in a tube category and show that they are in bijection with maximal rigid objects in the extension of the tube category containing the Prüfer and adic modules. We show that the annulus geometric model for the tube category can be extended to the larger category and interpret torsion pairs, maximal rigid objects and the bijection between them geometrically. We also give a similar geometric description in the case of the linear orientation of a Dynkin quiver of type A.

INTRODUCTION

Torsion pairs in abelian categories were introduced by Dickson [12] (see the introduction to [5] for further details). They play an important role in the study of localisation (see e.g. [5]) and in tilting theory (see e.g. [2, 5]). Also, torsion pairs in the case of triangulated categories, as defined in [17, 2.2], have been considered recently by a number of authors, e.g. [16, 17, 19, 21, 22, 28].

The main object of study of this article is the collection of torsion pairs in a tube category, which is a hereditary abelian category. Such categories arise as full subcategories of module categories over tame hereditary algebras, and are so-called because their Auslander-Reiten quivers have the shape of a tube.

A geometric model for tube categories has been given in [3, 27] (see also [7]). The indecomposable objects are parametrised by a collection of oriented arcs in an annulus with n marked points on one of its boundary components. The dimension of the Ext-group between two indecomposable objects coincides with the negative intersection number of the corresponding pair of arcs.

In this article, we classify the torsion pairs in a tube category \mathcal{T} . We build on results in [8] giving a bijection between torsion pairs in a tube category and cotilting objects in the category obtained by taking arbitrary direct limits of modules in the tube. We show that all torsion pairs in the tube category arise in this way or via a dual construction. Thus they are in bijection with maximal rigid objects in the category $\overline{\mathcal{T}}$ obtained from the tube by taking arbitrary direct or inverse limits of objects in the tube. We give an explicit description of this bijection and its inverse.

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We further show that the geometric model referred to above can be extended to the indecomposable objects in $\overline{\mathcal{T}}$, i.e. to include Prüfer and adic modules associated to the tube. These extra objects are represented by certain infinite arcs in the annulus which spiral in towards the inner boundary component. The result concerning the dimensions of Ext-groups extends to this case also. This enables us to give a characterisation of maximal rigid objects in \mathcal{T} in the geometric model. We also give a characterisation of torsion pairs in the geometric model in terms of certain closure properties corresponding to closure properties of the subcategories in a torsion pair. In particular, the collection of arcs corresponding to a subcategory in a torsion pair must form an *oriented Ptolemy diagram*, an oriented version in the annulus case of the Ptolemy diagrams appearing in [16, 22], as well as satisfying additional criteria. We also give a geometric description of the above bijection and its inverse.

In order to give the geometric interpretation, we first give a similar model for the linearly oriented quiver in type A (note that M. Warkentin [27] also suggests such a model) and show how to interpret tilting modules and torsion pairs in this model.

We remark that, in independent work, Holm and Jørgensen [15] have classified the torsion pairs in the cluster category associated to a tube (as opposed to the tube itself, which we study here). The torsion pairs in the cluster case are different, although we note that unoriented Ptolemy diagrams play a role in the cluster tube case, while oriented Ptolemy diagrams appear here.

In Section 1, we recall the definition and some of the properties of torsion pairs in abelian categories. In Section 2, we discuss the type A case. In Section 3, we recall the geometric model of the tube and show how it can be extended to include Prüfer and adic modules. We also discuss a certain reflection map (from [8]) on the indecomposable objects of the tube, and its properties. In Section 4 we classify the torsion pairs in the tube and prove that they are in bijection with maximal rigid objects in $\overline{\mathcal{T}}$, giving an explicit description of the bijection and its inverse. In Section 5, we give a geometric interpretation of maximal rigid objects in \mathcal{T} and torsion pairs in \mathcal{T} and the bijection between them.

1. PRELIMINARIES

We shall adopt the convention throughout that all subcategories considered are strictly full (i.e. full and closed under isomorphism). We shall also consider modules up to isomorphism. Let \mathcal{A} be an abelian category. If \mathcal{X} is a subcategory of \mathcal{A} , we define ${}^{\perp_H}\mathcal{X}$ (respectively, ${}^{\perp_E}\mathcal{X}$) to be the additive subcategory of \mathcal{A} whose objects are the objects Y satisfying $\text{Hom}_{\mathcal{A}}(Y, X) = 0$ (respectively, $\text{Ext}_{\mathcal{A}}^1(Y, X) = 0$) for all X in \mathcal{X} . For an object X of \mathcal{A} we define ${}^{\perp_H}X = {}^{\perp_H}\text{add } X$, and ${}^{\perp_E}X = {}^{\perp_E}\text{add } X$, where $\text{add } X$ denotes the additive subcategory of \mathcal{A} whose objects are finite direct sums of direct summands of X . We similarly define \mathcal{X}^{\perp_H} , \mathcal{X}^{\perp_E} , X^{\perp_H} and X^{\perp_E} .

If \mathcal{X} is an additive subcategory of \mathcal{A} , we write $\text{Gen } \mathcal{X}$ for the subcategory consisting of objects which are quotients of objects of \mathcal{X} and, for an object X , $\text{Gen } X = \text{Gen}(\text{add } X)$. Similarly, we write $\text{Cogen } \mathcal{X}$ for the subcategory consisting of objects which are subobjects of objects of \mathcal{X} and $\text{Cogen } X = \text{Cogen}(\text{add } X)$. We write $\text{ind } \mathcal{X}$ for the subcategory of \mathcal{X} whose objects are the indecomposable objects in \mathcal{X} .

We next recall some of the theory of torsion pairs.

Definition 1.1. [12, Sect. 1] *A pair $(\mathcal{F}, \mathcal{G})$ of subcategories of \mathcal{A} is said to be a torsion pair if*

- (i) $\text{Hom}(F, G) = 0$ for all objects F in \mathcal{F} and G in \mathcal{G} .
- (ii) For every object X in \mathcal{A} there is a short exact sequence

$$(1) \quad 0 \rightarrow F \rightarrow X \rightarrow G \rightarrow 0$$

with F in \mathcal{F} and G in \mathcal{G} .

We say that \mathcal{F} is the torsion part and \mathcal{G} is the torsion-free part of the pair.

Recall that an abelian category \mathcal{A} is said to be *finite length* if it is skeletally small and every object in it is artinian and noetherian. Equivalently, it is skeletally small and every object has a finite length composition series.

Thus, for example, the category $\text{mod } A$ of finite dimensional modules over a finite-dimensional algebra is a finite length category. The following is proved using arguments as in [12, Thm. 2.1] (see also [8, Lemma 1.7]).

Lemma 1.2. *Suppose that \mathcal{A} is a finite length abelian category.*

- (a) *Let \mathcal{F} be a subcategory of \mathcal{A} and let $\mathcal{G} = \mathcal{F}^{\perp_H}$. Then $(\mathcal{F}, \mathcal{G})$ is a torsion pair in \mathcal{A} if and only if \mathcal{F} is closed under quotients and extensions.*
- (b) *Let \mathcal{G} be a subcategory of \mathcal{A} and let $\mathcal{F} = {}^{\perp_H} \mathcal{G}$. Then $(\mathcal{F}, \mathcal{G})$ is a torsion pair in \mathcal{A} if and only if \mathcal{G} is closed under subobjects and extensions.*

This has the following well-known corollary.

Lemma 1.3. *Let \mathcal{A} be a finite length abelian category. Then a pair $(\mathcal{F}, \mathcal{G})$ of subcategories of \mathcal{A} is a torsion pair if and only if*

- (a) $\text{Hom}_{\mathcal{A}}(F, G) = 0$ for all F in \mathcal{F} and G in \mathcal{G} .
- (b) $\mathcal{F}^{\perp_H} = \mathcal{G}$.
- (c) ${}^{\perp_H} \mathcal{G} = \mathcal{F}$.

We fix an algebraically-closed field K and denote by D the vector space duality $\text{Hom}_K(-, K)$.

Next, consider a finite dimensional K -algebra Λ . We denote by $\text{Mod } \Lambda$ the category of all left Λ -modules, and by $\text{mod } \Lambda$ the subcategory of all finitely generated Λ -modules. Recall that a module T in $\text{mod } \Lambda$ is called *tilting* if

- (a) The projective dimension of T is at most 1;
- (b) $\text{Ext}^1(T, T) = 0$;
- (c) There is a short exact sequence

$$0 \rightarrow \Lambda \rightarrow T_0 \rightarrow T_1 \rightarrow 0$$

with T_0 and T_1 in $\text{add } T$.

Cotilting modules in $\text{mod } \Lambda$ are defined dually. We only consider basic tilting or cotilting modules, i.e. we assume that $T = \coprod T_i$, with T_i indecomposable and $T_i \not\cong T_j$ for $i \neq j$.

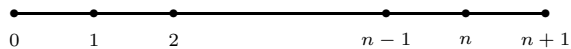
2. DYNKIN TYPE A

We consider a linearly oriented quiver Q of type A_n .

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n-1 \longleftarrow n$$

We write S_i , $i = 1, \dots, n$, for the simple KQ -modules. It is known that the indecomposable modules over KQ are in bijection with (ordered) pairs $[i, j]$ where $0 \leq i, j \leq n+1$ and $i \leq j-2$. Here $[i, j]$ represents the module M_{ij} with composition factors S_{i+1}, \dots, S_{j-1} (starting from the socle). If i lies in $\{j-1, j\}$ we regard M_{ij} as zero.

2.1. Geometric model. We now describe a geometric model for the indecomposable KQ -modules. Note that a similar such model is also suggested in [27, Remark 4.28]. Here we will indicate how this model can incorporate torsion theories in $\text{mod } KQ$ — later we will indicate how this can be done for tubes. We also give some more explicit information as preparation for this. We consider a line segment ℓ_n with marked points $0, 1, \dots, n+1$ in order along it:



and associate $[i, j]$ with the arc above ℓ_n from i to j (up to isotopy) oriented towards j , $0 \leq i < j-1 \leq n$. We write $[i, i+1]$ for the boundary arc with starting point i and ending point $i+1$, for $i = 0, 1, \dots, n$.

In this way, we see that the indecomposable KQ -modules are in bijection with arcs up to isotopy between marked points of ℓ_n , above ℓ_n , which are not isotopic to boundary arcs. Let $\mathcal{A}(\ell_n)$ be the set of such (non-boundary) arcs. The dimensions of Ext-spaces between indecomposable KQ -modules and their non-split extensions are well-known. In terms of the above model, we have

Proposition 2.1. (a) *Let $[i, j]$ and $[i', j']$ be in $\mathcal{A}(\ell_n)$. Then we have*

$$\text{Ext}^1(M_{ij}, M_{i'j'}) \cong \begin{cases} K & \text{if there is a negative crossing (see Figure 1) between } [i, j] \text{ and } [i', j']; \\ 0 & \text{else.} \end{cases}$$

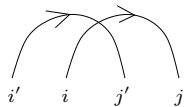


FIGURE 1. $[i, j]$ and $[i', j']$ have a negative crossing

(b) *In the case where $\text{Ext}^1(M_{ij}, M_{i'j'}) \cong K$, the non-split extension takes the form:*

$$0 \rightarrow M_{i'j'} \rightarrow \begin{array}{c} M_{i'j} \\ \amalg \\ M_{ij'} \end{array} \rightarrow M_{ij} \rightarrow 0$$

up to equivalence, if $j' > i + 1$. If $j' = i + 1$, it takes the form

$$0 \rightarrow M_{i'j'} \rightarrow M_{ij'} \rightarrow M_{ij} \rightarrow 0,$$

again up to equivalence. This can be interpreted geometrically as in Figure 2 where the indecomposable summands of the middle term of the short exact sequence are indicated by dotted lines.

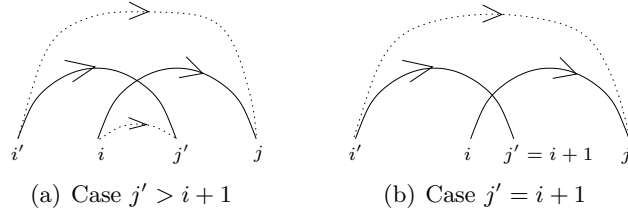


FIGURE 2. Non-split extension

Remark 2.2. If $0 \leq i \leq i+2 \leq j \leq n+1$, we have $\tau M_{ij} = \begin{cases} M_{i-1,j-1} & i \geq 1; \\ 0 & \text{otherwise,} \end{cases}$ where τ denotes the Auslander-Reiten translate.

It is also easy to see that we have the following (cf. Figure 3):

Lemma 2.3. (a) The indecomposable quotients of M_{ij} are the $M_{i'j}$ where $i \leq i' \leq j - 2$, i.e. the arcs in $\mathcal{A}(\ell_n)$ with the same ending point j and with starting point weakly to the right of i .
 (b) The indecomposable submodules of M_{ij} are the $M_{ij'}$ where $i + 2 \leq j' \leq j$, i.e. the arcs in $\mathcal{A}(\ell_n)$ with the same starting point i and with ending point weakly to the left of j .

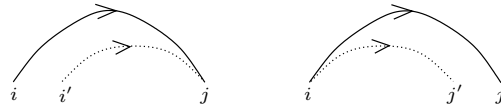


FIGURE 3. Quotients and submodules

We call the arcs corresponding to quotients of M_{ij} the *left-shortenings* of $[i, j]$ and the arcs corresponding to submodules of M_{ij} the *right-shortenings* of $[i, j]$.

Recall that by a result of Bongartz [6], a tilting KQ -module can be regarded as a maximal rigid KQ -module, or equivalently as a module $T = T_1 \amalg \cdots \amalg T_n$ with $\text{Ext}^1(T_i, T_j) = 0$ for all i, j . Since $M_{0,n+1}$ is projective-injective, it is a summand of every tilting module.

It follows from Proposition 2.1 that we have

Corollary 2.4. The bijection $[i, j] \mapsto M_{i,j}$ between $\mathcal{A}(\ell_n)$ and $\text{ind}(\text{mod } KQ)$ induces a bijection between triangulations of the polygon with $n + 2$ sides and tilting modules over KQ in type A_n .

Note that each internal arc in the triangulation receives an orientation from left to right, but this is ignored.

Definition 2.5. *We call a collection of arcs in $\mathcal{A}(\ell_n)$ an oriented Ptolemy diagram if, whenever $[i, j]$ and $[i', j']$ lie in the collection, with $i' < i < j' < j$, we have that $[i', j]$ (when $j' > i + 1$) and $[i, j']$ also lie in the collection. See Proposition 2.1(b).*

We note that Ptolemy diagrams of arcs (without orientation) arise in the geometric characterisation of torsion pairs for cluster categories of type A_∞ [22] (see Condition (i) in Defn. 0.3 there) and type A_n [16].

2.2. Torsion pairs. We recall the following Lemma, which is well-known; we include a few comments on the proof.

Lemma 2.6. *Let Λ be a finite-dimensional algebra.*

- (a) *Let $\varphi : M \rightarrow N$ be a map between indecomposable finitely-generated Λ -modules. If M has a simple socle then φ is a monomorphism if and only if $\varphi(\text{soc}(M)) \neq 0$. If N has a simple top then φ is an epimorphism if and only if the induced morphism from $\text{top}(M)$ to $\text{top}(N)$ is non-zero.*
- (b) *If $\varphi : M \rightarrow \coprod_i N_i$, where M and the N_i are (finitely many) indecomposable finitely-generated Λ -modules and M has simple socle, then φ is a monomorphism if and only if at least one of its components φ_i is a monomorphism.*
- (c) *If $\varphi : \coprod_i M_i \rightarrow N$, where N and the M_i are (finitely many) indecomposable finitely generated Λ -modules, and N has simple top, then φ is an epimorphism if and only if at least one of its components φ_i is an epimorphism.*

Proof. (a) It is clear that if f is a monomorphism it must be non-zero on the socle of M . Conversely, if $f(\text{soc}(M)) \neq 0$ and $\ker(f) \neq 0$ then $\text{soc}(\ker(f)) \neq 0$. Then, since $\text{soc}(M)$ is simple we have $\text{soc}(\ker(f)) = \text{soc}(M)$ and $f(\text{soc}(\ker(f))) = 0$, a contradiction. The proof of the statement for epimorphisms is similar.

(b) This follows from part (a).

(c) This is dual to (b) and the proof is similar. □

Recall that Q is assumed to be a linear orientation of the Dynkin diagram of type A_n . The following is also well-known.

Corollary 2.7. *An additive subcategory \mathcal{B} of $\text{mod } KQ$ generated (as an additive category) by a collection of indecomposable modules is closed under quotients (respectively, submodules), if and only if every indecomposable quotient (respectively, submodule) of an indecomposable module in \mathcal{B} lies in \mathcal{B} .*

It follows from Lemma 1.2 and Corollary 2.7 that we have:

- Proposition 2.8.**
- a) *A collection of arcs in $\mathcal{A}(\ell_n)$ corresponds to the torsion part of a torsion pair in $\text{mod } KQ$ if and only if*
 - (i) *It forms an oriented Ptolemy diagram;*
 - (ii) *It is closed under left-shortening.*
 - b) *A collection of arcs in $\mathcal{A}(\ell_n)$ corresponds to the torsion-free part of a torsion pair in $\text{mod } KQ$ if and only if*

- (i) *It forms an oriented Ptolemy diagram;*
- (ii) *It is closed under right-shortening.*

Given a tilting KQ -module T , the pair $(\text{Gen } T, \text{Cogen } \tau T) = (T^{\perp_E}, T^{\perp_H})$ is known to form a torsion pair (cf. Section VI.2 in [2]). In fact, the following are equivalent for a torsion pair $(\mathcal{F}, \mathcal{G})$:

Proposition 2.9. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair over KQ (or any finite dimensional hereditary algebra). Then the following are equivalent:*

- (a) *$(\mathcal{F}, \mathcal{G})$ arises in the above way from a tilting module,*
- (b) *\mathcal{F} contains all the indecomposable injective modules,*
- (c) *\mathcal{G} contains no indecomposable injective module.*

Proof. The proof of the equivalence of (a) and (b) is in Section VI.6 of [2]. The fact that (b) implies (c) follows from the fact that $\text{Hom}(F, G) = 0$ for all F in \mathcal{F} and G in \mathcal{G} . Suppose (c) holds, and let I be an indecomposable injective KQ -module. If G is in \mathcal{G} , G is not injective, so

$$0 = \text{Ext}^1(\tau^{-1}G, I) \cong D \text{Hom}(I, G)$$

by the Auslander-Reiten formula and the fact that I is injective, so $\text{Hom}(I, G) = 0$. Hence I lies in \mathcal{F} and we see that (c) implies (b). The proof is complete. \square

Corollary 2.10. *The map*

$$T \mapsto (\text{Gen } T, \text{Cogen } \tau T) = (T^{\perp_E}, T^{\perp_H})$$

gives a bijection between tilting KQ -modules and torsion pairs $(\mathcal{F}, \mathcal{G})$ for which \mathcal{F} contains all the indecomposable injective modules, or, equivalently, \mathcal{G} contains no indecomposable injective module.

Note that, since KQ is hereditary, a KQ -module is tilting if and only if it is cotilting. The dual of Corollary 2.10 is thus

Proposition 2.11. *The map*

$$T \mapsto (\text{Gen } \tau^{-1}T, \text{Cogen } T) = ({}^{\perp_H}T, {}^{\perp_E}T)$$

gives a bijection between tilting KQ -modules and torsion pairs $(\mathcal{F}, \mathcal{G})$ for which \mathcal{G} contains all the indecomposable projective modules, or, equivalently, \mathcal{F} contains no indecomposable projective module.

By Lemma 2.6(c), an indecomposable in $\text{mod } KQ$ is generated by a module U if and only if it is generated by an indecomposable summand of U . Hence, the map

$$T \mapsto (\text{Gen } T, \text{Cogen } \tau T)$$

in Corollary 2.10 can be interpreted in the geometric model: $\text{Gen } T$ is obtained from T by closure under left shortening and $\text{Cogen } \tau T$ is obtained from T by first shifting to the left one step (deleting any arcs starting at 0) and then closing under right shortening.

Conversely, if $(\mathcal{F}, \mathcal{G})$ is a torsion pair of the kind considered in Corollary 2.10, then T can be recovered as the direct sum of the indecomposable Ext-projectives in \mathcal{F} , i.e. the objects

$$(2) \quad \{X \in \text{ind}(\mathcal{F}) : \text{Ext}^1(X, F) = 0 \text{ for all } F \in \mathcal{F}\},$$

by [2, VI.2.5]. Geometrically, this means taking all of the arcs X in \mathcal{F} which do not intersect another arc F in \mathcal{F} as in Figure 4.

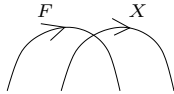


FIGURE 4. Intersection of arcs not allowed in recovering T from $(\mathcal{F}, \mathcal{G})$ of the kind considered in Corollary 2.10.

Dually, the map

$$T \mapsto (\text{Gen } \tau^{-1}T, \text{Cogen } T)$$

in Proposition 2.11 can be interpreted in the geometric model: $\text{Gen } \tau^{-1}T$ is obtained from T by first shifting to the right one step (deleting any arcs ending at $n+1$) and then closing under left shortening. $\text{Cogen } T$ is obtained from T by closure under right shortening.

Conversely, if $(\mathcal{F}, \mathcal{G})$ is a torsion pair of the kind considered in Proposition 2.11, T can be recovered as the direct sum of the indecomposable Ext-injectives in \mathcal{G} , i.e. the objects

$$\{X \in \text{ind}(\mathcal{G}) : \text{Ext}^1(G, X) = 0 \text{ for all } G \in \mathcal{G}\}.$$

That is, we take all the arcs Y in \mathcal{G} which do not intersect another arc G in \mathcal{G} as in Figure 5.

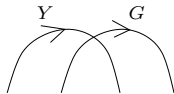


FIGURE 5. Intersection of arcs not allowed in recovering T from $(\mathcal{F}, \mathcal{G})$ of the kind considered in Proposition 2.11.

Finally, we remark that by [2, VI.1.11, VI.2.5], the direct sum of the non-projective summands of T can be recovered as $T' = \mathcal{F} \cap \tau^{-1}\mathcal{G}$ when $(\mathcal{F}, \mathcal{G}) = (\text{Gen } T, \text{Cogen } \tau T)$. Similarly, the non-injective summands in T can be recovered as $T'' = \tau\mathcal{F} \cap \mathcal{G}$ when $(\mathcal{F}, \mathcal{G}) = (\text{Gen } \tau^{-1}T, \text{Cogen } T)$. This gives an alternative route for computing the indecomposables in T' (or T'') geometrically.

Example 2.12. *Let $n = 6$. In Figure 6, we indicate $\mathcal{F} = \text{Gen } T$ and $\mathcal{G} = \text{Cogen } \tau T$ for the tilting module*

$$T = [0, 7] \amalg [0, 4] \amalg [1, 4] \amalg [1, 3] \amalg [4, 7] \amalg [5, 7].$$

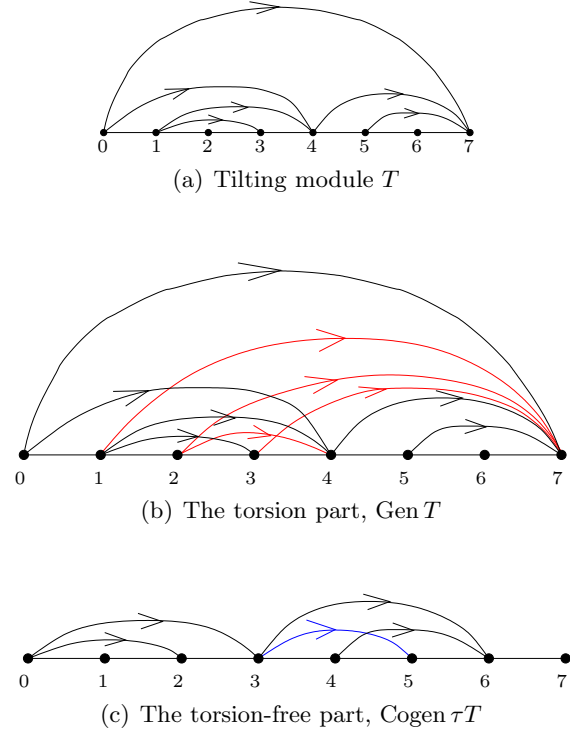
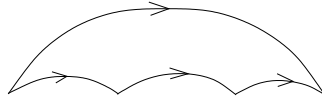


FIGURE 6. T , \mathcal{F} and \mathcal{G}

2.3. Exchange of complements. Let $T = \coprod_{i=1}^n T_i$ be a KQ -tilting module. Then $M_{0,n+1}$ is a direct summand of T . Let T_i be an indecomposable summand of T not isomorphic to $M_{0,n+1}$. Removing T_i from the diagram of arcs for T (cf. Corollary 2.4) leaves a quadrilateral:



where some or all of the arcs at the base of the diagram may just be boundary arcs. The non-boundary arcs correspond to some of the indecomposable summands T_j , $j \neq i$, of T . Then T_i must correspond to one of the two crossing diagonals of this quadrilateral, i.e. it must be either X or Y in Figure 7. Since \overline{T} is sincere, it follows

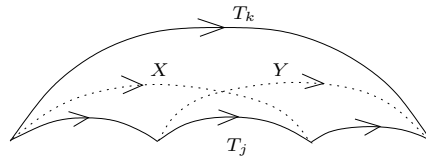


FIGURE 7. Sequence linking the complements of \overline{T}

from [14] that there is a non-split short exact sequence linking the two complements

X and Y of $\overline{T} = \coprod_{j \neq i} T_j$:

$$0 \rightarrow X \rightarrow T_j \amalg T_k \rightarrow Y \rightarrow 0$$

if T_j is not a boundary arc, or

$$0 \rightarrow X \rightarrow T_k \rightarrow Y \rightarrow 0$$

in the case where T_j is a boundary arc.

3. TUBES

3.1. Categorical description. Consider the quiver Q

$$(3) \quad \begin{array}{ccccccc} & & 2 & \longleftarrow & 3 & \longleftarrow & \cdots & \longleftarrow & n & & \\ & \swarrow & & & & & & & & \searrow & \\ 1 & & & & & & & & & & n+1 \\ & \longleftarrow & & & & & & & & & \end{array}$$

of Euclidean type $\tilde{A}_{1,n}$. The path algebra $\Lambda = KQ$ is tame hereditary, and the module category $\text{mod } KQ$ has an extension closed subcategory \mathcal{T}_n , which can be realized as the extension closure of the modules L, S_2, \dots, S_n , where as usual S_i denotes the simple corresponding to vertex i , and L denotes the unique indecomposable module with composition factors S_1 and S_{n+1} . The category \mathcal{T}_n is called a *tube* of rank n .

Actually \mathcal{T}_n is a hereditary finite length abelian category with n simple objects. For each pair of objects X, Y in \mathcal{T}_n , the spaces $\text{Hom}(X, Y)$ and $\text{Ext}^1(X, Y)$ have finite K -dimension, and there is an autoequivalence τ on \mathcal{T}_n , induced by the Auslander-Reiten translate on $\text{mod } \Lambda$, with the property that $\text{Hom}(Y, \tau X) \simeq D \text{Ext}^1(X, Y)$. Let $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$ be the map taking i to $i + n$. From now on we denote the simples in $\mathcal{T} = \mathcal{T}_n$ by $M_{i,i+2}$ for $i = 0, \dots, n - 1$, in such a way that $\tau M_{i,i+2} = M_{i-1,i+1}$, where we regard $M_{\sigma^k(i), \sigma^k(j)}$ as equal to $M_{i,j}$ for any integer k . The indecomposable objects in \mathcal{T}_n are uniserial and hence uniquely determined by their simple socle and their length. We denote by $M_{i,i+l+1}$ an indecomposable with socle $M_{i,i+2}$ and length l . Then we have that $\tau M_{i,i+l+1} = M_{i-1,i+l}$, and the AR-quiver of the tube \mathcal{T}_n is as in Figure 8 (with the columns on the left- and right-hand sides identified). Note that each indecomposable object has a unique name M_{ij} (with $j - i \geq 2$) if we insist that i lies in $\{0, 1, \dots, n - 1\}$.

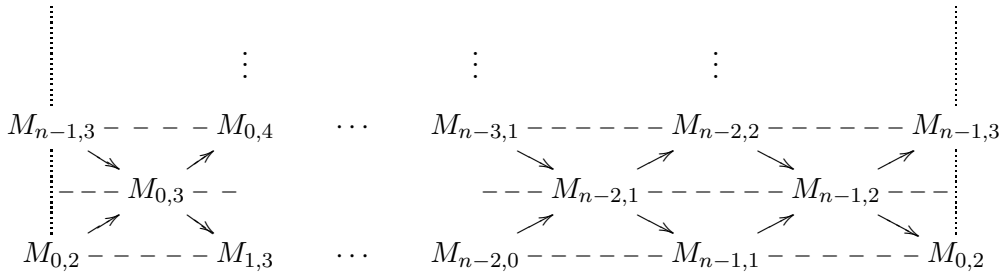


FIGURE 8. The AR-quiver of \mathcal{T}_n

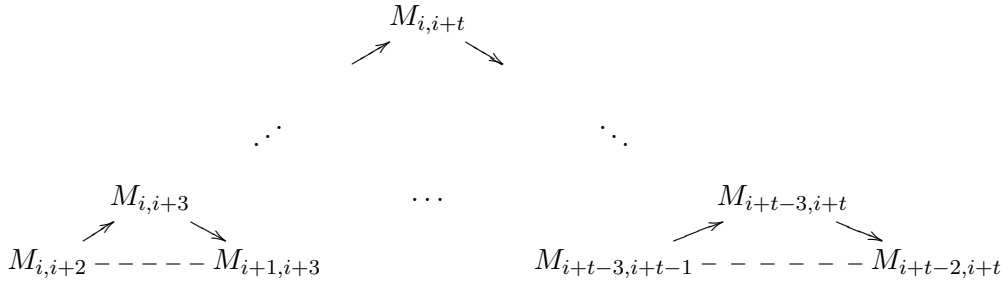


FIGURE 9. The wing $\mathcal{W}_{i,i+t}$.

An additive subcategory of \mathcal{T} is said to be of *finite type* if it contains only finitely many indecomposable objects. Otherwise it is said to be of *infinite type*. Some particular subcategories of \mathcal{T} are important for this paper. For each fixed i in $\{0, \dots, n\}$, we consider the subcategory whose objects are the indecomposable objects $M_{i,i+t}$, for all $t > 1$. This is called a *ray*, and is denoted \mathcal{R}_i . Dually, for each i in $\{0, \dots, n\}$, we consider the subcategory whose objects are all indecomposable objects $M_{i-u,i}$, for all $u > 1$. This is called a *coray*, and denoted \mathcal{C}_i .

For each i in $\{0, \dots, n-1\}$, and each $t > 1$, the *wing* $\mathcal{W}_{i,i+t}$ is the additive subcategory of \mathcal{T} whose indecomposable objects are the $M_{j,j+u}$ with $u \geq 2$, $i \leq j \leq i+t-2$ and $j+u \leq i+t$. It contains a unique indecomposable object $M_{i,i+t}$ of maximal quasilength. See Figure 9. For $t \leq 1$ we let $\mathcal{W}_{i,i+t}$ be the empty subcategory. We also denote the wing of an indecomposable object X by \mathcal{W}_X .

Due to the following well-known fact, we can apply results from the previous section in our analysis of \mathcal{T} .

Lemma 3.1. *For $u \leq n$ the wing $\mathcal{W}_{i,i+u}$ in \mathcal{T}_n is equivalent to the module category $\text{mod } KQ$, where Q is a linearly oriented quiver of Dynkin type A_{u-1} .*

3.2. Geometric model. Here we summarize the construction, given in [3], of a geometric model of \mathcal{T}_n . Note that such a geometric model also appears in [27].

Consider an annulus $\mathbb{A}(n)$ with n marked points on the outer boundary. The points are labelled $0, 1, \dots, n-1$, and arranged anticlockwise (see Figure 10).

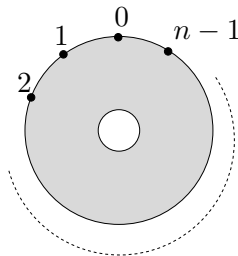


FIGURE 10. An annulus with n marked points on its outer boundary.

There is a bijection between the indecomposable objects in \mathcal{T}_n and a set of isotopy classes of arcs in $\mathbb{A}(n)$ (not allowing non-transverse or multiple intersections). To

describe this bijection, we consider the universal covering $\mathbb{U}(n)$ of $\mathbb{A}(n)$, with marked points corresponding to \mathbb{Z} (and with $0, 1, \dots, n-1$ lying in a fundamental domain). See Figure 11.

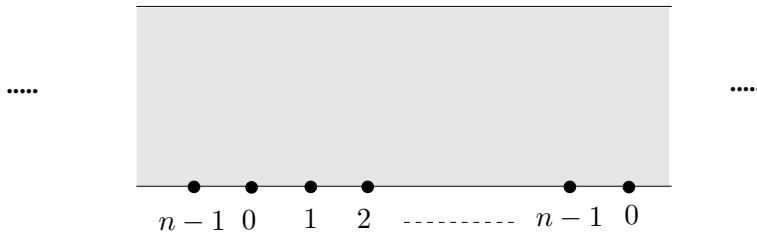


FIGURE 11. The universal cover of the annulus in Figure 10.

For integers i, j with $i+2 \leq j$, let $[i, j]$ denote the arc in $\mathbb{U}(n)$ with starting point i and ending point j , oriented from i to j .

Let $\pi_n([i, j])$ denote the corresponding arc in $\mathbb{A}(n)$ and let $\mathcal{A}(\mathbb{A}(n))$ denote the set of (isotopy classes) of such arcs.

Define a quiver with the elements in $\mathcal{A}(\mathbb{A}(n))$ as vertices and arrows:

$$\pi_n([i, j]) \rightarrow \pi_n([i, j+1])$$

and

$$\pi_n([i, j]) \rightarrow \pi_n([i+1, j]) \text{ (if } j \neq i+2)$$

Defining a translate using the formula $\tau(\pi_n([i, j])) = \pi_n([i-1, j-1])$, this becomes a translation quiver. We call this the (*translation*) *quiver of $\mathcal{A}(\mathbb{A}(n))$* .

For the following, see [3, Lemma 2.5], [27, 4.18] (or, using unoriented arcs, [13], [7, §3.4]).

Proposition 3.2. *The map $\psi : \pi_n([i, j]) \mapsto M_{i,j}$ induces an isomorphism between the translation quiver of $\mathcal{A}(\mathbb{A}(n))$ and the AR-quiver of \mathcal{T}_n .*

Note that the convention that $M_{\sigma^k(i), \sigma^k(j)} = M_{ij}$ corresponds exactly to the fact $\pi_n([\sigma^k(i), \sigma^k(j)]) = \pi_n([i, j])$ for any integer k .

For arcs α, β in $\mathcal{A}(\mathbb{A}(n))$, let $I(\alpha, \beta)$ be the minimum number of intersections between arcs in the isotopy classes α and β , not allowing non-transverse or multiple intersections. Similarly we let $I^+(\alpha, \beta)$ (resp. $I^-(\alpha, \beta)$) denote the number of positive (resp. negative) crossings between α and β . Figure 12 illustrates the two kinds of crossing. See also Figure 1.

The following is proved in [3, Thm. 3.7], [27, 4.23]. See also [7] for further results in this direction.

Theorem 3.3. *Let $\pi_n([i, j]), \pi_n([i', j'])$ be arcs in $\mathcal{A}(\mathbb{A}(n))$. Then*

$$\dim \text{Ext}_{\mathcal{T}}^1(M_{ij}, M_{i'j'}) = I^-(\pi_n([i, j]), \pi_n([i', j'])).$$

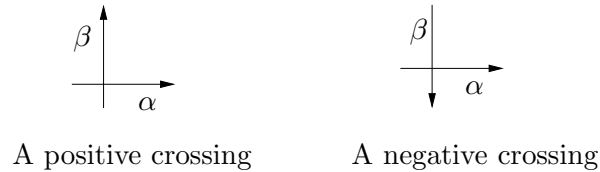


FIGURE 12. Crossing types between α and β

3.3. Extending the tubes. Let as before $\Lambda = KQ$, where Q is the quiver (3), and let \mathcal{T} be a tube of fixed rank $n > 1$, considered as a subcategory of $\text{mod } \Lambda$. Now we also consider $\text{Mod } \Lambda$, the category of all left Λ -modules. In particular, we consider subcategories $\varinjlim \mathcal{T}$, $\varprojlim \mathcal{T}$ and $\overline{\mathcal{T}}$ of $\text{Mod } \Lambda$. Here $\varinjlim \mathcal{T}$ is defined to be the subcategory of $\text{Mod } \Lambda$ whose objects are direct limits of filtered direct systems of objects in \mathcal{T} . Note that $\varinjlim \mathcal{T}$ contains the *Prüfer modules*. These are the n modules $M_{i,\infty}$ obtained as direct limits along the rays, i.e.

$$M_{i,\infty} = \varinjlim (M_{i,i+2} \rightarrow M_{i,i+3} \rightarrow M_{i,i+4} \rightarrow \cdots).$$

Let $\varprojlim \mathcal{T}$ be the subcategory of $\text{Mod } \Lambda$ whose objects are inverse limits of filtered inverse systems of objects in \mathcal{T} . This category contains the *adic modules*, which are obtained as inverse limits along a coray:

$$M_{-\infty,i} = \varprojlim (\cdots \rightarrow M_{i-4,i} \rightarrow M_{i-3,i} \rightarrow M_{i-2,i}).$$

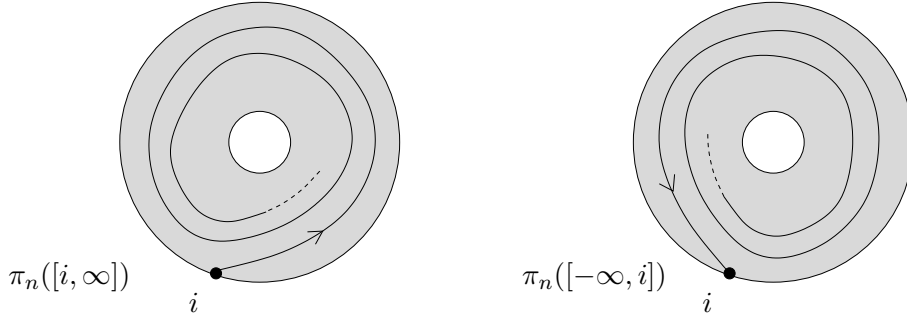
Let $\overline{\mathcal{T}}$ be the subcategory of $\text{Mod } \Lambda$ whose objects are all filtered direct limits or filtered inverse limits of objects in \mathcal{T} . This category clearly contains $\varinjlim \mathcal{T}$ and $\varprojlim \mathcal{T}$. We extend the definition of σ to Prüfer and adic modules with the convention that $\sigma(\pm\infty) = \pm\infty$: note that any Prüfer module has a unique name $M_{i,\infty}$ if we take i in $\{0, 1, \dots, n\}$, and similarly for adic modules.

Recall that a Λ -module M is *pure-injective* if the canonical map $M \rightarrow DDM$ is a split monomorphism. For background on pure-injective modules, and other definitions, see e.g. [18] or [24]. It can be shown (see [8]) that the category $\overline{\mathcal{T}}$ has the following properties:

- All objects are pure-injective as Λ -modules.
- Any object is determined by its indecomposable direct summands.
- The indecomposables in $\overline{\mathcal{T}}$ are exactly the indecomposables M_{ij} in \mathcal{T} , the Prüfer modules $M_{i,\infty}$ and the adic modules $M_{-\infty,i}$.

A module M in $\overline{\mathcal{T}}$ is called *rigid* if $\text{Ext}_\Lambda^1(M, M) = 0$. Note that since all objects in $\overline{\mathcal{T}}$ are pure-injective, this definition is equivalent to having $\text{Ext}_\Lambda^1(M', M'') = 0$ for all indecomposable direct summands M', M'' of M . Now, a rigid module M in $\overline{\mathcal{T}}$ is called *maximal rigid*, if $\text{Ext}_\Lambda^1(M \amalg X, M \amalg X) = 0$ for an indecomposable X in $\overline{\mathcal{T}}$ implies that X is isomorphic to a direct summand in M .

3.4. Extending the model. In this section, we show that the geometric model of \mathcal{T} discussed in Section 3.2 can be naturally extended to a geometric model for $\overline{\mathcal{T}}$.

FIGURE 13. Infinite arcs in $\mathbb{U}(n)$.FIGURE 14. Infinite arcs in $\mathbb{A}(n)$.

Consider the arc $[i, \infty]$ in $\mathbb{U}(n)$, i.e. the arc with starting point i which is oriented rightwards, away from i , and the arc $[-\infty, i]$ in $\mathbb{U}(n)$, i.e. the arc with ending point i and which is oriented leftwards, away from i . See Figure 13.

Consider also their images $\pi_n([i, \infty])$ and $\pi_n([-\infty, i])$ in $\mathbb{A}(n)$; see Figure 14.

Let $\tilde{\mathcal{A}}$ denote $\mathcal{A} = \mathcal{A}(\mathbb{A}(n))$ extended to include the homotopy classes of the arcs $\pi_n([i, \infty])$ and $\pi_n([-\infty, i])$, for i in $\{0, \dots, n-1\}$. It is clear that the bijection ψ from Section 3.2 extends to a bijection between the arcs in $\tilde{\mathcal{A}}$ and the indecomposable objects in $\overline{\mathcal{T}}$, with $\pi_n([i, \infty])$ and $\pi_n([-\infty, i])$ corresponding to $M_{i, \infty}$ and $M_{-\infty, i}$, respectively. We also denote this by ψ .

We aim to prove the following generalization of Theorem 3.3.

Theorem 3.4. *Given indecomposable objects M_{ij} and $M_{i'j'}$ in $\overline{\mathcal{T}}$. Then:*

$$\mathrm{Ext}^1(M_{ij}, M_{i'j'}) \cong \prod_{I^-(\pi_n([i,j]), \pi_n([i',j']))} K.$$

In the case where i, i', j, j' are all finite, the result is proved in [3]. To prove the remaining cases we will need the following well-known results.

Lemma 3.5. *Let X, Y be arbitrary Λ -modules, $(X_j)_j$ an arbitrary filtered direct system of modules and $(Y_j)_j$ an arbitrary filtered inverse system of modules.*

- (a) $\mathrm{Hom}(\varinjlim X_j, Y) \simeq \varinjlim \mathrm{Hom}(X_j, Y)$.
- (b) *If X is finitely generated, then $\mathrm{Hom}(X, \varinjlim Y_j) \simeq \varinjlim \mathrm{Hom}(X, Y_j)$*
- (c) *If the Y_j are finitely generated, then $\varinjlim Y_j \simeq D \varinjlim DY_j$.*
- (d) *If Y is pure-injective, then $\mathrm{Ext}^1(\varinjlim X_j, Y) \simeq \varinjlim \mathrm{Ext}^1(X_j, Y)$.*
- (e) *If the Y_j are finitely generated, then $\mathrm{Ext}^1(X, \varinjlim Y_j) \simeq \varinjlim \mathrm{Ext}^1(X, Y_j)$.*

Proof. For (a) see, for example, [26]. For (b), see [20, Lemma 1.6] or [11, Sect. 1.5]. For (c), we have, using part (a):

$$\begin{aligned} D \varinjlim DY_j &= \text{Hom}(\varinjlim \text{Hom}(Y_j, K), K) \\ &\simeq \varprojlim \text{Hom}(\text{Hom}(Y_j, K), K) \simeq \varprojlim DDY_j \simeq \varprojlim Y_j, \end{aligned}$$

as required. Part (d) is proved in [1, Prop. I.10.1]. For (e), we recall that $\text{Ext}^1(X, DY) \simeq \text{Ext}^1(Y, DX)$ for all modules X and Y . Using parts (c) and (d) and the fact [24, Prop. 4.3.29] that DX is pure-injective for any module X , we have:

$$\begin{aligned} \text{Ext}^1(X, \varprojlim Y_j) &\simeq \text{Ext}^1(X, D(\varinjlim DY_j)) \simeq \text{Ext}^1(\varinjlim DY_j, DX) \\ &\simeq \varprojlim \text{Ext}^1(DY_j, DX) \simeq \varprojlim \text{Ext}^1(X, DDY_j) \\ &\simeq \varprojlim \text{Ext}^1(X, Y_j), \end{aligned}$$

and (e) is shown. \square

We also note that the following holds (see e.g. [11, Sect. 3.1]).

Lemma 3.6. *For modules X and Y with X finitely generated, we have $D\text{Ext}^1(X, Y) \cong \text{Hom}(Y, \tau X)$ and $\text{Ext}^1(Y, X) \cong D\text{Hom}(\tau^{-1}X, Y)$.*

Note that if X, Y are finitely generated, then the first formula can also be written $\text{Ext}^1(X, Y) \cong D\text{Hom}(Y, \tau X)$.

Lemma 3.7. (a) *If X is a Prüfer module and Y is a finitely generated module then $\text{Hom}(X, Y) = 0$.*

(b) *If X is a finitely generated module and Y is an adic module then $\text{Hom}(X, Y) = 0$.*

Proof. See, for example, [23, p46]. \square

With arguments as in [3], the crossing numbers can be computed as follows. Recall that $\sigma: \mathbb{Z} \rightarrow \mathbb{Z}$ is the function $i \mapsto i + n$.

Proposition 3.8. *We have the following:*

- (a) $I^-(\pi_n([i, \infty]), \pi_n([a, b])) = |\{m \in \mathbb{Z}: a < \sigma^m(i) < b\}|$;
- (b) $I^-(\pi_n([a, b]), \pi_n([i, \infty])) = I^+(\pi_n([i, \infty]), \pi_n([a, b])) = 0$;
- (c) $I^-(\pi_n([-\infty, j]), \pi_n([a, b])) = 0$;
- (d) $I^-(\pi_n([a, b]), \pi_n([-\infty, j])) = I^+(\pi_n([-\infty, j]), \pi_n([a, b])) = |\{m \in \mathbb{Z}: a < \sigma^m(i) < b\}|$;
- (e) $I^-(\pi_n([i, \infty]), \pi_n([-\infty, i'])) = \aleph_0$, for all i, i' in $\{0, \dots, n-1\}$;
- (f) $I^-(\pi_n([i, \infty]), \pi_n([i', \infty])) = 0$ for all i, i' in $\{0, \dots, n-1\}$;
- (g) $I^-(\pi_n([-\infty, i]), \pi_n([i', \infty])) = 0$ for all i, i' in $\{0, \dots, n-1\}$;
- (h) $I^-(\pi_n([-\infty, i]), \pi_n([-\infty, i'])) = 0$ for all i, i' in $\{0, \dots, n-1\}$.

Proof of Theorem 3.4. We need to compute $\text{Ext}^1(X, Y)$ for all pairs of indecomposables X, Y in $\overline{\mathcal{T}}$, and compare these with the crossing-numbers from Proposition 3.8.

We first determine $\text{Ext}^1(M_{i, \infty}, M_{ab})$ using the AR formula. For large t , consider the indecomposable subobject $M_{i-1, i+b-a-1}$ of $M_{i-1, t}$ with length $(b-a-1)$, i.e.

equal to the length of $M_{a,b}$. We have that $\dim \text{Hom}(M_{a,b}, M_{i-1,t})$ equals the number of times the simple top $M_{b-2,b}$ of $M_{a,b}$ occurs as a composition factor of $M_{i-1,i+b-a-1}$. Using arguments as in [3], this number is $|\{n \in \mathbb{Z}: a < \sigma^n(i) < b\}|$. Hence, for large values of t , it is independent of t . It is also clear that the map

$$\text{Hom}(M_{a,b}, M_{i-1,t}) \rightarrow \text{Hom}(M_{a,b}, M_{i-1,t+1})$$

is an isomorphism for $t \gg i$.

Applying Lemma 3.5(d), we have

$$\begin{aligned} \text{Ext}^1(M_{i,\infty}, M_{ab}) &= \text{Ext}^1(\varinjlim M_{ij}, M_{ab}) \simeq \varprojlim \text{Ext}^1(M_{ij}, M_{ab}) \\ &\simeq \varprojlim D \text{Hom}(M_{ab}, \tau M_{ij}) = \varprojlim D \text{Hom}(M_{ab}, M_{i-1,j-1}) \end{aligned}$$

(adopting the convention in this proof that limits involve the variable j). Hence, we obtain that $\dim \text{Ext}^1(M_{i,\infty}, M_{ab}) = |\{n \in \mathbb{Z}: a < \sigma^n(i) < b\}|$.

Next, using Lemma 3.5(e), we have

$$\begin{aligned} \text{Ext}^1(M_{ab}, M_{-\infty,i}) &= \text{Ext}^1(M_{ab}, \varprojlim M_{ji}) \simeq \varinjlim \text{Ext}^1(M_{ab}, M_{ji}) \\ &\simeq \varinjlim D \text{Hom}(M_{ji}, \tau M_{ab}) = \varinjlim \text{Hom}(M_{ji}, M_{a-1,b-1}). \end{aligned}$$

For large j , it is easy to see that the map

$$\text{Hom}(M_{ji}, M_{a-1,b-1}) \rightarrow \text{Hom}(M_{j-1,i}, M_{a-1,b-1})$$

is an isomorphism and that $\dim \text{Hom}(M_{j,i}, M_{a-1,b-1})$ is equal to the number of times the simple top $M_{i-2,i}$ of M_{ji} appears as a composition factor of $M_{a-1,b-1}$, i.e.

$$|\{n \in \mathbb{Z}: a-1 < \sigma^n(i-1) < b-1\}|,$$

which equals

$$|\{n \in \mathbb{Z}: a < \sigma^n(i) < b\}|,$$

as required.

By Lemma 3.6,

$$D \text{Ext}^1(M_{ab}, M_{i,\infty}) \simeq \text{Hom}(M_{i,\infty}, M_{a-1,b-1}).$$

We see that $\text{Ext}^1(M_{ab}, M_{i,\infty}) = 0$ by Lemma 3.7(a).

Similarly, by Lemma 3.6, we have that

$$\text{Ext}^1(M_{-\infty,i}, M_{ab}) \simeq D \text{Hom}(\tau^{-1} M_{ab}, M_{-\infty,i}) = 0,$$

using Lemma 3.7(b).

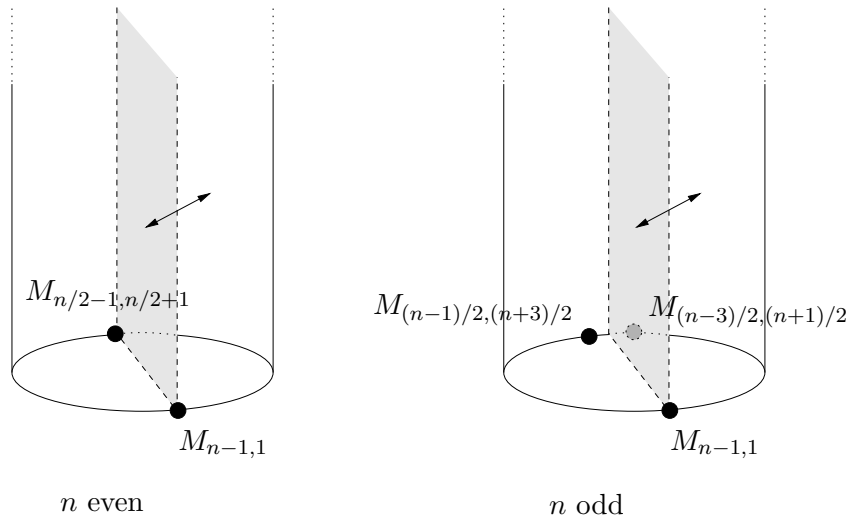
We next compute $\text{Ext}^1(M_{i,\infty}, M_{-\infty,i'})$. Using Lemma 3.5(e), we have that

$$\text{Ext}^1(M_{i,\infty}, M_{-\infty,i'}) = \text{Ext}^1(M_{i,\infty}, \varprojlim M_{j,i'}) \simeq \varinjlim \text{Ext}^1(M_{i,\infty}, M_{j,i'}).$$

The maps $\text{Ext}^1(M_{i,\infty}, M_{j,i'}) \rightarrow \text{Ext}^1(M_{i,\infty}, M_{j-1,i'})$ are surjective. As j tends to $-\infty$, the dimension of $\text{Ext}^1(M_{i,\infty}, M_{j,i'})$ is unbounded (by the above formula for it). Therefore, this limit evaluates to $\prod_{\mathbb{N}_0} K$.

It is shown in [8, Lemma 2.7], that, for all i, i' in $\{0, \dots, n-1\}$, all of $\text{Ext}^1(M_{-\infty,i}, M_{i',\infty})$, $\text{Ext}^1(M_{i,\infty}, M_{i',\infty})$ and $\text{Ext}^1(M_{-\infty,i}, M_{-\infty,i'})$ vanish.

Comparing the crossing numbers with the dimensions of the corresponding Ext-groups concludes the proof of the theorem. \square


 FIGURE 15. The map $M \mapsto M^\vee$.

3.5. Reflection. Recall (see [8]) that there is a map $M \mapsto M^\vee$, which gives a bijection on the indecomposable objects in $\overline{\mathcal{T}}$. With our notation, the map is given by $M_{ij} \mapsto M_{-j, -i}$ for i, j in $\mathbb{Z} \cup \{\pm\infty\}$.

We assume that the tube is drawn regularly, i.e. that the objects in a τ -orbit are equally spaced around a circle, with adjacent τ -orbits symmetrically interlaced. If n is even, the map $M \mapsto M^\vee$ arises from a reflection in a plane through the tube passing through $M_{n-1, 1}$ and $M_{n/2-1, n/2+1}$. If n is odd, this is a reflection in a plane through the tube passing through $M_{n-1, 1}$ and the mid-point through $M_{(n-3)/2, (n+1)/2}$ and $M_{(n-1)/2, (n+3)/2}$. See Figure 15.

The map has the following properties.

- Lemma 3.9.**
- (a) $\dim \operatorname{Hom}(X, Y) = \dim \operatorname{Hom}(Y^\vee, X^\vee)$ for X, Y in $\overline{\mathcal{T}}$;
 - (b) $\operatorname{Ext}^1(X, Y) = 0$ if and only if $\dim \operatorname{Ext}^1(Y^\vee, X^\vee) = 0$ for X, Y in $\overline{\mathcal{T}}$;
 - (c) $(\tau X)^\vee = \tau^{-1} X^\vee$ for all indecomposables X in \mathcal{T} .

As a direct consequence of Lemma 3.9, we have the following observation.

Lemma 3.10. For each X in $\overline{\mathcal{T}}$, we have

- (a) $(X^{\perp_H})^\vee = {}^{\perp_H}(X^\vee)$;
- (b) $(X^\vee)^{\perp_E} = ({}^{\perp_E}X)^\vee$.

For a subcategory \mathcal{X} of $\overline{\mathcal{T}}$, we let \mathcal{X}^\vee denote the subcategory $\{X^\vee \mid X \in \mathcal{X}\}$. We will also need the following.

Lemma 3.11. For a maximal rigid T in \mathcal{T} , we have

- (a) $(\operatorname{Gen} T)^\vee = \operatorname{Cogen}(T^\vee)$;
- (b) $(\operatorname{Cogen} T)^\vee = \operatorname{Gen}(T^\vee)$.

Proof. This follows directly from the definition of the map using Lemma 3.7 and the fact that an indecomposable X in \mathcal{T} is generated by a maximal rigid object T if and only if it is generated by an indecomposable direct summand in T (see also [9]). \square

4. TORSION PAIRS AND MAXIMAL RIGID OBJECTS

In this section we give an improvement of a result from [8], and discuss a combinatorial interpretation. The idea is to link maximal rigid objects in $\overline{\mathcal{T}}$ with torsion pairs in \mathcal{T} .

4.1. A bijection. A rigid object in $\overline{\mathcal{T}}$ is said to be of *Prüfer type* if it has a Prüfer module as a direct summand, and it is said to be of *adic type* if it has an adic module as a direct summand.

An object T in $\varinjlim \mathcal{T}$ is maximal rigid if and only if it is cotilting in the category $\varinjlim \mathcal{T}$, in the sense of Colpi [10]. This is proved in [9] (see [9, 1.10] and note that, by [9, Lemma 1.2], a finitely presented $\tilde{\Lambda}_n$ -module is maximal rigid if and only if it is a tilting module, where $\tilde{\Lambda}_n$ denotes the completion of the path algebra of an oriented n -cycle).

We consider equivalence classes of maximal rigid objects in $\overline{\mathcal{T}}$, where two maximal rigid objects are considered to be equivalent if they have the same indecomposable direct summands. The following is proved in [8].

Proposition 4.1. *Let T be a rigid object in $\overline{\mathcal{T}}$.*

- (a) *T is maximal rigid in $\overline{\mathcal{T}}$ if and only if it has n indecomposable direct summands.*
- (b) *If T is maximal rigid, it is either of Prüfer type or of adic type, but not both.*
- (c) *A rigid object of Prüfer type lies in the subcategory $\varinjlim \mathcal{T} \subset \overline{\mathcal{T}}$.*

We will give a parallel result concerning torsion pairs in \mathcal{T} . A torsion pair $(\mathcal{F}, \mathcal{G})$ in \mathcal{T} is said to be of *ray type* if \mathcal{G} contains at least one ray of \mathcal{T} . It is said to be of *coray type* if \mathcal{F} contains at least one coray of \mathcal{T} . A class of objects (or subcategory) \mathcal{S} of a category \mathcal{C} is said to be *generating*, if for each map f in \mathcal{C} , there is an object S in \mathcal{S} with $\text{Hom}(S, f) \neq 0$. Dually, one can define *cogenerating* classes of objects, i.e. \mathcal{S} is cogenerating if, for each map f in \mathcal{C} , there is an object S in \mathcal{S} with $\text{Hom}(f, S) \neq 0$.

Lemma 4.2. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair in \mathcal{T} . Then it is not the case that both \mathcal{F} and \mathcal{G} are of finite type.*

Proof. Note that subobjects or factor objects of indecomposable objects in \mathcal{T} are indecomposable. Assume there is an exact sequence

$$(4) \quad 0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0$$

with X in \mathcal{F} and Y in \mathcal{G} , and with M in \mathcal{T} indecomposable. Then also X and Y are indecomposable and we have $l(X) \leq l(M)$ and $l(Y) \leq l(M)$. There are indecomposable objects in \mathcal{T} of arbitrary (finite) length. For any indecomposable M there is an exact sequence (4), by the definition of a torsion pair. Hence, we cannot have that both \mathcal{F} and \mathcal{G} are of finite type. \square

We next prove that the torsion pairs with \mathcal{F} of infinite type are exactly those of coray type. For an indecomposable object X in \mathcal{T} we write \mathcal{C}_X for the coray containing X and \mathcal{R}_X for the ray containing X .

Lemma 4.3. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair in \mathcal{T} , where \mathcal{T} has rank n . Assume \mathcal{F} is of infinite type. Then the following hold:*

- (i) \mathcal{F} contains an indecomposable object X , with $l(X) = n$.
- (ii) Let X' be the (uniquely defined) indecomposable object in \mathcal{T} such that there is an irreducible monomorphism $X' \rightarrow X$. Then \mathcal{G} is contained in $\mathcal{W}_{X'}$. In particular, \mathcal{G} is of finite type.
- (iii) \mathcal{F} contains the coray \mathcal{C}_X .
- (iv) \mathcal{F} cogenerates \mathcal{T} .

Proof. First note that since \mathcal{F} is of infinite type, there is no limit on the length of the indecomposable objects in \mathcal{F} . Since \mathcal{F} is closed under factor objects, it must therefore contain an indecomposable object with $l(X) = n$. Hence (i) holds. Also since \mathcal{F} is closed under factor objects, the part of the coray \mathcal{C}_X below X is contained in \mathcal{F} ; more precisely $\mathcal{C}_X \cap \{Y \mid l(Y) \leq n\} \subset \mathcal{F}$. Since $l(X) = n$, we have that $\text{Hom}(X, Y) = 0$ for an indecomposable object Y in \mathcal{T} if and only if Y is in the wing $\mathcal{W}_{X'}$. Therefore, by Definition 1.1(i), we have that part (ii) holds. By Lemma 1.3, (iii) holds, while (iv) is a direct consequence of (iii). \square

We state the dual version of Lemma 4.3.

Lemma 4.4. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair in \mathcal{T} , where \mathcal{T} has rank n . Assume \mathcal{G} is of infinite type. Then the following holds:*

- (i) \mathcal{G} contains an indecomposable object X , with $l(X) = n$.
- (ii) Let X' be the (uniquely defined) indecomposable object in \mathcal{T} , such that there is an irreducible epimorphism $X \rightarrow X'$. Then \mathcal{F} is contained in $\mathcal{W}_{X'}$. In particular, \mathcal{F} is of finite type.
- (iii) \mathcal{G} contains the ray \mathcal{R}_X .
- (iv) \mathcal{G} generates \mathcal{T} .

Combining Lemma 4.2, 4.3 and 4.4, we obtain the following direct consequence.

Corollary 4.5. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair in \mathcal{T} .*

- (a) *The following are equivalent*
 - (i) \mathcal{F} is of infinite type;
 - (ii) \mathcal{F} contains a coray;
 - (iii) \mathcal{F} cogenerates \mathcal{T} ;
 - (iv) \mathcal{G} is of finite type.
- (b) *$(\mathcal{F}, \mathcal{G})$ is either of ray or of coray type (and not both).*

Moreover, by Lemma 1.3 and a direct application of Lemma 3.10, we obtain the following.

Lemma 4.6. *The map $M \rightarrow M^\vee$, maps a torsion pair $(\mathcal{F}, \mathcal{G})$ to a torsion pair $(\mathcal{G}^\vee, \mathcal{F}^\vee)$. Moreover if $(\mathcal{F}, \mathcal{G})$ is of ray-type, then $(\mathcal{G}^\vee, \mathcal{F}^\vee)$ is of coray-type and vice-versa.*

For a maximal rigid object T of Prüfer type, consider the subcategory

$$\mathcal{G}_T = {}^{\perp_E}T \cap \mathcal{T},$$

and let

$$\mathcal{F}_T = {}^{\perp_H}(\mathcal{G}_T) \cap \mathcal{T}.$$

For a maximal rigid object T of adic type, we define

$$\mathcal{F}_T = T^{\perp_E} \cap \mathcal{T}$$

and let

$$\mathcal{G}_T = (\mathcal{F}_T)^{\perp_H} \cap \mathcal{T}.$$

We have the following reformulation of a result of [9]:

Theorem 4.7. *The map $T \mapsto (\mathcal{F}_T, \mathcal{G}_T)$ gives a one-to-one correspondence between equivalence classes of maximal rigid objects in $\varinjlim \mathcal{T}$ and torsion pairs in \mathcal{T} with the property that \mathcal{G} generates \mathcal{T} .*

Now, using Lemma 3.10, we obtain a commutative square (*):

$$\begin{array}{ccc} \{\text{maximal rigid objects of Prüfer type in } \overline{\mathcal{T}}\} & \longrightarrow & \{\text{torsion pairs of ray-type in } \mathcal{T}\} \\ \downarrow & & \downarrow \\ \{\text{maximal rigid objects of adic type in } \overline{\mathcal{T}}\} & \longrightarrow & \{\text{torsion pairs of coray-type in } \mathcal{T}\} \end{array}$$

where the horizontal maps are given by $T \mapsto (\mathcal{F}_T, \mathcal{G}_T)$ and the vertical maps are induced by $M \mapsto M^\vee$.

As a direct consequence of Lemma 3.10 (as also observed in [8]), we have that the left vertical map in (*) is a bijection. We have already observed that the right vertical map is a bijection. The upper horizontal map is bijective by Theorem 4.7, combining with Proposition 4.1 and Lemma 4.4, and it follows that the lower horizontal map is bijective.

Combining the commutative diagram of bijections (*) with Corollary 4.5 we obtain the following improvement of Theorem 4.7.

Theorem 4.8. *The map $T \mapsto (\mathcal{F}_T, \mathcal{G}_T)$ gives a bijection between*

- *Torsion pairs $(\mathcal{F}, \mathcal{G})$ in \mathcal{T} , and*
- *Equivalence classes of maximal rigid objects in $\overline{\mathcal{T}}$*

Corollary 4.9. *The number of torsion pairs in \mathcal{T} is $2^{\binom{2n-1}{n-1}}$.*

Proof. By [9, 2.4, 5.2 and B.1], the number of cotilting objects in $\varinjlim \mathcal{T}$ is $\binom{2n-1}{n-1}$. The result then follows from Proposition 4.1, the above diagram of maps, and Theorem 4.8. \square

4.2. Alternative and explicit descriptions of \mathcal{F}_T and \mathcal{G}_T . We give alternative and more explicit descriptions for the subcategories \mathcal{F}_T and \mathcal{G}_T corresponding to a maximal rigid object T in $\overline{\mathcal{T}}$.

The following observation is useful.

Lemma 4.10. *Let \mathcal{T} be a tube of rank n . Then we have:*

- (a) ${}^{\perp E}M_{i,\infty} \cap \mathcal{T} = \mathcal{T}$;
- (b) $M_{i,\infty}^{\perp E} \cap \mathcal{T} = \mathcal{W}_{i,i+n}$

We now give an explicit description of the torsion pair $(\mathcal{F}_T, \mathcal{G}_T)$ corresponding to a maximal rigid module T . We first of all give a combinatorial lemma concerning wings, which is easy to check.

Lemma 4.11. *Let $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n-1$ be integers. Then*

- (i) *We have:*

$$\bigcap_{r=0}^{k-1} \mathcal{W}_{i_r, i_r+n} = \bigcup_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}}$$

(where the subscripts are interpreted modulo k and, for $r = k-1$, we interpret $i_{r+1} = i_0$ as $i_0 + n$).

- (ii) *The wing $\mathcal{W}_{i_r, i_{r+1}}$ is empty if and only if $i_{r+1} - i_r \equiv 1 \pmod{n}$.*
- (iii) *The wings $\mathcal{W}_{i_r, i_{r+1}}$ do not overlap, i.e. each indecomposable object in \mathcal{T} belongs to at most one $\mathcal{W}_{i_r, i_{r+1}}$.*
- (iv) *If X and Y lie in different wings among the $\mathcal{W}_{i_r, i_{r+1}}$, then $\text{Ext}^1(X, Y) = 0 = \text{Ext}^1(Y, X)$.*

For an example illustrating Lemma 4.11(i), with $n = 10$, $k = 4$, $i_1 = 0$, $i_2 = 4$, $i_3 = 7$ and $i_4 = 8$, see Figure 16.

In the following proposition and the sequel, we adopt the same convention for the wings as in Lemma 4.11(i).

Proposition 4.12. *Let T be maximal rigid in $\overline{\mathcal{T}}$.*

- (a) *If T is of Prüfer type, then*

$$(\mathcal{F}_T, \mathcal{G}_T) = ({}^{\perp H}T \cap \mathcal{T}, {}^{\perp E}T \cap \mathcal{T}) = (\tau^{-1}(\text{Gen } T \cap \mathcal{T}), \text{Cogen } T \cap \mathcal{T}).$$

- (b) *Assume T is of Prüfer type with Prüfer summands $M_{i_r, \infty}$ for $r = 0, \dots, k-1$ where $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n-1$. Then*

$$(\mathcal{F}_T, \mathcal{G}_T) = ((\bigcup_{r=0}^{k-1} \mathcal{F}_r), (\bigcup_{r=0}^{k-1} \mathcal{G}_r) \cup \mathcal{G}_{\infty}),$$

where $(\mathcal{F}_r, \mathcal{G}_r)$ is a torsion pair in $\mathcal{W}_{i_r, i_{r+1}+1}$ with \mathcal{G}_r containing all of the injective objects in $\mathcal{W}_{i_r, i_{r+1}+1}$ and $\mathcal{G}_{\infty} = \text{add}(\bigcup_{r=0}^{k-1} \mathcal{R}_{i_r})$.

- (c) *If T is of adic type, then*

$$(\mathcal{F}_T, \mathcal{G}_T) = (T^{\perp E} \cap \mathcal{T}, T^{\perp H} \cap \mathcal{T}) = (\text{Gen } T \cap \mathcal{T}, \tau(\text{Cogen } T \cap \mathcal{T})).$$

- (d) *Assume T is of adic type with adic summands $M_{i_r, \infty}$ for $r = 0, \dots, k-1$, where $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n-1$. Then*

$$(\mathcal{F}_T, \mathcal{G}_T) = ((\bigcup_{r=0}^{k-1} \mathcal{F}_r) \cup \mathcal{F}_{\infty}, (\bigcup_{r=0}^{k-1} \mathcal{G}_r)),$$

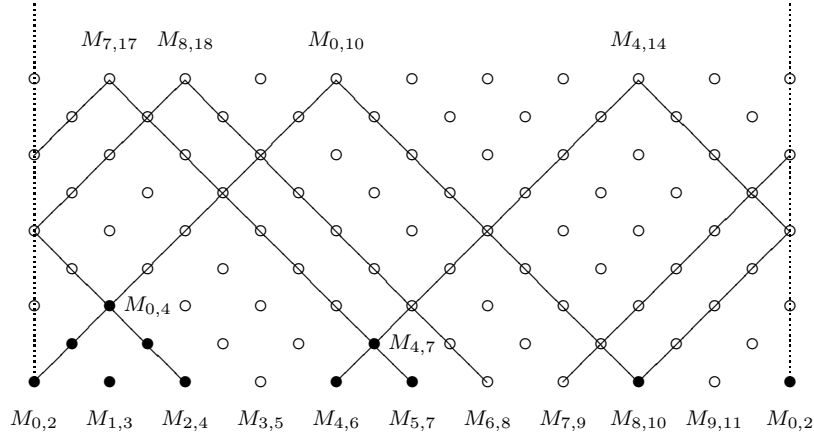


FIGURE 16. The indecomposable objects in the intersection of the four wings $\mathcal{W}_{0,10}, \mathcal{W}_{4,14}, \mathcal{W}_{7,17}$ and $\mathcal{W}_{8,18}$ in a tube of rank 10 corresponding to a maximal rigid object of Prüfer type. The objects lying in the intersection are indicated by filled-in circles. The intersection coincides with the union $\mathcal{W}_{0,4} \cup \mathcal{W}_{4,7} \cup \mathcal{W}_{8,10}$ (note that the wing $\mathcal{W}_{i_3, i_4} = \mathcal{W}_{7,8}$ is empty).

where $(\mathcal{F}_r, \mathcal{G}_r)$ is a torsion pair in $\mathcal{W}_{i_r, i_{r+1}+1}$ with \mathcal{F}_r containing all of the projective objects in $\mathcal{W}_{i_r, i_{r+1}+1}$ and $\mathcal{F}_\infty = \text{add}(\cup_{r=0}^{k-1} \mathcal{C}_{i_{r+1}+1})$.

Proof. We give the details for (a) and (b), while statements (c) and (d) can be proved similarly. Let T be maximal rigid of Prüfer type in $\overline{\mathcal{T}}$. Our aim is to compute $\mathcal{G}_T = {}^{\perp E} T \cap \mathcal{T}$ and then $\mathcal{F}_T = {}^{\perp H} \mathcal{G}_T \cap \mathcal{T}$. We use the Prüfer direct summands of \mathcal{T} in order to compute \mathcal{G}_T more precisely in terms of a set of wings in \mathcal{T} . We then use this to compute \mathcal{F}_T using the theory of torsion pairs in type A (see Section 2).

Let $T_{\mathcal{T}}$ be the direct sum of all indecomposable direct summands of T which are finitely generated. Let $M_{i_0, \infty}, \dots, M_{i_{k-1}, \infty}$ with $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n-1$ be the indecomposable Prüfer summands of T .

By Lemma 4.10, we then have that the finite part $T_{\mathcal{T}}$ lies in $(\cap_{r=0}^{k-1} \mathcal{W}_{i_r, i_r+n})$, which coincides with $(\cup_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}})$ by Lemma 4.11(i). We draw attention to the fact that the wings in the statement of the proposition are slightly larger than these — this will become clearer later in the proof.

Hence, we consider the decomposition $T_{\mathcal{T}} = \coprod_{r=0}^{k-1} T_r$, where each T_r is rigid in $\mathcal{W}_{i_r, i_{r+1}}$ (note that some of the T_r then might be 0, i.e. in the case when $i_{r+1} - i_r \equiv 1 \pmod{n}$). Then it follows easily from Lemma 4.11(iv) that T_r is maximal rigid in $\mathcal{W}_{i_r, i_{r+1}}$. Since T_r is maximal rigid, its restriction to the abelian category $\mathcal{W}_{i_r, i_{r+1}}$ is also cotilting. Our aim is to use this decomposition of T to compute \mathcal{G}_T .

If $\mathcal{W}_{i_r, i_{r+1}}$ is nontrivial, $M_{i_r, i_{r+1}}$ is necessarily an indecomposable direct summand of T_r (see Section 2.3). It is then straightforward to check that

$${}^{\perp E} (\coprod_{r=0}^{k-1} M_{i_r, i_{r+1}} \cap \mathcal{T}) = (\cup_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}}) \cup \text{add}(\cup_{r=0}^{k-1} \mathcal{R}_{i_r}),$$

and from this it follows that $\mathcal{G} \subset (\cup_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}}) \cup \text{add}(\cup_{r=0}^{k-1} \mathcal{R}_{i_r})$.

We also have that

$${}^{\perp E} T_{\mathcal{T}} \cap \mathcal{W}_{i_r, i_{r+1}} = {}^{\perp E} T_r \cap \mathcal{W}_{i_r, i_{r+1}},$$

which gives us the following description of \mathcal{G} :

$$\mathcal{G} = \cup_{r=0}^{k-1} (({}^{\perp E} T_r) \cap \mathcal{T}) \cup \text{add}(\cup_{r=0}^{k-1} \mathcal{R}_{i_r}).$$

Since T_r is cotilting in $\mathcal{W}_{i_r, i_{r+1}}$, we have that ${}^{\perp E} T_r \cap \mathcal{W}_{i_r, i_{r+1}} = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}}} T_r$.

From this it follows that

$$\mathcal{G} = \cup_{r=0}^{k-1} \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}}} T_r \cup \text{add}(\cup_{r=0}^{k-1} \mathcal{R}_{i_r}).$$

Noting that $\text{Cogen } M_{i, \infty} \cap \mathcal{T} = \text{add } \mathcal{R}_i$, we see that $\mathcal{G} = \text{Cogen } T \cap \mathcal{T}$ as claimed in (a).

Next, we consider slightly larger wings, $\mathcal{W}_{i_r, i_{r+1}+1}$ in order to obtain the description of \mathcal{G} in (b). For each r , define $\tilde{T}_r = T_r \amalg M_{i_r, i_{r+1}+1}$. Then \tilde{T}_r is a cotilting module in $\mathcal{W}_{i_r, i_{r+1}+1}$. Note that

$$\text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{T}_r = \text{add}(\text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}}} T_r \cup \{M_{i_r, i_{r+1}}\}).$$

We can thus rewrite \mathcal{G} as follows:

$$\mathcal{G} = \cup_{r=0}^{k-1} \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{T}_r \cup \text{add}(\cup_{r=0}^{k-1} \mathcal{R}_{i_r}).$$

So, setting $\mathcal{G}_r = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{T}_r$, we obtain a description of \mathcal{G} as claimed in (b).

Next, we compute \mathcal{F} . By definition, we have that $\mathcal{F} = {}^{\perp H} \mathcal{G} \cap \mathcal{T}$. Furthermore,

$${}^{\perp H} (\cup_{r=0}^{k-1} M_{i_r, i_{r+1}}) \cap \mathcal{T} = (\cup_{r=0}^{k-1} \tau^{-1} \mathcal{W}_{i_r, i_{r+1}}) \cup \text{add}(\cup_{r=0}^{k-1} \mathcal{C}_{i_{r+1}+1}).$$

Noting that $\text{Hom}(M_{l, i_{r+1}+1}, M_{i_r, i_{r+1}+1}) \neq 0$ for $l \leq i_r$, we see that

$$\mathcal{F} \subset \cup_{r=0}^{k-1} \tau^{-1} (\mathcal{W}_{i_r, i_{r+1}}) \subset \mathcal{W}_{i_r, i_{r+1}+1}.$$

From this, it follows that $\mathcal{F} = \cup ({}^{\perp H} \mathcal{G}_r \cap \mathcal{W}_{i_r, i_{r+1}+1})$, noting that

$$\mathcal{G}_r = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{T}_r.$$

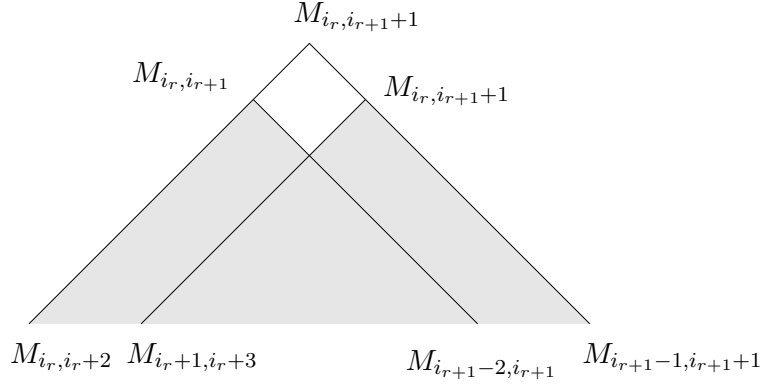
Using the fact that \tilde{T}_r is cotilting in $\mathcal{W}_{i_r, i_{r+1}+1}$, combined with Proposition 2.11, we see that $\mathcal{F} = \cup_{r=0}^{k-1} \mathcal{F}_r$, where

$$\mathcal{F}_r = {}^{\perp H} \tilde{T}_r \cap \mathcal{W}_{i_r, i_{r+1}+1} = \text{Gen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tau_{\mathcal{W}_{i_r, i_{r+1}+1}}^{-1} \tilde{T}_r$$

is the torsion part of the torsion pair in $\mathcal{W}_{i_r, i_{r+1}+1}$ with torsion-free part \mathcal{G}_r . We have that ${}^{\perp H} (\cup_{r=0}^{k-1} M_{i_r, \infty}) = \cup_{r=0}^{k-1} \mathcal{W}_{i_r, i_{r+1}}$, and therefore $\mathcal{F} = {}^{\perp H} T \cap \mathcal{T}$.

Since $\text{Gen } M_{i, \infty} \cap \mathcal{T}$ is empty for all i and since, in $\mathcal{W}_{i_r, i_{r+1}+1}$, we have $\tau^{-1} \tilde{T}_r = \tau^{-1} T_r$, we see that $\mathcal{F} = \tau^{-1} \text{Gen } T \cap \mathcal{T}$.

See Figure 17 for an illustration of one of the wings $\mathcal{W}_{i_r, i_{r+1}+1}$. We have that $\mathcal{G}_r = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tilde{T}_r$. The indecomposable objects in \mathcal{G}_r apart from $M_{i_r, i_{r+1}+1}$ lie in the left hand shaded triangle. Also, $\mathcal{F}_r = \text{Gen}_{\mathcal{W}_{i_r, i_{r+1}+1}} \tau_{\mathcal{W}_{i_r, i_{r+1}+1}}^{-1} \tilde{T}_r$, and the indecomposable objects in \mathcal{F}_r lie in the right hand shaded triangle. \square

FIGURE 17. A wing $\mathcal{W}_{i_r, i_{r+1}+1}$.

4.3. The maximal rigid module corresponding to a torsion pair. In this section we give an explicit description of the inverse of the bijection $T \mapsto (\mathcal{F}_T, \mathcal{G}_T)$ in Theorem 4.8 between maximal rigid objects in $\overline{\mathcal{T}}$ and torsion pairs in \mathcal{T} . We first need the following lemma.

Lemma 4.13. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair in \mathcal{T} . Then an indecomposable object in $\varinjlim(\text{ind } \mathcal{G})$ is either lies in $\text{ind } \mathcal{G}$ or it is a Prüfer module which is the direct limit of a ray in $\text{ind } \mathcal{G}$.*

Proof. It is clear that any indecomposable in \mathcal{G} or Prüfer module which is the direct limit of a ray in $\text{ind } \mathcal{G}$ lies in $\text{ind}(\varinjlim \mathcal{G})$. So suppose that $X = \varinjlim X_j$ is an indecomposable in $\overline{\mathcal{T}}$, where each X_j is in $\text{ind } \mathcal{G}$. Firstly note that X cannot be an adic module, since the adic modules do not lie in $\varinjlim \mathcal{T}$. Next, for any object F in \mathcal{F} , we have

$$\text{Hom}(F, X) = \text{Hom}(F, \varinjlim X_j) \simeq \varinjlim \text{Hom}(F, X_j) = 0,$$

using Lemma 3.5(b), since each X_j lies in \mathcal{G} . It follows that if X is in $\text{ind } \mathcal{T}$, we have that X lies in $\text{ind } \mathcal{G}$. The only other possibility is if X is a Prüfer module. If Y is any indecomposable object in the corresponding ray in \mathcal{T} , we have an embedding $Y \rightarrow X$. Hence $\text{Hom}(F, Y) = 0$ for any object F in \mathcal{F} , so Y lies in \mathcal{G} . The result follows. \square

Proposition 4.14. (a) *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair of ray type. Then $(\mathcal{F}, \mathcal{G})$ can be written in the form $(\mathcal{F}_T, \mathcal{G}_T)$, where T is the direct sum of the indecomposable objects in*

$$(5) \quad \varinjlim(\text{ind } \mathcal{G}) \cap (\varinjlim(\text{ind } \mathcal{G}))^{\perp E}.$$

Compare [8, Thm. 1.5].

(b) *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair of coray type. Then $(\mathcal{F}, \mathcal{G})$ can be written in the form $(\mathcal{F}_T, \mathcal{G}_T)$, where T is the direct sum of the indecomposable objects in*

$$(6) \quad \varinjlim(\text{ind } \mathcal{F}) \cap {}^{\perp E}(\varinjlim(\text{ind } \mathcal{F})).$$

Proof. We only consider (a); the proof of (b) is similar. By Theorem 4.8 and Proposition 4.12(b) and its proof, there is a maximal rigid object T in $\overline{\mathcal{T}}$ of Prüfer type such that $(\mathcal{F}, \mathcal{G}) = (\mathcal{F}_T, \mathcal{G}_T)$ and $\mathcal{F}_T, \mathcal{G}_T$ have the following description. Let the Prüfer summands of T be $M_{i_r, \infty}$ for $r = 0, \dots, k-1$ where $0 \leq i_0 < i_1 < \dots < i_{k-1} \leq n-1$. Then

$$(\mathcal{F}_T, \mathcal{G}_T) = (\cup_{r=0}^{k-1} \mathcal{F}_r, (\cup_{r=0}^{k-1} \mathcal{G}'_r) \cup \mathcal{G}_\infty),$$

where $\mathcal{F}_r, \mathcal{G}_\infty$ are as in Proposition 4.12(b) and $\mathcal{G}'_r = \text{Cogen}_{\mathcal{W}_{i_r, i_{r+1}}} T_r$, where $T|_{\mathcal{T}} = \coprod_{r=0}^{k-1} T_r$ with T_r in $\mathcal{W}_{i_r, i_{r+1}}$.

By (2) in Section 2,

$$\begin{aligned} \text{ind } T_r &= \{X \in \text{ind } \mathcal{G}'_r : \text{Ext}^1(Y, X) = 0 \text{ for all } Y \in \mathcal{G}'_r\} \\ &= \{X \in \text{ind } \mathcal{G}'_r : \text{Ext}^1(Y, X) = 0 \text{ for all } Y \in \text{ind } \mathcal{G}\}. \end{aligned}$$

Note that $\mathcal{G}_\infty = \text{add}(\cup_{r=0}^{k-1} \mathcal{R}_{i_r}) \subset \text{ind } \mathcal{G}$. It follows that any object X in \mathcal{T} which satisfies $\text{Ext}^1(Y, X) = 0$ for all Y in $\text{ind } \mathcal{G}$ must lie inside some wing $\mathcal{W}_{i_r, i_{r+1}}$, and hence in some \mathcal{G}'_r . Therefore

$$\begin{aligned} \text{ind } T|_{\mathcal{T}} &= \text{ind}(\coprod_{r=0}^{k-1} T_r) \\ &= \{X \in \text{ind } \mathcal{G} : \text{Ext}^1(Y, X) = 0 \text{ for all } Y \in \text{ind } \mathcal{G}\}. \end{aligned}$$

Next, by Lemma 4.13, an indecomposable object in $\varinjlim(\text{ind } \mathcal{G})$ either lies in $\text{ind } \mathcal{G}$ or is the direct limit of a ray in \mathcal{G} , i.e. it is a Prüfer summand $M_{i_r, \infty}$ of T . If Y is one of the latter summands and X lies in $\text{ind } \mathcal{G}$, we have

$$D \text{Ext}^1(Y, X) \simeq \text{Hom}(X, \tau Y) = 0,$$

by Lemmas 3.6 and 3.7. Since the Prüfer summands themselves satisfy

$$\text{Ext}^1(M_{i_r, \infty}, M_{i_{r'}, \infty}) = 0$$

for all r, r' , the result follows. \square

5. GEOMETRIC INTERPRETATION IN THE TUBE CASE

5.1. Short exact sequences. Let $[i, j], [i', j']$ be arcs in $\mathcal{A}(\mathbb{U}(n))$, giving rise to arcs $\pi_n([i, j]), \pi_n([i', j'])$ in $\mathcal{A}(\mathbb{A}(n))$ with corresponding indecomposable objects $M_{ij}, M_{i'j'}$ in \mathcal{T} . Suppose that $I_{\mathbb{U}(n)}^-([i, j], [i' + kn, j' + kn])$ (i.e. the intersection number between arcs $[i, j]$ and $[i' + kn, j' + kn]$ in $\mathbb{U}(n)$) is equal to 1 for some k in \mathbb{Z} . Then, if $j' + kn > i + 1$, we have four objects in the Auslander-Reiten quiver of \mathcal{T} as shown in Figure 18(a); if $j' + kn = i + 1$, we have three objects as shown in Figure 18(b).

In the case $j' + kn > i + 1$, this corresponds to a non-split short exact sequence:

$$(7) \quad 0 \rightarrow M_{i', j'} \xrightarrow{f} M_{i'+kn, j} \amalg M_{i, j'+kn} \xrightarrow{g} M_{ij} \rightarrow 0,$$

and in the case $j' + kn = i + 1$, it corresponds to a short exact sequence:

$$(8) \quad 0 \rightarrow M_{i', j'} \xrightarrow{f} M_{i'+kn, j} \xrightarrow{g} M_{ij} \rightarrow 0$$

in \mathcal{T} . As in [27, Remark 4.25], these can be interpreted geometrically in $\mathbb{U}(n)$: see Figure 19.

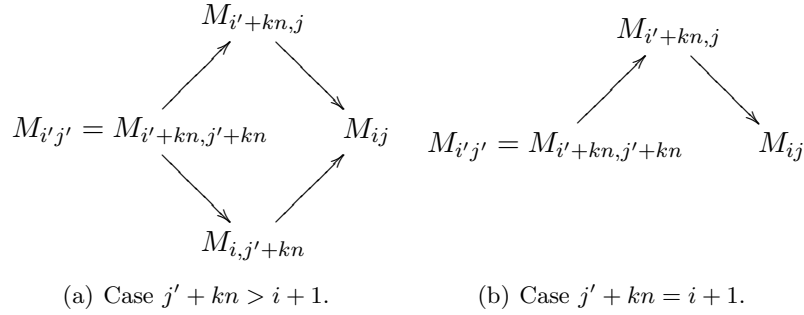


FIGURE 18. Objects and paths in the Auslander-Reiten quiver of \mathcal{T} corresponding to an intersection in $\mathbb{U}(n)$ between arcs $[i, j]$ and $[i' + kn, j' + kn]$.

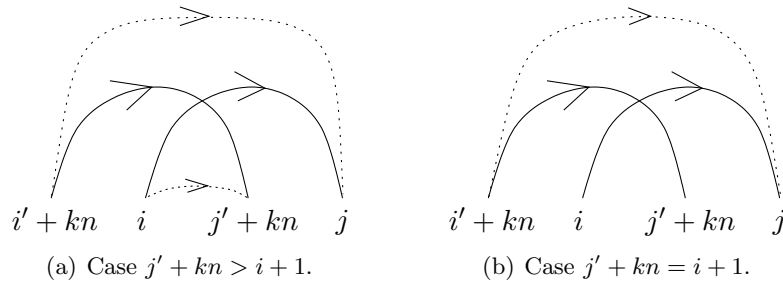


FIGURE 19. Non-split extensions in \mathcal{T} represented geometrically.

If $j' + kn > i + 1$, write $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = (g_1 \ g_2)$. Then f_1 and g_2 are monomorphisms and f_2 and g_1 are epimorphisms. The f_i and g_i are uniquely determined up to a choice of scalars. If $j' + kn = i + 1$, f is a monomorphism and g is an epimorphism, again uniquely determined up to a choice of scalars.

We note that Lemma 2.6 holds for \mathcal{T} (with the same proof).

Lemma 5.1. *Any non-split short exact sequence with first term $M_{i'j'}$ and last term M_{ij} has the same form as (7) or (8).*

Proof. Let

$$(9) \quad 0 \rightarrow M_{i'j'} \xrightarrow{u} E \xrightarrow{v} M_{ij} \rightarrow 0$$

be an arbitrary non-split short exact sequence with first term $M_{i'j'}$ and last term M_{ij} . By Lemma 2.6(b) we may write $E = E_1 \amalg E_2$ where E_1 is indecomposable and such that we have that, decomposing $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = (v_1, v_2)$, we have that u_1 a monomorphism. Since the sequence is not split, u_1 is not an isomorphism, so, denoting the length of an object M in \mathcal{T} by $\ell(M)$, we have

$$\ell(E_2) = \ell(M_{ij}) + \ell(M_{i'j'}) - \ell(E_1) < \ell(M_{ij}),$$

so v_1 must be an epimorphism by Lemma 2.6(c).

Case (i): Suppose first that $v_1 u_1 \neq 0$. If an integer k is such that $i' + kn \leq i$ and $i + 2 \leq j' + kn \leq j$, there is a homomorphism in \mathcal{T} from $M_{i'j'}$ to M_{ij} obtained by composing an epimorphism from $M_{i'j'}$ to $M_{i,j'+kn}$ with a monomorphism from $M_{i,j'+kn}$ to M_{ij} (i.e. the two maps in the lower edges of the diamond in Figure 18).

It is easy to check that the homomorphisms of this kind (allowing k to vary) form a basis of $\text{Hom}(M_{i'j'}, M_{ij})$. Since $vu = 0$ and $v_1 u_1 \neq 0$ and $v_1 u_1$ is a scalar multiple of such a basis element, there must be an indecomposable summand X of E_2 such that $v_X u_X$ is a scalar multiple of $v_1 u_1$ (where u_X, v_X are the corresponding components of u_2, v_2). But

$$(10) \quad \ell(X) \leq \ell(M_{ij}) + \ell(M_{i'j'}) - \ell(E_1),$$

and there is a unique path from $M_{i'j'}$ to M_{ij} through such an X giving rise to $g_1 f_1$ (i.e. with $X = M_{i,j'+kn}$) from which it follows that we have equality in (10) and thus that $E_2 = X$ is indecomposable and u_2 is an epimorphism and v_2 is a monomorphism. It follows that the short exact sequence (9) is of the form (7) up to a choice of scalars.

Case (ii): Now assume that $v_1 u_1 = 0$. This implies that

$$\ell(E_1) \geq \ell(M_{ij}) + \ell(M_{i'j'}),$$

but we also have

$$\ell(E_1) \leq \ell(E_1 \amalg E_2) = \ell(M_{ij}) + \ell(M_{i'j'}),$$

so we must have equality and $E = E_1$ is indecomposable. It follows that (9) is of the form (8) up to a choice of scalars. The proof is complete. \square

Definition 5.2. *We call a collection \mathcal{S} of arcs in $\mathcal{A}(\mathbb{A}(n))$ an oriented Ptolemy diagram (in $\mathbb{A}(n)$) if, whenever $\pi_n([i, j])$ and $\pi_n([i', j'])$ lie in \mathcal{S} with $i' < i < j' < j$ then $\pi_n([i, j'])$ (when $j' > i + 1$) and $\pi_n([i', j])$ also lie in \mathcal{S} . (Compare with Definition 2.5 in the type A case).*

Corollary 5.3. *A collection of indecomposable objects in \mathcal{T} is closed under extensions if and only if the corresponding collection of arcs is an oriented Ptolemy diagram in $\mathbb{A}(n)$.*

The following is easy to check (see Lemma 2.3 for the type A case).

- Lemma 5.4.**
- (a) *Let $\pi_n([i, j])$ be an arc in $\mathcal{A}(\mathbb{A}(n))$. Then the indecomposable quotients of M_{ij} are the $M_{i'j}$ where $i \leq i' \leq j - 2$, i.e. arcs in $\mathbb{A}(n)$ corresponding to arcs in $\mathbb{U}(n)$ with the same ending point and with starting point weakly to the right of i . Call these the left-shortenings of $\pi_n([i, j])$.*
 - (b) *Let $\pi_n([i, j])$ be an arc in $\mathcal{A}(\mathbb{A}(n))$. Then the indecomposable submodules of M_{ij} are the $M_{i'j}$ where $i + 2 \leq j' \leq j$, i.e. arcs in $\mathbb{A}(n)$ corresponding to arcs in $\mathbb{U}(n)$ with the same starting point and with ending point weakly to the left of j . Call these the right-shortenings of $\pi_n([i, j])$.*

We make the following remark, which follows from Proposition 3.8. Recall that $\tilde{\mathcal{A}}(\mathbb{A}(n))$ denotes $\mathcal{A} = \mathcal{A}(\mathbb{A}(n))$ extended to include the homotopy classes of the arcs $\pi_n([i, \infty])$ and $\pi_n([-\infty, i])$.

Remark 5.5. *The bijection $\psi : \pi_n([i, j]) \mapsto M_{ij}$ between $\widetilde{\mathcal{A}}(\mathbb{A}(n))$ and $\text{ind}(\overline{\mathcal{T}})$ induces a bijection between maximal noncrossing collections of arcs in $\mathbb{A}(n)$ (including the infinite arcs) and maximal rigid objects in $\overline{\mathcal{T}}$.*

We can now describe the conditions on collections of arcs appearing in torsion pairs in \mathcal{T} .

- Proposition 5.6.** (a) *A collection \mathcal{S} of arcs in $\mathcal{A}(\mathbb{A}(n))$ corresponds to the torsion part of a torsion pair in \mathcal{T} if and only if*
- (i) *\mathcal{S} is an oriented Ptolemy diagram in $\mathbb{A}(n)$.*
 - (ii) *\mathcal{S} is closed under left-shortening.*
- (b) *A collection \mathcal{S} of arcs in $\mathcal{A}(\mathbb{A}(n))$ corresponds to the torsion-free part of a torsion pair in \mathcal{T} if and only if*
- (i) *\mathcal{S} is an oriented Ptolemy diagram in $\mathbb{A}(n)$.*
 - (ii) *\mathcal{S} is closed under right-shortening.*

Proof. This follows from Lemmas 1.2 and 5.4 and Corollaries 2.7 and 5.3. \square

We also note that Proposition 4.12 has the following geometric interpretation:

Lemma 5.7. *Let T be a maximal rigid object in $\overline{\mathcal{T}}$.*

- (a) *If T is of Prüfer type, then $\psi(\text{ind } \mathcal{F}_T)$ can be obtained by taking the closure of the collection of arcs corresponding to finitely generated indecomposable summands of T under left shortening and rotating all resulting arcs one step to the right (adding one to their end-points). We obtain $\psi(\text{ind } \mathcal{G}_T)$ by taking the closure of the collection of arcs corresponding to finitely generated indecomposable summands of T under right shortening.*
- (b) *If T is of adic type, then $\psi(\text{ind } \mathcal{F}_T)$ can be obtained by taking the closure of the collection of arcs corresponding to finitely generated indecomposable summands of T under left shortening. We obtain $\psi(\text{ind } \mathcal{G}_T)$ by taking the closure of the collection of arcs corresponding to finitely generated indecomposable summands of T under right shortening and then translating all resulting arcs one step to the left (subtracting one from their end-points).*

Similarly, Proposition 4.14 has the following geometric interpretation:

Lemma 5.8. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair.*

- (a) *If \mathcal{G} generates \mathcal{T} , then $\psi(\varinjlim(\text{ind } \mathcal{G}))$ can be obtained from $\psi(\text{ind } \mathcal{G})$ by first adding any infinite arc $\pi_n([i, \infty])$ for which all arcs $\pi_n([i, j])$ for $j \geq a$ for some a lie in $\psi(\text{ind } \mathcal{G})$. Then T is the direct sum of the indecomposable objects corresponding to the arcs α in $\psi(\varinjlim(\text{ind } \mathcal{G}))$ such that the pair (β, α) of arcs has no negative intersections for all β in $\psi(\varinjlim(\text{ind } \mathcal{G}))$.*
- (b) *If \mathcal{F} cogenerates \mathcal{T} , then $\psi(\varprojlim(\text{ind } \mathcal{F}))$ can be obtained from $\psi(\text{ind } \mathcal{F})$ by first adding any infinite arc $\pi_n([-\infty, i])$ for which all arcs $\pi_n([j, i])$ for $j \leq a$ for some a lie in $\psi(\text{ind } \mathcal{F})$. Then T is the direct sum of the indecomposable objects corresponding to the arcs α in $\psi(\varprojlim(\text{ind } \mathcal{F}))$ such that the pair (α, β) of arcs has no negative intersections for all β in $\psi(\varprojlim(\text{ind } \mathcal{F}))$.*

Finally, we give an example in a tube of rank $n = 14$ to illustrate Proposition 4.12 and the results in this section. The arcs corresponding to the indecomposable direct summands of T are displayed in Figure 20 (only the beginnings of the infinite arcs are shown). Note that the Prüfer modules which are indecomposable summands of T are $M_{0,\infty}, M_{6,\infty}, M_{10,\infty}$ and $M_{13,\infty}$, so $i_0 = 0, i_1 = 6, i_2 = 10$ and $i_3 = 13$.

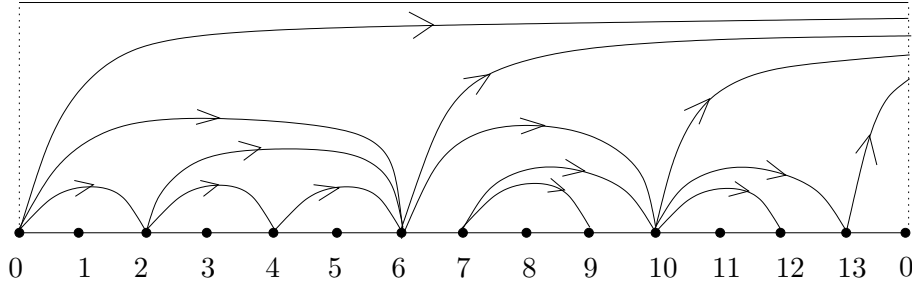


FIGURE 20. The maximal rigid object T

The arcs corresponding to the indecomposable objects in \mathcal{F}_T are displayed in Figure 21.

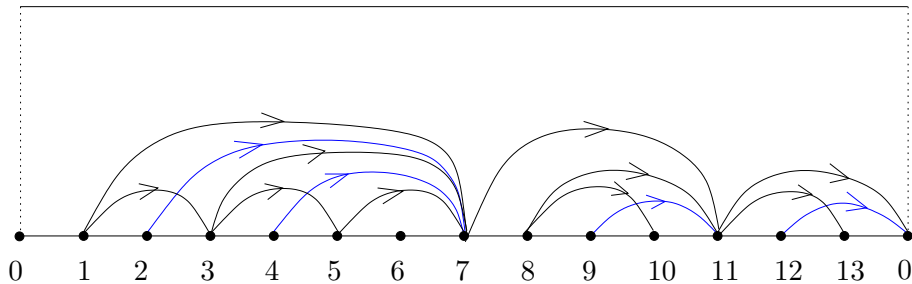


FIGURE 21. The torsion part, $\mathcal{F}_T = \tau^{-1}(\text{Gen } T \cap \mathcal{T})$, of the torsion theory corresponding to T . The arcs corresponding to indecomposable objects not in $\tau^{-1}(\text{add } T \cap \mathcal{T})$ are drawn in blue.

The arcs corresponding to the indecomposable objects in the $\mathcal{G}_T \cap \mathcal{W}_{i_r, i_{r+1}+1}$ are displayed in Figure 22 (with dotted arcs indicating the indecomposable summands of T which are not in \mathcal{T} (or $\text{ind } \mathcal{G}$)). Note that there are infinitely many additional arcs not displayed, corresponding to indecomposables in \mathcal{G}_∞ but not in any of the \mathcal{G}_r . The missing arcs are $\pi_{14}([0, j])$ for $j \geq 8$, $\pi_{14}([6, j])$ for $j \geq 12$, $\pi_{14}([10, j])$ for $j \geq 15$ and $\pi_{14}([13, j])$ for $j \geq 16$. Note that, as indicated by Proposition 4.12, the intersections of \mathcal{F}_T and \mathcal{G}_T with the wings $\mathcal{W}_{i_r, i_{r+1}+1}$ (which are $\mathcal{W}_{0,7}, \mathcal{W}_{6,11}, \mathcal{W}_{10,14}$ and $\mathcal{W}_{13,15}$) are torsion pairs.

In Figure 23, we show the indecomposable summands of \mathcal{T} and the indecomposable objects in \mathcal{F}_T and \mathcal{G}_T in the AR-quiver of the tube.

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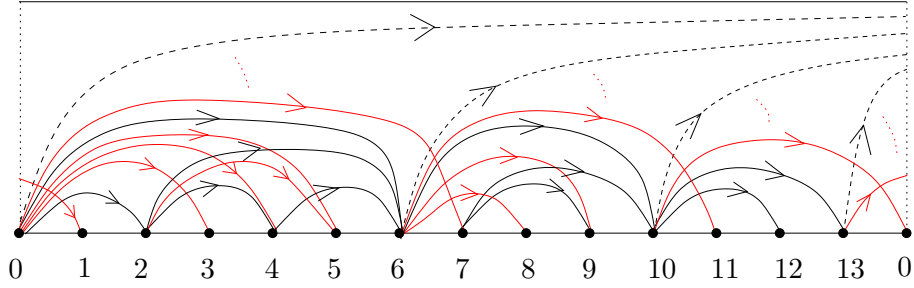


FIGURE 22. The torsion-free part, $\mathcal{G}_T = \text{Cogen } T \cap \mathcal{T}$, of the torsion theory corresponding to T . The arcs not in $\text{add } T$ are drawn in red. The dashed arcs are the indecomposable summands of T which are not of finite length (and thus not in \mathcal{G}_T). The arcs $\pi_{14}([0, j])$ for $j \geq 8$, $\pi_{14}([6, j])$ for $j \geq 12$, $\pi_{14}([10, j])$ for $j \geq 15$ and $\pi_{14}([13, j])$ for $j \geq 16$ have been omitted for clarity.

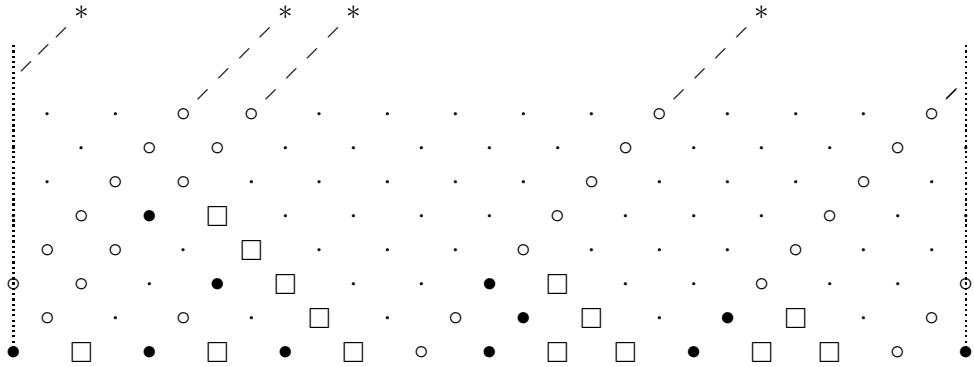


FIGURE 23. The AR-quiver of the tube, showing the indecomposable summands of T (\bullet or $*$), $\text{ind}(\mathcal{F}_T)$ (\square) and $\text{ind}(\mathcal{G}_T)$ (\circ or \bullet). The Prüfer direct summands of T are shown (symbolically) at the top of the diagram as asterisks.

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INSTITUT FÜR MATHEMATIK UND WISSENSCHAFTLICHES RECHNEN, UNIVERSITÄT GRAZ, HEINRICHSTRASSE 36, A-8010 GRAZ, AUSTRIA
E-mail address: `baurk@uni-graz.at`

DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, N-7491 TRONDHEIM, NORWAY
E-mail address: `aslakb@math.ntnu.no`

SCHOOL OF MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, ENGLAND
E-mail address: `marsh@maths.leeds.ac.uk`