

CATEGORIFICATION OF A FRIEZE PATTERN DETERMINANT

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ABSTRACT. Broline, Crowe and Isaacs have computed the determinant of a matrix associated to a Conway-Coxeter frieze pattern. We generalise their result to the corresponding frieze pattern of cluster variables arising from the Fomin-Zelevinsky cluster algebra of type A . We give a representation-theoretic interpretation of this result in terms of certain configurations of indecomposable objects in the root category of type A .

1. INTRODUCTION

Consider a generic $2 \times n$ matrix, i.e. a matrix

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

whose entries are indeterminates. For a choice of two columns of X , $1 \leq i, j \leq n$, let

$$\Delta_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

be the corresponding minor of X (so $\Delta_{ii} = 0$). Let A be the symmetric matrix with entries A_{ij} given by:

$$A_{ij} = \begin{cases} \Delta_{ij} & \text{if } i \leq j; \\ \Delta_{ji} & \text{if } i > j. \end{cases}$$

Our main result is the following.

Theorem 1.1.

$$\det(A) = (-2)^{n-2} \Delta_{12} \Delta_{23} \cdots \Delta_{n-1,n} \Delta_{n1}.$$

Our motivation comes from a result of Broline, Crowe and Isaacs [BCI] concerning *frieze patterns* of integers. Theorem 1.1 can be regarded as a generalisation of this result, which we now describe.

Frieze patterns of integers in the plane were considered in [Cox, CoCox1, CoCox2] by Conway and Coxeter. Such a frieze pattern consists of a finite number of infinite rows of integers, with each row interlacing its neighbouring rows, and satisfies

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the *unimodular rule*, which states that for every four adjacent numbers forming a square:

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

the relation $ad - bc = 1$ is satisfied. The entries in the first and last row are zero; the entries in the second and penultimate rows are 1, and all other entries should be positive.

Fix an integer $n \geq 3$. A frieze pattern is said to be of *order* n if it has $n + 1$ rows. We fix a regular n -sided polygon \mathcal{P}_n , with vertices $1, 2, \dots, n$ numbered in cyclic order, arranged clockwise around the boundary (we work with the vertices modulo n , with representatives in $\{1, 2, \dots, n\}$). In [CoCox1, CoCox2] it is shown that a frieze pattern can be obtained from a triangulation π of \mathcal{P}_n in the following way. For each pair of integers $i, j \in \{1, 2, \dots, n\}$, define an integer m_{ij} as follows. Set $m_{ii} = 0$ and $m_{i, i+1} = 1$ for all i . Then let $m_{i-1, i+1}$ be the number of triangles in π incident with vertex i . Define m_{ij} for all $i < j$ inductively using the formula

$$(1) \quad m_{i-1, j+1} = \frac{m_{i-1, j} m_{i, j+1} - 1}{m_{ij}}.$$

Then set $m_{ji} = m_{ij}$ for all $i < j$. The numbers m_{ij} are all positive integers and, when arranged as in Figure 1, form a frieze pattern. Furthermore, every frieze pattern of order n arises from a triangulation of \mathcal{P}_n in this way.

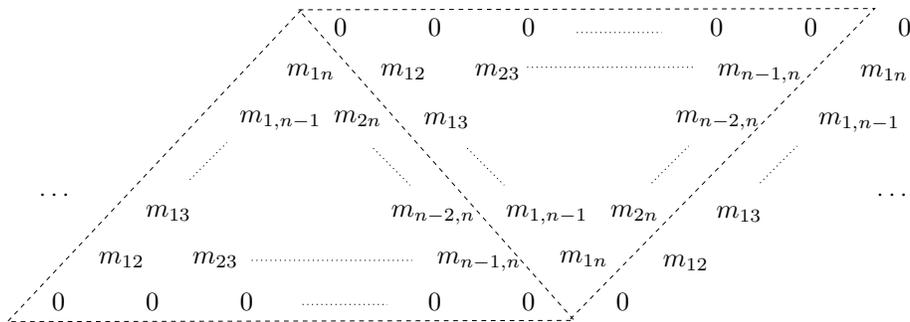


FIGURE 1. Rule for writing out a frieze pattern

Note that the entries lying in a triangle below $m_{1, n}$ form a fundamental domain for a glide reflection preserving the pattern. This fundamental domain is indicated by a dashed line triangle (together with its shift to the right).

For example, the frieze pattern corresponding to the triangulation in Figure 2 is shown in Figure 3. The middle triangle indicates the fundamental domain mentioned above; its images under the glide reflection and its inverse are also displayed.

The result of Broline, Crowe and Isaacs can be stated as follows.

Theorem 1.2. [BCI] *Let $n \geq 3$, let π be a triangulation of \mathcal{P}_n and let $M(\pi) = (m_{ij})$ be the symmetric matrix defined above. Then $\det(M) = -(-2)^{n-2}$.*

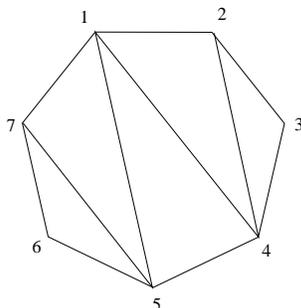


FIGURE 2. A triangulation of \mathcal{P}_7

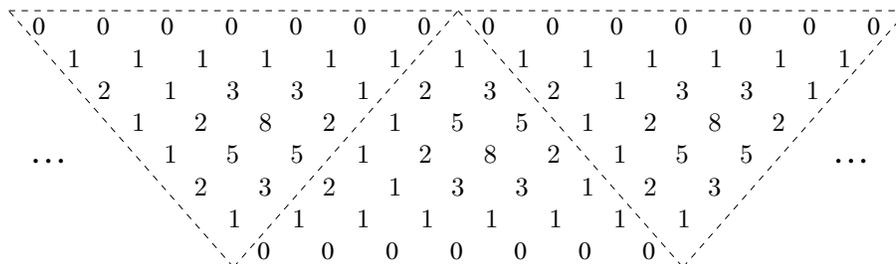


FIGURE 3. The frieze pattern corresponding to the triangulation in Figure 2.

For the example in Figure 2, we have

$$M(\pi) = \begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 2 & 5 & 3 \\ 2 & 1 & 0 & 1 & 3 & 8 & 5 \\ 1 & 1 & 1 & 0 & 1 & 3 & 2 \\ 1 & 2 & 3 & 1 & 0 & 1 & 1 \\ 2 & 5 & 8 & 3 & 1 & 0 & 1 \\ 1 & 3 & 5 & 2 & 1 & 1 & 0 \end{pmatrix}$$

which has determinant $-(-2)^5 = 32$. In addition, a geometric interpretation of all of the entries in the frieze pattern in terms of π is given in [BCI].

We note that frieze patterns of integers of various kinds have been studied recently; see, for example [ARS, BM2, BR, G, KS, M-G, MOT, P].

A connection between cluster algebras and frieze patterns was established in the article [CaCh], which showed that the frieze patterns above can be obtained from cluster algebras of type A .

The homogeneous coordinate ring of the Grassmannian of 2-planes in a n -dimensional vector space is a cluster algebra of type A_{n-3} [FZ2, 12.6] (see also [FZ1, §1]). The cluster variables are in bijection with the diagonals of \mathcal{P}_n . If i, j are the end-points of such a diagonal, we write u_{ij} for the corresponding cluster variable (so $u_{ij} = u_{ji}$). It has stable variables u_{ij} where i, j are the end-points of a boundary edge. We set $u_{ii} = 0$ for all i and $u_{ij} = u_{ji}$ for $i > j$.

Via the above bijection, the clusters are in bijection with the triangulations of \mathcal{P}_n . We fix such a triangulation π and corresponding cluster. By the Laurent Phenomenon [FZ1, 3.1], each cluster variable can be written as a Laurent polynomial in the elements of the cluster with the coefficients of the polynomial given by polynomials in the stable variables.

When the variables in the cluster and the stable variables are specialised to 1, the resulting integers, when arranged correctly, produce the corresponding frieze pattern.

Theorem 1.1 can be reinterpreted in terms of this cluster algebra as follows. In this way we see that it is in fact a generalisation of Theorem 1.2.

Theorem 1.3. *Let π be a triangulation of \mathcal{P}_n . Let $U(\pi) = (u_{ij})$. Then*

$$\det(U(\pi)) = -(-2)^{n-2} u_{12} u_{23} \cdots u_{n-1, n} u_{n1}.$$

We go on to show that this result can be given a categorical interpretation in terms of the root category of type A_{n-1} . By interpreting this category as a category of oriented edges between vertices of \mathcal{P}_n (using methods similar to those in [CaChS]) we show that the above determinant can be reinterpreted as a sum over configurations of indecomposable objects in the root category. Each configuration is a maximal collection of indecomposable objects such that no object lies in the frame of the other (see Section 7 for the definition of frame) which is also of maximal cardinality.

2. A DETERMINANTAL RESULT

In this section, we prove our main result:

Theorem 2.1. *Let $A = (A_{ij})$ be the matrix of minors of X defined above. Then*

$$\det(A) = (-2)^{n-2} \Delta_{12} \Delta_{23} \cdots \Delta_{n-1, n} \Delta_{n1}.$$

Proof. The key point is that the minors Δ_{ij} satisfy the Plücker relations, i.e., whenever $1 \leq p < q < r < s \leq n$ are vertices of \mathcal{P}_n , we have:

$$(2) \quad \Delta_{pq} \Delta_{rs} + \Delta_{qr} \Delta_{ps} = \Delta_{pr} \Delta_{qs}.$$

Noting that A is symmetric, it follows that, whenever i, j, k, l are arranged clockwise around the boundary of \mathcal{P}_n , we have:

$$(3) \quad A_{ij} A_{kl} + A_{jk} A_{il} = A_{ik} A_{jl}.$$

We use induction on n , showing that the Plücker relations are sufficient to imply the result. For $n = 3$ we have:

$$A = \begin{pmatrix} 0 & \Delta_{12} & \Delta_{13} \\ \Delta_{12} & 0 & \Delta_{23} \\ \Delta_{13} & \Delta_{23} & 0 \end{pmatrix}$$

which has determinant $-2\Delta_{12}\Delta_{23}\Delta_{31}$ as required. Now suppose that $n \geq 4$ and that the result is true for $n - 1$. Fix vertices a, b of \mathcal{P}_n such that b is distinct from $a - 1, a, a + 1$. From the quadrilateral in Figure 4 we have the Plücker relation:

$$A_{a-1, a} A_{a+1, b} + A_{a, a+1} A_{a-1, b} = A_{a-1, a+1} A_{ab}.$$

We note that $A_{a-1, a+1} \neq 0$. Thus it follows from the above that:

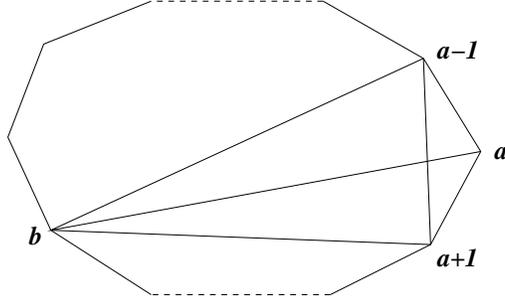
$$\frac{A_{a-1, a}}{A_{a-1, a+1}} A_{a+1, b} + \frac{A_{a, a+1}}{A_{a-1, a+1}} A_{a-1, b} = A_{ab}.$$

Let R_a denote the a th row of A . From the above, it follows that if we replace R_a with the row:

$$R_a - \frac{A_{a-1, a}}{A_{a-1, a+1}} R_{a+1} - \frac{A_{a, a+1}}{A_{a-1, a+1}} R_{a-1},$$

then the a, b entry A'_{ab} in the new matrix A' will be 0 for any $b \neq a - 1, a$ or $a + 1$. Noting that $A_{a-1, a-1} = 0$, we also have:

$$A'_{a, a-1} = A_{a, a-1} - \frac{A_{a-1, a}}{A_{a-1, a+1}} A_{a+1, a-1} - \frac{A_{a, a+1}}{A_{a-1, a+1}} A_{a-1, a-1} = 0,$$


 FIGURE 4. A quadrilateral in \mathcal{P}_n

and, similarly, $A'_{a,a+1} = 0$, while

$$A'_{aa} = A_{aa} - \frac{A_{a-1,a}}{A_{a-1,a+1}} A_{a+1,a} - \frac{A_{a,a+1}}{A_{a-1,a+1}} A_{a-1,a} = \frac{-2A_{a-1,a}A_{a,a+1}}{A_{a-1,a+1}}.$$

Expanding along the a th row, we obtain that the determinant of A is

$$(4) \quad \det(A) = (-1)^{2a} \frac{(-2)\Delta_{a-1,a}\Delta_{a,a+1}}{\Delta_{a-1,a+1}} \det(A(a)),$$

where $A(a)$ denotes the matrix obtained by removing the a th row and the a th column from A . Note that the entries of $A(a)$ satisfy the relations (3) for the polygon \mathcal{P}_{n-1} with vertices parametrised clockwise using the numbers $\{1, 2, \dots, n\} \setminus \{a\}$. By the induction hypothesis,

$$\det(A(a)) = (-2)^{n-3} \Delta_{a-1,a+1} \Delta_{12} \Delta_{23} \cdots \hat{\Delta}_{a-1,a} \hat{\Delta}_{a,a+1} \cdots \Delta_{n-1,n} \Delta_{n1},$$

where the hats indicate omission. Combining this with equation (4) we obtain that

$$\det(A) = (-2)^{n-2} \Delta_{12} \Delta_{23} \cdots \Delta_{n-1,n} \Delta_{n1},$$

as required. \square

Note that the identity in Theorem 2.1 can be restated as:

$$(5) \quad \det(A) = -(-2)^{n-2} A_{12} A_{23} \cdots A_{n-1,n} A_{n1}$$

since $A_{n1} = \Delta_{1n} = -\Delta_{n1}$.

3. FRIEZE PATTERNS AND CLUSTER ALGEBRAS

We consider a cluster algebra \mathcal{A} of type A_{n-3} associated to \mathcal{P}_n , defined over the complex numbers. This cluster algebra appears in [FZ1, §1] and is described in detail in [FZ3, 12.2]; see alternatively [P, 3.2] (using a perfect matching model for frieze patterns due to Gabriel Carroll and Gregory Price; see [loc. cit.] for details).

Let π be a triangulation of \mathcal{P}_n and let \mathbb{F} be the field of rational functions in the variables u_{ij} where i, j are the end-points of a diagonal in π or a boundary edge of \mathcal{P}_n (regarding u_{ij} and u_{ji} as equal). Define elements $u_{ij} \in \mathbb{F}$, for i, j the end-points of an arbitrary diagonal of \mathcal{P}_n , inductively as follows (again with $u_{ij} = u_{ji}$). If i, j, k, l are vertices of \mathcal{P}_n , arranged clockwise around the boundary, and $u_{ij}, u_{jk}, u_{kl}, u_{li}$ and u_{ik} are defined but u_{jl} is not, define u_{jl} by the following *exchange relation*:

$$(6) \quad u_{ij} u_{kl} + u_{jk} u_{il} = u_{ik} u_{jl}.$$

It turns out that the elements u_{ij} are well-defined. The cluster algebra \mathcal{A} is the \mathbb{C} -subalgebra of \mathbb{F} generated by the u_{ij} for i, j the end-points of any diagonal or boundary edge of \mathcal{P}_n . The generators corresponding to diagonals are known as

cluster variables and those corresponding to boundary edges are known as *stable variables*. For $n = 3$ there are no cluster variables, only stable variables. The exchange relation (6) holds for any choice of i, j, k, l arranged clockwise around the boundary of \mathcal{P}_n .

The diagonals in a triangulation of \mathcal{P}_n determine a corresponding set of cluster variables of \mathcal{A} known as a *cluster*. The cluster corresponding to π is the *initial cluster*. By [FZ1, 3.1], the cluster variables can be written as Laurent polynomials in any fixed cluster, with coefficients given by polynomials in the stable variables.

The cluster algebra \mathcal{A} is independent (up to isomorphism) of the choice of π . In fact, it is isomorphic to the homogeneous coordinate ring $\mathbb{C}[Gr_2(n)]$ of the Grassmannian of 2-dimensional subspaces of an n -dimensional vector space; see [FZ2, 12.6]. Such a subspace can be described by a $2 \times n$ matrix, with rows given by a choice of spanning vectors, and the 2×2 minors of the matrix give homogeneous coordinates on the Grassmannian.

The homogeneous coordinate ring is generated by these minors, subject to the Plücker relations, and under the isomorphism, u_{ij} , for $i < j$, maps to the minor associated to columns i and j of the matrix. The exchange relations above map to the Plücker relations.

Fix a triangulation π of \mathcal{P}_n . Setting $u_{ii} = 0$ for all i , we consider the symmetric matrix $U(\pi) = (u_{ij})$, where the u_{ij} are regarded as Laurent polynomials in the u_{ij} for i, j the end-points of an edge in π with coefficients given by polynomials in the stable variables.

We note the following:

Proposition 3.1. [CaCh, 5.2] (see also [P, §2,§3]) *If the u_{ij} , for i, j the end-points of a diagonal in π , and the stable variables are all specialised to 1, then $U(\pi)$ becomes the matrix $M(\pi)$ defined above.*

Caldero-Chapoton prove this by first showing that, for each i , $u_{i-1, i+1}$ specialises to the number of triangles in π incident with vertex i , which coincides with $m_{i-1, i+1}$ in the above definition. The result then follows from a comparison of formulas (1) and (6).

We can now restate Theorem 2.1 in this context. The proof of this version is the same, using the exchange relations (6) (and noting equation (5)).

Theorem 3.2. *Let π be a triangulation of \mathcal{P}_n . Let $U(\pi) = (u_{ij})$ be the matrix defined above. Then*

$$\det(U(\pi)) = -(-2)^{n-2} u_{12} u_{23} \cdots u_{n-1, n} u_{n1}.$$

We have thus interpreted Theorem 2.1 as a generalisation of Theorem 1.2.

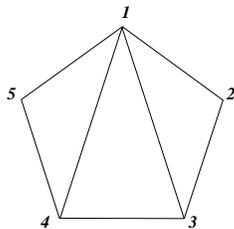
4. AN EXAMPLE

Here we give an example of the result in the previous section. Let π be the triangulation of a pentagon shown in Figure 5. Then the corresponding matrix is given by:

$$U(\pi) = \begin{pmatrix} 0 & u_{12} & u_{13} & u_{14} & u_{15} \\ u_{12} & 0 & u_{23} & \frac{u_{12}u_{34} + u_{23}u_{14}}{u_{13}} & v \\ u_{13} & u_{23} & 0 & u_{34} & \frac{u_{13}u_{45} + u_{15}u_{34}}{u_{14}} \\ u_{14} & \frac{u_{12}u_{34} + u_{23}u_{14}}{u_{13}} & u_{34} & 0 & u_{45} \\ u_{15} & v & \frac{u_{13}u_{45} + u_{15}u_{34}}{u_{14}} & u_{45} & 0 \end{pmatrix}$$

where

$$v = \frac{u_{12}u_{13}u_{45} + u_{12}u_{15}u_{34} + u_{14}u_{15}u_{23}}{u_{13}u_{14}}.$$


 FIGURE 5. A triangulation of \mathcal{P}_5 .

By Theorem 3.2, we have that

$$\det(u) = 8u_{12}u_{23}u_{34}u_{45}u_{51}.$$

5. A GEOMETRIC MODEL OF THE ROOT CATEGORY

Let $n \geq 3$ be an integer and let Q be a quiver of type A_{n-1} . Let k be an algebraically closed field. Let $D^b(kQ)$ denote the bounded derived category of modules over kQ , with shift functor $[1]$. Let $\mathcal{R}_n = D^b(kQ)/[2]$ denote the quotient of $D^b(kQ)$ by the square $[2]$ of the shift. In this section we shall exhibit a geometric construction of this category (along the lines of [CaChS]). We remark that this category is sometimes referred to as the *root category* (of type A) since its objects can be put into one-to-one correspondence with the roots in the corresponding root system (by Gabriel's Theorem). It was considered in [H1].

We now consider *oriented* edges between vertices of \mathcal{P}_n , denoting the edge oriented from i to j by $[i, j]$, for any $1 \leq i, j \leq n$ with $i \neq j$ (thus boundary edges are included).

Recall that a *stable translation quiver* is a pair (Γ, τ) where Γ is a locally finite quiver and $\tau : \Gamma_0 \rightarrow \Gamma_0$ is a bijection such that for any $X, Y \in \Gamma_0$, the number of arrows from X to Y is the same as the number of arrows from $\tau(Y)$ to X .

Let $\Gamma = \Gamma(n)$ be the quiver defined as follows. The set of vertices, Γ_0 , is the set of all possible oriented edges between distinct vertices of \mathcal{P}_n as above. The arrows, Γ_1 , are of the form $[i, j] \rightarrow [i, j+1]$ and $[i, j] \rightarrow [i+1, j]$ (where $j+1$ is interpreted as 1 if $j = n+1$ and similarly for $i+1$), whenever $[i, j]$ and $[i, j+1]$ (respectively, $[i, j]$ and $[i+1, j]$) are vertices of Γ . Thus an arrow comes from rotating an oriented edge clockwise about one of its end-points so that the other end-point moves to an adjacent vertex on the boundary of \mathcal{P}_n .

Let τ be the automorphism of Γ obtained by rotating \mathcal{P}_n through $2\pi/n$ anticlockwise; thus $\tau([i, j]) = [i-1, j-1]$.

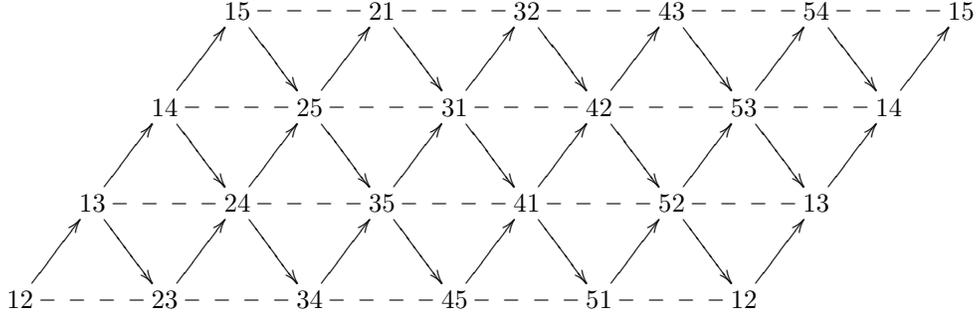
Lemma 5.1. *The pair (Γ, τ) is a stable translation quiver.*

The proof is as in [BM1, 2.2]: note that this proof also works in the oriented case we have here.

Example 5.2. *We consider the case when $n = 5$, so \mathcal{P}_n is a pentagon. The translation quiver $(\Gamma(5), \tau)$ is given in Figure 6.*

By [PX, 2.3] (see also [K, 9.9]), \mathcal{R}_n is a triangulated category, and, by [BMRRT, 1.3], it has Auslander-Reiten triangles and its Auslander-Reiten quiver, $\Gamma(\mathcal{R}_n)$ is the quotient of the Auslander-Reiten quiver of $D^b(kQ)$ by the automorphism induced by $[2]$. We have the following:

Proposition 5.3. *The translation quiver $\Gamma(n)$ is isomorphic to $\Gamma(\mathcal{R}_n)$.*

FIGURE 6. The translation quiver $\Gamma(5)$.

Proof. Suppose that Q is a linearly oriented quiver of type A_{n-1} , with arrows $i \leftarrow i+1$, $1 \leq i \leq n-2$. Then, up to isomorphism, the indecomposable modules for kQ are of the form M_{ij} , $1 \leq i < j \leq n$, where M_{ij} has socle S_i (the simple module corresponding to vertex i) and length $j-i$. So $M_{i,i+1} = S_i$. For $i < j$, we also set $M_{ji} = M_{ij}[1]$. Then the map $[i, j] \mapsto M_{ij}$, for $1 \leq i, j \leq n$, $i \neq j$, gives a bijection between oriented edges between vertices of \mathcal{P}_n and isomorphism classes of indecomposable objects of \mathcal{R}_n . The fact that this is an isomorphism of translation quivers follows from the description of the Auslander-Reiten quiver of $D^b(kQ)$ in [H2]. We just need to check that the mesh beginning at corresponding vertices is the same in each quiver. The only non-trivial cases are the meshes beginning with M_{ij} where $j = n$ or $i = n$. In the first case, the mesh in $\Gamma(\mathcal{R}_n)$ is:

$$\begin{array}{ccc}
 & M_{i,1} = P_{i-1}[1] & \\
 & \nearrow & \searrow \\
 M_{i,n} = I_i & \text{-----} & M_{i+1,1}[1] = P_i[1] \\
 & \searrow & \nearrow \\
 & M_{i+1,n} = I_{i+1} &
 \end{array}
 ,$$

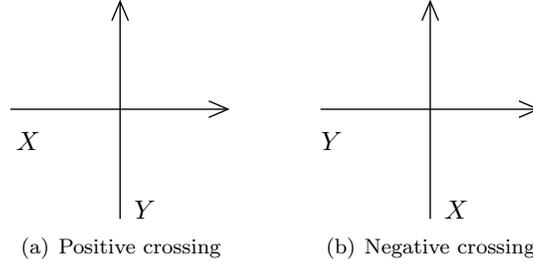
and in the second case, the mesh in $\Gamma(\mathcal{R}_n)$ is:

$$\begin{array}{ccc}
 & M_{n,i+1} = I_{i+1}[1] & \\
 & \nearrow & \searrow \\
 M_{n,i} = I_i[1] & \text{-----} & M_{1,i+1} = P_i \\
 & \searrow & \nearrow \\
 & M_{1,i} = P_{i-1} &
 \end{array}
 ,$$

noting that in \mathcal{R}_n , $X \cong X[2]$ for any object X . These meshes are the images of the corresponding meshes in $\Gamma(n)$, so we are done. \square

Remark 5.4. We note that the induced subquiver of $\Gamma(n)$ on vertices of form $[i, j]$ with $i < j$ (with $\tau([i, j])$ undefined if $i = 1$) is isomorphic to the Auslander-Reiten quiver of $kQ - \text{mod}$.

We note that, as for the cluster category (see [BMRRT, §1]), the category \mathcal{R}_n is standard. We thus have the following corollary of Proposition 5.3, giving a geometric realisation of \mathcal{R}_n .


 FIGURE 7. The two types of crossing between X and Y .

Corollary 5.5. *The root category \mathcal{R}_n is equivalent to the additive hull of the mesh category of $\Gamma(n)$.*

We shall identify indecomposable objects in \mathcal{R}_n , up to isomorphism, with the corresponding oriented edges between vertices of \mathcal{P}_n , in the sequel, and we shall freely switch between objects and oriented edges between vertices in \mathcal{P}_n .

6. DIMENSIONS OF EXTENSION GROUPS

In this section we indicate how the dimensions of Ext^1 -groups between indecomposable objects of \mathcal{R}_n can be read off from the geometric model. We fix i, j with $1 \leq i, j \leq n$, $i \neq j$, and consider the corresponding indecomposable object $[i, j]$.

Consider the two rectangles $R_B = R_B(i, j)$, with corners $[j, i]$, $[j, j-1]$, $[i-1, j-1]$ and $[i-1, i]$, and $R_F = R_F(i, j)$, with corners $[i+1, j+1]$, $[i+1, i]$, $[j, i]$ and $[j, j+1]$, in \mathcal{R}_n .

Note that the Auslander-Reiten formula holds in $D^b(kQ)$. This states that, for two indecomposable objects X, Y in $D^b(kQ)$,

$$\text{Ext}_{D^b(kQ)}^1(X, Y) \cong D \text{Hom}_{D^b(kQ)}(Y, \tau X),$$

where $D = \text{Hom}_k(-, k)$. This passes down to the root category \mathcal{R}_n and it follows that the dimensions of $\text{Ext}_{\mathcal{R}_n}^1(X, Y)$ and $\text{Hom}_{\mathcal{R}_n}(Y, \tau X)$ coincide for any two indecomposable objects X, Y in \mathcal{R}_n .

Using [B] or the mesh relations and the Auslander-Reiten formula in \mathcal{R}_n directly, we see that:

Lemma 6.1. *Let X and Y be indecomposable objects in \mathcal{R}_n . Then:*

- (a) *The space $\text{Ext}_{\mathcal{R}_n}^1([i, j], Y)$ is non-zero if and only if Y lies in R_B . If it is non-zero then it is one-dimensional.*
- (b) *The space $\text{Ext}_{\mathcal{R}_n}^1(X, [i, j])$ is non-zero if and only if X lies in R_F . If it is non-zero then it is one-dimensional.*

As an example, we show in Figure 8 (by underlining) those indecomposable objects Y such that $\text{Ext}_{\mathcal{R}_5}^1([3, 1], Y) \neq 0$ for the case $n = 5$, and (by overlining) those indecomposable objects X such that $\text{Ext}_{\mathcal{R}_5}^1(X, [3, 1]) \neq 0$. In each case the objects in question are written in bold font. Note that $[1, 3]$ is the only object satisfying both conditions.

Let X, Y be two indecomposable objects of \mathcal{R}_n , regarded as oriented edges between vertices of \mathcal{P}_n . Then X and Y may cross each other in two different ways. If the tangents to X, Y (in that order) form a pair of axes corresponding to the usual orientation on \mathbb{R}^2 we say that the crossing of X and Y is positive, otherwise negative. See Figure 7.

Lemma 6.1 can be reinterpreted geometrically as follows.

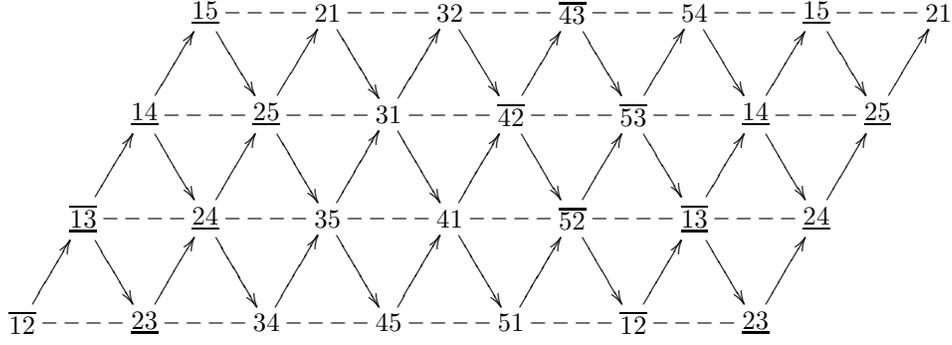


FIGURE 8. Objects with non-trivial extensions with $[3, 1]$ in either direction, in \mathcal{R}_5 .

Proposition 6.2. *Let X, X' be indecomposable objects in \mathcal{R}_n . Then $\dim \text{Ext}_{\mathcal{R}_n}^1(X, X')$ is equal to 1 if and only if one of the following conditions holds, and is zero otherwise:*

- (a) *The crossing of X, X' is positive;*
- (b) *The terminal vertex of X coincides with the initial vertex of X' and X' lies to the left of X in \mathcal{P}_n ;*
- (c) *The initial vertex of X coincides with the terminal vertex of X' and X' lies to the right of X in \mathcal{P}_n ;*
- (d) *X' is the reverse of X .*

As an example, consider the objects Y such that $\text{Ext}_{\mathcal{R}_5}^1([3, 1], Y) \neq 0$, displayed in Figure 8 (by underlining). Note that the crossing of $[3, 1]$ with $[2, 5]$ or with $[2, 4]$ is positive; both $[1, 4]$ and $[1, 5]$ start at the terminal vertex of $[3, 1]$ and lie to its left; $[2, 3]$ has terminal vertex coinciding with the initial vertex of $[3, 1]$ and lies to its right, and $[1, 3]$ is the reverse of $[3, 1]$.

7. STARTING AND ENDING FRAMES

In this section we consider starting and ending frames of indecomposable objects in \mathcal{R}_n (following [BMRRT, 8.4]). Let $\text{ind}(\mathcal{R}_n)$ denote the set of (isomorphism classes of) indecomposable objects of \mathcal{R}_n . Let X be an indecomposable object in \mathcal{R}_n . Then the *starting frame* $S(X)$ of X is the set

$$S(X) = \{Y \in \text{ind}(\mathcal{R}_n) : \text{Hom}_{\mathcal{R}_n}(X, Y) \neq 0, \text{Ext}_{\mathcal{R}_n}^1(Y, X) = 0\}.$$

The *ending frame* $E(X)$ of X is the set

$$E(X) = \{Y \in \text{ind}(\mathcal{R}_n) : \text{Hom}_{\mathcal{R}_n}(Y, X) \neq 0, \text{Ext}_{\mathcal{R}_n}^1(X, Y) = 0\}.$$

We define the *frame* $F(X)$ of X to be the union:

$$F(X) = S(X) \cup E(X).$$

The following can be checked by direct calculation or using Proposition 6.2 and the Auslander-Reiten formula.

We also note that, if \overline{X} denotes the reverse of X , then

$$\text{Hom}_{\mathcal{R}_n}(\overline{X}, Y) \cong \text{Hom}_{\mathcal{R}_n}(X[1], Y) \cong \text{Hom}_{\mathcal{R}_n}(X, Y[1]) \cong \text{Ext}_{\mathcal{R}_n}(X, Y)$$

and

$$\text{Ext}_{\mathcal{R}_n}(\overline{X}, Y) \cong \text{Ext}_{\mathcal{R}_n}(X[1], Y) \cong \text{Ext}_{\mathcal{R}_n}(X, Y[-1]) \cong \text{Hom}_{\mathcal{R}_n}(X, Y).$$

- Proposition 7.1.** (a) $Y \in S(X)$ if and only if Y and X share a common terminal vertex and Y lies to the left of X , or Y and X share a common initial vertex and Y lies to the right of X , or $Y = X$.
- (b) $Y \in E(X)$ if and only if Y and X share a common initial vertex and Y lies to the left of X , or Y and X share a common terminal vertex and Y lies to the right of X , or $Y = X$.
- (c) $Y \in F(X)$ if and only if X and Y share a common initial vertex or share a common terminal vertex (or both).

8. CATEGORIFICATION OF THE DETERMINANTAL RESULT

Fix again a triangulation π of \mathcal{P}_n . For $1 \leq i, j \leq n$ with $i \neq j$, we associate the indecomposable object $[i, j]$ (or oriented edge) with the i, j position in the $n \times n$ matrix $U(\pi)$ considered in Section 1. The indecomposable kQ -modules (with $i < j$) correspond to the part of $U(\pi)$ above the leading diagonal and their shifts (with $i > j$) correspond to the part of $U(\pi)$ below the leading diagonal.

Reinterpreting Proposition 7.1 in these terms, we obtain:

Lemma 8.1. *The frame of an indecomposable object X corresponds to the union of the row and column of $U(\pi)$ containing X (apart from the diagonal entries).*

Define a *frame-free* configuration of \mathcal{R}_n to be a maximal collection C of (isomorphism classes of) indecomposable objects of $\text{ind } \mathcal{R}_n$ such that $Y \notin F(X)$ for all X, Y in C . Thus frame-free configurations of \mathcal{R}_n correspond to maximal collections of positions in $U(\pi)$ which do not lie in the same row or column as each other and contain no diagonal entries.

Lemma 8.2. *Let C be a frame-free configuration of \mathcal{R}_n . Then the cardinality of C is either $n - 1$ or n .*

Proof. Since frame-free configurations cannot have objects in the same row or column, the maximum cardinality is n . If the cardinality of a configuration is $n - k$ where $k \geq 2$, there are at least two rows and two columns of $U(\pi)$ containing no elements of the configuration, a contradiction to its maximality (as at least two elements could be added to the configuration, at the non-diagonal intersections of the empty rows and columns). The result follows. \square

Given a fixed-point free permutation, σ (sometimes known as a *derangement*), let $C(\sigma)$ be the set of objects $[i, \sigma(i)]$ for $1 \leq i \leq n$. Since σ is fixed-point free, it follows from Proposition 7.1(c) that $C(\sigma)$ is a frame-free configuration; it has cardinality n . It is clear that this gives a bijection between fixed-point free permutations and frame-free configurations of cardinality n .

Thus, as a collection of oriented edges between vertices of \mathcal{P}_n , a frame-free configuration in \mathcal{R}_n of cardinality n is a union of oriented cycles (with no cycles of cardinality 1).

Similarly, a permutation σ with one fixed point, p , say, corresponds to a configuration of cardinality $n - 1$ consisting of the objects $[i, \sigma(i)]$ for $i \neq p$. This gives a bijection between the permutations with a single fixed point and the frame-free configurations of cardinality $n - 1$.

The number of permutations with a given number of fixed points is well-known (see e.g. [S, A008290]). We thus have the following:

Lemma 8.3. *The number of frame-free configurations of \mathcal{R}_n of cardinality n is*

$$!n := n! \sum_{k=0}^n \frac{(-1)^k}{k!},$$

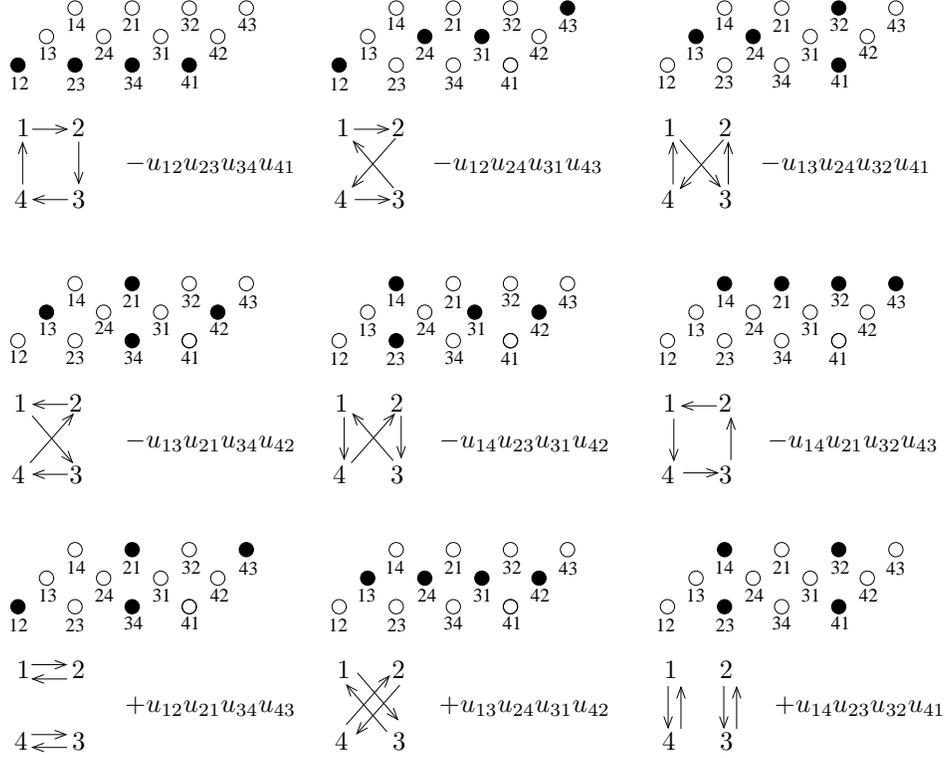


FIGURE 9. The 9 frame-free configurations of cardinality 4 in the root category of type A_3 and the corresponding terms in the expansion of $\det(U(\pi))$.

(known as the subfactorial of n) while the number of frame-free configurations of \mathcal{R}_n of cardinality $n - 1$ is $!n + (-1)^{n-1}$.

Thus the number of frame-free configurations in \mathcal{R}_n of cardinality n for $n = 2, 3, \dots$, is 1, 2, 9, 44, 265, 1854 (see [S, A000166]) and the number of frame-free configurations in \mathcal{R}_n of cardinality $n - 1$ is 0, 3, 8, 45, 264, 1855 (see [S, A000240]). The 9 frame-free configurations of \mathcal{R}_4 of cardinality 4 are shown in Figure 9 as collections of vertices in the AR-quiver (filled in vertices indicate those indecomposable objects in the collection) and as collections of oriented arcs between vertices of a square.

We see that the frame-free configurations of \mathcal{R}_n of cardinality n correspond bijectively to the non-zero terms in the expansion

$$\det(U(\pi)) = \sum_{\sigma \in \Sigma_n} (-1)^{\ell(\sigma)} u_{1,\sigma(1)} \cdots u_{n,\sigma(n)}$$

of the determinant of $U(\pi)$ (since $U(\pi)$ has zeros along its leading diagonal).

Given a frame-free configuration C of \mathcal{R}_n of cardinality n , define its *sign* $\varepsilon(C)$ to be

$$\varepsilon(C) = \prod_{\gamma} (-1)^{\ell(\gamma)-1}$$

where the product is over the oriented cycles in the representation of C as a collection of oriented edges between vertices of \mathcal{P}_n , and $\ell(\gamma)$ is equal to the number of vertices in γ for a cycle γ . It is easy to see that this is equal to the sign of

the corresponding permutation. Set $\alpha(C)$ to be the product of the entries of $U(\pi)$ corresponding to the elements of C .

We therefore have:

$$\det(U(\pi)) = \sum_C \varepsilon(C) \alpha(C),$$

where the sum is over all frame-free configurations of \mathcal{R}_n of cardinality n . The monomials in this expansion are shown for each frame-free configuration in the example in Figure 9.

We can reinterpret Theorem 3.2 representation-theoretically as follows. Recall that \mathcal{A} is a cluster algebra of type A_{n-3} with coefficients corresponding to the boundary edges of \mathcal{P}_n .

Theorem 8.4. *Let $n \geq 3$, let π be a triangulation of \mathcal{P}_n , and let $U(\pi) = (u_{ij})$ be the matrix of cluster variables in \mathcal{A} regarded as Laurent polynomials in the u_{ij} for i, j end-points of diagonals in π with coefficients given by polynomials in the stable variables. Then we have:*

$$\sum_C \varepsilon(C) \alpha(C) = -(-2)^{n-2} u_{12} u_{23} \cdots u_{n-1, n} u_{n, 1},$$

where the sum is over all frame-free configurations of $\mathcal{R}_n = D^b(kQ)/[2]$ of maximum cardinality.

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