CLUSTER STRUCTURES FROM 2-CALABI-YAU CATEGORIES WITH LOOPS

ASLAK BAKKE BUAN, ROBERT J. MARSH, AND DAGFINN F. VATNE

Abstract. We generalise the notion of *cluster structures* from the work of Buan-Iyama-Reiten-Scott to include situations where the endomorphism rings of the clusters may have loops. We show that in a Hom-finite 2-Calabi-Yau category, the set of maximal rigid objects satisfies these axioms whenever there are no 2-cycles in the quivers of their endomorphism rings.

We apply this result to the cluster category of a tube, and show that this category forms a good model for the combinatorics of a type B cluster algebra.

Introduction

Since the introduction of cluster algebras by Fomin and Zelevinsky [FZ1], relationships between such algebras and interesting topics in several branches of mathematics have emerged.

The project of modelling cluster algebras in a representation theoretic setting was initiated in [MRZ]. Inspired by this, cluster categories were defined in [BMRRT] to be certain orbit categories obtained from the derived category of Hom-finite hereditary abelian categories. These categories have been widely studied for the case when the initial hereditary category is the category of finite dimensional representations of an acyclic quiver Q. (When Q is a quiver with underlying graph A_n, the cluster category was independently defined in [CCS].) It has then been shown that the indecomposable rigid objects are in bijection with the cluster variables in the cluster algebra $\mathcal{A}_Q$ associated with the same quiver, and under this bijection the clusters correspond to the maximal rigid objects ([CK] based on [CC], see also [BCKMRT]). Moreover, by [BMR], the quiver of the endomorphism ring of a maximal rigid object is the same as the quiver of the corresponding cluster.

The last phenomenon also appears for maximal rigid modules in the stable module category of preprojective algebras of simply laced Dynkin type [GLS], which is another example of a Hom-finite 2-Calabi-Yau triangulated category. Inspired by this and [IY], [KR], an axiomatic framework for mutation in 2-CY categories was defined in [BIRS]. The essential features were considered to be: the unique exchange of indecomposable summands; the fact that exchange pairs were related by approximation triangles; and the fact that on the level of endomorphism rings, exchange of indecomposable summands led to Fomin-Zelevinsky quiver mutation on the Gabriel quivers. For the third of these features to make sense, one must require that the endomorphism rings have no loops or 2-cycles in their quivers. In [BIRS] it was also shown that the collection of maximal rigid objects in any Hom-finite 2-CY triangulated category fulfils these axioms for cluster structures whenever the quivers of their endomorphism rings do not have loops or 2-cycles.

The cluster structures from [BIRS] have two limitations: Firstly, there exist Hom-finite 2-CY triangulated categories where the endomorphism rings of maximal rigid objects do have loops and 2-cycles in their quivers. The unique exchange property holds also in these categories [IY], but FZ quiver mutation does not make sense in this setting. Secondly, the cluster algebras which can be modelled from the cases studied in [BIRS] are the ones defined from quivers (equivalently, skew-symmetric matrices), while cluster algebras can be defined also from more general matrices.

The aim of this paper is to extend the notion of cluster structures from [BIRS]. We show that the set of maximal rigid objects in a Hom-finite 2-CY triangulated category satisfies this new definition of cluster structure regardless of whether the endomorphism rings have loops or not. (We must however assume that the quivers do not have 2-cycles.) One effect of this is that we will also relax the second limitation, since the cluster algebras which can be modelled in this new setting but not in the setting of [BIRS] are defined from matrices which are not necessarily skew-symmetric.

While previous investigations of cluster structures have regarded the quiver of the endomorphism ring as the essential combinatorial data, our approach is to emphasize the exchange triangles instead.

We collect the information from the exchange triangles in a matrix and require for our cluster structure that exchange of indecomposable objects leads to FZ matrix mutation of this matrix. In the no-loop situation considered in [BIRS], the matrix defined here is the same as the one describing the quiver of the endomorphism ring, so our definition is an extension of the definition in [BIRS].

In the cluster category defined from the module category of a hereditary algebra, the maximal rigid objects are the same as the *cluster-tilting* objects. In general, however, the cluster-tilting condition is stronger, and there exist categories where the maximal rigid objects are not cluster-tilting. Examples of this in a geometrical setting are given in [BIKR]. In this paper we present a class of examples from a purely representation-theoretical setting, namely the cluster categories $\mathcal{C}_{\mathcal{T}_n}$ defined from *tubes* $\mathcal{T}_n$. Here, the tube $\mathcal{T}_n$ is the category of nilpotent representations of a quiver with underlying graph $\tilde{A}_{n-1}$ and with cyclic orientation. The category $\mathcal{C}_{\mathcal{T}_n}$ has also been studied in [BKL1, BKL2].

An interesting aspect of cluster categories from tubes is that the endomorphism rings of the maximal rigid objects have quivers with loops, but not 2-cycles. Thus they are covered by the definition of cluster structures in this paper, but not by the one in [BIRS]. We will show that the set of rigid objects in $\mathcal{C}_{\mathcal{T}_n}$ is a model of the exchange combinatorics of a type B cluster algebra. To this end, we give a bijection between the indecomposable rigid objects in $\mathcal{C}_{\mathcal{T}_n}$ and the cluster variables in a type B_{n-1} cluster algebra such that the maximal rigid objects correspond to the clusters. This bijection is via the *cyclohedron*, or Bott-Taubes polytope [BT]. Also, we will see that the matrix defined from the exchange triangles associated to a maximal rigid object is the same as the one belonging to the corresponding cluster in the cluster algebra.

The article is organised as follows. In Section 1 we give the new definition of cluster structures and show that the set of maximal rigid objects in a Hom-finite 2-CY category satisfies this definition whenever there are no 2-cycles in the quivers of their endomorphism rings. In Section 2 we give a complete description of the maximal rigid objects in the cluster category of a tube. Finally, in Section 3, we show that the cluster structure in a cluster tube forms a good model of the combinatorics of a type B cluster algebra.

1. Cluster structures

In this section we generalise the notion of *cluster structures* from [BIRS] to include situations where the quivers of the clusters may have loops. We then proceed to show that the set of maximal rigid objects in a Hom-finite 2-Calabi-Yau category admits a cluster structure with this new definition, under the assumption that the Gabriel quivers of their endomorphism rings do not have 2-cycles.

Let k be some algebraically closed field. By a Hom-finite k-category we will mean a category $\mathcal{C}$ where $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is a finite dimensional k-vector space for all pairs of indecomposable objects X and Y. We will normally suppress the field k. A triangulated category $\mathcal{C}$ is said to be Calabi-Yau of dimension 2, or 2-CY, if $D\operatorname{Ext}^i_{\mathcal{C}}(X,Y) \simeq \operatorname{Ext}^{2-i}_{\mathcal{C}}(Y,X)$ for all objects X,Y of $\mathcal{C}$ and all i in $\mathbb{Z}$. For the remainder of this section, $\mathcal{C}$ will denote a Hom-finite 2-Calabi-Yau triangulated category.

We now recall the definition of a *weak cluster structure* from [BIRS]. Let $\mathscr{T}$ be a non-empty collection of sets of non-isomorphic indecomposable objects of $\mathcal{C}$. Each set T of indecomposables which is an element of $\mathscr{T}$ is called a *precluster*. The collection $\mathscr{T}$ is said to have a *weak cluster structure* if the following two conditions are met:

(a) For each precluster $T = \overline{T}\dot{\cup}\{M\}$ with M indecomposable, there exists a unique indecomposable $M^* \not\simeq M$ such that the disjoint union $T^* = \overline{T} \dot{\cup} \{M^*\}$ is a precluster.

(b) M and M^* are related by triangles

$$M^* \xrightarrow{f} U_{M,\overline{T}} \xrightarrow{g} M \to \quad \text{and} \quad M \xrightarrow{s} U'_{M,\overline{T}} \xrightarrow{t} M^* \to$$

where f and s are minimal left $\operatorname{add}\overline{T}$-approximations and g and t are minimal right $\operatorname{add}\overline{T}$-approximations. These triangles are called the *exchange triangles* of M (equivalently, of M^*) with respect to $\overline{T}$.

We will not distinguish between a precluster and the object obtained by taking the direct sum of the indecomposable objects which form the precluster. The same goes for subsets of preclusters.

Assume now that $\mathscr{T}$ has a weak cluster structure. For each $T = \{T_i\}_{i \in I}$ in $\mathscr{T}$ we define a (possible infinite) matrix $B_T = (b_{ij})$ by

$$b_{ij} = \alpha_{U'_{T_i,T/T_i}} T_j - \alpha_{U_{T_i,T/T_i}} T_j$$

where $\alpha_Y X$ denotes the multiplicity of X as a direct summand of Y. So the matrix B_T records, for each $T_i, T_j \in T$, the difference between the multiplicities of T_j in the two exchange triangles for T_i. Note that $b_{ii} = 0$ for all i, since T_i does not appear as a summand in the target (resp. source) of a left (resp. right) $\operatorname{add}(T/T_i)$-approximation.

Recall Fomin-Zelevinsky *matrix mutation*, defined in [FZ1]: For a matrix $B = (b_{ij})$ the mutation at k is given by $\mu_k(B) = (b^*_{ij})$, where

$$b^*_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & i, j \neq k \end{cases}$$

Note in particular that for $i, j \neq k$, we have $b_{ij} \neq b^*_{ij}$ if and only if b_{ik} and b_{kj} are both positive or both negative.

We say that $\mathscr{T}$ has a *cluster structure* if $\mathscr{T}$ has a weak cluster structure, and in addition the following conditions are satisfied:

(c) For each $T \in \mathscr{T}$ and each $T_i \in T$, the objects $U_{T_i,T/T_i}$ and $U'_{T_i,T/T_i}$ have no common direct summands.

(d) If T_k is an indecomposable object and $T = \overline{T} \cup \{T_k\}$ and $T^* = \overline{T} \cup \{T^*_k\}$ are preclusters, then B_T and B_{T^*} are related by Fomin-Zelevinsky matrix mutation at k.

In this case, we call the elements of $\mathscr{T}$ *clusters*.

An interpretation of condition (c) above is that the endomorphism rings of the clusters have Gabriel quivers which do not have 2-cycles. It should also be noted that if there are no loops at the vertices corresponding to T_i and T_j in these quivers, then the multiplicity of T_j as a summand in the exchange triangle for T_i will equal the number of arrows between these two vertices. In particular, if there are no loops at any of the vertices, B_T is skew-symmetric and can be considered as a record of the quiver. Condition (d) then reduces to FZ quiver mutation, so our definition coincides with the definition in [BIRS] in this case.

An object T in a triangulated category $\mathcal{K}$ is said to be *rigid* if $\operatorname{Ext}^1_{\mathcal{K}}(T, T) = 0$. It is called *maximal rigid* if it is maximal with this property, that is, $\operatorname{Ext}^1_{\mathcal{K}}(T \amalg X, T \amalg X) = 0$ implies that $X \in \operatorname{add} T$.

If there exist maximal rigid objects in a Hom-finite 2-CY triangulated category, then the collection of such objects has a weak cluster structure. This follows from [IY], as stated in Theorem I.1.10 (a) of [BIRS]. We can now prove a stronger version of part (b) of the same theorem:

Theorem 1.1. *Let $\mathcal{C}$ be a Hom-finite 2-Calabi-Yau triangulated category, and let $\mathscr{T}$ be the collection of maximal rigid objects in $\mathcal{C}$. Assume that $\mathscr{T}$ is nonempty, and that it satisfies condition (c) above. Then $\mathscr{T}$ has a cluster structure.*

The proof of Theorem 1.1 follows the same lines as the proof of Theorem I.1.6. in [BIRS]. We need to show that when an indecomposable summand of a maximal rigid object is exchanged, the change in the matrix is given by Fomin-Zelevinsky matrix mutation. The fact that the matrices B_T for maximal rigid T are not necessarily skew-symmetric forces us to prove different cases separately. For the proof we will need the following lemma:

Lemma 1.2. *In the situation of the theorem, for any maximal rigid object T, B_T is sign skew symmetric, i.e., for all i, j, we have $b_{ij} < 0$ if and only if $b_{ji} > 0$.*

Proof. As remarked after the definition of the matrix B_T, the diagonal entries b_{ii} vanish, so the statement in the lemma is clearly true for these.

Now assume $b_{ij} < 0$ for some $i \neq j$. Then T_j is a summand of the middle term of the exchange triangle

$$T^*_i \to U_{T_i,T/T_i} \to T_i \to$$

Since the second map is a minimal right $\operatorname{add}(T/T_i)$-approximation, this means that there exists a map $T_j \to T_i$ which does not factor through any other object in $\operatorname{add}(T/(T_i \amalg T_j))$. In other words, there is an arrow $T_j \to T_i$ in the quiver of $\mathrm{End}_{\mathcal{C}}(T)$. This in turn implies that T_i is a summand of the middle term in the exchange triangle

$$T_j \to U'_{T_j, T/T_j} \to T_j^* \to$$

and thus $b_{ji} > 0$. □

Proof of Theorem 1.1. Let $T = \amalg_{i=1}^n T_i$ be a maximal rigid object in $\mathcal{C}$. Suppose we want to exchange the indecomposable summand T_k. Consider $\overline{T} = \amalg_{i \neq k} T_i$ and the maximal rigid object $T^* = \overline{T} \amalg T_k^*$. We want to show that the matrices $B_T = (b_{ij})$ and $B_{T^*} = (b^*_{ij})$ are related by FZ matrix mutation. Pick two indecomposable summands $T_i \not\simeq T_j$ of $\overline{T}$, so i, j and k are all distinct.

For the purposes of this proof we introduce some notation. Let

$$\alpha_a(T_b) = \alpha_{U_{T_a, T/T_a}} T_b$$

that is, the multiplicity of T_b as a direct summand of the middle term in the exchange triangle ending in T_a with respect to T/T_a. Similarly, we denote the multiplicity of T_b in the other triangle by

$$\alpha'_a(T_b) = \alpha_{U'_{T_a, T/T_a}} T_b$$

Note that under the assumption of condition (c), at least one of these two numbers will be zero for any choice of a, b.

The exchange triangles for T_k with respect to $\overline{T}$ are

$$T_k^* \to T_i^{\alpha_k(T_i)} \amalg T_j^{\alpha_k(T_j)} \amalg V_k \to T_k \to \tag{1}$$

$$T_k \to T_i^{\alpha'_k(T_i)} \amalg T_j^{\alpha'_k(T_j)} \amalg V'_k \to T_k^* \to \tag{2}$$

Note that V_k and V'_k do not have T_i or T_j as direct summand. These are also the exchange triangles for T_k^* with respect to $\overline{T}$, but the roles of the middle terms are interchanged. It follows immediately that $b^*_{ki} = -b_{ki}$, and the FZ formula holds for row k of the matrix.

In the rest of the proof we study the changes in row i, where $i \neq k$. For this we also need the exchange triangles for T_i with respect to T/T_i, which are

$$T_i^* \xrightarrow{\phi_1} T_j^{\alpha_i(T_j)} \amalg T_k^{\alpha_i(T_k)} \amalg V_i \xrightarrow{\phi_2} T_i \to \tag{3}$$

$$T_i \xrightarrow{\phi_3} T_j^{\alpha'_i(T_j)} \amalg T_k^{\alpha'_i(T_k)} \amalg V'_i \xrightarrow{\phi_4} T_i^* \to \tag{4}$$

Again, note that V_i and V'_i do not have T_j or T_k as direct summand. From these triangles we must collect information about the exchange triangles of T_i with respect to $\widetilde{T} = T/(T_i \amalg T_k) \amalg T_k^*$, since these determine the entries in row i of B_{T^*}.

We will consider three different cases, depending on whether b_{ik} is positive, negative or zero.

Case I: Assume $b_{ik} = 0$. Because of (c) this means that T_k does not appear in any of the exchange triangles for T_i. Then, by Lemma 1.2, we have $b_{ki} = 0$ as well, and by the above, $b^*_{ki} = 0$. By appealing once more to Lemma 1.2, we see that T_k^* does not appear in the exchange triangles for T_i with respect to $\widetilde{T}$. This is enough to establish that the map ϕ_1 in triangle (3) is also a minimal left $\mathrm{add}\,\widetilde{T}$-approximation. Similarly, the map ϕ_4 in triangle (4) is a minimal right $\mathrm{add}\,\widetilde{T}$-approximation. This means that the triangles (3) and (4) are also the exchange triangles for T_i with respect to $\widetilde{T}$. Thus the entries in row i remain unchanged and behave according to the FZ rule in this situation. (Note also that this proves that T_i^* is the complement of T_i both before and after we have exchanged a summand T_k with $b_{ik} = 0$.)

Case II: Suppose now that $b_{ik} < 0$, which means that T_k appears as a summand in (3), while $\alpha'_i(T_k) = 0$. By Lemma 1.2, $b_{ki} > 0$, which means that T_i appears in (2), not in (1). Our strategy is to construct redundant versions of the exchange triangles of T_i with respect to $\widetilde{T}$. We will use the triangle

$$(T_k^*)^{\alpha_i(T_k)} \longrightarrow \begin{matrix} \left(T_j^{\alpha_i(T_j)} \amalg V_i\right) \\ \amalg \\ \left(T_j^{\alpha_k(T_j)} \amalg V_k\right)^{\alpha_i(T_k)} \end{matrix} \xrightarrow{\phi'_1} \begin{matrix} \left(T_j^{\alpha_i(T_j)} \amalg V_i\right) \\ \amalg \\ T_k^{\alpha_i(T_k)} \end{matrix} \longrightarrow \tag{5}$$

which is the direct sum of $\alpha_i(T_k)$ copies of (1) and the identity map of $T_j^{\alpha_i(T_j)} \amalg V_i$. Applying the octahedral axiom to the composition of the map ϕ_1' in (5) and the map ϕ_2 in (3) yields the following commutative diagram in which the middle two rows and middle two columns are triangles.

$$
\begin{array}{ccccccc}
 & & & & (5) & & \\
 & & (T_k^*)^{\alpha_i(T_k)}[1] & = & (T_k^*)^{\alpha_i(T_k)}[1] & & \\
 & & \uparrow{\scriptstyle \chi_1} & & \uparrow & & \\
T_i[-1] & \longrightarrow & T_i^* & \longrightarrow & \left(T_j^{\alpha_i(T_j)} \amalg V_i\right) \amalg T_k^{\alpha_i(T_k)} & \xrightarrow{\phi_2} & T_i \qquad (3) \\
\| & & \uparrow & & \uparrow{\scriptstyle \phi_1'} & & \| \\
T_i[-1] & \longrightarrow & X & \longrightarrow & \left(T_j^{\alpha_i(T_j)} \amalg V_i\right) \amalg \left(T_j^{\alpha_k(T_j)} \amalg V_k\right)^{\alpha_i(T_k)} & \xrightarrow{\phi} & T_i \\
 & & \uparrow & & \uparrow & & \\
 & & (T_k^*)^{\alpha_i(T_k)} & = & (T_k^*)^{\alpha_i(T_k)} & &
\end{array}
$$

We now want to show that the map $\phi = \phi_2\phi_1'$ is a (not necessarily minimal) right add $\widetilde{T}$-approximation. Any map $f : T_t \to T_i$ where $t \neq i$ will factor through ϕ_2 since this is a right $\operatorname{add}(T/T_i)$-approximation, so $f = \phi_2 f_1$. But since ϕ_1' is a right $\operatorname{add}(\overline{T})$-approximation, f_1 factors through ϕ_1'. Thus f factors through $\phi_2\phi_1' = \phi$.

Suppose instead that we have a map $f : T_k^* \to T_i$. Let $h : T_k^* \to T_j^{\alpha_k(T_j)} \amalg V_k$ be the minimal left add $\overline{T}$-approximation for T_k^*. Then $f = gh$ for some map $g : T_j^{\alpha_k(T_j)} \amalg V_k \to T_i$. Since T_i is not a summand of V_k, we have that $T_j^{\alpha_k(T_j)} \amalg V_k$ is in $\operatorname{add}(\widetilde{T}/T_k^*)$, and by the above, g factors through ϕ. We conclude that ϕ is a right add $\widetilde{T}$-approximation.

Similarly we now construct a second commutative diagram. We use the octahedral axiom on the composition of the map ϕ_4 in (4) and the map χ_1 in the triangle

$$
(T_k^*)^{\alpha_i(T_k)} \longrightarrow X \longrightarrow T_i^* \xrightarrow{\chi_1} (T_k^*)^{\alpha_i(T_k)}[1] \tag{6}
$$

from the second column of the previous diagram. (Note that by our assumption, T_k does not appear in (4).)

$$
\begin{array}{ccccccc}
 & & (T_k^*)^{\alpha_i(T_k)}[1] & = & (T_k^*)^{\alpha_i(T_k)}[1] & & \\
 & & \uparrow{\scriptstyle \chi} & & \uparrow{\scriptstyle \chi_1} & & \\
T_i & \longrightarrow & T_j^{\alpha_i'(T_j)} \amalg V_i' & \xrightarrow{\phi_4} & T_i^* & \longrightarrow & T_i[1] \\
\| & & \uparrow & & \uparrow & & \| \\
T_i & \longrightarrow & Y & \xrightarrow{\psi} & X & \xrightarrow{\psi_1} & T_i[1] \\
 & & \uparrow & & \uparrow & & \\
 & & (T_k^*)^{\alpha_i(T_k)} & = & (T_k^*)^{\alpha_i(T_k)} & &
\end{array}
$$

We notice that in this second diagram, the map χ must be zero, since $\operatorname{Ext}^1_{\mathcal{C}}(T_j, T_k^*) = \operatorname{Ext}^1_{\mathcal{C}}(V_i', T_k^*) = 0$. Therefore, the triangle splits, and

$$
Y \simeq \left(T_j^{\alpha_i'(T_j)} \amalg V_i'\right) \amalg (T_k^*)^{\alpha_i(T_k)}
$$

We see that ψ in the diagram is a (not necessarily minimal) right $\operatorname{add}\widetilde{T}$-approximation as well: For any map $f : U \to X$ where $U \in \operatorname{add}\widetilde{T}$, the composition $\psi_1 f$ is zero since $\operatorname{Ext}^1_{\mathcal{C}}(U, T_i) = 0$, which again implies that f factors through ψ.

Since ϕ is a right $\operatorname{add}\widetilde{T}$-approximation, $X = Z \amalg T_i^!$, where $T_i^! \not\simeq T_i$ is the unique other indecomposable that completes $\widetilde{T}$ to a maximal rigid object. $T_i^!$ exists and is unique since $\mathscr{T}$ has a weak cluster structure and so satisfies (a). Consider the triangles we have constructed:

$$Z \amalg T_i^! \longrightarrow \left(T_j^{\alpha_i(T_j)} \amalg V_i\right) \amalg \left(T_j^{\alpha_k(T_j)} \amalg V_k\right)^{\alpha_i(T_k)} \xrightarrow{\phi} T_i \longrightarrow \tag{7}$$

and

$$T_i \longrightarrow \left(T_j^{\alpha'_i(T_j)} \amalg V'_i\right) \amalg (T_k^*)^{\alpha_i(T_k)} \xrightarrow{\psi} Z \amalg T_i^! \longrightarrow \tag{8}$$

Since ϕ and ψ are right $\operatorname{add}(\widetilde{T})$-approximations, we see that an automorphism of Z splits off in both triangles, and the remaining parts are the exchange triangles for T_i and $T_i^!$ with respect to $\widetilde{T}$. So to find the entry b^*_{ij} we calculate the difference of the multiplicities of T_j in the two triangles (7) and (8), since the difference is not affected when Z is split off from both triangles. So if we denote by $\alpha_{(7)}(T_j)$ the multiplicity of T_j in the middle term of triangle (7) and similarly for triangle (8) we get

$$\begin{aligned}
b^*_{ij} &= \alpha_{(8)}(T_j) - \alpha_{(7)}(T_j) \\
&= \alpha'_i(T_j) - (\alpha_i(T_j) + \alpha_k(T_j)\alpha_i(T_k)) \\
&= (\alpha'_i(T_j) - \alpha_i(T_j)) - \alpha_k(T_j)\alpha_i(T_k) \\
&= \begin{cases} b_{ij} & \text{when } \alpha_k(T_j) = 0, \text{ i.e. when } b_{kj} \geq 0 \\ b_{ij} - (-b_{kj})(-b_{ik}) = b_{ij} - b_{kj}b_{ik} & \text{when } \alpha_k(T_j) > 0, \text{ i.e. when } b_{kj} < 0 \end{cases}
\end{aligned}$$

Also, it is clear from (7) and (8) that $b^*_{ik} = \alpha_i(T_k) = -b_{ik}$. Summarising, we see that the entries in the ith row change as required by the FZ rule in this case.

Case III: Finally we consider the case where $b_{ik} > 0$. This means that T_k appears in (4), but not in (3). Furthermore, by Lemma 1.2, T_i appears in (1), but not in (2). The argument follows the same lines as in Case II. Instead of (5), we use the following triangle:

$$(T_k^*)^{\alpha'_i(T_k)}[-1] \longrightarrow \begin{array}{c} \left(T_j^{\alpha'_i(T_j)} \amalg V'_i\right) \\ \amalg \\ T_k^{\alpha'_i(T_k)} \end{array} \xrightarrow{\phi'_2} \begin{array}{c} \left(T_j^{\alpha'_i(T_j)} \amalg V'_i\right) \\ \amalg \\ \left(T_j^{\alpha'_k(T_j)} \amalg V'_k\right)^{\alpha'_i(T_k)} \end{array} \longrightarrow (T_k^*)^{\alpha'_i(T_k)} \tag{9}$$

which is the direct sum of $\alpha'_i(T_k)$ copies of (2) and the identity map of $T_j^{\alpha'_i(T_j)} \amalg V'_i$. By the octahedral axiom, applied to the composition of the map ϕ_3 in (4) and the map ϕ'_2 in (9), we get this commutative diagram:

$$\begin{array}{ccccccc}
 & & (T_k^*)^{\alpha'_i(T_k)} & = & (T_k^*)^{\alpha'_i(T_k)} & & \\
 & & \uparrow & & \uparrow & & \\
T_i & \xrightarrow{\phi'} & \left(T_j^{\alpha'_i(T_j)} \amalg V'_i\right) \amalg \left(T_j^{\alpha'_k(T_j)} \amalg V'_k\right)^{\alpha'_i(T_k)} & \longrightarrow & X & \longrightarrow & T_i[1] \\
\| & & \uparrow{\scriptstyle \phi'_2} & & \uparrow & & \| \\
T_i & \xrightarrow{\phi_3} & \left(T_j^{\alpha'_i(T_j)} \amalg V'_i\right) \amalg T_k^{\alpha'_i(T_k)} & \longrightarrow & T_i^* & \longrightarrow & T_i[1] \\
 & & \uparrow & & \uparrow & & \\
 & & (T_k^*)^{\alpha'_i(T_k)}[-1] & = & (T_k^*)^{\alpha'_i(T_k)}[-1] & &
\end{array}$$

By arguments dual to those in Case II, we can check that ϕ' is a left (not necessarily minimal) $\operatorname{add}\widetilde{T}$-approximation. Details are left to the reader.

Now the octahedral axiom applied to the composition in the left square below gives a new diagram and a new object Y, where the second column is the same triangle as the third column in the previous diagram:

$$\begin{array}{ccccccc}
 & & (T_k^*)^{\alpha'_i(T_k)} & = = = & (T_k^*)^{\alpha'_i(T_k)} & & \\
 & & \uparrow & & \uparrow & & \\
T_i[-1] & \longrightarrow & X & \xrightarrow{\psi'} & Y & \longrightarrow & T_i \\
\| & & \uparrow & & \uparrow & & \| \\
T_i[-1] & \longrightarrow & T_i^* & \longrightarrow & T_j^{\alpha_i(T_j)} \amalg V_i & \longrightarrow & T_i \\
 & & \uparrow & & \uparrow{\scriptstyle \chi'} & & \\
 & & (T_k^*)^{\alpha'_i(T_k)}[-1] & = = = & (T_k^*)^{\alpha'_i(T_k)}[-1] & &
\end{array}$$

Once again, by the fact that T_k is not a summand of V_i, and the vanishing of $\operatorname{Ext}^1_{\mathcal{C}}$-groups, χ' in this diagram is zero, the triangle splits, and

$$Y \simeq (T_k^*)^{\alpha'_i(T_k)} \amalg T_j^{\alpha_i(T_j)} \amalg V_i$$

and we may also conclude that ψ' is a left $\operatorname{add}\widetilde{T}$-approximation as in the previous case.

As in Case II, we can find the entry b^*_{ij} by subtracting the multiplicity of T_j in the triangle involving ψ' from its multiplicity in the triangle involving ϕ':

$$\begin{aligned}
b^*_{ij} &= (\alpha'_i(T_j) + \alpha'_k(T_j)\alpha'_i(T_k)) - \alpha_i(T_j) \\
&= (\alpha'_i(T_j) - \alpha_i(T_j)) + \alpha'_k(T_j)\alpha'_i(T_k) \\
&= \begin{cases} b_{ij} & \text{when } \alpha'_k(T_j) = 0, \text{ i.e. when } b_{kj} \leq 0 \\ b_{ij} + b_{kj}b_{ik} & \text{when } \alpha'_k(T_j) > 0, \text{ i.e. when } b_{kj} > 0 \end{cases}
\end{aligned}$$

Also, arguing in a similar way to Case II, we obtain $b^*_{ik} = -\alpha'_i(T_k) = -b_{ik}$. We have shown that b^*_{ij} is obtained from b_{ij} using the FZ mutation rule, so the proof is complete. □

Recall that a collection $\mathcal{R}$ of non-isomorphic indecomposable objects in a triangulated category $\mathcal{K}$ is said to be *rigid* if $\operatorname{Ext}^1_{\mathcal{K}}(U, V) = 0$ for all objects U and V in $\mathcal{R}$. It is called maximal rigid if it is maximal with respect to this property, that is, whenever X is an indecomposable object of $\mathcal{K}$ such that $\operatorname{Ext}^1_{\mathcal{K}}(U \amalg X, U \amalg X) = 0$ for all objects U in $\mathcal{R}$, then X is isomorphic to an object of $\mathcal{R}$. We say that $\mathcal{R}$ is functorially finite if the full additive subcategory with objects given by direct sums of objects of $\mathcal{R}$ is functorially finite in the sense of [AS].

The following more general statement can be proved using the same proof as that for Theorem 1.1.

Theorem 1.3. *Let $\mathcal{C}$ be a Hom-finite 2-Calabi-Yau triangulated category, and let $\mathscr{T}$ be the collection of functorially finite maximal rigid collections in $\mathcal{C}$. Assume $\mathscr{T}$ is non-empty and satisfies condition (c) above. Then $\mathscr{T}$ has a cluster structure.*

2. Maximal rigid objects in cluster categories of tubes

In this section we will give a complete description of the maximal rigid objects in the cluster category of a tube, as defined in [BMRRT]. It turns out that none of these are cluster-tilting objects. In Section 3, we will apply the main result in Section 1 to show that this category provides a model for the combinatorics of a type B cluster algebra.

We will denote by $\mathcal{T}_n$ the *tube of rank n*, where n is always understood to be at least 2. One realization of this category is as the category of nilpotent representations of a quiver with underlying graph $\tilde{A}_{n-1}$ and cyclic orientation. We will also think of of $\mathcal{T}_n$ as the full exact subcategory generated by the tube of rank n in the module category $\operatorname{mod} H$, where H is the path algebra of the quiver with underlying graph $\tilde{A}_n$ and n arrows oriented in clockwise direction and one arrow oriented in anticlockwise direction. We then know that for instance the AR-formula holds in $\mathcal{T}_n$:

$$\operatorname{Ext}^1_{\mathcal{T}_n}(X, Y) \simeq D\operatorname{Hom}_{\mathcal{T}_n}(Y, \tau X)$$

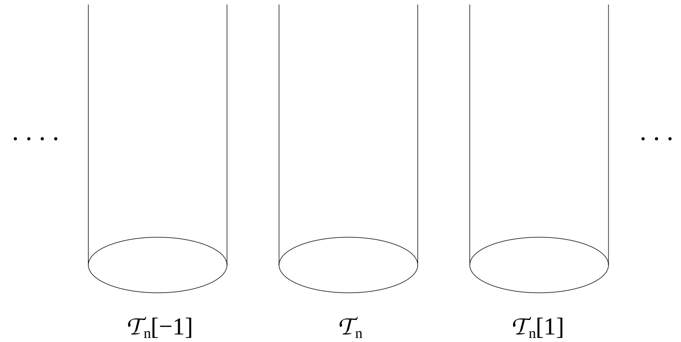


FIGURE 1. The AR-quiver of the derived category of $\mathcal{T}_n$; a countable set of disconnected tubes. There exist maps from indecomposables in each copy to the next copy on the right, corresponding to extensions in the tube itself.

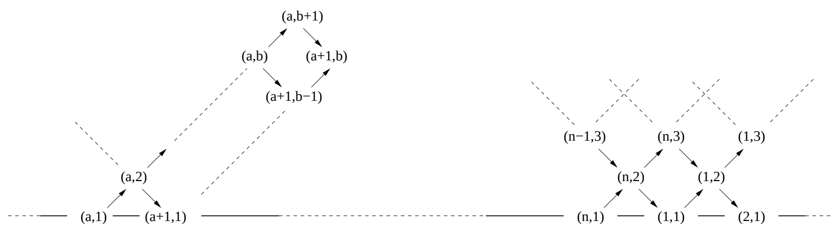


FIGURE 2. Coordinate system for the indecomposable objects in the tube.

We will write just $\mathcal{T}$ for this category if the actual value of n is not important. The category $\mathcal{T}$ is a Hom-finite hereditary abelian category, and we can therefore apply the definition from [BMRRT] to form its cluster category.

The AR-quiver of the bounded derived category $\mathcal{D}^b(\mathcal{T})$ of $\mathcal{T}$ is a countable collection of copies of the tube, one for each shift. See Figure 1. The only maps in $\mathcal{D}^b(\mathcal{T})$ which are not visible as a composition of finitely many maps in the AR-quiver are the maps from each $\mathcal{T}[i]$ to $\mathcal{T}[i+1]$ which correspond to the extensions in $\mathcal{T}$.

The cluster category is now defined as the orbit category $\mathcal{C}_{\mathcal{T}_n} = \mathcal{D}^b(\mathcal{T}_n)/\tau^{-1}[1]$ where τ is the AR translation and [1] is the shift functor. Again, we will sometimes write just $\mathcal{C}_{\mathcal{T}}$. There is a 1-1 correspondence between the indecomposable objects in $\mathcal{C}_{\mathcal{T}_n}$ and those of $\mathcal{T}_n$, since $\operatorname{ind}\mathcal{T}_n$ is itself a fundamental domain for the action of $\tau^{-1}[1]$. We will often denote both an object in $\mathcal{T}_n$ and its orbit as an object in $\mathcal{C}_{\mathcal{T}_n}$ by the same symbol, and we will sometimes refer to the category $\mathcal{C}_{\mathcal{T}_n}$ as a *cluster tube*.

Since $\mathcal{T}$ does not have tilting objects, it does not follow directly from Keller's theorem [K] that $\mathcal{C}_{\mathcal{T}}$ is triangulated. However, $\mathcal{C}_{\mathcal{T}}$ is a thick subcategory of $\mathcal{C}_H$, or, as in [BKL1], a subcategory of the category of sheaves over a weighted projective line. It follows that $\mathcal{C}_{\mathcal{T}}$ is triangulated, and that the canonical functor $\mathcal{D}^b(\mathcal{T}_n) \to C_{\mathcal{T}_n}$ is a triangle functor. It also follows that $\mathcal{C}_{\mathcal{T}}$ is a Hom-finite 2-Calabi-Yau category, since cluster categories of hereditary algebras are, and that we have an AR-formula for the cluster tube as well.

We will use a coordinate system on the indecomposable objects. We will let (a, b) be the unique object with socle $(a, 1)$ and quasi-length b, where the simples are arranged such that $\tau(a, 1) = (a-1, 1)$ for $1 \le a \le n$. Throughout, when we write equations and inequalities which involve first coordinates outside the domain $1, ..., n$, we will implicitly assume identification modulo n. See Figure 2.

Lemma 2.1. *If X and Y are indecomposables in $\mathcal{T}$, we have*

$$\operatorname{Hom}_{\mathcal{C}_{\mathcal{T}}}(X, Y) \simeq D \operatorname{Hom}_{\mathcal{T}}(Y, \tau^2 X) \amalg \operatorname{Hom}_{\mathcal{T}}(X, Y)$$

where D denotes the k-vector space duality $\operatorname{Hom}_k(-, k)$.

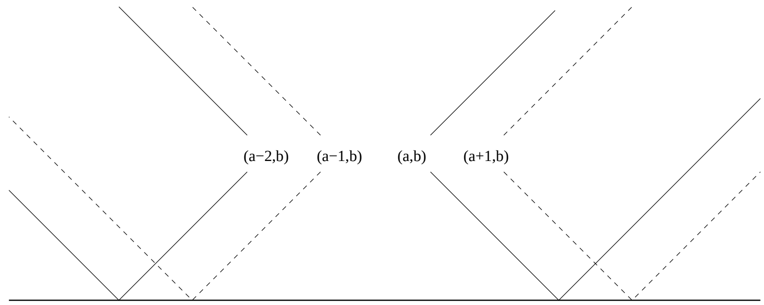


FIGURE 3. For an indecomposable $X = (a, b)$, the Hom-hammock is illustrated by the full lines. Shifting it one to the right, we get the Ext-hammock. The backwards and forwards hammocks will overlap, depending on the rank of the tube.

Proof. By the definition of orbit categories,

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(X,Y) = \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(\mathcal{T})}(\tau^{-i}X[i],Y)$$

Since $\mathcal{T}$ is hereditary, the only possible contribution can be for $i = -1, 0$:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(X,Y) &= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{T})}(\tau X[-1],Y) \amalg \mathrm{Hom}_{\mathcal{D}^b(\mathcal{T})}(X,Y)\\ &\simeq \mathrm{Hom}_{\mathcal{D}^b(\mathcal{T})}(\tau X,Y[1]) \amalg \mathrm{Hom}_{\mathcal{D}^b(\mathcal{T})}(X,Y)\\ &= \mathrm{Ext}^1_{\mathcal{T}}(\tau X,Y) \amalg \mathrm{Hom}_{\mathcal{T}}(X,Y)\\ &\simeq D\,\mathrm{Hom}_{\mathcal{T}}(Y,\tau^2X) \amalg \mathrm{Hom}_{\mathcal{T}}(X,Y).\end{aligned}$$

In the last step, we have used the AR-formula for $\mathcal{T}$. □

The Hom- and Ext-hammocks of an indecomposable object X (that is, the supports of $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(X,-)$ and $\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}}}(X,-)$) are illustrated in Figure 3. For two indecomposables X and Y in $\mathcal{C}_{\mathcal{T}}$, let $\widehat{X}$ and $\widehat{Y}$ be their preimages in $\mathcal{T}$. Then the images in $\mathcal{C}_{\mathcal{T}}$ of maps $\widehat{X} \to \widehat{Y}$ will be called $\mathcal{T}$-maps and the images of maps $\widehat{X} \to \tau^{-1}\widehat{Y}[1]$ will be called $\mathcal{D}$-maps. We will repeatedly use the fact that the existence of a $\mathcal{D}$-map $X \to Y$ is equivalent to the existence of a $\mathcal{T}$-map $Y \to \tau^2X$.

The following lemma is necessary for understanding endomorphism rings of objects in $\mathcal{C}_{\mathcal{T}}$. For any indecomposable $X \in \mathcal{C}_{\mathcal{T}}$ the *ray* starting in $X = (a, b)$ is the infinite sequence of irreducible $\mathcal{T}$-maps

$$\mathbf{R}_X : (a,b) \to (a,b+1) \to \cdots \to (a,b+i) \to \cdots .$$

Similarly, the *coray* ending in X is the infinite sequence of irreducible $\mathcal{T}$-maps

$$\mathbf{C}_X : \cdots \to (a-i,b+i) \to \cdots \to (a-1,b+1) \to (a,b).$$

Note that the sum of the coordinates is constant (as always, modulo n) in a coray.

Lemma 2.2. *Let X and Y be two indecomposable objects in $\mathcal{C}_{\mathcal{T}}$. A $\mathcal{D}$-map in $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(X,Y)$ factors through the ray $\mathbf{R}_X$ starting in X and the coray $\mathbf{C}_Y$ ending in Y.*

Proof. Let X and Y be objects in $\mathcal{T}$ which correspond to X and Y in $\mathcal{C}_{\mathcal{T}}$. Let $\widetilde{Y} = \tau^{-1}Y[1]$ in $\mathcal{D}^b(\mathcal{T})$. We will prove that a map $f \in \mathrm{Hom}_{\mathcal{D}^b(\mathcal{T})}(X,\widetilde{Y})$ factors through the ray starting in X in the AR-quiver of $\mathcal{D}^b(\mathcal{T})$. This will imply that the image of f in $\mathcal{C}_{\mathcal{T}}$ factors through the ray starting in X in the AR-quiver of $\mathcal{C}_{\mathcal{T}}$.

Since $f : X \to \widetilde{Y}$ is not an isomorphism in $\mathcal{D}^b(\mathcal{T})$ for any choice of X and Y, it is enough to show that f factors through the irreducible map g which forms the start of the ray, and we can do this by induction on the quasi-length of X. If $\mathrm{ql}\,X = 1$, then g is a left almost split map in $\mathcal{D}^b(\mathcal{T})$, and the claim holds.

Suppose now that $\operatorname{ql} X \geq 2$. There is an almost split triangle in $\mathcal{D}^b(\mathcal{T})$

$$\begin{array}{ccccc} & & Z & & \\ & \overset{g}{\nearrow} & & \overset{h}{\searrow} & \\ X & & \amalg & & \tau^{-1}X \longrightarrow \\ & \underset{g'}{\searrow} & & \underset{h'}{\nearrow} & \\ & & Z' & & \end{array}$$

Since f is not an isomorphism, f factors through the left almost split map $g \amalg g'$. We have that $\operatorname{ql} Z' = \operatorname{ql} X - 1$, so by induction we can assume that any map in $\operatorname{Hom}_{\mathcal{D}^b(\mathcal{T})}(Z', \widetilde{Y})$ factors through h'. By the mesh relation, $h' \circ g'$ factors through g. Thus f factors through g.

The other assertion is proved dually. □

From now on, suppose T is a maximal rigid object in $\mathcal{C}_{\mathcal{T}_n}$. We will give a complete description of the possible configurations of indecomposable summands of T. As a consequence, we will find that there are no cluster-tilting objects in $\mathcal{C}_{\mathcal{T}}$.

Lemma 2.3. *All summands T' of T must have* $\operatorname{ql} T' \leq n - 1$.

Proof. If $\operatorname{ql} T' \geq n$, then there is a $\mathcal{T}$-map $T' \to \tau T' = T'[1]$, so in particular $\operatorname{Ext}^1_{\mathcal{C}_{\mathcal{T}_n}}(T', T') = \operatorname{Hom}_{\mathcal{C}_{\mathcal{T}}}(T', T'[1]) \neq 0$, and T' is not a summand of a rigid object. □

In what follows, for any indecomposable $X = (a, b)$ in $\mathcal{C}_{\mathcal{T}}$, the *wing determined by* X will mean the set of indecomposables whose position in the AR-quiver is in the triangle which has X on top, that is, (a', b') such that $a' \geq a$ and $a' + b' \leq a + b$. We will use the notation $\mathcal{W}_X$ for this.

Lemma 2.4. *If T_0 is an indecomposable summand of T such that no other summand of T has higher quasi-length, then all the summands of T are in the wing $\mathcal{W}_{T_0}$ determined by T_0.*

Proof. We assume, without loss of generality, that T_0 has coordinates $T_0 = (1, q)$. Let $T_1 = (s, m)$ be a summand of T which maximises the sum $x + y$ with (x, y) the coordinates of summands of T. Recall that such sums are constant modulo n on corays, so this means that all summands of T are in the region bounded by the ray $\mathbf{R}_{(1,1)}$ and the coray $\mathbf{C}_{(s+m-1,1)}$ which passes through T_1 and ends in the quasisimple $(s + m - 1, 1)$.

We note that for $n = 2$, one sees immediately that maximal rigid objects have only one indecomposable summand, and then $T_1 = T_0$ and the claim is trivial. So for the rest of the proof, let $n \geq 3$.

Assume that T_1 is not in $\mathcal{W}_{T_0}$. See Figure 4. Since $\operatorname{Ext}^1_{\mathcal{C}_{\mathcal{T}}}(T_0, T_1) = 0$, we have $q + 2 \leq s$ and $s + m \leq n$. Therefore all summands of T sit inside the wing $\mathcal{W}_X$ where $X = (1, s + m - 1)$. Furthermore, since $s + m - 1 \leq n - 1$, X has no self-extensions and X has no extensions with T, so $\operatorname{Ext}^1_{\mathcal{C}_{\mathcal{T}}}(T \amalg X, T \amalg X) = 0$. Hence X is a summand of T. Since the quasi-length of X is at least the quasi-length of T_0, and X and T_0 are on the same ray, we must have $X = T_0$, but this contradicts the fact that T_1 is not in $\mathcal{W}_{T_0}$. We conclude that all summands are in $\mathcal{W}_{T_0}$. □

For our maximal rigid object T we will in the rest of this section denote by T_0 the unique summand of maximal quasi-length, and we will sometimes call it the top summand.

Lemma 2.5. *The quasi-length of T_0 is $n - 1$.*

Proof. Suppose $\operatorname{ql} T_0 = l_0 < n-1$. Then $T_0 = (m, l_0)$ for some $m \in \{1, ..., n\}$. The object $Y = (m, n-1)$ will satisfy $\operatorname{Ext}^1_{\mathcal{C}_{\mathcal{T}_n}}(T \amalg Y, T \amalg Y) = 0$. But this contradicts the fact that T is maximal rigid. □

With the preceeding series of lemmas at our disposal, we get the following.

Proposition 2.6. *There is a natural bijection between the set of maximal rigid objects in $\mathcal{C}_{\mathcal{T}_n}$ and the set*

$$\left\{ \textit{tilting modules over } \vec{A} \right\} \times \{1, ..., n\}$$

where $\vec{A}$ is a linearly oriented quiver of Dynkin type A_{n-1}.

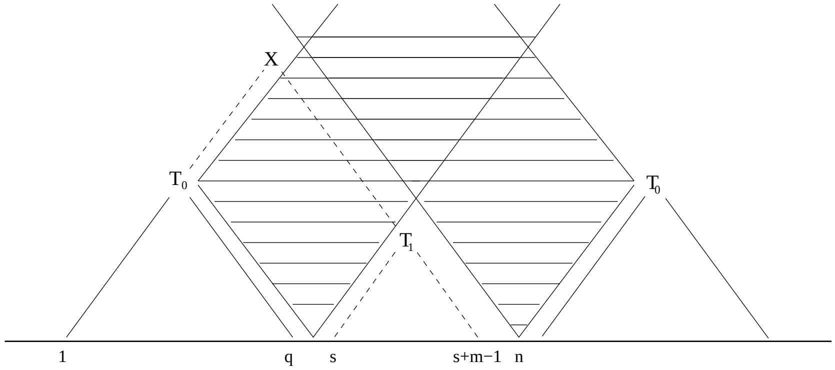


FIGURE 4. The shaded region is the Ext-hammock of T_0. In the proof of Lemma 2.4, if T_1 is outside the wing determined by T_0, then $T \amalg X$ is rigid. Note that $\mathrm{ql}\, X = s + m - 1 \leq n - 1$, since $\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}_n}}(T_0, T_1) = 0$.

Proof. We have already established that all the summands of T must be in the wing determined by the summand T_0 of maximal quasi-length. This wing has exactly the same shape as the AR-quiver of $k\vec{A}$. One can see easily that for an indecomposable object in one such wing, the restriction of the Ext-hammock to the wing exactly matches the (forwards and backwards) Ext-hammocks of the corresponding indecomposable module in the AR-quiver of $k\vec{A}$.

Thus the possible arrangements of pairwise Ext-orthogonal indecomposable objects inside the wing match the possible arrangements of pairwise Ext-orthogonal indecomposable modules in the AR-quiver of $k\vec{A}$.

Since we have n choices for the top summand, we get the bijection by mapping a maximal rigid object in the cluster tube to the pair consisting of the corresponding tilting module over $k\vec{A}$ and the first coordinate of its top summand. □

A rigid object C in a triangulated 2-CY category $\mathcal{C}$ is called *cluster-tilting* if $\mathrm{Ext}^1_{\mathcal{C}}(C, X) = 0$ implies that $X \in \mathrm{add}\, C$. In particular, all cluster tilting objects are maximal rigid. For cluster categories arising from module categories of finite dimensional hereditary algebras, the opposite implication is also true, namely that all maximal rigid objects are cluster-tilting. The cluster tubes provide examples in which this is not the case.

Corollary 2.7. *The category $\mathcal{C}_{\mathcal{T}}$ has no cluster-tilting objects.*

First a technical lemma:

Lemma 2.8. *Let T be a maximal rigid object with top summand T_0, and X an indecomposable which is not in $\mathcal{W}_{T_0}$ and not in $\mathcal{W}_{\tau T_0}$. Then $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T, X) = 0$ if and only if $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T_0, X) = 0$.*

Proof. For any indecomposable object A and an indecomposable B in the wing $\mathcal{W}_A$ determined by A, the restriction of the Hom-hammock of B to $\mathrm{ind}\, \mathcal{C}_{\mathcal{T}} \setminus (\mathcal{W}_A \cup \mathcal{W}_{\tau A})$ is contained in the Hom-hammock of A. (See Figure 3.) □

Proof of Corollary 2.7. Let T be maximal rigid, and assume without loss of generality that $T_0 = (1, n-1)$ is the top summand. Then from the shape of the Hom-hammock of T_0 we see that there are no $\mathcal{T}$-maps from T_0 to any indecomposable object on the ray $\mathbf{R}_{(n,1)}$. Similarly, there are no $\mathcal{D}$-maps from T_0 to any of the indecomposables on the coray $\mathbf{C}_{(n-2,1)}$.

Consider the object $X = (n, 2n-1)$. Then X sits on $\mathbf{R}_{(n,1)}$ and the coray $\mathbf{C}_{(n-2,1)}$, as we see since the sum of the coordinates is congruent to $n-1$ mod n. So there are no maps in $\mathcal{C}_{\mathcal{T}}$ from T_0 to X. Therefore, by Lemma 2.8, we have $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T, X) = 0$. Consequently, if $Y = \tau^{-1} X$, then

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}}}(T, Y) = \mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T, X) = 0.$$

But $Y \notin \mathcal{W}_{T_0}$, so $Y \notin \mathrm{add}\, T$, so T is not cluster-tilting. □

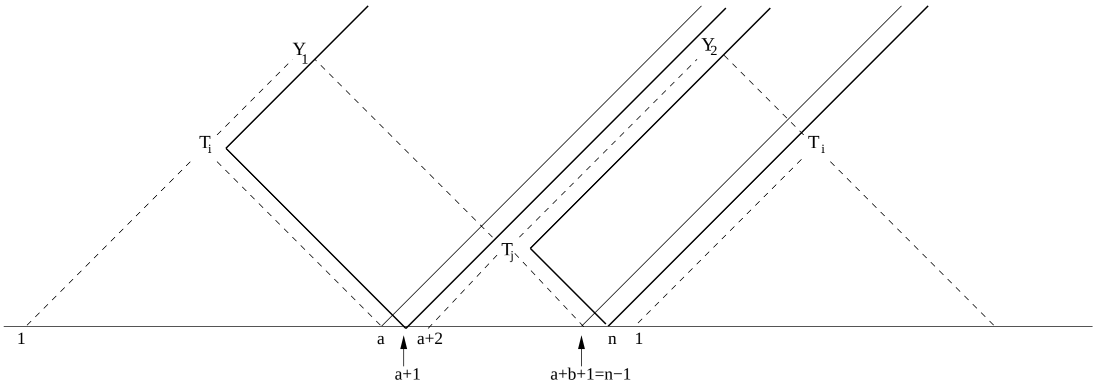


FIGURE 5. A situation from the proof of Proposition 2.9. When $T_i = (1, a)$, the existence of a $\mathcal{T}$-map to $\tau^2 T_j$ implies that $T_j = (a+2, b)$. By the same reasoning, if there is a $\mathcal{D}$-map also in the other direction, then $a+b+1 = n-1$. The only possible positions for T_0 are $Y_1 = (1, n-1)$ and $Y_2 = (a+2, n-1)$.

As remarked after the definition of cluster structures in Section 1, an interpretation of condition (c) in the definition is that the quiver of the endomorphism ring of the maximal rigid object does not have 2-cycles. This is the case here:

Proposition 2.9. *The set of maximal rigid objects in $\mathcal{C}_{\mathcal{T}}$ has a cluster structure.*

Proof. Since $n \geq 2$ there are rigid indecomposables, and since there are only a finite number of rigid indecomposables by Lemma 2.3, the set of maximal rigid objects is non-empty.

By the fact that $\mathcal{C}_{\mathcal{T}}$ is a Hom-finite 2-Calabi-Yau triangulated category, and Theorem 1.1, it is only necessary to show that for any maximal rigid T in $\mathcal{C}_{\mathcal{T}}$, there are no 2-cycles in the quiver of $\mathrm{End}_{\mathcal{C}_{\mathcal{T}}}(T)$. Recall that the vertices of the quiver are in correspondence with the indecomposable summands of T, and the arrows correspond to maps that are irreducible in $\operatorname{add} T$.

We first recall from Lemma 2.4 that all the summands of T lie in the wing $\mathcal{W}_{T_0}$, where T_0 is the top summand of T, and from Lemma 2.5 that T_0 has quasilength $n-1$. Let T_i and T_j be two non-isomorphic indecomposable summands of T.

Assume first that one of these is isomorphic to T_0, say $T_i \simeq T_0$. If X is an indecomposable in $\mathcal{W}_{T_0}$ such that $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(X, T_0) \neq 0$, then T_0 must be on the ray $\mathbf{R}_X$. But then $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T_0, X) = 0$ unless $X \simeq T_0$. So in particular it is impossible that $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T_0, T_j)$ and $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T_j, T_0)$ are both non-zero, and therefore there is no 2-cycle in the quiver of $\mathrm{End}_{\mathcal{C}_{\mathcal{T}}}(T)$ traversing the vertex corresponding to T_0.

We now consider the case when neither T_i nor T_j is isomorphic to T_0. Recall that the existence of a non-zero $\mathcal{D}$-map $X \to Y$ is equivalent to the existence of a non-zero $\mathcal{T}$-map $Y \to \tau^2 X$.

Suppose first that there is a non-zero $\mathcal{T}$-map $T_i \to T_j$. If there was also a non-zero $\mathcal{T}$-map $T_i \to \tau^2 T_j$, then there would be a non-zero $\mathcal{T}$-map $T_i \to \tau T_j$. But this is impossible, since $\mathrm{Ext}^1_{\mathcal{T}}(T_j, T_i) = 0$. So if there is a non-zero $\mathcal{T}$-map $T_i \to T_j$, there cannot be a non-zero $\mathcal{D}$-map $T_j \to T_i$.

If there is a non-zero $\mathcal{T}$-map $T_i \to T_j$, then, since T_i and T_j both sit in a wing of height $n-1$, there cannot be a non-zero $\mathcal{T}$-map $T_j \to T_i$. We therefore conclude that if there is a 2-cycle traversing the vertices corresponding to T_i and T_j, then one of the arrows corresponds to a $\mathcal{D}$-map $T_i \to T_j$, and the other corresponds to a $\mathcal{D}$-map $T_j \to T_i$, and both these maps are irreducible in $\operatorname{add} T$.

So it remains to show that if there are non-zero $\mathcal{T}$-maps both $T_i \to \tau^2 T_j$ and $T_j \to \tau^2 T_i$, then at least one of the corresponding $\mathcal{D}$-maps must be reducible in $\operatorname{add} T$. Assume therefore that two such maps exist. Without loss of generality we can also assume that $T_i = (1, a)$ for some $a \leq n-2$, by if necessary redefining the coordinates. Then, since there is a $\mathcal{T}$-map $T_i \to \tau^2 T_j$, and there are no maps $T_i \to \tau T_j$, we find that T_j must necessarily have coordinates $(a+2, b)$ for some $b \leq n-1$. So T_j sits on the coray $\mathbf{C}_{(a+b+1,1)}$. See Figure 5.

Similarly, since there is also a $\mathcal{T}$-map $T_j \to \tau^2 T_i$, the summand T_i must be on the neighbouring ray beneath the Ext-hammock of T_j, so since $T_i = (1, a)$, we must have that $a+b+1 = n-1$, and in particular that T_j sits on the coray $\mathbf{C}_{(n-1,1)}$. See again Figure 5.

Note now that $\mathcal{W}_{(1,n-1)}$ contains both T_i and T_j, and T_i sits on the left hand edge and T_j sits on the right hand edge. By symmetry, $\mathcal{W}_{(a+2,n-1)}$ also contains both T_i and T_j, with T_j on the left hand edge and T_i on the right hand edge. It follows that no other wing of height $n-1$ contains both T_i and T_j, and thus $T_0 = (1, n-1)$ or $T_0 = (a+2, n-1)$. If we assume that $T_0 = (1, n-1)$, then by Lemma 2.2, the $\mathcal{D}$-map $T_i \to T_j$ factors through both the ray $\mathbf{R}_{T_i}$ and the coray $\mathbf{C}_{T_j}$. In particular, it factors through the $\mathcal{T}$-map from T_i to T_0, and does not correspond to an arrow in the quiver of $\mathrm{End}_{\mathcal{C}_\mathcal{T}}(T)$. Similarly, if $T_0 = (a+2, n-1)$, then the $\mathcal{D}$-map $T_j \to T_i$ does not correspond to an arrow in the quiver. In any case, one of the $\mathcal{D}$-maps is reducible.

We conclude that there are no 2-cycles in the quiver. □

Remark 2.10. One should notice that this set does not satisfy the definition of cluster structures in [BIRS]. The reason for this is that the quiver of $\mathrm{End}_{\mathcal{C}_\mathcal{T}}(T)$ will have a loop for all maximal rigid T. This is because there is a non-zero $\mathcal{D}$-map from T_0 to itself. In addition, we have that the only indecomposable objects in the wing of T_0 which T_0 has non-zero maps to in $\mathcal{C}_\mathcal{T}$ are those on the right hand edge of the wing, and the only indecomposable objects in the wing of T_0 which have non-zero maps in $\mathcal{C}_\mathcal{T}$ to T_0 are those on the left hand edge of the wing. It follows that the $\mathcal{D}$-map from T_0 to itself does not factor through any other indecomposable object in the wing, and therefore it does not factor through any other indecomposable direct summand of T.

3. Relationship to type B cluster algebras

Cluster algebras were introduced in [FZ1]; see, for example, [FR] or [FZ4] for an introduction.

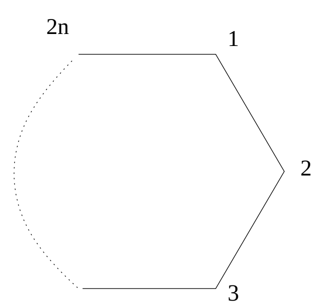
A simplicial complex was associated in [FZ3] to any finite root system, and it was conjectured there and later proved in [CFZ] that these simplicial complexes are the face complexes of certain polytopes which were called *generalised associahedra*. In the finite type classification of cluster algebras [FZ2], it was shown that a cluster algebra is of finite type if and only if its cluster complex is one of these simplicial complexes.

The generalised associahedron associated to a type B root system turned out to be the *cyclohedron*, also known as the Bott-Taubes polytope [BT]. This polytope was independently discovered by Simion [S].

We will now recall the description of the exchange graph from [FZ3] (which corresponds to the geometric description of the corresponding polytope in [S]). Let $\mathcal{G}_n$ denote a regular $2n$-gon. The set of cluster variables in a type B_{n-1} cluster algebra is in bijection with the set $\mathscr{D}_n$ of *centrally symmetric pairs of diagonals* of $\mathcal{G}_n$, where the diameters are included as degenerate pairs. Under this bijection, the clusters correspond to the centrally symmetric triangulations of $\mathcal{G}_n$, and exchange of a cluster variable corresponds to flipping either a pair of centrally symmetric diagonals or a diameter.

In this section we define a bijection from the set of indecomposable rigid objects in the cluster tube $\mathcal{C}_{\mathcal{T}_n}$ to the set $\mathscr{D}_n$. This map induces a correspondence between the maximal rigid objects and centrally symmetric triangulations which is compatible with exchange. Thus rigid objects in $\mathcal{C}_{\mathcal{T}_n}$ model the cluster combinatorics of type B_{n-1} cluster algebras.

We label the corners of $\mathcal{G}_n$ clockwise, say, from 1 to $2n$:



For two corners labelled a and b (which are neither equal nor neighbours), we denote by $[a, b]$ the corresponding diagonal. Thus $[a, b] = [b, a]$, and the centrally symmetric pairs of diagonals are given as $([a, b], [a+n, b+n])$. (Here and in what follows, we reduce modulo $2n$ if necessary.) Since this pair

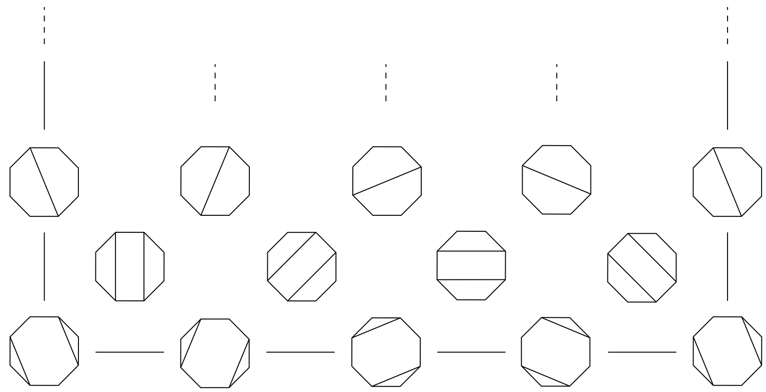


FIGURE 6. The AR-quiver of $\mathcal{C}_{\mathcal{T}_4}$, with the indecomposable rigid objects replaced by their images under the map δ.

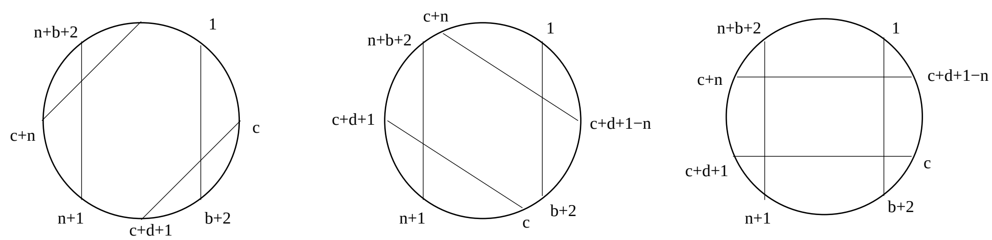


FIGURE 7. Three cases in the proof of Proposition 3.2. Left, condition (i) only holds; middle, condition (ii) only holds; right, conditions (i) and (ii) both hold.

is uniquely determined by either of the two diagonals, we will sometimes denote the pair by one of its representatives.

Now, for each indecomposable rigid object in $\mathcal{C}_{\mathcal{T}_n}$ we assign a centrally symmetric pair of diagonals as follows:

$$(a,b) \overset{\delta}{\longmapsto} ([a, a+b+1], [a+n, a+b+1+n])$$

Note that the pairs of diagonals assigned to indecomposable objects in the same ray or coray of $\mathcal{C}_{\mathcal{T}_n}$ all share a centrally symmetric pair of corners. Objects of quasi-length 1 correspond to the shortest diagonals, while the objects of quasi-length $n-1$ correspond to the diameters. See Figure 6.

The following fact is readily verified.

Lemma 3.1. *The map δ defined above is a bijection from the set of indecomposable rigid objects in $\mathcal{C}_{\mathcal{T}_n}$ to the set $\mathscr{D}_n$ of centrally symmetric pairs of diagonals of $\mathcal{G}_n$.*

Proposition 3.2. *Let $T_1 = (a,b)$ and $T_2 = (c,d)$ be indecomposable rigid objects in $\mathcal{C}_{\mathcal{T}_n}$. Then the number of crossing points of $\delta(a,b)$ and $\delta(c,d)$ is equal to $2 \dim \operatorname{Ext}^1_{\mathcal{C}_{\mathcal{T}}}(T_1, T_2)$.*

Proof. Without loss of generality, we may assume that $a = 1$. We have

$$\begin{aligned} \delta(1,b) &= ([1, b+2], [n+1, b+n+2]) \\ \delta(c,d) &= ([c, c+d+1], [c+n, c+d+n+2]). \end{aligned}$$

It is easy to check that $\delta(1,b)$ and $\delta(c,d)$ cross if and only if one of the following two conditions holds:

(i) $1 < c < b+2$ and $c+d > b+1$;
(ii) $1 < c+d+1-n < b+2$ and $1 < c < n+1$

Furthermore, the number of crossing points is 4 when both conditions hold, and is 2 if only one holds. See Figure 7.

From the structure of the tube, it can be checked that condition (i) holds if and only if $\operatorname{Hom}_{\mathcal{T}}(T_1, \tau T_2) \neq 0$, and in this case $\dim \operatorname{Hom}_{\mathcal{T}}(T_1, \tau T_2) = 1$. Similarly, condition (ii) holds if and only if $\operatorname{Hom}_{\mathcal{T}}(T_2, \tau T_1) \neq$

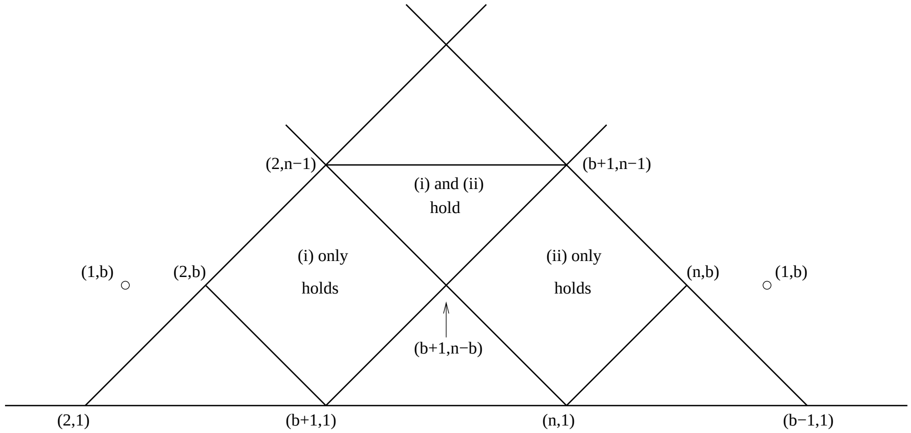


FIGURE 8. Three situations for extensions with the object $(1, b)$, as in the proof of Proposition 3.2

0, and in this case $\dim \mathrm{Hom}_{\mathcal{T}}(T_2, \tau T_1) = 1$. By Lemma 2.1 and the Auslander-Reiten formula,

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}}}(T_1, T_2) = D\,\mathrm{Hom}_{\mathcal{T}}(T_2, \tau T_1) \amalg \mathrm{Hom}_{\mathcal{T}}(T_1, \tau T_2)$$

and the result follows. □

Corollary 3.3. *The map δ induces a bijection between the indecomposable rigid objects in $\mathcal{C}_{\mathcal{T}_n}$ and the cluster variables in a type B_{n-1} cluster algebra, and under this bijection, the maximal rigid objects correspond to the clusters.*

Proof. By the work of [FZ2, FZ3], we need to show that the image under δ of the summands of a maximal rigid object coincides with a set of pairs of diagonals which form a centrally symmetric triangulation of $\mathcal{G}_n$. This is clear from Lemma 3.1 and Proposition 3.2. □

Now let T_{init} be the zig-zag maximal rigid object

$$T_{\mathrm{init}} = \coprod_{i=1}^{t} (i, n-2i+1) \amalg (i, n-2i)$$

where $t = \frac{n}{2}$ for n even and $t = \frac{n-1}{2}$ for n odd, and any expression with zero in the last coordinate is to be disregarded.

Proposition 3.4. *The Cartan counterpart of the matrix $B_{T_{init}}$ is the Cartan matrix for the root system of type B_{n-1}.*

Proof. We recall that the Cartan counterpart of a matrix $B = (b_{ij})$ is the matrix $A(B) = (a_{ij})$ given by $a_{ii} = 2$ and $a_{ij} = -|b_{ij}|$ when $i \neq j$.

For this proof, we set T_i to be the summand of T_{init} with quasi-length $n - i$, for $i = 1, ..., n-1$. In particular the top summand is T_1.

For convenience, we define $T_n = 0$. For each T_i with i even and $2 \leq i \leq n-1$, the exchange triangles are

$$T_i^* \to 0 \to T_i \to$$
$$T_i \to T_{i-1} \amalg T_{i+1} \to T_i^* \to$$

while for i odd and $1 < i \leq n-1$, the exchange triangles are

$$T_i^* \to T_{i-1} \amalg T_{i+1} \to T_i \to$$
$$T_i \to 0 \to T_i^* \to$$

In the quiver of $\mathrm{End}_{\mathcal{C}_{\mathcal{T}}}(T_{\mathrm{init}})$, there is a loop on the vertex corresponding to the summand T_1. (See Remark 2.10.) However, twice around this loop is a zero relation, so the exchange triangles for this summand are

$$T_1^* \to T_2 \amalg T_2 \to T_1 \to$$

$$T_1 \to 0 \to T_1^* \to$$

So the matrix of the exchange triangles is

$$B_{T_{\mathrm{init}}} = \begin{pmatrix} 0 & -2 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & -1 & & 0 \\ 0 & 0 & 1 & 0 & \ddots & \\ \vdots & \vdots & & \ddots & & (-1)^{n-2} \\ 0 & 0 & & & (-1)^{n-1} & 0 \end{pmatrix}$$

and the Cartan counterpart is

$$A(B_{T_{\mathrm{init}}}) = \begin{pmatrix} 2 & -2 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & & 0 \\ 0 & 0 & -1 & 2 & \ddots & \\ \vdots & \vdots & & \ddots & & -1 \\ 0 & 0 & & & -1 & 2 \end{pmatrix}$$

□

Noting that for a cluster algebra of type B, a cluster determines its seed [FZ2], we have thus proved the following theorem:

Theorem 3.5. *There is a bijection between the indecomposable rigid objects of a cluster tube of rank n and the cluster variables of a cluster algebra of type B_{n-1}, inducing a bijection between the maximal rigid objects of the cluster tube and the clusters of the cluster algebra. Furthermore, the exchange matrix of a seed coincides with the matrix associated to the corresponding maximal rigid object in Section 1.*

Proof. The first part of the theorem has been shown above (Corollary 3.3). The second part follows from Theorem 1.1 and Proposition 3.4, noting that the exchange matrix corresponding to the initial root cluster $\{-\alpha_1, -\alpha_2, \ldots, -\alpha_{n-1}\}$ of type B_{n-1} in [FZ2] is the matrix $B_{T_{\mathrm{init}}}$ arising from the exchange triangles corresponding to the maximal rigid object T_{init} in Proposition 3.4. (See also Figure 5 in [FZ3]). □

Acknowledgements

The first and third named author wish to thank Robert Marsh and the School of Mathematics at the University of Leeds for their kind hospitality in the spring of 2008.

The first author is supported by an NFR Storforsk-grant. The second author acknowledges support from the EPSRC, grant number EP/C01040X/2, and is also currently an EPSRC Leadership Fellow, grant number EP/G007497/1.

We would also like to thank the referee for valuable comments.

References

[ABS] I. Assem, T. Brüstle, R. Schiffler *Cluster-tilted algebras as trivial extensions*, Bull. London Math. Soc. **40** (1), 151-162 (2008)

[AS] M. Auslander, S. Smalø *Preprojective modules over Artin algebras*, J. Algebra **66** (1), 61-122 (1980)

[BKL1] M. Barot, D. Kussin, H. Lenzing *The Grothendieck group of a cluster category*, J. Pure Appl. Algebra **212** (1), 33-46 (2008)

[BKL2] M. Barot, D. Kussin, H. Lenzing *The cluster category of a canonical algebra*, to appear in Trans. Amer. Math. Soc, preprint v. 3 arxiv:math.RT/0801.4540 (2008)

[BT] R. Bott, C. Taubes *On the self-linking of knots. Topology and physics*, J. Math. Phys. **35** (10), 5247-5287 (1994)

[BIRS] A. B. Buan, O. Iyama, I. Reiten, J. Scott *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, to appear in Composito Math., preprint v.3 arxiv:math/0701557 (2007)

[BMR] A. B. Buan, R. Marsh, I. Reiten *Cluster mutation via quiver representations*, Comment. Math. Helv. **83** (1), 143-177 (2008)

[BMRRT] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2), 572-618 (2006)

[BCKMRT] A. B. Buan, P. Caldero, B. Keller, R. Marsh, I. Reiten, G. Todorov *Appendix to Clusters and seeds for acyclic cluster algebras*, Proc. Amer. Math. Soc. **135** (10), 3049-3060 (2007)

[BIKR] I. Burban, O. Iyama, B. Keller, I. Reiten *Cluster tilting for one-dimensional hypersurface singularities*, Adv. Math. **217** (6), 2443-2484 (2008)

[CC] P. Caldero, F. Chapoton *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (3), 595-616 (2006)

[CCS] P. Caldero, F. Chapoton, R. Schiffler *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. **358** (3), 1347-1364 (2006)

[CK] P. Caldero, B. Keller *From triangulated categories to cluster algebras II*, Ann. Sci. École Norm. Sup. (4) **39** (6), 983-1009 (2006)

[CFZ] F. Chapoton, S. Fomin, A. Zelevinsky *Polytopal realizations of generalized associahedra*, Canad. Math. Bull. **45** (4), 537-566 (2002)

[FR] S. Fomin, N. Reading *Root systems and generalized associahedra*, IAS/Park City Math. Ser. **13**, 63-131 (2004)

[FZ1] S. Fomin, A. Zelevinsky *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2), 497-529 (2002)

[FZ2] S. Fomin, A. Zelevinsky *Cluster algebras II: Finite type classification*, Invent. Math. **154** (1), 63-121 (2003)

[FZ3] S. Fomin, A. Zelevinsky *Y-systems and generalized associahedra*, Ann. Math. (2) **158** (3), 977-1018 (2003)

[FZ4] S. Fomin, A. Zelevinsky *Cluster algebras: Notes for the CDM-03 conference*, CDM 2003: Current Developments in Mathematics, International Press (2004)

[GLS] C. Geiß, B. Leclerc, J. Schröer *Rigid modules over preprojective algebras*, Invent. Math. **165** (3), 589-632 (2006)

[IY] O. Iyama, Y. Yoshino *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (1), 117-168 (2008)

[K] B. Keller *On triangulated orbit categories*, Doc. Math. **10**, 551-581 (2005)

[KR] B. Keller, I. Reiten *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (1), 123-151 (2007)

[MRZ] R. Marsh, M. Reineke, A. Zelevinsky *Generalized associahedra via quiver representations*, Trans. Amer. Math. Soc. **355** (10), 4171-4186 (2003)

[S] R. Simion *A type-B associahedron*, Adv. in Appl. Math. **30**, 2-25 (2003)

Institutt for matematiske fag, Norges teknisk-naturvitenskapelige universitet, N-7491 Trondheim, Norway

E-mail address: aslakb@math.ntnu.no

Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England

E-mail address: marsh@maths.leeds.ac.uk

Institutt for matematiske fag, Norges teknisk-naturvitenskapelige universitet, N-7491 Trondheim, Norway

E-mail address: dvatne@math.ntnu.no