Algorithms to Obtain the Canonical Basis

1 Introduction

Let $\mathbf{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, and let U be the q-analogue of its universal enveloping algebra defined by Drinfel'd [3] and Jimbo [4]. According to [7, 3.5.6, 6.2.3 & 6.3.4], for each dominant weight λ in the weight lattice of $\mathbf{g}$ there is an irreducible, finite-dimensional highest weight U-module $V(\lambda)$ with highest weight λ. Kashiwara [5] and Lusztig [7, 14.4.12] have independently shown the existence of a certain canonical basis $\mathbf{B}(\lambda)$ for $V(\lambda)$. In [5], Kashiwara proves the existence of a *crystal basis* associated with $V(\lambda)$ using certain operators, and defines a graph (the *crystal graph*) which encodes how these operators act on the crystal basis. Using this, he defines $\mathbf{B}(\lambda)$. In Section 2, we give a description of the crystal basis and some of its properties. In [6], Kashiwara and Nakashima give explicit descriptions of the crystal basis for $V(\lambda)$ when $\mathbf{g}$ is of type A, B, C, or D. In Section 3, we note that if we can use these descriptions to find a sequence of Kashiwara operators (see Definition 2.1) satisfying certain properties, for each element of the crystal basis, then the canonical basis can be written in a very nice way in terms of these sequences. In sections 4, 5, 6 and 7, we present algorithms which generate sequences with the required properties and thus find the canonical basis, for most of the fundamental modules in types A, C, D and B, respectively (see Theorems 4.2, 5.1, 6.1 and 7.2). For the remaining cases see [8]. Furthermore, we show that with our choice of sequences there is a close link between the crystal and canonical bases of the fundamental modules in type C_n $(n \geq 3)$ and the corresponding bases in type A_{2n-1} (see Proposition 5.2) and also between the crystal and canonical bases of the fundamental modules in type D_n $(n \geq 4)$ and the corresponding bases in type B_{n-1} (see Proposition 7.3).

We use the treatment in [7, §§1-3]. Let $\mathbf{g}$ be a semisimple Lie algebra, with root system Φ, simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, and Killing form $(\ , \ )$. Let $h_1, h_2, \ldots, h_n$ be a basis for a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, satisfying $(h_i, h) = \alpha_i^*(h)$ for all h in $\mathbf{h}$ and all $i \in I = \{1, 2, \ldots, n\}$. Let Y be the $\mathbb{Z}$-lattice spanned by $h_1, h_2, \ldots, h_n$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the fundamental weights of $\mathbf{g}$, defined by $\omega_i(h_j) = \delta_{ij}$, and let X be the $\mathbb{Z}$-lattice spanned by them (the weight lattice). Let d be the minimal positive integer so that $d(\alpha_i, \alpha_j)$ is always an integer and $d(\alpha_i, \alpha_i)$ is always even. If the highest common factor of the $d(\alpha_i, \alpha_j)$ and the $\frac{1}{2}d(\alpha_i, \alpha_i)$ is not 1, then replace d by d divided by this highest common factor. We then define $i \cdot j$ to be $d(\alpha_i, \alpha_j)$ for each $i, j \in I$, so $(I, \cdot)$ is a Cartan datum as in [7, 1.1.1]. For $\mu \in Y$ and $\lambda \in X$, define $\langle \mu, \lambda \rangle$ to be $\lambda(\mu)$. Define an imbedding of I into Y by $i \mapsto h_i$ and into X by $i \mapsto \alpha_i$ for all $i \in I$. We then have a root datum of type $(I, \cdot)$ as in [7, 2.2.1], with $\langle h_i, \alpha_j \rangle = \alpha_j(h_i) = A_{ij}$ the corresponding symmetrizable Cartan matrix. For each $i \in I$, we define d_i to be the integer $\frac{1}{2}d(\alpha_i, \alpha_i)$. Then $d_i A_{ij} = \frac{1}{2}d(\alpha_i, \alpha_i)\left(\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right) = d(\alpha_i, \alpha_j)$ for each $i, j \in I$, and is thus a symmetric matrix over $\mathbb{Z}$. We use the same numbering as [2, Planches 1 to IX].

Let $\mathbb{Q}(v)$ be the field of rational functions in an indeterminate v, and $\mathcal{A} \subseteq \mathbb{Q}(v)$ the ring $\mathbb{Z}[v, v^{-1}]$. For $N, M \in \mathbb{N}$ and $i \in I$ we put $v_i = v^{d_i}$ and define the following (which all lie in $\mathcal{A}$):

$$[N]_i = \frac{v_i^N - v_i^{-N}}{v_i - v_i^{-1}}, \quad [N]_i^! = [N]_i[N-1]_i \cdots [1]_i, \quad \left[\begin{array}{c} M \\ N \end{array}\right]_i = \frac{[M]_i^!}{[N]_i^![M-N]_i^!}.$$

These are referred to as quantized integers, quantized factorials and quantized binomial coefficients, respectively. If v is specialised to 1 they specialise to the usual integers, factorials and binomial coefficients.

We define the quantized enveloping algebra U corresponding to the above data (as in [7, 3.1.1 & 33.1.5]) to be the $\mathbb{Q}(v)$-algebra U with generators $1, E_1, E_2, \ldots, E_n$, $F_1, F_2, \ldots, F_n$, and K_μ for $\mu \in Y$, subject to the relations: (for each $i, j \in I$ and $\mu, \mu' \in Y$)

$$K_0 = 1,$$

$$K_\mu K_{\mu'} = K_{\mu+\mu'},$$

$$K_\mu E_i = v^{\alpha_i(\mu)} E_i K_\mu,$$

$$K_\mu F_i = v^{-\alpha_i(\mu)} F_i K_\mu,$$

$$E_i F_i - F_i E_i = \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v_i - v_i^{-1}},$$

$$E_i F_j - F_j E_i = 0, \quad i \neq j,$$

$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \left[\begin{array}{c} 1 - A_{ij} \\ p' \end{array}\right]_i E_i^p E_j E_i^{p'} = 0, \quad i \neq j,$$

$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \left[\begin{array}{c} 1 - A_{ij} \\ p' \end{array}\right]_i F_i^p F_j F_i^{p'} = 0, \quad i \neq j,$$

(where, for $i \in I$, we put $\tilde{K}_i = K_{d_i h_i}$ and $\tilde{K}_i^{-1} = K_{-d_i h_i}$). In the last two summations, p and p' are restricted to the non-negative integers.

We make the following definitions (see [7, 3.1.1 & 3.1.13]). For $M \in \mathbb{N}$, and $i \in I$, we put $E_i^{(M)} = E_i^M/[M]_i^!$, and $F_i^{(M)} = F_i^M/[M]_i^!$, which are called *divided powers*. We also put $K_i =$

K_{h_i} and $K_i^{-1} = K_{-h_i}$ for $i \in I$. Let $U_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of U generated by the elements $E_i^{(N)}, F_i^{(N)}, K_\mu$ for $i \in I$, $N \in \mathbb{N}$ and $\mu \in Y$. It is called the *integral form* of U. Let U^+ be the $\mathbb{Q}(v)$-subalgebra of U generated by the E_i, $i \in I$, and $U^+_{\mathcal{A}}$ the $\mathcal{A}$-subalgebra of U generated by $E_i^{(N)}$, $i \in I, N \in \mathbb{N}$. Let U^- be the $\mathbb{Q}(v)$-subalgebra of U generated by the F_i, $i \in I$, and $U^-_{\mathcal{A}}$ the $\mathcal{A}$-subalgebra of U generated by $F_i^{(N)}$, $i \in I, N \in \mathbb{N}$. Let U^0 be the $\mathbb{Q}(v)$-subalgebra generated by the K_μ, $\mu \in Y$.

Let W be the Weyl group of $\mathbf{g}$. So W is the group:

$$W = \langle s_1, s_2, \dots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \ (i \neq j) \rangle$$

where $m_{ij} = 2, 3, 4, 6$ if $A_{ij}A_{ji} = 0, 1, 2, 3$, respectively. For $r \in I$, let W^r be the set of distinguished left coset representatives of the parabolic subgroup W_r of W generated by $\{s_1, s_2, \dots, s_n\} \setminus \{s_r\}$.

Let $X^+ \subseteq X$ be the set of dominant weights, i. e. those of the form $\lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n \in X$ where $\omega_1, \omega_2, \dots, \omega_n$ are the fundamental weights of $\mathbf{g}$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{N}$. If L is a U-module, $x \in L$ and $\lambda \in X$, we say that x has weight λ if $K_\mu x = v^{\lambda(\mu)} x$ for all $\mu \in Y$. We call the subspace of L consisting of all of the elements of weight λ the λ-*weight space* of L. As in [7, 3.4.1], we restrict our attention to U-modules which are direct sums of their weight spaces. We say that $x \in L$, $x \neq 0$, is a *highest* (respectively, *lowest*) weight vector if x has weight λ, for some $\lambda \in X$, $E_i x = 0$ (respectively, $F_i x = 0$) for each $i \in I$ and $U^- x = L$ (respectively, $U^+ x = L$). Such a vector is uniquely determined up to a non-zero scalar multiple. We say that L is a highest weight module with highest weight λ if it contains a highest weight vector of weight λ. Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n$ be a dominant weight. We follow the construction in [7, 3.4.5 & 3.5.6]. Let J be the left ideal of U generated by the elements E_i for $i \in I$ and the elements $K_\mu - v^{(\lambda(\mu))}$ for $\mu \in Y$. Then the map from U^- to U/J taking $x \in U^-$ to $x + J$ is a $\mathbb{Q}(v)$-vector space isomorphism, which can be used to transfer the left U-module structure of U/J to U^-. The resulting U-module we denote by $M(\lambda)$; it is the called a *Verma module*. Let $T(\lambda)$ be the left ideal of $M(\lambda)$ (as a $\mathbb{Q}(v)$-algebra) generated by the elements $F_i^{\lambda_i+1}$, for $i \in I$, and let $V(\lambda)$ be the quotient module $M(\lambda)/T(\lambda)$. Then, by [7, 6.2.3 & 6.3.4], $V(\lambda)$ is an irreducible, finite-dimensional highest weight U-module with highest weight λ, unique up to isomorphism. We fix x_1 as the image of $1 \in M(\lambda)$ under the natural map from $M(\lambda)$ to $V(\lambda)$. Then x_1 is a highest weight vector for $V(\lambda)$. If λ and λ' are any two distinct dominant weights, then $V(\lambda)$ and $V(\lambda')$ are not isomorphic (see [7, 6.2.3(b)]). It is known that $V(\lambda)$ is the direct sum of its weight spaces (see [7, 3.4.1 & 3.5.6]). We also write $V(\lambda)_{\mathcal{A}} = U^-_{\mathcal{A}} x_1$, the integral form of $V(\lambda)$ (see [7, 19.3.1]). For $\mu \in Y$, we have $K_\mu V(\lambda)_{\mathcal{A}} \subseteq V(\lambda)_{\mathcal{A}}$, since K_μ always acts as an integral power of v on an element in a weight space and $V(\lambda)_{\mathcal{A}}$ is the direct sum of its weight spaces (see [7, 19.3.1]). Therefore, by [7, 19.3.2], $V(\lambda)_{\mathcal{A}}$ is a $U_{\mathcal{A}}$-module. For each $r \in I$ we denote by V_r the module $V(\omega_r)$ with highest weight ω_r. This is called the r-th *fundamental module* for U.

2 The Crystal Basis

We shall need the following definition of the Kashiwara operators (see [5, §2.2]):

Definition 2.1 Suppose that λ is a dominant weight, and $V(\lambda)$ is the corresponding U-module as above. Fix $i \in [1, n]$. Any element $m \in V(\lambda)$ can be written uniquely $m = \sum_{0 \leq k \leq k'} F_i^{(k)} x_{k,k'}$, where the $x_{k,k'}$ satisfy $E_i x_{k,k'} = 0$ and $K_i x_{k,k'} = v^{k'} x_{k,k'}$. Then define $\widetilde{F}_i(m) = \sum_{0 \leq k \leq k'} F_i^{(k+1)} x_{k,k'}$, and $\widetilde{E}_i(m) = \sum_{1 \leq k \leq k'} F_i^{(k-1)} x_{k,k'}$.

Following Kashiwara [5, 2.3.1] we make the following definition:

Definition 2.2 Suppose that M is an integrable U-module. Let R be the ring of rational functions in v regular at v specialised to 0. A pair $(\mathcal{L}, B)$ is called a *crystal basis* of M if the following conditions hold:

(1) $\mathcal{L}$ is a free R-submodule of M, and $\mathbb{Q}(v) \otimes_R \mathcal{L} \cong M$.

(2) B is a basis of the $\mathbb{Q}$-vector space $\mathcal{L}/v\mathcal{L}$.

(3) $\mathcal{L} = \bigoplus_\nu \mathcal{L}_\nu$ and $B = \sqcup_\nu B_\nu$, where M_ν is the ν-weight space of M, $\mathcal{L}_\nu = \mathcal{L} \cap M_\nu$, and $B_\nu = B \cap (\mathcal{L}_\nu / v\mathcal{L}_\nu)$.

(4) $\widetilde{F}_i \mathcal{L} \subseteq \mathcal{L}$ and $\widetilde{E}_i \mathcal{L} \subseteq \mathcal{L}$, for all $i \in [1, n]$, where $\widetilde{E}_i$ and $\widetilde{F}_i$ are the Kashiwara operators (see Definition 2.1).

(5) $\widetilde{F}_i B \subseteq B \cup \{0\}$ and $\widetilde{E}_i B \subseteq B \cup \{0\}$, for all $i \in [1, n]$.

(6) For $b_1, b_2 \in B$ and $i \in [1, n]$, $b_1 = \widetilde{E}_i b_2$ if and only if $b_2 = \widetilde{F}_i b_1$.

We use (6) to draw the corresponding crystal graph, which is an indication of how the Kashiwara operators act on the crystal basis. There is one vertex corresponding to each element of B. If b_1 and

b_2 are as in (6), we draw an edge from the vertex corresponding to b_1 to the vertex corresponding to b_2, with the arrow from b_1 to b_2, and the label i on it.

The modules $V(\lambda)$, for λ a dominant weight, always possess a crystal basis:

Theorem 2.3 *Suppose that λ is a dominant weight and $V(\lambda)$ is the irreducible finite-dimensional highest weight U-module with highest weight λ, generated by our fixed highest weight vector x_1, as above. Let $L(\lambda)$ be the R-submodule generated by the vectors of the form $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}x_1$ and let $B(\lambda)$ be the subset of $L(\lambda)/vL(\lambda)$ consisting of the non-zero images under the natural projection $L(\lambda)\to L(\lambda)/vL(\lambda)$ of these vectors. Then $(L(\lambda),B(\lambda))$ is a crystal basis of $V(\lambda)$.*

Proof: See [5, §2.6]. □

Theorem 2.4 *Suppose λ is a dominant weight, $V(\lambda)$ is the corresponding U-module, and $(\mathcal{L},B)$ is a crystal basis for $V(\lambda)$. Then there is an automorphism of $V(\lambda)$ taking $(\mathcal{L},B)$ to $(L(\lambda),B(\lambda))$.*

Proof: This follows from Theorem 3 in [5, §2.6], also noting [7, 6.2.3(b)]. □

If ϕ is an automorphism of $V(\lambda)$ as a U-module, and x_1 is our fixed highest weight vector in $V(\lambda)$, then it is easy to see that $\phi(x_1)$ is again a highest weight vector, and therefore must be a non-zero scalar multiple of x_1. Since x_1 generates $V(\lambda)$ as a U^--module, the automorphism ϕ must be merely multiplication by this scalar. So, while the crystal basis is not unique, different crystal bases are very closely related. Note also that, by Theorem 2.4, the crystal graph corresponding to any crystal basis of $V(\lambda)$ is isomorphic (as an oriented, coloured graph) to the crystal graph corresponding to any other crystal basis, so the crystal graph of such a module can be defined to be the crystal graph corresponding to any crystal basis.

Definition 2.5 There is a comultiplication Δ on U, which is a $\mathbb{Q}(v)$-algebra homomorphism from

U to $U \otimes U$, having the following effect on the generators of U:

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes \tilde{K}_i^{-1} + 1 \otimes E_i, \\
\Delta(F_i) &= F_i \otimes 1 + \tilde{K}_i \otimes F_i, \\
\Delta(K_\mu) &= K_\mu \otimes K_\mu, \\
\Delta(1) &= 1 \otimes 1.
\end{aligned}$$

(where $i \in [1, n]$ and $\mu \in Y$). This is the comultiplication Δ_- as in [5, §1.4].

We write $\Delta^{(2)} = (\Delta \otimes 1)\Delta$, $\Delta^{(3)} = (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)\Delta$ and similarly $\Delta^{(p)}$ for any positive integer p. So $\Delta^{(p-1)} : U \to U^{\otimes p}$. If M_j is a U-module, $j = 1, 2, \ldots, p$, we can make $M_1 \otimes M_2 \otimes \cdots \otimes M_p$ into a U-module by defining for $u \in U, m_j \in M_j$, $j = 1, 2, \ldots, p$,

$$u.(m_1 \otimes m_2 \otimes \cdots \otimes m_p) = \Delta^{(p-1)}(u)(m_1 \otimes m_2 \otimes \cdots \otimes m_p),$$

and extending linearly to the whole of $M_1 \otimes M_2 \otimes \cdots \otimes M_p$. Here we use the natural action of $U^{\otimes p}$ on $M_1 \otimes M_2 \otimes \cdots \otimes M_p$.

Theorem 2.6 *Suppose $(\mathcal{L}_j, B_j)$ is a crystal basis of a U-module M_j, for $j = 1, 2$, where each M_j is a module $V(\lambda)$ for some dominant weight λ.*

(a) Set $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2 \subseteq M_1 \otimes M_2$ and $B = \{b_1 \otimes b_2 \ : \ b_j \in B_j, \ j = 1, 2\} \subseteq \mathcal{L}/v\mathcal{L}$. Then $(\mathcal{L}, B)$ is a crystal basis of $M_1 \otimes M_2$.

(b) For $b_1 \in B_1$, $b_2 \in B_2$ and $i \in [1, n]$, we have:

$$\widetilde{F}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{F}_i b_1 \otimes b_2 & \text{if there exists } m \geq 1 \text{ such that } \widetilde{F}_i^m b_1 \neq 0 \\ & \quad \text{and } \widetilde{E}_i^m b_2 = 0; \\ b_1 \otimes \widetilde{F}_i b_2 & \text{otherwise.} \end{cases}$$

$$\widetilde{E}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{E}_i b_2 & \text{if there exists } m \geq 1 \text{ such that } \widetilde{E}_i^m b_2 \neq 0 \\ & \quad \text{and } \widetilde{F}_i^m b_1 = 0; \\ \widetilde{E}_i b_1 \otimes b_2 & \text{otherwise.} \end{cases}$$

Proof: See [5, §2.4]. □

Following Kashiwara and Nakashima (see [6, 2.1.2]), we make the following definition:

Definition 2.7 Fix $i \in [1, n]$. Let λ be a dominant weight, and $V(\lambda)$ the corresponding U-module, with a crystal basis $(\mathcal{L}, B)$, and $b_1, b_2 \in B$. Suppose further that $\widetilde{E}_i b_1 = \widetilde{F}_i b_2 = 0$ and $\widetilde{F}_i b_1 = b_2$. We say that b_1 is *of type u_+ for i* and that b_2 is *of type u_- for i*. If $b \in B$ and $\widetilde{E}_i b = \widetilde{F}_i b = 0$, we say that b is *of type u_0 for i*.

Suppose instead that $b_1, b_2, b_3 \in B$, and that $\widetilde{E}_i b_1 = \widetilde{F}_i b_3 = 0$, $\widetilde{F}_i b_1 = b_2$ and $\widetilde{F}_i b_2 = b_3$. We say, for $j = 1, 2, 3$, that b_j is *of type u_j for i*.

Proposition 2.8 *Fix $i \in [1, n]$. Suppose that M_j, $j = 1, 2, \ldots, p$, are each U-modules of the form $V(\lambda)$, where λ is some dominant weight, and that $(\mathcal{L}_j, B_j)$ is a crystal basis for each M_j, with $b_j \in B_j, j = 1, 2, \ldots, p$. Suppose also that each b_j is of type u_+, u_- or u_0 for i. Then, by Theorem 2.6, $b_1 \otimes b_2 \otimes \cdots \otimes b_p$ lies in a crystal basis for $M_1 \otimes M_2 \otimes \cdots \otimes M_p$. The following procedure describes how $\widetilde{E}_i$ and $\widetilde{F}_i$ act on $b_1 \otimes b_2 \otimes \cdots \otimes b_p$.*

(a) Rewrite the tensor formally by replacing each b_j with u_+ if it is of type u_+ for i, or with u_- or u_0 similarly.

*(b) Delete u_0 and also any pair $u_+ \otimes u_-$. Repeat this process until there are no pairs $u_+ \otimes u_-$ left. The new tensor t obtained in this manner is called the i-*reduced form *of b.*

(c) To apply $\widetilde{E}_i$: Change the rightmost u_- in t to u_+. (If there is no u_- in the i-reduced form, then $\widetilde{E}_i$ acts as zero on the original crystal basis element.)

To apply $\widetilde{F}_i$: Change the leftmost u_+ in t to u_-. (If there is no u_+ in the i-reduced form, then $\widetilde{F}_i$ acts as zero on the original crystal basis element.)

(d) Return to t all of the elements deleted in (b). Then replace each u_+, u_- or u_0 with the b_j it was originally. This is well-defined except when a u_+ or u_- has been changed as in step (c). In the case when a u_- has been replaced with a u_+, suppose that originally the u_- was b_j. Replace the u_+ in t with $\widetilde{E}_i b_j$. Similarly if a u_+ has been replaced with a u_- and the u_+ originally was b_j, replace the u_- in t with $\widetilde{F}_i b_j$.

Suppose instead that each b_j is of type u_0, u_1, u_2, u_3, u_+ or u_- for i. The following procedure describes how $\widetilde{E}_i$ and $\widetilde{F}_i$ act on $b_1 \otimes b_2 \otimes \cdots \otimes b_p$: rewrite the tensor formally by replacing elements of type u_0, u_- or u_+ for i with u_0, u_- or u_+ respectively, and by replacing elements of type u_1

with $u_+ \otimes u_+$, those of type u_2 with $u_- \otimes u_+$ and those of type u_3 with $u_- \otimes u_-$. Apply step (b) above. Again we call this the i-reduced form *of b. Then follow step (c) above. Finally, return all the elements deleted to t, and replace each u_+, u_- or u_0 with the b_j it was originally, if it was a single b_j originally. For a pair, if it is unchanged, replace it with the b_j it was originally. If $u_- \otimes u_-$ has become $u_- \otimes u_+$ or $u_- \otimes u_+$ has become $u_+ \otimes u_+$, replace the pair with $\widetilde{E}_i b_j$, where b_j is what the pair was originally. If $u_+ \otimes u_+$ has become $u_- \otimes u_+$ or $u_- \otimes u_+$ has become $u_- \otimes u_-$, replace the pair with $\widetilde{F}_i b_j$, where b_j is what the pair was originally.*

Proof: See Remarks 2.1.2 and 2.1.3 in [6]. □

Let $\bar{\ }$ be the $\mathbb{Q}$-algebra automorphism from U to U taking E_i to E_i, F_i to F_i, and K_μ to $K_{-\mu}$, for each $i \in [1, n]$ and $\mu \in Y$, and v to v^{-1} (see [7, 3.1.12]). There is an induced $\mathbb{Q}$-linear automorphism (also denoted $\bar{\ }$) of any module $V(\lambda)$ for U defined by $\overline{ux_1} = \overline{u}x_1$ for any $u \in U^-$ (see [7, 19.3.4]). Note that every element of $V(\lambda)$ is of the form ux_1 for some $u \in U^-$.

Theorem 2.9 *Suppose $V(\lambda)$, $B(\lambda)$ and $L(\lambda)$ are as in Theorem 2.3. Then for each $b \in B(\lambda)$ there is a unique $\widetilde{b} \in L(\lambda)$ such that $\widetilde{b} \mapsto b$ under the canonical projection $L(\lambda) \to L(\lambda)/vL(\lambda)$ and $\widetilde{b} \in L(\lambda) \cap \overline{L(\lambda)}$. Furthermore, the set $\{\widetilde{b} \ : \ b \in B(\lambda)\}$ forms a $\mathbb{Q}(v)$-basis $\mathbf{B}(\lambda)$ for $V(\lambda)$, which is called the canonical basis for $V(\lambda)$.*

Proof: See [5, §0]. □

Note: Throughout this paper the explicit description of the crystal basis for the fundamental modules for quantized enveloping algebras of classical type given by Kashiwara and Nakashima in [6] will be used.

3 Main Ideas

Suppose that $\mathbf{g}$ is of type A, B, C, or D, that ω_r is a fundamental weight and that $V(\omega_r)$ is a fundamental module for U, the corresponding quantized enveloping algebra. Let $B(\omega_r)$ and $L(\omega_r)$ be as in Theorem 2.3. Let x_1 be our fixed highest weight vector in $V(\omega_r)$. By Theorem 2.3, the following is true:

Remark 3.1 *Suppose $b \in B(\omega_r)$. Then there is a sequence $i_1, i_2, \ldots, i_k$ in $[1, n]$ such that $b = \widetilde{F}_{i_k} \widetilde{F}_{i_{k-1}} \cdots \widetilde{F}_{i_1} \widehat{x_1}$, where $\widehat{x_1}$ is the unique element in $B(\omega_r)$ with the same weight as x_1 (that is, the highest weight). So by Definition 2.2 (6), we have $\widetilde{E}_{i_1} \widetilde{E}_{i_2} \cdots \widetilde{E}_{i_k} b = \widehat{x_1}$.*

We shall see that, for the modules we are interested in, it is possible to find such sequences so that the following is true: write $b = \widetilde{F}_{j_1}^{r_1} \widetilde{F}_{j_2}^{r_2} \cdots \widetilde{F}_{j_l}^{r_l} \widehat{x_1}$, where equal $\widetilde{F}_i$'s which are adjacent in the product have been grouped together, so $j_1 \neq j_2$, $j_2 \neq j_3, \ldots$ and $j_{l-1} \neq j_l$. Put $\widetilde{b} = \widetilde{F}_{j_1}^{r_1} \widetilde{F}_{j_2}^{r_2} \cdots \widetilde{F}_{j_l}^{r_l} x_1 \in V(\omega_r)$. Then $\widetilde{b} = F_{j_1}^{(r_1)} F_{j_2}^{(r_2)} \cdots F_{j_l}^{(r_l)} x_1$. If we can do this, it is clear that $\widetilde{b} \mapsto b$ under the canonical projection $L(\omega_r) \to L(\omega_r)/vL(\omega_r)$, and that $\overline{\widetilde{b}} = \widetilde{b}$, whence (by Theorem 2.9) the canonical basis of $V(\omega_r)$ is given by:

$$\mathbf{B}(\omega_r) = \{\widetilde{b} \ : \ b \in B(\omega_r)\}.$$

To accomplish this, in each case, we shall construct sequences as above satisfying the following conditions:

(1) The maximum length of a constant subsequence of $i_1, i_2, \ldots, i_k$ is 2.

(2) For any u for which i_u is distinct from its neighbours in $i_1, i_2, \ldots, i_k$, we have:

$$\begin{array}{lrcl} \text{(a)} & K_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} x_1 & = & v^c \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} x_1, \quad \text{(for some } c \in \{1,2\}\text{), and} \\ \text{(b)} & E_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} x_1 & = & 0. \end{array}$$

(3) Suppose that for some u, $i_u = i_{u+1}$, but $i_{u-1} \neq i_u$ (if $u \neq 1$) and $i_{u+2} \neq i_{u+1}$ (if $u + 1 \neq k$). Then:

$$\begin{array}{lrcl} \text{(a)} & K_{i_u} \widetilde{F}_{i_{u+2}} \cdots \widetilde{F}_{i_k} x_1 & = & v^2 \widetilde{F}_{i_{u+2}} \cdots \widetilde{F}_{i_k} x_1, \quad \text{and} \\ \text{(b)} & E_{i_u} \widetilde{F}_{i_{u+2}} \cdots \widetilde{F}_{i_k} x_1 & = & 0. \end{array}$$

We have in (2), by Definition 2.1,

$$\widetilde{F}_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} x_1 = F_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} x_1$$

and in (3),

$$\begin{array}{rcl} \widetilde{F}_{i_{u+1}} \widetilde{F}_{i_{u+2}} \cdots \widetilde{F}_{i_k} x_1 & = & F_{i_{u+1}} \widetilde{F}_{i_{u+2}} \cdots \widetilde{F}_{i_k} x_1 \\ \& \ \ \widetilde{F}_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} x_1 & = & \dfrac{F_{i_u} F_{i_{u+1}}}{[2]_{i_u}} \widetilde{F}_{i_{u+2}} \cdots \widetilde{F}_{i_k} x_1, \end{array}$$

which is enough to achieve the desired result.

Remark 3.2 Note that it is clear, from the definition of the basic module V_1 and the crystal basis in each case given in [6, §§3.2, 4.2, 5.2 & 6.2], that if b lies in the crystal basis given there, and $i \in [1, n]$, then b is of type u_+, u_- or u_0 for i. Also that, if M one of the U-modules $V(\lambda)$:

(1) If $x \in M$ has the same weight as a crystal basis element of type u_+ for i, then $K_i x = vx$.

(2) If $x \in M$ has the same weight as a crystal basis element of type u_- for i, then $K_i x = v^{-1}x$.

(3) If $x \in M$ has the same weight as a crystal basis element of type u_0 for i, then $K_i x = x$.

(4) If $x \in M$ has the same weight as a crystal basis element of type u_1 for i, then $K_i x = v^2 x$.

(5) If $x \in M$ has the same weight as a crystal basis element of type u_2 for i, then $K_i x = x$.

(6) If $x \in M$ has the same weight as a crystal basis element of type u_3 for i, then $K_i x = v^{-2}x$.

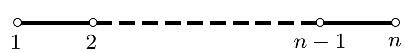
If $x \in M$ has weight ν, then $K_i x = v^c x$, where $c = (\nu, \alpha_i^*)$. This is because $K_i x = K_{h_i} x = v^{\nu(h_i)} x$ (by the definition of weight), so $K_i x = v^{(h_\nu, h_i)} x = v^{(h_\nu, h_{\alpha_i^*})} x = v^{(\nu, \alpha_i^*)} x$. Note that we have $h_i = h_{\alpha_i^*}$ since $(h_i, h_{\alpha_j}) = \alpha_j(h_i) = A_{ij} = (\alpha_i^*, \alpha_j) = (h_{\alpha_i^*}, h_{\alpha_j})$ for all $j \in [1, n]$, $h_{\alpha_1}, h_{\alpha_2}, \dots h_{\alpha_n}$ are a $\mathbb{C}$-basis for $\mathbf{h}$ and the Killing form is non-degenerate on $\mathbf{h}$. Therefore, if ν is the weight of x in each case and $c = (\nu, \alpha_i^*)$, then in (1), $c = 1$, in (2), $c = -1$, in (3) & (5), $c = 0$, in (4), $c = 2$, and in (6) $c = -2$.

For each $r \in [1, n]$ denote by $(\mathcal{L}_r, B_r)$ a crystal basis of V_r as in Theorem 2.3. We shall use this notation throughout. In each case, A_n, B_n, C_n and D_n, we shall use Kashiwara and Nakashima's description of B_r in [6, §§3.3, 5.3, 4.3 & 6.3].

4 Type A_n

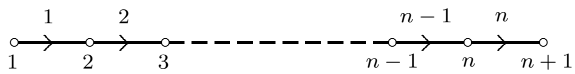
For completeness we consider here type A_n (see also Chapter 2), which enables us to emphasise the connection between A_{2n-1} and C_n. We recall here the Dynkin diagram:

A_n



The first fundamental module, V_1, has dimension $n+1$, so B_1 has $n+1$ elements, which we label $e_1, e_2, \dots, e_{n+1}$. The crystal graph is:

The edge labels are written above the edges, and vertex j corresponds to the crystal basis



element e_j (see [6, §3.2]).

For $r \in [1,n]$, B_r is contained in $B_1^{\otimes r}$. It consists of the set:

$$\{e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \;:\; 1 \leq j_1 < j_2 < \cdots < j_r \leq n+1\}.$$

(See [6, §3.3]). We shall now describe a recursive algorithm which, for each $b \in B_r$, will produce a sequence $i_1, i_2, \ldots, i_k$, as in Section 3, such that $b = \widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{x_1}$. We shall define these sequences in such a way that in Section 5 we will be able to establish a close connection with the modules in type C.

An Algorithm for Type A

Suppose $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$, and suppose that $b \neq \widehat{x_1}$. Note that we have $\widehat{x_1} = e_1 \otimes e_2 \otimes \cdots \otimes e_r$, since each $\widetilde{E}_j$ kills this vector. Suppose we have defined a sequence for all crystal basis elements of this form with smaller sum $\sum_{s=1}^{r} j_s$. We say that e_{j_m} is *movable* in b if there exists $i \in [1,n]$ such that $\widetilde{E}_i e_{j_m}$ is not e_{j_s} for any s and is not zero. Note that by condition (5) of Definition 2.2, $\widetilde{E}_i e_{j_m}$ lies in $B_1 \cup \{0\}$. At least one e_{j_m} is movable in b since $b \neq \widehat{x_1}$. Let e_{j_m} be the vector in b with m minimal such that e_{j_m} is movable. We call this the *leftmost movable vector* in b.

Case (a): Suppose e_{n+3-j_m} is also in the expression for b, is movable in b, and $j_m < (n+1)/2$. We put $i = n+3-j_m-1$, which is the label on the edge to the left of $n+3-j_m$.

Case (b): If otherwise, we put $i = j_m - 1$.

The idea here is to ensure the algorithm for type A_{2n-1} is compatible with that for type C_n — see Section 5, in particular, Proposition 5.2. We shall consider the element $\widetilde{E}_i b$ for this value of i.

Note that, if $k \in [1,n]$ then in B_1, e_k is of type u_+ for k, e_{k+1} is of type u_- for k, and all other e_p's are of type u_0 for k. Since e_{j_m} is movable in b, e_{j_m-1} does not occur in b, so the i-reduced form of b (see Proposition 2.8, step (b)) must be u_-. Following the steps through in Proposition 2.8, we see that $\widetilde{E}_i b$ is not zero and that in case (a) or (b) here, $\widetilde{E}_i b$ is the tensor obtained from b by replacing e_{i+1} with e_i. It is clear that $\widetilde{E}_i b$ has smaller sum $\sum_{s=1}^{r} j_s$, so we have already defined a sequence for it, by assumption, say $\widetilde{E}_i b = \widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{x_1}$ and therefore we have a sequence for b given by $b = \widetilde{F}_i\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{x_1}$, using condition (6) of Definition 2.2.

Example We consider the case A_5 when $r = 2$ and take the crystal basis element $b = e_{j_1} \otimes e_{j_2} = e_2 \otimes e_6$. For the first step, the leftmost movable vector is e_2, as $\widetilde{E}_1(e_2) = e_1$. We are in case (a), since $e_{n+3-j_1} = e_6$ is also in the expression for b, so the algorithm tells us to apply $\widetilde{E}_5$, which takes b to $e_2 \otimes e_5$. Here is a summary of all the steps:

Step	Leftmost movable vector	Case	Kashiwara operator, $\widetilde{E}_i$	b
1	e_2	(a)	$\widetilde{E}_5$	$e_2 \otimes e_6$
2	e_2	(b)	$\widetilde{E}_1$	$e_2 \otimes e_5$
3	e_5	(b)	$\widetilde{E}_4$	$e_1 \otimes e_5$
4	e_4	(b)	$\widetilde{E}_3$	$e_1 \otimes e_4$
5	e_3	(b)	$\widetilde{E}_2$	$e_1 \otimes e_3$

We thus obtain the expression $\widetilde{E}_2\widetilde{E}_3\widetilde{E}_4\widetilde{E}_1\widetilde{E}_5 b = e_1 \otimes e_2 = \widehat{x_1}$, whence $b = \widetilde{F}_5\widetilde{F}_1\widetilde{F}_4\widetilde{F}_3\widetilde{F}_2\widehat{x_1}$.

We claim that these sequences will satisfy the requirements of Section 3. Suppose $b \in B_r$, and $i \in [1, n]$. Then, by [6, 3.3.2], $\widetilde{F}_i^2 b = 0$. Thus no subsequence of the form i_u, i_{u+1} with $i_u = i_{u+1}$ will occur in the sequences generated by the algorithm, since the algorithm never produces zero (it always generates an expression for an element of B_r), and condition (1) in Section 3 is seen to hold.

Suppose $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$. Let $b = \widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{x_1}$ be the expression defined above. By the remarks in the definition of the algorithm, we see that b has i_1-reduced form u_- and $\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{x_1}$ has i_1-reduced form u_+. We shall need the following lemma:

Lemma 4.1 *Fix $i \in [1, n]$. We define the following map π from formal tensors in the symbols u_+, u_- and u_0 to $\mathbb{Z}$. Firstly, put $\pi(u_+) = 1$, $\pi(u_-) = -1$ and $\pi(u_0) = 0$. If $u^1 \otimes u^2 \otimes \cdots \otimes u^s$ is a formal tensor in u_+, u_- and u_0, put $\pi(u^1 \otimes u^2 \otimes \cdots \otimes u^s) = \sum_{j=1}^{s} \pi(u^j)$. Suppose now $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$ has weight ν and i-reduced form $u^1 \otimes u^2 \otimes \cdots \otimes u^s$, where each u^j is one of u_+, u_- or u_0. Let $c = \pi(u^1 \otimes u^2 \otimes \cdots \otimes u^s)$. If $b' \in V_r$ has the same weight as b, then $K_i b' = v^c b'$.*

Proof: Let $t = t_1 \otimes t_2 \otimes \cdots \otimes t_r$ be the formal tensor obtained from b by replacing each e_{j_k} by t_k, where e_{j_k} is of type t_k for i (so each $t_k \in \{u_\pm, u_0\}$). Then $\pi(t) = c$, since in obtaining the i-reduced form from t, only u_0's and pairs $u_+ \otimes u_-$ (upon which π takes the value zero) are removed. The weight of b' is the weight of b and therefore is the sum of the weights of the vectors $e_{j_1}, e_{j_2}, \ldots, e_{j_r}$. Let ν_k be the weight of e_{j_k}, $k = 1, 2, \ldots, r$. Each e_{j_k} is of type t_k for i and therefore by Remark 3.2, $(\nu_k, \alpha_i^*) = \pi(t_k)$. Therefore, as $\nu = \sum_{k=1}^{r} \nu_k$, we have $(\nu, \alpha_i^*) = \pi(t)$. Thus, by Remark 3.2, $K_i b' = v^{\pi(t)} b'$ as required, since $\pi(t) = c$. □

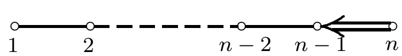
Since $\widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1$ has the same weight as $\widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$, and $\pi(u_+) = 1$, we have $K_i \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1 = v \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1$ and we have shown condition 2(a) of Section 3. Suppose that $E_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1 \neq 0$. Its weight is then a weight of V_r and there is an element b_1 in B_r with the same weight. Its i_1-reduced form can have at most one u_+ in it and therefore, by Lemma 4.1, $K_{i_1} E_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1 = v^c E_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1$ for some $c \leq 1$. But $K_i \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1 = v \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1$, whence $K_i E_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1 = v^3 E_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1$, a contradiction. Hence $E_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} x_1 = 0$. Therefore all of the conditions we required of the algorithm in Section 3 are satisfied and we have proved:

Theorem 4.2 *(Type A) Let $(\mathcal{L}_r, B_r)$ be a crystal basis for V_r as in Theorem 2.3. Suppose that $b \in B_r$ and let $b = \widetilde{F}_{j_1}^{r_1} \widetilde{F}_{j_2}^{r_2} \cdots \widetilde{F}_{j_k}^{r_k} \widehat{x_1}$ be the expression given by the algorithm above, with equal $\widetilde{F}_j$'s adjacent in the product gathered together. Then each $r_p = 1$, for $p = 1, 2, \ldots, k$. Let $\widetilde{b} = \widetilde{F}_{j_1} \widetilde{F}_{j_2} \cdots \widetilde{F}_{j_k} x_1$. Then $\widetilde{b} = F_{j_1} F_{j_2} \cdots F_{j_k} x_1$ and $\widetilde{b}$ is the canonical basis element corresponding to b.* □

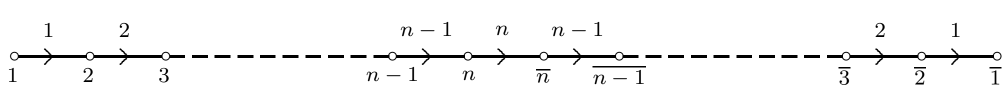
5 Type C_n

Note that we include here the case C_2. We recall here the Dynkin diagram:

C_n



The first fundamental module, V_1, has dimension $2n$, so B_1 has $2n$ elements, which we label $e_1, e_2, \ldots, e_n, e_{\overline{1}}, e_{\overline{2}}, \ldots, e_{\overline{n}}$. The crystal graph is:



The edge labels are written above the edges, and vertex j corresponds to the crystal basis element e_j (see [6, §4.2]). Define a total order $\preceq$ on $\{1, 2, \dots, n, \overline{1}, \overline{2}, \dots, \overline{n}\}$ by putting:

$$1 \preceq 2 \preceq \cdots \preceq n \preceq \overline{n} \preceq \overline{n-1} \preceq \cdots \preceq \overline{1}.$$

For $r \in [1, n]$, B_r is contained in $B_1^{\otimes r}$. It consists of the set:

$$\left\{ e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \quad : \quad \begin{array}{c} 1 \preceq j_1 \prec j_2 \prec \cdots \prec j_r \preceq \overline{1} \\ \text{and if } j_l = \overline{j_k} \text{ for some } k, l, \\ \text{then } k + r + 1 - l \leq j_k. \end{array} \right\}.$$

(See [6, §4.3]). We shall now describe a recursive algorithm which, for each $b \in B_r$, will produce a sequence $i_1, i_2, \dots, i_k$, as in Section 3, such that $b = \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$.

An Algorithm for Type C

Suppose $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$, and suppose that $b \neq \widehat{x_1}$. Note that we have $\widehat{x_1} = e_1 \otimes e_2 \otimes \cdots \otimes e_r$, since each $\widetilde{E}_j$ kills this vector. Suppose we have defined a sequence for all crystal basis elements of this form with smaller sum $\sum_{s=1}^{r} \varepsilon(j_s)$ (where $\varepsilon(1) = 1$, $\varepsilon(2) = 2, \dots, \varepsilon(n) = n$, $\varepsilon(\overline{n}) = n+1$, $\varepsilon(\overline{n-1}) = n+2, \dots, \varepsilon(\overline{1}) = 2n$). As in case A, we say that e_{j_m} is *movable* in b if there exists $i \in [1, n]$ such that $\widetilde{E}_i e_{j_m}$ is not e_{j_s} for any s and is not zero. Note that by condition (5) of Definition 2.2, $\widetilde{E}_i e_{j_m}$ lies in $B_1 \cup \{0\}$. At least one e_{j_m} is movable in b since $b \neq \widehat{x_1}$. Let e_{j_m} be the vector in b with m minimal such that e_{j_m} is movable. We call this the *leftmost movable vector* in b. Let i be the label on the edge to the left of j_m on the crystal graph. So if $j_m \in [2, n]$, $i = j_m - 1$, and if $j_m = \overline{j}$ for some $j \in [1, n]$, then $i = j$. We shall consider the element $\widetilde{E}_i b$ for this value of i.

Note that, if $k \in [1, n-1]$ then in B_1, e_k and $e_{\overline{k+1}}$ are of type u_+ for k, e_{k+1} and $e_{\overline{k}}$ are of type u_- for k, and all other e_p's are of type u_0 for k. For n, e_n is of type u_+, $e_{\overline{n}}$ is of type u_-, and all other e_p's are of type u_0. Since e_{j_m} is the leftmost movable vector in b, $\widetilde{E}_i e_{j_m}$ (the corresponding vector of type u_+, for i) does not occur in b, so the i-reduced form of b (see Proposition 2.8, step (b)) must be either u_-, $u_- \otimes u_+$ or $u_- \otimes u_-$, with the first u_- in each case corresponding to e_{j_m}. Note that the case $u_+ \otimes u_-$ cannot occur: since e_{j_m} is movable, $\widetilde{E}_i e_{j_m}$ does not occur in b so, looking at the crystal graph, the only way this could happen would be for $j_m = \overline{j}$ for j in $[1, n-1]$ and for e_j to occur also in b (say as e_{j_l}). Since e_{j_m} is the *leftmost* movable vector in b, e_j cannot be movable, whence e_{j-1} must also occur in b, to the left of e_j. Similarly, $e_{j-2}, \dots, e_1$ must also occur in b. Thus $l \geq j$. Since $m \leq r$ we have $l + (r + 1 - m) \geq j + 1 > j$, contradicting the above condition for $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$ to lie in B_r, so this situation cannot occur. We call this argument the *domino argument*, and it will be used again later.

Following the steps through in Proposition 2.8, we see that $\widetilde{E}_i b$ is not zero and has i-reduced form as follows:

	i-reduced form of b	i-reduced form of $\widetilde{E}_i b$
(a)	u_-	u_+
(b)	$u_- \otimes u_+$	$u_+ \otimes u_+$
(c)	$u_- \otimes u_-$	$u_- \otimes u_+$

It is clear that $\widetilde{E}_i b$ has smaller sum $\sum_{s=1}^{r} \varepsilon(j_s)$, so we have already defined a sequence for it by assumption, say $\widetilde{E}_i b = \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ and therefore we have a sequence for b given by $b = \widetilde{F}_i \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$, using condition (6) of Definition 2.2.

Example We consider the case C_3 when $r = 2$ and take the crystal basis element $b = e_{j_1} \otimes e_{j_2} = e_2 \otimes e_{\overline{1}}$. For the first step, the leftmost movable vector is e_2, as $\widetilde{E}_1(e_2) = e_1$. The algorithm tells us to apply $\widetilde{E}_1$, which takes b to $e_2 \otimes e_{\overline{2}}$. Here is a summary of all the steps:

Step	Leftmost movable vector	Kashiwara operator, $\widetilde{E}_i$	b	i-reduced form of b
1	e_2	$\widetilde{E}_1$	$e_2 \otimes e_{\overline{1}}$	$u_- \otimes u_-$
2	e_2	$\widetilde{E}_1$	$e_2 \otimes e_{\overline{2}}$	$u_- \otimes u_+$
3	$e_{\overline{2}}$	$\widetilde{E}_2$	$e_1 \otimes e_{\overline{2}}$	u_-
4	$e_{\overline{3}}$	$\widetilde{E}_3$	$e_1 \otimes e_{\overline{3}}$	u_-
5	e_3	$\widetilde{E}_2$	$e_1 \otimes e_3$	u_-

We thus obtain the expression $\widetilde{E}_2 \widetilde{E}_3 \widetilde{E}_2 \widetilde{E}_1 \widetilde{E}_1 b = e_1 \otimes e_2 = \widehat{x_1}$, whence $b = \widetilde{F}_1 \widetilde{F}_1 \widetilde{F}_2 \widetilde{F}_3 \widetilde{F}_2 \widehat{x_1}$. (Note the similarity of this example to the example in type A_5 given in the previous section; see Proposition 5.2 for more details).

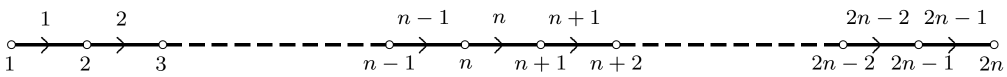
We claim that these sequences will satisfy the requirements of Section 3. Suppose $b \in B_r$, and $i \in [1, n]$. Then, by [6, 4.3.3], $\widetilde{F}_i^3 b = 0$. Therefore, as the algorithm never produces zero (it always generates an expression for an element of B_r), condition (1) (in Section 3) is seen to hold. Suppose $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$. Let $b = \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ be the expression defined above. We consider first the case when $i_u = i_{u+1}$ for some u. The only way this can happen is for $\widetilde{F}_{i_{u+2}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ to have i_u-reduced form $u_+ \otimes u_+$ (since the algorithm will have told us to apply $\widetilde{E}_{i_u}$ twice to $\widetilde{F}_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}\widehat{x_1}$; see the table of i-reduced forms above). Lemma 4.1 applies equally well to this case as to type A, so we have $K_{i_u}\widetilde{F}_{i_{u+2}}\cdots\widetilde{F}_{i_k}x_1 = v^2\widetilde{F}_{i_{u+2}}\cdots\widetilde{F}_{i_k}x_1$. Suppose that $E_{i_u}\widetilde{F}_{i_{u+2}}\cdots\widetilde{F}_{i_k}x_1 \neq 0$. Its weight is then a weight of V_r and there is an element b_1 in B_r with the same weight. Its i_u-reduced form can have at most two u_+'s in it and therefore, by Lemma 4.1, $K_{i_u}E_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1 = v^cE_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1$ for some $c \leq 2$. But $K_{i_u}\widetilde{F}_{i_{u+2}}\cdots\widetilde{F}_{i_k}x_1 = v^2\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1$, whence $K_{i_u}E_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1 = v^4E_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1$, a contradiction. Thus $E_{i_u}\widetilde{F}_{i_{u+2}}\cdots\widetilde{F}_{i_k}x_1 = 0$ and condition (3) of Section 3 is seen to hold.

Next consider the case when i_u is distinct from its neighbours. The only way this can happen is for $\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}\widehat{x_1}$ to have i_u-reduced form u_+, $u_-\otimes u_+$ or $u_+\otimes u_+$. (Note that $\widetilde{F}_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}\widehat{x_1}$ is a crystal basis element that the algorithm specifies we should apply $\widetilde{E}_{i_u}$ to as a first step. The table above tells us the only possible i_u-reduced forms of $\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}\widehat{x_1}$.) Suppose it was $u_-\otimes u_+$. Then consider the definition of the algorithm. Let e_{j_m} be the leftmost movable vector in $\widetilde{F}_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}\widehat{x_1}$, which must correspond to the first u_- in the i_u-reduced form $u_-\otimes u_-$ for $\widetilde{F}_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}\widehat{x_1}$, because it is leftmost. Then applying $\widetilde{E}_{i_u}$ to $\widetilde{F}_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}\widehat{x_1}$ does not alter e_{j_m} — it is the vector corresponding to the second u_- which is altered. (Recall that in Proposition 2.8(c), to apply $\widetilde{E}_i$, the rightmost u_- is changed to a u_+). Therefore e_{j_m} is also the leftmost movable vector in $\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}\widehat{x_1}$. Thus, according to the algorithm, we must have $i_u = i_{u+1}$, a contradiction, so this case cannot occur. We thus have, applying Lemma 4.1, $K_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1 = v^c\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1$ for some $c \in \{1,2\}$, and condition 2(a) of Section 3 is shown to hold. Suppose that $E_{i_u}\widetilde{F}_{i_{u+2}}\cdots\widetilde{F}_{i_k}x_1 \neq 0$. Its weight is then a weight of V_r and there is an element b_1 in B_r with the same weight. Its i_u-reduced form can have at most two u_+'s in it and therefore, by Lemma 4.1, $K_{i_u}E_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1 = v^cE_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1$ for some $c \leq 2$. But $K_{i_u}\widetilde{F}_{i_{u+2}}\cdots\widetilde{F}_{i_k}x_1 = v^e\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1$, where $e = 1$ or 2, whence $K_{i_u}E_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1 = v^{e+2}E_{i_u}\widetilde{F}_{i_{u+1}}\cdots\widetilde{F}_{i_k}x_1$, a contradiction. Thus we have $E_{i_u}\widetilde{F}_{i_{u+2}}\cdots\widetilde{F}_{i_k}x_1 = 0$ and condition (2) of Section 3 is seen to hold.

Therefore all of the conditions we required of the algorithm in Section 3 are satisfied and we have proved:

Theorem 5.1 *(Type C) Let $(\mathcal{L}_r, B_r)$ be a crystal basis for V_r as in Theorem 2.3. Suppose that $b \in B_r$ and let $b = \widetilde{F}_{j_1}^{r_1}\widetilde{F}_{j_2}^{r_2}\cdots\widetilde{F}_{j_k}^{r_k}\widehat{x_1}$ be the expression given by the algorithm above, with equal $\widetilde{F}_j$'s adjacent in the product gathered together. Let $\widetilde{b} = \widetilde{F}_{j_1}^{r_1}\widetilde{F}_{j_2}^{r_2}\cdots\widetilde{F}_{j_k}^{r_k}x_1$. Then $\widetilde{b} = F_{j_1}^{(r_1)}F_{j_2}^{(r_2)}\cdots F_{j_k}^{(r_k)}x_1$ and $\widetilde{b}$ is the canonical basis element corresponding to b.* □

We now look at the connection with A_{2n-1}. Note that the crystal graph for the basic module for A_{2n-1} is as follows:



It is the same graph as that for the basic module for C_n, with different labels. The inverse of the map ε defined at the start of the definition of the algorithm takes the vertices of the graph in case A_{2n-1} onto the vertices of the graph in case C_n bijectively. Also, we define a map $\tau : \{1, 2, \ldots, 2n-1\} \to \{1, 2, \ldots, n\}$ by $\tau(m) = \min\{m, 2n-m\}$, which takes the edges of the graph in case A_{2n-1} onto the edges of the graph in case C_n. Let L_r be the r-th fundamental module in case A_{2n-1}, with our fixed highest weight vector y_1 and crystal basis element $\widehat{y_1}$ of highest weight. (Here we are always referring to the crystal bases as in Theorem 2.3).

We would like to see a connection between the bases in L_r, the module in type A_{2n-1}, and V_r, the module in type C_n (the examples in this section and the previous one give an example of this). Note that the dimension of L_r is $\binom{2n}{r}$, while the dimension of V_r is only $\binom{2n}{r} - \binom{2n}{r-2}$ (see [1, Table 2,p214]), so L_r has more crystal basis elements than V_r, and we can't hope to get a one to one correspondence between these crystal bases. The solution is to take a restricted subset of the crystal basis for L_r. The following Proposition describes exactly what happens, and also the corresponding results for the canonical basis.

Proposition 5.2 *Let $n \geq 2$ and $r \leq n$. Suppose that $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{y_1} = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$ is a crystal basis element, as generated by the algorithm, for the module L_r in case A_{2n-1} (we use the notation from Section 4 except that we denote the type A_{2n-1} fundamental module by L_r) with the property that if $j_k = j \leq n$ and $j_l = 2n+1-j$ for some k, l, then $k + r + 1 - l \leq j$. Then $\widetilde{F}_{\tau(i_1)}\widetilde{F}_{\tau(i_2)}\cdots\widetilde{F}_{\tau(i_k)}\widehat{x_1}$ is a crystal basis element for the module V_r in case C_n. Furthermore, every crystal basis element in V_r arises exactly once in this way. Similarly, if $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}y_1$ is the corresponding type A_{2n-1} canonical basis element (so $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{y_1}$ satisfies the above restrictions), then $\widetilde{F}_{\tau(i_1)}\widetilde{F}_{\tau(i_2)}\cdots\widetilde{F}_{\tau(i_k)}x_1$ lies in the canonical basis for V_r in type C_n. Each canonical basis element for V_r in type C_n will arise exactly once in this way.*

Proof: Before starting the proof we give an example to illustrate the significance of the restriction above. Suppose that $n = r = 2$. Then the dimension of L_r is $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$, while the dimension of V_r is $\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 6 - 1 = 5$. Thus 5 of the crystal basis elements for L_r will give the crystal basis elements for V_r. The only crystal basis element in L_r that does not satisfy the restriction is $e_1 \otimes e_4$ (take $k = 1, l = 2$; then $k + r + 1 - l = 1 + 2 + 1 - 2 = 2 > j = 1$). The element $e_{\varepsilon^{-1}(1)} \otimes e_{\varepsilon^{-1}(4)} = e_1 \otimes e_{\overline{1}}$ does not lie in the crystal basis for V_r.

Using the notation in Section 4, let $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$ be a crystal basis element for L_r in type A_{2n-1}, satisfying the given restrictions. These restrictions, and the similarity of the crystal graphs of L_1 (in type A_{2n-1}) and V_1 (in type C_n) and the corresponding total orders defined on the vertices (in A_{2n-1} just the natural ordering of the integers $\{1, 2, \ldots, 2n\}$) imply that $e_{\varepsilon^{-1}(j_1)} \otimes e_{\varepsilon^{-1}(j_2)} \otimes \cdots \otimes e_{\varepsilon^{-1}(j_r)}$, with the notation of this Section, is a crystal basis element for V_r in type C_n. It is clear that every crystal basis element of V_r will arise in this way exactly once, because the restriction in the Proposition corresponds exactly to the restriction in the description of the crystal basis of V_r (see the start of this section). The statement in the proposition about crystal bases will follow from the fact that if $\widetilde{E}_i$ is a Kashiwara operator in case A and the definition of the case A algorithm specifies we should apply $\widetilde{E}_i$ to $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$ to get $e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_r}$, then the case C algorithm specifies that we should apply $\widetilde{E}_{\tau(i)}$ to $e_{\varepsilon^{-1}(j_1)} \otimes e_{\varepsilon^{-1}(j_2)} \otimes \cdots \otimes e_{\varepsilon^{-1}(j_r)}$ to get $e_{\varepsilon^{-1}(k_1)} \otimes e_{\varepsilon^{-1}(k_2)} \otimes \cdots \otimes e_{\varepsilon^{-1}(k_r)}$. We shall now verify this fact. The algorithm in case A was defined exactly so this would happen.

Let e_{j_m} be the leftmost movable vector in b. Suppose first that e_{2n+2-j_m} is in the expression for b, is also movable in b and $j_m < n$. The case A_{2n-1} algorithm specifies that we apply $\widetilde{E}_{2n+2-j_m-1}$ to b, and this will change e_{2n+2-j_m} to e_{2n+2-j_m-1}. Now $\varepsilon^{-1}(j_m) = j_m$ and $\varepsilon^{-1}(2n+2-j_m) = \overline{j_m - 1}$. Each of these will be movable in $e_{\varepsilon^{-1}(j_1)} \otimes e_{\varepsilon^{-1}(j_2)} \otimes \cdots \otimes e_{\varepsilon^{-1}(j_r)}$. The leftmost movable vector here will be e_{j_m}, since e_{j_m} is leftmost movable in b, and therefore the case C_n algorithm will specify we should apply $\widetilde{E}_{j_m-1} = \widetilde{E}_{\tau(j_m-1)}$, since $\tau(j_m - 1) = \min\{j_m - 1, 2n - (j_m - 1)\}$ and $j_m - 1 < n$. The $(j_m - 1)$-reduced form of $e_{\varepsilon^{-1}(j_1)} \otimes e_{\varepsilon^{-1}(j_2)} \otimes \cdots \otimes e_{\varepsilon^{-1}(j_r)}$ must be $u_- \otimes u_-$. This is because both e_{j_m} and $e_{\overline{j_m-1}}$ are in it — each reduces to u_- — and each one is movable, so there can be no u_+ in the reduced form. Therefore applying $\widetilde{E}_{j_m-1}$ will affect the rightmost u_- and change $e_{\overline{j_m-1}}$ to $e_{\overline{j_m}}$ (that is,$e_{\varepsilon^{-1}(2n+2-j_m)}$ to $e_{\varepsilon^{-1}(2n+1-j_m)}$), corresponding to what the type A_{2n-1} algorithm did to b above.

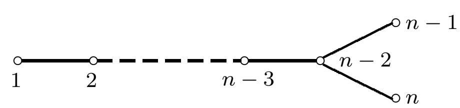
Next, suppose that e_{2n+2-j_m} is not in the expression for b, or that it is in b but is not movable in b. If it was in the expression for b and was movable but we had $j_m \geq n+1$ then e_{j_m} would not be the leftmost movable vector so this could not happen. The case A_{2n-1} algorithm specifies that we should apply $\widetilde{E}_{j_m-1}$ to b and this will change e_{j_m} to e_{j_m-1}. Now $e_{\varepsilon^{-1}(j_m)}$ will certainly be the leftmost movable vector in $e_{\varepsilon^{-1}(j_1)} \otimes e_{\varepsilon^{-1}(j_2)} \otimes \cdots \otimes e_{\varepsilon^{-1}(j_r)}$. First, suppose that $j_m \leq n$, so $\varepsilon^{-1}(j_m) = j_m$ and the type C_n algorithm specifies we should apply $\widetilde{E}_{j_m-1}$ which will change e_{j_m} to e_{j_m-1} (that is, $e_{\varepsilon^{-1}(j_m)}$ to $e_{\varepsilon^{-1}(j_m-1)}$), corresponding to what to the type A_{2n-1} algorithm did to b above. If $j_m = n+1$, then $\varepsilon^{-1}(j_m) = \overline{n}$ and the type C_n algorithm specifies we should apply $\widetilde{E}_n$ which will change $e_{\overline{n}}$ to e_n (that is, $e_{\varepsilon^{-1}(n+1)}$ to $e_{\varepsilon^{-1}n}$), corresponding to what the type A_{2n-1} algorithm did to b above. Finally, suppose that $j_m > n+1$, so $\varepsilon^{-1}(j_m) = \overline{2n+1-j_m}$ and the type C_n algorithm specifies that we should apply $\widetilde{E}_{2n+1-j_m}$ which will change $e_{\overline{2n+1-j_m}}$ to $e_{\overline{2n+2-j_m}}$ (that is, $e_{\varepsilon^{-1}(j_m)}$ to $e_{\varepsilon^{-1}(j_m-1)}$), corresponding to what the type A_{2n-1} algorithm did to b above. Note that in each of these cases, if the type C_n algorithm specifies $\widetilde{E}_i$, then the i-reduced form of $e_{\varepsilon^{-1}(j_1)} \otimes e_{\varepsilon^{-1}(j_2)} \otimes \cdots \otimes e_{\varepsilon^{-1}(j_r)}$ is either u_- or $u_- \otimes u_+$ since we have assumed e_{2n+2-j_m} is not in the expression for b or that it is in b but is not movable in b. This means that the $\widetilde{E}_i$ does act as claimed in each case.

We have shown that the statement regarding crystal bases is true, and the statement about canonical bases follows from the statement about crystal bases and Theorems 4.2 and 5.1. □

6 Type D_n

We recall here the Dynkin diagram:

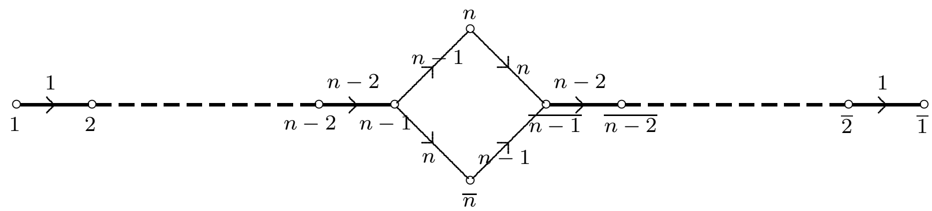
D_n



The first fundamental module, V_1, has dimension $2n$, so B_1 has $2n$ elements, which we label $e_1, e_2, \ldots, e_n, e_{\overline{1}}, e_{\overline{2}}, \ldots, e_{\overline{n}}$. The crystal graph is:

Vertex j corresponds to the crystal basis element e_j (see [6, §6.2]). Define a partial order $\preceq$ on $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$ by putting:

$$1 \preceq 2 \preceq \cdots \preceq n-1 \preceq \begin{matrix} n \\ \overline{n} \end{matrix} \preceq \overline{n-1} \preceq \cdots \preceq \overline{2} \preceq \overline{1}.$$



We shall assume in this section that $r \in [1, n-2]$ (see Chapter 3 for consideration of the case $r = n-1$ or n). For $r \in [1, n-2]$, B_r is contained in $B_1^{\otimes r}$ (see [6, §6.3]). It consists of the set:

$$\left\{ e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \quad : \quad \begin{array}{c} 1 \preceq j_1 \prec j_2 \prec \cdots \prec j_r \preceq \overline{1} \\ \text{(except } (j_m, j_{m+1}) = (n, \overline{n}) \text{ or } (\overline{n}, n) \text{ are both allowed)} \\ \text{and if } j_l = \overline{j_k} \text{ for some } k, l, \text{ then } (k + r + 1 - l) \leq j_k. \end{array} \right\}.$$

(See [6, §6.3]). We shall now describe a recursive algorithm which, for each $b \in B_r$, will produce a sequence $i_1, i_2, \dots, i_k$, as in Section 3, such that $b = \widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{x_1}$.

An Algorithm for Type D

Suppose $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$, and suppose that $b \neq \widehat{x_1}$. We have $\widehat{x_1} = e_1 \otimes e_2 \otimes \cdots \otimes e_r$ since each $\widetilde{E}_j$ kills this vector. Suppose we have defined a sequence for all crystal basis elements of this form with smaller sum $\sum_{s=1}^{r} \kappa(j_s)$ (where $\kappa(1) = 1, \kappa(2) = 2, \dots, \kappa(n) = n, \kappa(\overline{n}) = n, \kappa(\overline{n-1}) = n+1, \dots, \kappa(\overline{1}) = 2n-1$). As in case A, we say that e_{j_m} is *movable* in b if there exists $i \in [1, n]$ such that $\widetilde{E}_i e_{j_m}$ is not e_{j_s} for any s and is not zero. Note that by condition (5) of Definition 2.2, $\widetilde{E}_i e_{j_m}$ lies in $B_1 \cup \{0\}$. At least one e_{j_m} is movable in b since $b \neq \widehat{x_1}$. Let e_{j_m} be the vector in b with m minimal such that e_{j_m} is movable. We call this the *leftmost movable vector* in b. We define $i \in [1, n]$ in the following way:

Case (I): Suppose that $j_m = \overline{n-1}$ and that neither e_n nor $e_{\overline{n}}$ occurs in the expression for b. We put $i = n-1$. We could equally well put $i = n$ here but we make a choice for definiteness.

Case (II): Suppose that $j_m = \overline{n}$ and the part of the expression for b immediately after e_{j_m} (including e_{j_m}) is of the form:

$$e_{\overline{n}} \otimes e_n \otimes \cdots \otimes e_{\overline{n}} \otimes e_n \otimes e_{\overline{n-1}},$$

where $e_{\overline{n}} \otimes e_n$ appears at least once. We put $i = n-1$.

Case (III): Suppose that $j_m = n$ and the expression for b immediately after e_{j_m} (including e_{j_m}) is of the form:

$$e_n \otimes e_{\overline{n}} \otimes \cdots \otimes e_n \otimes e_{\overline{n}} \otimes e_{\overline{n-1}},$$

where $e_n \otimes e_{\overline{n}}$ appears at least once in the expression. We put $i = n$.

Case (IV): Suppose that none of the above cases occur. Let i be such that $\widetilde{E}_i(e_{j_m})$ is not zero and is not e_{j_k} for any k. This is uniquely defined; case (I) deals with the only case where such an i is not unique.

We do not treat cases (II) and (III) in the same way as in case (IV) in order to make clear a nice connection with the type B_{n-1} case (see the next section). However the algorithm would still work if we made this simplification. We shall consider the element $\widetilde{E}_i b$ for this value of i.

Note that, if $k \in [1, n-1]$ then in B_1, e_k and $e_{\overline{k+1}}$ are of type u_+ for k, e_{k+1} and $e_{\overline{k}}$ are of type u_- for k, and all other e_p's are of type u_0 for k. For n, e_n and e_{n-1} are of type u_+, $e_{\overline{n-1}}$ and $e_{\overline{n}}$ are of type u_-, and all other e_p's are of type u_0. The following are clear, using the fact that e_{j_m} is the leftmost movable vector in b:
In case (I), (II) or (III), the i-reduced form of b must be u_-. In case (IV), the i-reduced form of b must be u_-, $u_- \otimes u_+$ or $u_- \otimes u_-$. *Note*: we eliminate the case $u_+ \otimes u_-$ for (I) and (IV) with the domino argument as in the algorithm for type C_n. The subsequences of the form $n, \overline{n}, n, \overline{n}, n, \ldots$ (with or without the initial n) do not introduce more u_+'s or u_-'s than this into the reduced form since for $n-1$ the reduced form of $e_{\overline{n}} \otimes e_n$ is empty, and for n the reduced form of $e_n \otimes e_{\overline{n}}$ is empty; for any other k their i-reduced forms are clearly empty. Following the steps through in Proposition 2.8, we see that $\widetilde{E}_i b$ is not zero and has i-reduced form as follows:

	i-reduced form of b	i-reduced form of $\widetilde{E}_i b$
(a)	u_-	u_+
(b)	$u_- \otimes u_+$	$u_+ \otimes u_+$
(c)	$u_- \otimes u_-$	$u_- \otimes u_+$

Since applying $\widetilde{E}_i$ to b replaces an e_k in b with a vector with suffix further to the left in the crystal graph, that is, smaller in the given partial order, $\widetilde{E}_i b$ has smaller sum $\sum_{s=1}^{r} \kappa(j_s)$ so we have already defined a sequence for it by assumption, say $\widetilde{E}_i b = \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ and therefore we have a sequence for b given by $b = \widetilde{F}_i \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$, using condition (6) of Definition 2.2.

Example We consider the case D_4 when $r = 3$ and take the crystal basis element $b = e_{j_1} \otimes e_{j_2} \otimes j_3 = e_{\overline{4}} \otimes e_4 \otimes e_{\overline{3}}$. For the first step, the leftmost movable vector is $e_{\overline{4}}$, as $\widetilde{E}_4(e_{\overline{4}}) = e_3$. We are in case (II), because of the form of b, so the algorithm tells us to apply $\widetilde{E}_3$, which takes b to $e_{\overline{4}} \otimes e_4 \otimes e_{\overline{4}}$. Here is a summary of all the steps:

Step	Leftmost movable vector	Case	Kashiwara operator, $\widetilde{E}_i$	b	i-reduced form of b
1	$e_{\overline{4}}$	(II)	$\widetilde{E}_3$	$e_{\overline{4}} \otimes e_4 \otimes e_{\overline{3}}$	u_-
2	$e_{\overline{4}}$	(IV)	$\widetilde{E}_4$	$e_{\overline{4}} \otimes e_4 \otimes e_{\overline{4}}$	u_-
3	e_3	(IV)	$\widetilde{E}_2$	$e_3 \otimes e_4 \otimes e_{\overline{4}}$	u_-
4	e_2	(IV)	$\widetilde{E}_1$	$e_2 \otimes e_4 \otimes e_{\overline{4}}$	u_-
5	e_4	(IV)	$\widetilde{E}_3$	$e_1 \otimes e_4 \otimes e_{\overline{4}}$	$u_- \otimes u_+$
6	e_3	(IV)	$\widetilde{E}_2$	$e_1 \otimes e_3 \otimes e_{\overline{4}}$	u_-
7	$e_{\overline{4}}$	(IV)	$\widetilde{E}_4$	$e_1 \otimes e_2 \otimes e_{\overline{4}}$	u_-

We thus obtain the expression $\widetilde{E}_4\widetilde{E}_2\widetilde{E}_3\widetilde{E}_1\widetilde{E}_2\widetilde{E}_4\widetilde{E}_3 b = e_1 \otimes e_2 \otimes e_3 = \widehat{x_1}$, whence $b = \widetilde{F}_3\widetilde{F}_4\widetilde{F}_2\widetilde{F}_1\widetilde{F}_3\widetilde{F}_2\widetilde{F}_4\widehat{x_1}$.

We claim that these sequences will satisfy the requirements of Section 3. Suppose $b \in B_r$, and $i \in [1, n]$. Then, by [6, 6.3.5], we have $\widetilde{F}_i^3 b = 0$. Therefore, as the algorithm never produces zero (it always generates an expression for an element of B_r), condition (1) (in Section 3) is seen to hold. Arguing in exactly the same way as in type C_n, we see that conditions (2) and (3) of Section 3 hold.

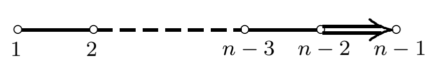
Therefore all of the conditions we required of the algorithm in Section 3 are satisfied and we have proved:

Theorem 6.1 *(Type D) Let $(\mathcal{L}_r, B_r)$ be a crystal basis for V_r as in Theorem 2.3. Suppose that $b \in B_r$ and let $b = \widetilde{F}_{j_1}^{r_1}\widetilde{F}_{j_2}^{r_2}\cdots\widetilde{F}_{j_k}^{r_k}\widehat{x_1}$ be the expression given by the algorithm above, with equal $\widetilde{F}_j$'s adjacent in the product gathered together. Let $\widetilde{b} = \widetilde{F}_{j_1}^{r_1}\widetilde{F}_{j_2}^{r_2}\cdots\widetilde{F}_{j_k}^{r_k}x_1$. Then $\widetilde{b} = F_{j_1}^{(r_1)}F_{j_2}^{(r_2)}\cdots F_{j_k}^{(r_k)}x_1$ and $\widetilde{b}$ is the canonical basis element corresponding to b.* □

7 Type B_{n-1}

We recall here the Dynkin diagram:

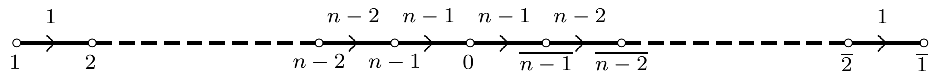
B_{n-1}



Note: throughout we assume the rank to be $n-1$, where $n \in \mathbb{N}$, $n \geq 3$; later we will compare with

D_n.

The first fundamental module, V_1, has dimension $2n-1$, so B_1 has $2n-1$ elements, which we label $e_1, e_2, \ldots, e_{n-1}, e_0, e_{\overline{1}}, e_{\overline{2}}, \ldots, e_{\overline{n-1}}$. The crystal graph is:



The edge labels are written above the edges, and vertex j corresponds to the crystal basis element e_j (see [6, §5.2]). Define a total order $\preceq$ on $\{1, 2, \ldots, n-1, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$ by putting:

$$1 \preceq 2 \preceq \cdots \preceq n-1 \preceq 0 \preceq \overline{n-1} \preceq \cdots \preceq \overline{2} \preceq \overline{1}.$$

We shall assume in this section that $r \in [1, n-2]$ (see Chapter 3 for consideration of the case $r = n-1$). For $r \in [1, n-2]$, B_r is contained in $B_1^{\otimes r}$ (see [6, §5.3]). It consists of the set:

$$\left\{ e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \quad : \quad \begin{array}{c} 1 \preceq j_1 \prec j_2 \prec \cdots \prec j_r \preceq \overline{1} \\ \text{(except that } (j_m, j_{m+1}) = (0,0) \text{ is allowed)} \\ \text{and if } j_l = \overline{j_k} \text{ for some } k, l, \text{ then } (k + r + 1 - l) \leq j_k. \end{array} \right\}.$$

(See [6, §5.3]). We shall now describe a recursive algorithm which, for each $b \in B_r$, will produce a sequence $i_1, i_2, \ldots, i_k$, as in Section 3, such that $b = \widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{x_1}$.

An Algorithm for Type B

Suppose $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$, and suppose that $b \neq \widehat{x_1}$. We have $\widehat{x_1} = e_1 \otimes e_2 \otimes \cdots \otimes e_r$ since each $\widetilde{E}_j$ kills this vector. Suppose we have defined a sequence for all crystal basis elements of this form with smaller sum $\sum_{s=1}^{r} \gamma(j_s)$ (where $\gamma(1) = 1, \gamma(2) = 2, \ldots, \gamma(n-1) = n-1, \gamma(0) = n, \gamma(\overline{n-1}) = n+1, \gamma(\overline{n-2}) = n+2, \ldots, \gamma(\overline{1}) = 2n-1$). As in case A, we say that e_{j_m} is *movable* in b if there exists $i \in [1, n]$ such that $\widetilde{E}_i e_{j_m}$ is not e_{j_s} for any s and is not zero. Note that by condition (5) of Definition 2.2, $\widetilde{E}_i e_{j_m}$ lies in $B_1 \cup \{0\}$. At least one e_{j_m} is movable in b since $b \neq \widehat{x_1}$. Let e_{j_m} be the vector in b with m minimal such that e_{j_m} is movable. We call this the *leftmost movable vector* in b. Let i be the label on the edge to the left of j_m on the crystal graph. So, if $j_m \in [1, n-1]$, $i = j_m - 1$, if $j_m = 0$, $i = n-1$ and if $j_m = \overline{j}$ for some $j \in [1, n-1]$ then $i = j$. We shall consider the element $\widetilde{E}_i b$ for this value of i.

Note that, if $k \in [1, n-2]$ then in B_1, e_k and $e_{\overline{k+1}}$ are of type u_+ for k, e_{k+1} and $e_{\overline{k}}$ are of type u_- for k, and all other e_p's are of type u_0 for k. For $n-1$, e_{n-1} is of type u_1, e_0 is of type u_2, $e_{\overline{n-1}}$ is of type u_3 and all other e_p's are of type u_0. Recall that in Kashiwara and Nakashima's description of the way the operators $\widetilde{E}_i$ and $\widetilde{F}_i$ act on the crystal basis (see Proposition 2.8), elements of type u_1, u_2 and u_3 are replaced by $u_+ \otimes u_+$, $u_- \otimes u_+$ and $u_- \otimes u_-$, respectively.

The following are clear, using the fact that e_{j_m} is the leftmost movable vector in b:
If $i \neq n-1$, the only possibilities for the i-reduced form of b are u_-, $u_- \otimes u_+$ and $u_- \otimes u_-$. We eliminate the possibility of i-reduced form $u_+ \otimes u_-$ by using the domino argument — see the definition of the algorithm for type C, near the start of Section 5. If $i = n-1$ then $j_m = \overline{n-1}$ or 0. If it is $\overline{n-1}$ there can be no $e_0's$ in the expression for b and, again using the domino argument, we see that the only possibility for the i-reduced form of b is $u_- \otimes u_-$. If $j_m = 0$ then e_{n-1} cannot occur in b, as e_{j_m} is movable. If $e_{j_m} \otimes e_{j_m+1} \otimes \cdots \otimes e_{j_r} = e_0 \otimes e_0 \otimes \cdots \otimes e_0$ then the i-reduced form of b is $u_- \otimes u_+$. Otherwise, the expression for b includes $e_0 \otimes \cdots \otimes e_0 \otimes e_k$ for some $k \neq 0$ (where the first e_0 is e_{j_m}). If $k \neq \overline{n-1}$ then the i-reduced form of b is $u_- \otimes u_+$. If $k = \overline{n-1}$ then the i-reduced form of b is $u_- \otimes u_-$. So, overall, the only possibilities for the i-reduced form of b are u_-, $u_- \otimes u_+$ and $u_- \otimes u_-$. Following the steps through in Proposition 2.8, we see that $\widetilde{E}_i b$ is not zero and has i-reduced form as follows:

	i-reduced form of b	i-reduced form of $\widetilde{E}_i b$
(a)	u_-	u_+
(b)	$u_- \otimes u_+$	$u_+ \otimes u_+$
(c)	$u_- \otimes u_-$	$u_- \otimes u_+$

Since applying $\widetilde{E}_i$ to b replaces an e_k in b with a vector with suffix further to the left in the crystal graph, that is, smaller in the given total order, $\widetilde{E}_i b$ has smaller sum $\sum_{s=1}^{r} \gamma(j_s)$, so we have already defined a sequence for it by assumption, say $\widetilde{E}_i b = \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ and therefore we have a sequence for b given by $b = \widetilde{F}_i \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$, using condition (6) of Definition 2.2.

Example We consider the case B_3 when $r = 3$ and take the crystal basis element $b = e_{j_1} \otimes e_{j_2} \otimes e_{j_3} = e_0 \otimes e_0 \otimes e_{\overline{3}}$. For the first step, the leftmost movable vector is e_0, as $\widetilde{E}_3(e_0) = e_3$. The algorithm tells us to apply $\widetilde{E}_3$, which takes b to $e_0 \otimes e_0 \otimes e_0$. Here is a summary of all the steps:

Step	Leftmost movable vector	Kashiwara operator, $\widetilde{E}_i$	b	i-reduced form of b
1	e_0	$\widetilde{E}_3$	$e_0 \otimes e_0 \otimes e_{\overline{3}}$	$u_- \otimes u_-$
2	e_0	$\widetilde{E}_3$	$e_0 \otimes e_0 \otimes e_0$	$u_- \otimes u_+$
3	e_3	$\widetilde{E}_2$	$e_3 \otimes e_0 \otimes e_0$	u_-
4	e_2	$\widetilde{E}_1$	$e_2 \otimes e_0 \otimes e_0$	u_-
5	e_0	$\widetilde{E}_3$	$e_1 \otimes e_0 \otimes e_0$	$u_- \otimes u_+$
6	e_3	$\widetilde{E}_2$	$e_1 \otimes e_3 \otimes e_0$	u_-
7	e_0	$\widetilde{E}_3$	$e_1 \otimes e_2 \otimes e_0$	$u_- \otimes u_+$

We thus obtain the expression $\widetilde{E}_3\widetilde{E}_2\widetilde{E}_3\widetilde{E}_1\widetilde{E}_2\widetilde{E}_3\widetilde{E}_3 b = e_1 \otimes e_2 \otimes e_3 = \widehat{x_1}$, whence $b = \widetilde{F}_3\widetilde{F}_3\widetilde{F}_2\widetilde{F}_1\widetilde{F}_3\widetilde{F}_2\widetilde{F}_3\widehat{x_1}$. (Note the similarity of this example to the example in type D_4 given in the previous section; see Proposition 7.3 for more details).

We claim that these sequences will satisfy the requirements of Section 3. Suppose $b \in B_r$, and $i \in [1, n]$. Then, by [6, 5.3.2], $\widetilde{F}_i^3 b = 0$. Therefore, as the algorithm never produces zero (it always generates an expression for an element of B_r), condition (1) (in Section 3) is seen to hold. We shall need the following lemma for the type B case:

Lemma 7.1 *Fix $i \in [1, n]$. Define the following map π on formal tensors in u_+, u_- and u_0. Firstly, put $\pi(u_+) = 1$, $\pi(u_-) = -1$ and $\pi(u_0) = 0$. If $u^1 \otimes u^2 \otimes \cdots \otimes u^s$ is a formal tensor in u_+, u_- and u_0, put $\pi(u^1 \otimes u^2 \otimes \cdots \otimes u^s) = \sum_{j=1}^{s} \pi(u^j)$. Suppose now $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$ has weight ν and i-reduced form $u^1 \otimes u^2 \otimes \cdots \otimes u^s$. Let $c = \pi(u^1 \otimes u^2 \otimes \cdots \otimes u^s)$. If $b' \in V_r$ has the same weight as b, then $K_i b' = v^c b'$.*

Proof: Let $t = t_1 \otimes t_2 \otimes \cdots \otimes t_r$ be the formal tensor obtained from b by replacing each e_{j_k} by t_k, where e_{j_k} is of type t_k for i (so t_k could, for example, be $u_+ \otimes u_+$). Then $\pi(t) = c$, since in obtaining the i-reduced form from t, only u_0's and pairs $u_+ \otimes u_-$ (upon which π takes the value zero) are removed. The weight of b' is the weight of b and therefore is the sum of the weights of the vectors $e_{j_1}, e_{j_2}, \ldots, e_{j_r}$. Let ν_k be the weight of e_{j_k}, $k = 1, 2, \ldots, r$. Each e_{j_k} is of type t_k for i and therefore by Remark 3.2, $(\nu_k, \alpha_i^*) = \pi(t_k)$. Therefore, as $\nu = \sum_{k=1}^{r} \nu_k$, $(\nu, \alpha_i^*) = \pi(t)$, whence, by Remark 3.2, $K_i b' = v^{\pi(t)} b'$ as required, since $\pi(t) = c$. □

Condition (3) of Section 3 holds for the B_{n-1} case; the argument given for case C_n goes through in the same way, using the above lemma.

Now suppose that $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \in B_r$ and $b = \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ is the expression for b given by the algorithm, with i_u distinct from its neighbours. The only way for this to happen is for $\widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ to have i_u-reduced form u_+, $u_+ \otimes u_+$ or $u_- \otimes u_+$. (Note that $\widetilde{F}_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ is a crystal basis element that the algorithm specifies we should apply to $\widetilde{E}_{i_u}$ to as a first step. The table above tells us the only possible i_u-reduced forms of $\widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$.)

We show that the case $u_- \otimes u_+$ cannot occur. Let e_{j_m} be the leftmost movable vector in $\widetilde{F}_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$. Note that because e_{j_m} is a leftmost movable vector, it cannot correspond to the second u_- in the i_u-reduced form $u_- \otimes u_-$ for $\widetilde{F}_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$, so suppose that it corresponds to the first u_-. Then (by Proposition 2.8(c)), applying $\widetilde{E}_{i_u}$ to $\widetilde{F}_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ does not alter e_{j_m} — it is the vector corresponding to the second u_- which is altered. Therefore e_{j_m} is also the leftmost movable vector in $\widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$. Thus, according to the algorithm, we must have $i_u = i_{u+1}$, contrary to assumption, so this case cannot occur. There is one other possibility arising here, that doesn't arise in the type C_n case. We may have that $i_u = n-1$ and that $u_- \otimes u_-$ corresponds to $e_{\overline{n-1}}$. Note that as $i_u = n-1$ we only have to consider appearances of e_{n-1}, e_0 and $e_{\overline{n-1}}$; all other possibilities are of type u_0 for $n-1$. We are therefore in the case when a subsequence of the form $e_0 \otimes e_0 \otimes \cdots \otimes e_0 \otimes e_{\overline{n-1}}$ occurs in $\widetilde{F}_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$, with no occurrence of e_{n-1} and possibly no e_0's. (If e_{n-1} appeared, the i_u-reduced form of $\widetilde{F}_{i_u} \widetilde{F}_{i_{u+1}} \cdots \widetilde{F}_{i_k} \widehat{x_1}$ would not be $u_- \otimes u_-$.) According to the algorithm, we must again have $i_u = i_{u+1} = n-1$ (following the steps through), contrary to assumption.

Therefore all of the conditions we required of the algorithm in Section 3 are satisfied and we have proved:

Theorem 7.2 *(Type B) Let $(\mathcal{L}_r, B_r)$ be a crystal basis for V_r as in Theorem 2.3. Suppose that $b \in B_r$ and let $b = \widetilde{F}_{j_1}^{r_1} \widetilde{F}_{j_2}^{r_2} \cdots \widetilde{F}_{j_k}^{r_k} \widehat{x_1}$ be the expression given by the algorithm above, with equal $\widetilde{F}_j$'s adjacent in the product gathered together. Let $\widetilde{b} = \widetilde{F}_{j_1}^{r_1} \widetilde{F}_{j_2}^{r_2} \cdots \widetilde{F}_{j_k}^{r_k} x_1$. Then $\widetilde{b} = F_{j_1}^{(r_1)} F_{j_2}^{(r_2)} \cdots F_{j_k}^{(r_k)} x_1$ and $\widetilde{b}$ is the canonical basis element corresponding to b.* □

We now look at the connection with D_n. Noting the crystal graph for the basic module in case D_n (see the start of Section 6), we make the following definitions: define the map $\delta : \{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\} \to \{1, 2, \ldots, n-1, 0, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$ by $\delta(m) = m$ if $m \neq n$ or $\overline{n}$, and $\delta(n) = \delta(\overline{n}) = 0$. This map takes the vertices of the crystal graph for the basic module in case D_n to those of the crystal graph for the basic module in case B_{n-1}. For a map which operates similarly on the edges we use $\beta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n-1\}$ which takes $1, 2, \ldots, n-1$ to themselves, and n to $n-1$. Let L_r be the r-th fundamental module in case D_n, with highest weight vector y_1, and crystal basis element of highest weight, $\widehat{y_1}$. (Here we are always referring to the crystal bases as in Theorem 2.3).

We would like to see a connection between the bases in L_r, the module in type D_n, and V_r, the module in type B_{n-1}. Note that the dimension of L_r is $\binom{2n}{r}$, while the dimension of V_r is only $\binom{2n-1}{r}$ (see [1, Table 2,p214]), so L_r has more crystal basis elements than V_r, and we can't get a one to one correspondence. In this case, what happens is that we can get a crystal basis element for V_r from every crystal basis element for L_r but there will be some duplication; we should restrict to taking one crystal basis element from each orbit of a certain involution σ of the crystal basis of L_r (see below). The following Proposition describes exactly what happens, and also the corresponding results for the canonical basis.

Proposition 7.3 *Suppose that* $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{y_1} = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$ *is a crystal basis element, as generated by the algorithm, for the module* L_r *in case* D_n *(we use the notation from Section 6 except that we denote the type* D_n *fundamental module by* L_r*). Then* $\widetilde{F}_{\beta(i_1)}\widetilde{F}_{\beta(i_2)}\cdots\widetilde{F}_{\beta(i_k)}\widehat{x_1}$ *is a crystal basis element for the module* V_r *in case* B_{n-1}*. Furthermore, every crystal basis element in* V_r *arises in this way. Similarly, if* $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}y_1$ *is the corresponding type* D_n *canonical basis element (so* $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_k}\widehat{y_1}$ *satisfies the above restrictions), then* $\widetilde{F}_{\beta(i_1)}\widetilde{F}_{\beta(i_2)}\cdots\widetilde{F}_{\beta(i_k)}x_1$ *lies in the canonical basis for* V_r *in type* B_{n-1}*. Each canonical basis element for* V_r *in type* B_{n-1} *will arise in this way.*

The involution σ *of* $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$ *which swaps* n *and* $\overline{n}$ *but fixes everything else induces an involution on the crystal basis for* L_r *in case* D_n*, by taking* $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$ *to* $e_{\sigma(j_1)} \otimes e_{\sigma(j_2)} \otimes \cdots \otimes e_{\sigma(j_r)}$*. Moreover,* $\sigma(b)$ *always lies in the crystal basis if* b *does. Also, if* $b \in L_r$*,* b *and* $\sigma(b)$ *will give rise to the same type* B_{n-1} *crystal basis element. If we repeat the above starting only with one element from each orbit of* σ *(each orbit is of the form* $\{b\}$ *or* $\{b, \sigma(b)\}$*), each crystal basis element in type* B_{n-1} *will arise exactly once. Similarly, if we start with the subset of the canonical basis in type* D_n *corresponding to this subset, we shall obtain each element of the canonical basis in type* B_{n-1} *exactly once.*

Proof: Before giving the proof, we give an example of the duplication mentioned above. Suppose that $n = 3$ and $r = 1$, and consider the crystal basis elements e_3 and $e_{\overline{3}}$ of L_1. These both generate the crystal basis element $e_0 = e_{\delta(3)} = e_{\delta(\overline{3})}$ of V_1.

Using the notation in Section 6, let $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$ be a crystal basis element for L_r in type D_n. It follows from the definitions given above for the crystal bases in types D_n and B_{n-1} that $e_{\delta(j_1)} \otimes e_{\delta(j_2)} \otimes \ldots \otimes e_{\delta(j_r)}$ is a crystal basis element of V_r in case B_{n-1}. (Note that the restrictions given in the descriptions of the crystal bases are very similar). It is clear that every crystal basis element of V_r will arise in this way, and exactly once if we use the subset in the proposition. (Note that $\delta(n) = \delta(\overline{n}) = 0$). The statements in the proposition about crystal bases will follow from the fact that if $\widetilde{E}_i$ is a Kashiwara operator in case D and the definition of the case D algorithm specifies we should apply $\widetilde{E}_i$ to $b = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$ to get $e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_r}$, then the case B algorithm specifies that we should apply $\widetilde{E}_{\beta(i)}$ to $b^* = e_{\delta(j_1)} \otimes e_{\delta(j_2)} \otimes \cdots \otimes e_{\delta(j_r)}$ to get $e_{\delta(k_1)} \otimes e_{\delta(k_2)} \otimes \cdots \otimes e_{\delta(k_r)}$. We shall now verify this fact. The algorithm in case D was defined exactly so this would happen.

Let e_{j_m} be the leftmost movable vector in b. We consider the four cases separated out in the definition of the algorithm for case D_n, and check that this is what happens in each case.

Case (I): Suppose that $j_m = \overline{n-1}$ and that neither e_n nor $e_{\overline{n}}$ occurs in the expression for b. The type D_n algorithm specifies that in this situation we should apply $\widetilde{F}_{n-1}$ to b, and this changes $e_{\overline{n-1}}$ to $e_{\overline{n}}$. Since neither e_n nor $e_{\overline{n}}$ occurs in the expression for b, b^* is the same expression as b. Again e_{j_m} is the leftmost movable vector in b^* and the type B_{n-1} algorithm specifies we should apply $\widetilde{E}_{n-1} = \widetilde{E}_{\beta(n-1)}$ to b^* and this changes $e_{\overline{n-1}}$ to e_0, that is, $e_{\delta(\overline{n-1})}$ to $e_{\delta(\overline{n})}$, which corresponds exactly to what the type D algorithm did to b.

Case (II): Suppose that $j_m = \overline{n}$ and the expression for b immediately after e_{j_m} (including e_{j_m}) is of the form:

$$e_{\overline{n}} \otimes e_n \otimes \cdots \otimes e_{\overline{n}} \otimes e_n \otimes e_{\overline{n-1}},$$

where $e_{\overline{n}} \otimes e_n$ appears at least once in the expression. The type D_n algorithm specifies that we should apply $\widetilde{E}_{n-1}$ to b, and this changes $e_{\overline{n}} \otimes e_n \otimes \cdots \otimes e_{\overline{n}} \otimes e_n \otimes e_{\overline{n-1}}$ to $e_{\overline{n}} \otimes e_n \otimes \cdots \otimes e_{\overline{n}} \otimes e_n \otimes e_{\overline{n}}$. The corresponding part of b^* is $e_0 \otimes e_0 \otimes \cdots \otimes e_0 \otimes e_0 \otimes e_{\overline{n-1}}$, with one e_0 corresponding to each e_n or $e_{\overline{n}}$. The type B_{n-1} algorithm specifies that we should apply $\widetilde{E}_{n-1}$ to b^*, since e_0 will be the leftmost movable vector in b^*. Note that e_{n-1} cannot appear in b as $e_{\overline{n}}$ is movable, so it cannot appear in b^* either. The result after applying $\widetilde{E}_{n-1}$ to b^* is $e_0 \otimes e_0 \otimes \cdots \otimes e_0 \otimes e_0$, with one extra

e_0. That is, $e_{\delta(\overline{n})}\otimes e_{\delta(n)}\otimes\cdots\otimes e_{\delta(\overline{n})}\otimes e_{\delta(n)}\otimes e_{\delta(\overline{n-1})}$ becomes $e_{\delta(\overline{n})}\otimes e_{\delta(n)}\otimes\cdots\otimes e_{\delta(\overline{n})}\otimes e_{\delta(n)}\otimes e_{\delta(\overline{n})}$, which corresponds exactly to what the type D algorithm did to b.

Case (III): Suppose that $j_m = n$ and the expression for b immediately after e_{j_m} (including e_{j_m}) is of the form:

$$e_n \otimes e_{\overline{n}} \otimes \cdots \otimes e_n \otimes e_{\overline{n}} \otimes e_{\overline{n-1}},$$

where $e_n \otimes e_{\overline{n}}$ appears at least once in the expression. We argue in a similar way to case (II).

Case (IV): Suppose that none of the above cases occur. In this case there is a unique i such that $\widetilde{E}_i e_{j_m}$ is not zero and is not e_{j_k} for any k, and the case D_n algorithm specifies that we should apply this $\widetilde{E}_i$ to b, which changes e_{j_m} to $\widetilde{E}_i e_{j_m}$. It is clear that $e_{\delta(j_m)}$ will be the leftmost movable vector in b^*. So the type B_{n-1} algorithm specifies we should apply $\widetilde{E}_{i'}$ to b^*, where i' is the label on the edge immediately to the left of $\delta(j_m)$ on the crystal graph for the basic module in type B_{n-1}. It is clear, from the definitions of δ and β, that we must have $i' = \beta(i)$. Suppose first that $j_m \in [1, n-1]$. Then $\widetilde{E}_i$ (in type D_n) takes e_{j_m} to $e_{j_m - 1}$ in b and $\widetilde{E}_{i'}$ (in type B_{n-1}) takes e_{j_m} to e_{j_m-1} in b^* (that is, it takes $e_{\delta(j_m)}$ to $e_{\delta(j_m-1)}$). A similar thing happens when $j_m = \overline{j}$ for some $j \in [1, n-2]$. If $j_m = \overline{n-1}$ then e_{n-1} cannot occur in b (by the domino argument). Since we are not in case (I), (II) or (III), we must have $j_{m-1} = n$ or $\overline{n}$. We have $i = n-1$ in the first case and $i = n$ in the latter. In the first case $e_n \otimes e_{\overline{n-1}}$ becomes $e_n \otimes e_{\overline{n}}$ in b. In b^*, we have $e_0 \otimes e_{\overline{n-1}}$ which becomes $e_0 \otimes e_0$ after applying $\widetilde{E}_{i'}$, which corresponds to this; the second case is similar. Finally, suppose $j_m = n$ (the case $j_m = \overline{n}$ is similar). After e_{j_m} (and including it), b must look like $e_n \otimes e_{\overline{n}} \otimes \cdots \otimes e_n \otimes e_{\overline{n}}$ (with or without the $e_{\overline{n}}$ at the end), and after this there is no $e_{\overline{n-1}}$ since we are not in case (III). We have $i = n-1$ and in b, $\widetilde{E}_i$ takes $e_n \otimes e_{\overline{n}} \otimes \cdots \otimes e_n \otimes e_{\overline{n}}$ to $e_{n-1} \otimes e_{\overline{n}} \otimes e_n \otimes e_{\overline{n}} \cdots \otimes e_n \otimes e_{\overline{n}}$ (with or without the $e_{\overline{n}}$ at the end). The corresponding part of b^* is $e_0 \otimes e_0 \otimes \cdots \otimes e_0$, $i' = n-1$, and $\widetilde{E}_{i'}$ takes $e_0 \otimes e_0 \cdots \otimes e_0$ to $e_{n-1} \otimes e_0 \otimes \cdots \otimes e_0$ in b^*, which corresponds to what happens in the D_n case.

We have shown that the statement regarding crystal bases is true, and the statement about canonical bases follows from the statement about crystal bases and Theorems 6.1 and 7.2. □

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