DENOMINATORS IN CLUSTER ALGEBRAS OF AFFINE TYPE

ASLAK BAKKE BUAN AND ROBERT J. MARSH

ABSTRACT. The Fomin-Zelevinsky *Laurent phenomenon* states that every cluster variable in a cluster algebra can be expressed as a Laurent polynomial in the variables lying in an arbitrary initial cluster. We give representation-theoretic formulas for the denominators of cluster variables in cluster algebras of affine type. The formulas are in terms of the dimensions of spaces of homomorphisms in the corresponding cluster category, and hold for any choice of initial cluster.

INTRODUCTION

Cluster algebras were introduced by Fomin and Zelevinsky in [FZ1]. They have strong links with the representation theory of finite dimensional algebras (see e.g. the survey articles [BM, Kel2]), with semisimple algebraic groups and the dual semicanonical basis of a quantum group (see e.g. the survey article [GLS]), and with many other areas (see e.g. the survey article [FZ2]); these articles contain many further references.

Here we consider acyclic coefficient-free cluster algebras of affine type, i.e. those which can be given by an extended Dynkin quiver. We give a formula expressing the denominators of cluster variables in terms of any given initial cluster in terms of dimensions of certain Hom-spaces in the corresponding cluster category. The representation theory, and hence the cluster category, is well understood in the tame case. Thus, the formula can be used to compute the denominators explicitly.

We assume that k is an algebraically closed field. Caldero and Keller [CK2] (see also [BCKMRT]) have shown, using the Caldero-Chapoton map [CC], that for an acyclic quiver Q, the cluster variables of the acyclic cluster algebra $\mathcal{A}_Q$ are in bijection with the indecomposable exceptional objects in the cluster category $\mathcal{C}_H$, where $H = kQ$ is the path algebra of Q. Furthermore, under this correspondence the clusters correspond to cluster-tilting objects. We denote by x_M the cluster variable corresponding to the exceptional indecomposable M in $\mathcal{C}_{kQ}$.

Recall that an indecomposable regular H-module X lies in a connected component of the AR-quiver of H known as a tube, which we denote by $\mathcal{T}_X$. For a regular indecomposable exceptional module X, we let $\mathcal{W}_X$ denote the

wing of X inside $\mathcal{T}_X$, i.e. the category of subfactors of X inside $\mathcal{T}_X$. We let τ denote the Auslander-Reiten translate.

We prove the following theorem.

Theorem A. *Let Q be an extended Dynkin quiver. Let H be the path algebra of Q, and let $\{y_1, \ldots, y_n\} = \{x_{\tau T_1}, \ldots, x_{\tau T_n}\}$ be an arbitrary initial seed of the cluster algebra $\mathcal{A}_Q$, where $T = \amalg_i T_i$ is a cluster-tilting object in $\mathcal{C}_{kQ}$. Let X be an exceptional object of $\mathcal{C}$ not isomorphic to τT_i for any i. Then, in the expression $x_X = f/m$ in reduced form we have $m = \prod_i y_i^{d_i}$, where*

$$
d_i = \begin{cases} \dim \operatorname{Hom}_{\mathcal{C}}(T_i, X) - 1 & \textit{if there is a tube of rank } t + 1 \geq 2 \\ & \textit{containing } T_i \textit{ and } X, \text{ q.l.} T_i = t \textit{ and} \\ & X \notin \mathcal{W}_{\tau T_i}, \\ \dim \operatorname{Hom}_{\mathcal{C}}(T_i, X) & \textit{otherwise.} \end{cases}
$$

We remark that representation-theoretic expressions for denominators of cluster variables for an arbitrary initial seed were given in [CCS1] for type A and for any simply-laced Dynkin quiver in [CCS2, RT]. In the general case, for an initial seed with acyclic exchange quiver, it was shown in [BMRT, CK2] that denominators of cluster variables are given by dimension vectors (see the next section for more details). The general case for an arbitrary initial seed was studied in [BMR2]. In particular, it was shown that for an affine cluster algebra, provided the cluster-tilting object corresponding to the initial seed contains no regular summand of maximal quasilength in its tube, the denominators of all cluster variables are given by dimension vectors. Cluster variables in affine cluster algebras of rank 2 have been studied in [CZ, MP, SZ, Ze]. The present article completes the denominator picture (for an arbitrary initial seed), in terms of dimension vectors, for affine (coefficient-free) cluster algebras.

In [FK, 6.6] it is shown that for any cluster category (and in fact in a wider context), the dimension vector of a module coincides with the corresponding f-*vector* in the associated cluster algebra with principal coefficients. See [FK, Sec. 6] for the definition of f-vectors.

Thus our results determine when Conjecture 7.17 of [FZ3] holds for affine cluster algebras. We also remark that in Theorem A, each exponent in the denominator is less than or equal to the corresponding entry in the dimension vector, in agreement with [FK, 5.8] and [DWZ].

Representation-theoretic expressions for cluster variables have been widely studied; see for example [CK1, D, Hu, Pal, XX1, XX2, XX3, Zh]. See in particular [BKL, D] for other aspects of cluster combinatorics associated with tubes, and see e.g. [M, Par, Pr, Sc, ST, YZ] for related combinatorial constructions.

In Section 1, we recall some of the results described in the previous paragraph. In Section 2, we recall some standard facts about tame hereditary algebras. In Section 3, we study the transjective component of the cluster category, to prepare for the proof of Theorem A. In Section 4, we study regular objects in the cluster category, and then in Section 5 we prove the main theorem, and in Section 6 we give a small example to illustrate it.

1. Preliminaries

Let Q be a finite connected acyclic quiver and k an algebraically closed field. Then $H = kQ$ denotes the (finite dimensional) path algebra of Q over k. Let $D^b(H)$ be the bounded derived category of finite dimensional left H-modules. The category $D^b(H)$ is a triangulated category with a suspension functor [1] (the shift). Since H is hereditary, the category $D^b(H)$ has almost split triangles; see [Ha], and thus has an autoequivalence τ, the Auslander-Reiten translate. Let $\mathcal{C} = \mathcal{C}_H = D^b(H)/\tau^{-1}[1]$ be the cluster category of H (introduced in [CCS1] for type A and in [BMRRT] in general). Keller [Kel1] has shown that $\mathcal{C}$ is triangulated. For more information about the representation theory of finite dimensional algebras, see [ARS, ASS], and see [Ha] for basic properties of derived categories.

We regard H-modules as objects of $\mathcal{C} = \mathcal{C}_H$ via the natural embedding of the module category of H in $D^b(H)$. For a vertex i of Q, let P_i denote the corresponding indecomposable projective $H = kQ$-module. Note that every indecomposable object of $\mathcal{C}$ is either an indecomposable H-module or of the form $P_i[1]$ for some i.

We denote homomorphisms in $\mathcal{C}$ simply by $\mathrm{Hom}(\ ,\ )$, while $\mathrm{Hom}_H(\ ,\ )$ denotes homomorphisms in $\mathrm{mod}\, H$ (or $D^b(H)$). For a fixed H, we say that a map $X \to Y$ in $\mathcal{C}_H$ is an F-map if it is induced by a map $X \to \tau^{-1}Y[1]$ in $D^b(H)$, where X, Y are direct sums of H-modules or objects of the form $P_i[1]$. Note that the composition of two F-maps is zero.

An H-module T is a called a *partial tilting module* if $\mathrm{Ext}^1_H(T, T) = 0$, an *almost complete tilting module* if in addition it has $n - 1$ nonisomorphic indecomposable summands, and a *tilting module* if it has n such summands (by a result of Bongartz [Bo] this is equivalent to the usual notion of a tilting module over H). We shall assume throughout that all such modules are *basic*, i.e. no indecomposable summand appears with multiplicity greater than 1. For more information on tilting theory see [AHK].

The corresponding notions of *cluster-tilting object*, *partial cluster-tilting object* and *almost complete cluster-tilting object* in $\mathcal{C}$ can be defined similarly with reference to the property $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$; see [BMRRT]. Note that every cluster-tilting object in $\mathcal{C}$ is induced from a tilting module over some hereditary algebra derived equivalent to H [BMRRT, 3.3].

If $\overline{T}$ is a partial tilting module (respectively, a partial cluster-tilting object) and $\overline{T} \amalg X$ is a tilting module (respectively, a cluster-tilting object), then X is called a *complement* of $\overline{T}$.

Let $\mathcal{A} = \mathcal{A}(Q) \subseteq \mathbb{F} = \mathbb{Q}(x_1, x_2, \ldots, x_n)$ be the (acyclic, coefficient-free) cluster algebra defined using the initial seed $(\mathbf{x}, Q)$, where $\mathbf{x}$ is a free generating set $\{x_1, x_2, \ldots, x_n\}$ for $\mathbb{F}$; see [FZ1].

For an object X of $\mathcal{C}$, let $c_X = \prod_{i=1}^n x_i^{\dim \mathrm{Hom}_C(P_i, X)}$. The following gives a connection between cluster categories and acylic cluster algebras.

Theorem 1.1. *(a)* [BMRT, 2.3] *There is a surjective map*

$\alpha \colon \{cluster\ variables\ of\ \mathcal{A}(Q)\} \to \{indecomposable\ exceptional\ objects\ in\ \mathcal{C}\}.$

It induces a surjective map

$$\bar{\alpha} \colon \{clusters\} \to \{cluster\text{-}tilting\ objects\}.$$

(b) [CK2] *There is a bijection* $\beta\colon X \to x_X$ *from indecomposable excep-
tional objects of* $\mathcal{C}$ *to cluster variables of* $\mathcal{A}$ *such that for any indecomposable
exceptional* kQ-*module* X, *we have* $x_X = f/c_X$ *as a quotient of integral poly-
nomials in the* x_i *in reduced form, where* $c_X = \prod_{i=1}^{n} x_i^{\dim \operatorname{Hom}_{\mathcal{C}}(P_i,X)}$.
(c) [BCKMRT] *The maps* α *and* β *are mutual inverses.*

We now recall some results and definitions from [BMR2]. Assume Γ is a
quiver which is mutation-equivalent to Q. By the above theorem there is a
seed $(\mathbf{y}, \Gamma)$ of $\mathcal{A}$, where $\mathbf{y} = \{y_1, y_2, \dots, y_n\}$ is a free generating set of $\mathbb{F}$ over
$\mathbb{Q}$. Let $T_i = \tau^{-1}\alpha(y_i)$ for $i = 1, 2, \dots, n$, so that we have $\alpha(y_i) = \tau T_i$. Then
$\amalg_{i=1}^{n} \tau T_i$ is a cluster-tilting object in $\mathcal{C}$ and Γ is the quiver of $\operatorname{End}_{\mathcal{C}}(\tau T)^{\mathrm{op}} \simeq$
$\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ by [BMR1].

For a polynomial $f = f(z_1, z_2, \dots, z_n)$, we say that f satisfies the *positivity
condition* if $f(e_i) > 0$ for $i = 1, 2, \dots, n$, where $e_i = (1, ..., 1, 0, 1, ..., 1)$ (with
a 0 in the ith position).

Definition 1.2. [BMR2] *Let* x *be a cluster variable of* $\mathcal{A}$ *with* $\alpha(x) = X$ *for
some exceptional indecomposable object* X *of* $\mathcal{C}$. *We say that* x *expressed in
terms of the cluster* $\mathbf{y}$ *has a* T-*denominator if either:*

(I) *We have that* X *is not isomorphic to* τT_i *for any* i, *and* $x = f/t_X$,
where f *satisfies the positivity condition and* $t_X = \prod_{i=1}^{n} y_i^{\dim \operatorname{Hom}_{\mathcal{C}}(T_i,X)}$,
or

(II) *We have that* $X \simeq \tau T_i$ *for some* i *and* $x = y_i$.

Note that when the choice of cluster $\mathbf{y}$ is clear we also say that an ex-
ceptional indecomposable object X of $\mathcal{C}$ has a T-denominator, when its
corresponding cluster variable has. We also recall that in (I), the expression
for x must be in reduced form. (For a contradiction, suppose that f and t_X
have a common monomial factor u. Suppose that the variable y_i divides u.
Then $f(e_i) = 0$, since $u(e_i) = 0$, contradicting the fact that f satisfies the
positivity condition.)

Here, in addition, we make the following definition:

Definition 1.3. *Let* x *be a cluster variable with* $\alpha(x) = X$ *for a regular
object* X *of* $\mathcal{C}$, *and assume* $x = f/\prod_{i=1}^{n} y_i^{d_i}$ *for some cluster* $\mathbf{y}$, *where* f
satisfies the positivity condition. We say x *(or* X*) has a* diminished T-
denominator *if* $d_i = \dim \operatorname{Hom}_{\mathcal{C}}(T_i, X) - 1$ *for all regular summands* T_i *of
T with* $\mathcal{T}_X = \mathcal{T}_{T_i}$ *and* q.l. $T_i = t_i$, *where* $t_i + 1$ *is the rank of* $\mathcal{T}_{T_i}$, *and
$d_j = \dim \operatorname{Hom}_{\mathcal{C}}(T_j, X)$ *for all other summands* T_j.

Theorem 1.4. [BMR2] *Let* $T = \amalg_{i=1}^{n} T_i$ *be a cluster-tilting object in* $\mathcal{C} = \mathcal{C}_{kQ}$
for an acyclic quiver Q *and let* $\mathcal{A} = \mathcal{A}(Q)$ *be the cluster algebra associated
to* Q. *Then:*

(a) *If no indecomposable direct summand of* T *is regular then every clus-
ter variable of* $\mathcal{A}$ *has a* T-*denominator.*

(b) *If every cluster variable of* $\mathcal{A}$ *has a* T-*denominator, then* $\operatorname{End}_{\mathcal{C}}(T_i) \simeq$
k *for all* i.

Suppose in addition that kQ *is a tame algebra. Then the following are
equivalent:*

(i) *Every cluster variable of* $\mathcal{A}$ *has a* T-*denominator.*

(ii) *No regular summand T_i of quasi-length t lies in a tube of rank $t+1$.*
(iii) *For all i, $\mathrm{End}_{\mathcal{C}}(T_i) \simeq k$.*

The main result (Theorem A) of this paper gives a precise descripton of the denominators of all cluster variables for the tame case, i.e. also including the case when T has a regular summand T_i of quasi-length t lying in a tube of rank $t+1$.

Fix an almost complete (basic) cluster-tilting object $\overline{T}'$ in $\mathcal{C}$. Let X, X^* be the two complements of $\overline{T}'$, so that $T' = \overline{T}' \amalg X$ and $T'' = \overline{T}' \amalg X^*$ are cluster-tilting objects (see [BMRRT, 5.1]). Let

$$X^* \xrightarrow{f} B \xrightarrow{g} X \xrightarrow{h} \tag{1}$$

$$X \xrightarrow{f'} B' \xrightarrow{g'} X^* \xrightarrow{h'} \tag{2}$$

be the exchange triangles corresponding to X and X^* (see [BMRRT, §6]), so that $B \to X$ is a minimal right $\mathrm{add}(\overline{T}')$-approximation of X in $\mathcal{C}$ and $B' \to X^*$ is a minimal right $\mathrm{add}(\overline{T}')$-approximation of X^* in $\mathcal{C}$. The following definition is crucial:

Definition 1.5. [BMR2] *Let M be an exceptional indecomposable object of $\mathcal{C}$. We say that M is* compatible *with an exchange pair (X, X^*), if either $X \simeq \tau M$, $X^* \simeq \tau M$, or, if neither of these holds,*

$$\dim \mathrm{Hom}_{\mathcal{C}}(M, X) + \dim \mathrm{Hom}_{\mathcal{C}}(M, X^*)$$
$$= \max(\dim \mathrm{Hom}_{\mathcal{C}}(M, B), \dim \mathrm{Hom}_{\mathcal{C}}(M, B')).$$

If M is compatible with every exchange pair (X, X^*) in $\mathcal{C}$ we call M exchange compatible.

We also have:

Proposition 1.6. [BMR2]
(a) *Suppose that (X, X^*) is an exchange pair such that neither X nor X^* is isomorphic to τM. Then the following are equivalent:*
(i) *M is compatible with the exchange pair (X, X^*).*
(ii) *Either the sequence*

$$0 \to \mathrm{Hom}_{\mathcal{C}}(M, X^*) \to \mathrm{Hom}_{\mathcal{C}}(M, B) \to \mathrm{Hom}_{\mathcal{C}}(M, X) \to 0 \tag{3}$$

is exact, or the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(M, X) \to \mathrm{Hom}_{\mathcal{C}}(M, B') \to \mathrm{Hom}_{\mathcal{C}}(M, X^*) \to 0 \tag{4}$$

is exact.
(b) *Let M be an exceptional indecomposable object of $\mathcal{C}$ and suppose that $X \simeq \tau M$ or $X^* \simeq \tau M$. Then we have that*

$$\dim \mathrm{Hom}_{\mathcal{C}}(M, X) + \dim \mathrm{Hom}_{\mathcal{C}}(M, X^*) =$$
$$\max(\dim \mathrm{Hom}_{\mathcal{C}}(M, B), \dim \mathrm{Hom}_{\mathcal{C}}(M, B')) + 1$$

(c) *Let $(\mathbf{x}', Q')$ be a seed, with $\mathbf{x}' = \{x_1', \dots, x_n'\}$, and assume that each x_i' has a T-denominator. Let $T_i' = \alpha(x_i')$ for $i = 1, 2, \dots, n$ and $T' = \amalg_{j=1}^n T_j'$. Mutating $(\mathbf{x}', Q')$ at x_k' we obtain a new cluster variable $(x_k')^*$. Let $\overline{T}' = \amalg_{j \neq k} T_j'$. and let X^* be the unique indecomposable*

object in $\mathcal{C}$ with $X^* \not\simeq T'_k = X$ such that $\overline{T}' \amalg X^*$ is a cluster-tilting object. Then the cluster variable $x_{X^*} = (x'_k)^*$ has a T-denominator if each summand T_i of T is compatible with the exchange pair (X, X^*).

Note that (c) is used as an induction step in [BMR2] for showing that cluster variables have T-denominators. Also, in [BMR2] it is shown that in (c) the cluster variable $x_{X^*} = (x'_k)^*$ has a T-denominator if and only if each summand T_i of T is compatible with the exchange pair (X, X^*), but we shall not need this stronger statement.

Proposition 1.7. [BMR2] *Let H be a tame hereditary algebra, and let M be an indecomposable exceptional object in $\mathcal{C}$. Then M is exchange compatible if and only if $\mathrm{End}_{\mathcal{C}}(M) \simeq k$.*

2. Tame hereditary algebras

In this section we review some facts about tame hereditary algebras, cluster categories and cluster algebras.

We fix a connected extended Dynkin quiver Q. The category $\mathrm{mod}\, kQ$ of finite dimensional modules over the tame hereditary algebra $H = kQ$ is well understood; see [R]. Let τ denote the Auslander-Reiten translate. All indecomposable kQ-modules X are either preprojective, i.e. $\tau^m X$ is projective for some $m \geq 0$; preinjective, i.e. $\tau^{-m} X$ is injective for some $m \geq 0$; or regular, i.e. not preprojective or preinjective.

The Auslander-Reiten quiver of H consists of:

  (i) the preprojective component, consisting exactly of the indecomposable preprojective modules;
 (ii) the preinjective component, consisting exactly of the indecomposable preinjective modules;
(iii) a finite number d of regular components called non-homogeneous (or exceptional) tubes, $\mathcal{T}_1, \ldots \mathcal{T}_d$;
 (iv) an infinite set of regular components called homogeneous tubes.

For a fixed tube $\mathcal{T}$, there is a number m, such that $\tau^m X = X$ for all indecomposable objects in $\mathcal{T}$. The minimal such m is the *rank* of $\mathcal{T}$. If $m = 1$ then $\mathcal{T}$ is said to be homogeneous.

We will also use the following facts about maps in $\mathrm{mod}\, H$. Let P (respectively, I and R) be preprojective (respectively, preinjective and regular) indecomposable modules, and a R' a regular indecomposable module with $\mathcal{T}_R \neq \mathcal{T}_{R'}$. Then we have that $\mathrm{Hom}_H(I, R) = \mathrm{Hom}_H(I, P) = \mathrm{Hom}_H(R, P) = \mathrm{Hom}_H(R, R') = 0$.

3. The transjective component

We will call an indecomposable object in the cluster category *transjective* if it is not induced by a regular module. Note that the transjective objects form a component of the Auslander-Reiten quiver of $\mathcal{C}$. One of our aims is to show that every transjective object has a T-denominator for tame hereditary algebras. In this section, we show that for this it is sufficient to find one transjective cluster-tilting object all of whose summands have a T-denominator. Note that the results in this section do not require H to be tame, but hold for all finite dimensional hereditary algebras.

Remark 3.1. *We remark that, given a finite set of indecomposable transjective objects in the cluster category, we can, by replacing the hereditary algebra H with a derived equivalent hereditary algebra, assume that all of the objects in the set are preprojective* [BMRRT, 3.3]. *We shall make use of this in what follows.*

We start with the following observation

Lemma 3.2. *Assume $(X, \tau X)$ is an exchange pair.*
- (a) *The AR-triangle $\tau X \to E \to X \to$ is an exchange triangle.*
- (b) *Any exceptional object M is compatible with the exchange pair $(X, \tau X)$.*

Proof. Part (a) is well-known. By [BMRRT, 7.5], we know that $\mathrm{Ext}^1_{\mathcal{C}}(X, \tau X) \simeq k$ when $(X, \tau X)$ is an exchange pair. Hence, the AR-triangle must be isomorphic to the exchange triangle.

For part (b) we can assume that $\tau X \to E \to X \to$ is induced by an almost split sequence in $\mathrm{mod}\, H'$ with $\mathcal{C}_H = \mathcal{C}_{H'}$, by Remark 3.1. Then we use Lemma [BMR2, 5.1] to obtain that $\mathrm{Hom}_{\mathcal{C}}(M, \ )$ applied to the AR-triangle $\tau X \to E \to X \to$ gives an exact sequence. The claim then follows from Proposition 1.6. □

The following summarizes some facts that will be useful later.

Proposition 3.3. *Let H be a hereditary algebra and U a tilting H-module.*
- (a) *If U is a tilting module such that $U \not\simeq H$ then there is an indecomposable direct summand U_i of U which is generated by $\bar{U} = U/U_i$.*
- (b) *Furthermore, if $\bar{U} = U/U_i$ generates U_i, and $B \to U_i$ is the (necessarily surjective) minimal right $\mathrm{add}\,\bar{U}$-approximation of U_i and*
$$0 \to U^* \to B \to U_i \to 0 \tag{5}$$
  is the induced exact sequence in $\mathrm{mod}\, H$ then the H-module $\bar{U} \amalg U^*$ is a tilting module in $\mathrm{mod}\, H$.
- (c) *The exact sequence (5) induces an exchange triangle in $\mathcal{C}_H$.*
- (d) *If U is preprojective, then so is U^*.*

Proof. Part (a) is a theorem of Riedtmann and Schofield [RS]. Part (b) is a special case of a theorem by Happel and Unger [HU]. Part (c) is contained in [BMRRT] and part (d) is obvious. □

The following is also well-known and holds for any finite dimensional hereditary algebra H. Note that for H of finite representation type, all modules are by definition preprojective.

Lemma 3.4. *For every preprojective tilting module U in $\mathrm{mod}\, H$ there is a finite sequence of preprojective tilting modules*
$$U = W_0, W_1, W_2, \ldots, W_r = H,$$
and exact sequences
$$0 \to M_j^* \to B_j \to M_j \to 0 \tag{6}$$
with $B_j \to M_j$ a minimal right $\mathrm{add}\, W_j/M_j$-approximation of the indecomposable direct summand M_j of W_j, and
$$W_{j+1} = (W_j/M_j) \amalg M_j^*.$$

Proof. We use the fact that the preprojective component is directed, so there is an induced partial order on the indecomposable modules, generated by $X \preceq Y$ if $\mathrm{Hom}(X, Y) \neq 0$. For the above exchange sequences we have $M_j^* \nsucceq M_j$. The result now follows directly from Proposition 3.3. $\qquad\square$

Next we consider transjective exchange pairs.

Lemma 3.5. *Let (X, X^*) be an exchange pair, where both X and X^* are transjective. Then any regular indecomposable exceptional M is compatible with (X, X^*).*

Proof. We choose a hereditary algebra H' derived equivalent to H such that both X and X^* correspond to preprojective H'-modules (see Remark 3.1). Hence one of the exchange triangles, say

$$X^* \to B \to X \to$$

is induced by a short exact sequence, by [BMRRT]. It is clear that the middle term B is also induced by a preprojective module. Note that we have $\mathcal{C}_H \simeq \mathcal{C}_{H'}$.

We want to show that we get a short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(M, X^*) \to \mathrm{Hom}_{\mathcal{C}}(M, B) \to \mathrm{Hom}_{\mathcal{C}}(M, X) \to 0. \qquad (7)$$

Since there is a path of H'-maps from X^* to X in the preprojective component of H', and this component is directed, we have that there is no H'-map $X \to \tau X^*$. Hence the nonzero map $X \to \tau X^*$ induced from the exchange triangle is an $F' = F_{H'}$-map. Any map $M \to X$ is also an F'-map, using that there are no H'-maps from regular objects to preprojective objects. But any composition of two F'-maps is zero. Hence every map $M \to X$ will factor through $B \to X$, so the sequence (7) is right exact.

Assume there is a map $M \to X^*$. Then this map must be an F'-map. Assume the composition $M \to X^* \to B$ is zero, so that $M \to X^*$ factors through $\tau^{-1} X \to X^*$.

$$
\begin{array}{ccccccc}
 & & M & & & & \\
 & \swarrow & \downarrow & & & & \\
\tau^{-1}X & \longrightarrow & X^* & \longrightarrow & B & \longrightarrow & X
\end{array}
$$

Then both maps $M \to \tau^{-1} X$ and $\tau^{-1} X \to X^*$ are F'-maps, and hence the composition is zero. Hence the map $M \to X^*$ is zero, and we have shown left-exactness of (7). This finishes the proof by Proposition 1.6. $\qquad\square$

A *slice* in $\mathrm{mod}\, H$ (see [R]), is a tilting module V with a hereditary endomorphism ring. Note that $\mathrm{End}_{\mathcal{C}}(V)$ is hereditary if and only if $\mathrm{End}_H(V)$ is hereditary by [ABS].

Lemma 3.6. *Assume there is a slice $V = \amalg_i V_i$ such that each indecomposable direct summand V_i has a T-denominator. Then every transjective indecomposable object has a T-denominator.*

Proof. This follows from combining Lemma 3.2 with Proposition 1.6. $\qquad\square$

Lemma 3.7. *Assume there is a transjective cluster-tilting object $U = \amalg_i U_i$ such that each indecomposable direct summand U_i has a T-denominator. Then there is a slice $V = \amalg_i V_i$ such that each indecomposable direct summand V_i has a T-denominator.*

Proof. We choose a hereditary algebra H' derived equivalent to H, so that all the U_i are preprojective modules in $\operatorname{mod} H'$ and hence U is a preprojective tilting module in $\operatorname{mod} H'$ (see Remark 3.1)

It is clear that each W_j in Lemma 3.4 is a cluster-tilting object in $\mathcal{C}_H$, and that the object H' forms a slice in $\mathcal{C}_H$. Also it is is clear that the short exact sequences (6) are exchange triangles in $\mathcal{C}_H = \mathcal{C}_{H'}$, with transjective end-terms. So the claim follows from Propositions 1.6 and 1.7 and Lemma 3.5. □

We can now state the main result of this section.

Proposition 3.8. *Assume that there is a transjective cluster-tilting object $U = \amalg_i U_i$ such that each indecomposable direct summand U_i has a T-denominator. Then every transjective indecomposable object has a T-denominator.*

Proof. This follows directly from combining Lemmas 3.6 and 3.7. □

4. Wings

For this section assume that H is a tame hereditary algebra. We state some properties and results concerning regular objects in the cluster category of H.

Recall that a module M over an algebra A is known as a *brick* if it is exceptional and $\operatorname{End}_A(M) = k$. In fact, it is known that if A is hereditary, every exceptional A-module is a brick. We say that an object M in the cluster category $\mathcal{C}$ is a *$\mathcal{C}$-brick* if M is exceptional with $\operatorname{End}_{\mathcal{C}}(M) = k$.

For an indecomposable exceptional regular module M, we can consider the full subcategory $\mathcal{W}_M$ of the abelian category $\mathcal{T}_M$, where the objects are all subfactors of M formed inside $\mathcal{T}_M$. This is called the *wing* of M. The indecomposable objects in $\mathcal{W}_M$ form a full subquiver of the AR-quiver shaped as a triangle with vertices given by the unique quasi-simple with a non-zero map to M, the unique quasi-simple which M has a non-zero map to, and M itself.

Suppose that q.l. $M = t$. We consider $\mathcal{W}_M$ as an abelian category equivalent to $\operatorname{mod} \Lambda_t$, where Λ_t is the hereditary algebra given as the path algebra of a quiver of Dynkin type A_t, with linear orientation; see [R]. The module M is a projective and injective object in $\mathcal{W}_M$, and a tilting object in $\mathcal{W}_M$ has exactly t indecomposable direct summands.

The following lemma summarizes some well-known facts, including the fact that there are bricks in the cluster category of H which are not $\mathcal{C}$-bricks.

Lemma 4.1. *Let M, N be regular exceptional indecomposable modules in a tube $\mathcal{T}$ of rank $t + 1 > 1$ in $\mathcal{C}_H$.*
  (a) *[BMR2] The object M is not a $\mathcal{C}$-brick if and only if q.l. $M = t$*

(b) *Any cluster-tilting object in $\mathcal{C}_H$ contains at most one object from each tube which is not a $\mathcal{C}$-brick.*

(c) *If* q.l. $M = t$ *then the following are equivalent:*

   (i) $\mathrm{Hom}_H(M, N) \neq 0$

   (ii) $\dim \mathrm{Hom}_H(M, N) = 1$

   (iii) $\dim \mathrm{Hom}_{\mathcal{C}}(M, N) = 2$

   (iv) $N \notin \mathcal{W}_{\tau M}$

Proof. For (c), see [R] for the fact that $\dim \mathrm{Hom}_H(M, N) \leq 1$, and the fact that $\mathrm{Hom}_H(M, N) \neq 0$ if and only if $\mathrm{Hom}_H(N, \tau^2 M) \neq 0$ if and only if $N \notin \mathcal{W}_{\tau M}$. We have

$$\mathrm{Hom}_{\mathcal{C}}(M, N) = \mathrm{Hom}_H(M, N) \amalg \mathrm{Hom}_{\mathcal{D}}(M, \tau^{-1} N[1])$$

and

$$\mathrm{Hom}_{\mathcal{D}}(M, \tau^{-1} N[1]) \simeq D \mathrm{Hom}_{\mathcal{D}}(N, \tau^2 M),$$

so the equivalence in (c) follows and (a) follows.

Part (b) is well-known and easy to see. $\qquad\square$

The following is well-known by [St].

Lemma 4.2. *Assume that a cluster-tilting object T in $\mathcal{C}_H$ has a regular summand M. Then the summands of T lying in $\mathcal{W}_M$ form a tilting object in $\mathcal{W}_M$.*

We recall the notion of a Bongartz complement [Bo]:

Lemma 4.3. *Let N be a partial tilting module with no projective direct summands. Then there exists a complement E, known as the Bongartz complement of N, with the following properties:*

(a) *The module E satisfies the following properties*

   (B1) $\mathrm{Ext}_H^1(N, A) = 0$ *implies* $\mathrm{Ext}_H^1(E, A) = 0$ *for any A in* $\mathrm{mod}\, H$.

   (B2) $\mathrm{Hom}_H(N, E) = 0$.

(b) *If a complement E' of a partial tilting module X satisfies (B1) and (B2), then $E' \simeq E$, where E is the Bongartz complement.*

Proof. See [Ha]. $\qquad\square$

We are especially interested in the Bongartz complements of certain regular modules.

Lemma 4.4. *Let $X = X_t$ be an exceptional regular indecomposable module with* q.l. $X = t$. *For $i = 1, \ldots, t-1$, let X_i be the regular indecomposable exceptional module such that there is an irreducible monomorphism $X_i \to X_{i+1}$. Then there is a preprojective module $\widetilde{Q}$ such that:*

(a) *The Bongartz complement of $X = X_t$ is $X_1 \amalg \cdots \amalg X_{t-1} \amalg \widetilde{Q}$.*

(b) *The Bongartz complement of $\widehat{X} = X_1 \amalg \cdots \amalg X_{t-1} \amalg X_t$ is $\widetilde{Q}$.*

(c) *All partial tilting modules Y such that Y is a tilting object in $\mathcal{W}_X$ have Bongartz complement $\widetilde{Q}$.*

Proof. For (a), first note that $\mathrm{Ext}_H^1(X, \tau A) = 0$ while $\mathrm{Ext}_H^1(A, \tau A) \neq 0$ for any indecomposable module A which is either preinjective or regular with $\mathcal{T}_A \neq \mathcal{T}_X$. Hence by (B1) the summands in $\widetilde{Q}$ are either preprojective or regular and lie in $\mathcal{T}_X$. The property (B2) shows that any regular summand

of the Bongartz complement E of X must be in $\mathcal{W}_{\tau X}$, by Lemma 4.1. The fact that E is a complement implies that any regular summand must be in $\mathcal{W}_{X'}$, where $X' \to X$ is an irreducible monomorphism, since an object Z in $\mathcal{W}_{\tau X} \setminus \mathcal{W}_{X'}$ has $\mathrm{Ext}(X, Z) \neq 0$. We claim that for any indecomposable regular summand E' of E there is a monomorphism $E' \to X$. Assume E' is an indecomposable regular summand of E. Then, if E' is in $\mathcal{W}_{X'}$, but there is no monomorphism to X, the module $\tau E'$ will satisfy $\mathrm{Ext}^1(X, \tau E') = 0$, while $\mathrm{Ext}^1(E', \tau E') \neq 0$, a contradiction to (B1). Since $X \amalg E$ is a cluster-tilting object in $\mathcal{C}$, it follows from Lemma 4.2 that all indecomposable regular objects in the tube of X with monomorphisms to X are summands of E.

Part (b) is easily verified, noting that (B1) and (B2) are satisfied.

For (c) we show that if a module A satisfies $\mathrm{Ext}^1_H(Y, A) = 0$, then it satisfies $\mathrm{Ext}^1_H(\widehat{X}, A) = 0$. Then it follows that $\mathrm{Ext}^1_H(\widetilde{Q}, A) = 0$, which implies that $\widetilde{Q}$ satisfies (B1); (B2) is clearly satisfied.

To see that $\mathrm{Ext}^1_H(\widehat{X}, A) = 0$ we use that $\mathcal{W}_X$ is equivalent to $\mathrm{mod}\,\Lambda_t$, where Λ_t is the path algebra of the Dynkin quiver A_t with linear orientation. Now let Y_i be a direct summand in Y which is generated by Y/Y_i, and consider the exact sequence $0 \to Y^* \to B \to Y \to 0$, where $B \to Y$ is the minimal right $\mathrm{add}\,Y/Y_i$-approximation. Then $\mathrm{Ext}^1_H(Y^*, A) = 0$, since we have an epimorphism $\mathrm{Ext}^1_H(B, A) \to \mathrm{Ext}^1_H(Y^*, A)$. Iterating this sufficiently many times, which is possible by Lemma 3.4, we get that $\mathrm{Ext}^1_H(\widehat{X}, A) = 0$. $\square$

Lemma 4.5. *Let $X_1, \ldots, X_t = X$ be as in Lemma 4.4. Then there is preprojective module Q such that:*

(a) *All partial tilting modules Y such that Y is a tilting object in $\mathcal{W}_X$ have complement Q.*

(b) *Q generates X.*

Proof. Let $\widetilde{Q}$ be as in Lemma 4.4, and let r be the rank of the tube containing X. We have that $Q = \tau^{-kr}\widetilde{Q}$ is sincere for k large enough; see [PA]. Since τ^r preserves $\mathcal{T}$, (a) follows from Lemma 4.4(c) (but note that it may no longer be the case that Q is the Bongartz complement). Let $Y_1 = X, Y_2, \ldots, Y_t$ be exceptional regular indecomposable modules such that there is an irreducible epimorphism $Y_i \to Y_{i+1}$ for all i. By (a), Q is a complement of $Y_1 \amalg \cdots \amalg Y_t$. Hence $X = Y_1$ is a complement of $\overline{U} = Q \amalg Y_2 \amalg \cdots \amalg Y_t$. Since Q is sincere, so is the almost complete tilting module $\overline{U}$. By [HU], $\overline{U}$ has exactly two nonisomorphic indecomposable complements, Z and Z', and there is a short exact sequence

$$0 \to Z \to U' \to Z' \to 0$$

where $Z \to U'$ is a minimal left $\mathrm{add}(\overline{U})$-approximation in $\mathrm{mod}\,H$. Thus exactly one of Z, Z' is isomorphic to X; we claim it is Z'. Since Q is preprojective, $\mathrm{Hom}_A(X, Q) = 0$. Since the module maps from X to the Y_i all factor through Y_{t-1}, the minimal left $\mathrm{add}(\overline{U})$-approximation of X is a non-zero map $X \to Y_{t-1}$ and is therefore not a monomorphism. It follows that $Z' \cong X$ and thus that $\overline{U}$ generates X. Since $\mathrm{Hom}_A(Y_i, X) = 0$ for $2 \leq i \leq t$ (from the structure of the tube containing X), we see that Q generates X as required. $\square$
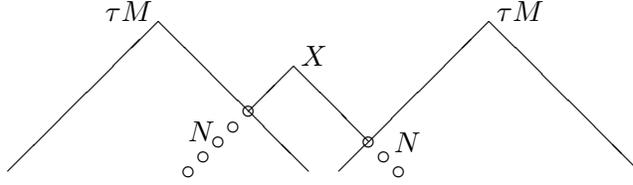
FIGURE 1. A complement N of X in $\mathcal{W}_X$ with summands (indicated by $\circ$) in $\mathcal{W}_{\tau M}$: see Lemma 4.6(a).

Lemma 4.6. *Let $\mathcal{T}$ be a tube of rank $t + 1$ and M an exceptional object in $\mathcal{T}$ which is not a $\mathcal{C}$-brick. Let $X = X_s$ be an exceptional indecomposable with q.l. $X = s \leq t$.*

  (a) *There is a complement N of X in $\mathcal{W}_X$ all of whose summands lie in $\mathcal{W}_{\tau M}$.*
  (b) *The partial tilting module $X \amalg N$ has a preprojective complement Q which generates X.*

Proof. See Figure 1 for a pictorial representation of this lemma. We can assume $X \notin \mathcal{W}_{\tau M}$, since otherwise the result follows directly from Lemma 4.5.

For (a) consider the relative projective tilting object in $\mathcal{W}_X$ given by $X_1 \amalg \cdots \amalg X_{s-1} \amalg X_s$, with q.l. $X_j = j$. If $X_{s-1} \notin \mathcal{W}_{\tau M}$, consider the non-split exact sequence $0 \to X_{s-1} \to X \to X'_{s-1} \to 0$. We claim that X'_{s-1} is in $\mathcal{W}_{\tau M}$. For this, apply $\mathrm{Hom}_H(M, \ )$ to the above exact sequence. Since by assumption $\mathrm{Hom}_H(M, X_{s-1}) \neq 0$, we have that $\mathrm{Hom}_H(M, X_{s-1}) \to \mathrm{Hom}_H(M, X)$ is surjective. The map $(\mathrm{Ext}^1_H(M, X_{s-1}) \to \mathrm{Ext}^1_H(M, X)) \simeq (D\,\mathrm{Hom}_H(X_{s-1}, \tau M) \to D\,\mathrm{Hom}_H(X, \tau M))$ is a monomorphism, since

$$\mathrm{Hom}_H(X, \tau M) \to \mathrm{Hom}_H(X_{s-1}, \tau M)$$

is an epimorphism. The last statement follows since τM is not a factor of X_{s-1}.

We also claim that X'_{s-1} is a complement of $X_1 \amalg \cdots \amalg X_{s-2} \amalg X_s$ in $\mathcal{W}_X$. This follows from the fact that the map $X_{s-1} \to X$ is a minimal left add $X_1 \amalg \cdots \amalg X_{s-2} \amalg X_s$-approximation, together with Proposition 3.3.

Now, if necessary, we exchange X_{s-2} using the minimal add $X_1 \amalg \cdots \amalg X_{s-3} \amalg X'_{s-1} \amalg X_s$-approximation $X_{s-2} \to X$. The same argument as above shows that the cokernel of this map gives us a complement in $\mathcal{W}_{\tau M}$. We iterate this at most $s - 1$ times, until we obtain a complement

$$N = X_1 \amalg \cdots \amalg X_k \amalg X'_{k+1} \amalg \cdots \amalg X'_{s-1}$$

for X in $\mathcal{W}_X$, with $0 \leq k \leq s - 1$, all of whose summands lie in $\mathcal{W}_{\tau M}$, as required.

Since $X \amalg N$ is a tilting object in $\mathcal{W}_X$, part (b) follows immediately from Lemma 4.5. $\qquad\square$

5. The main result

In this section, we show the main theorem. The proof will follow from a series of lemmas. Throughout this section, let T be a cluster-tilting object in the cluster category $\mathcal{C}_H$ of a tame hereditary algebra H. We assume that

T has a summand which is not a $\mathcal{C}$-brick. We have the following preliminary results.

Lemma 5.1. *Let Z be an exceptional indecomposable regular module. Let $X \amalg Y$ be a tilting object in $\mathcal{W}_Z$, with X indecomposable. Assume $U \amalg X \amalg Y$ is a tilting module in $\operatorname{mod} H$, where U has no preinjective summands.*

    (a) *Let $B \to X$ be the minimal right $\operatorname{add} Y$-approximation in $\mathcal{W}_Z$ and assume there is an exchange sequence $0 \to X^* \to B \to X \to 0$ in $\mathcal{W}_Z$. Then $B \to X$ is a right $\operatorname{add} U \amalg Y$-approximation.*

    (b) *Let $X \to B'$ be the minimal left $\operatorname{add} Y$-approximation in $\mathcal{W}_Z$ and assume there is an exchange sequence $0 \to X \to B' \to X^* \to 0$ in $\mathcal{W}_Z$. Then $X \to B'$ is a left $\operatorname{add} U \amalg Y$-approximation.*

Proof. We prove (a), the proof of (b) is similar. Let $U = U_p \amalg U_r$ where U_p is preprojective and U_r is regular. By assumption U_r has no summands in $\mathcal{W}_Z$.

We have $\operatorname{Hom}_H(U_p, B) \to \operatorname{Hom}_H(U_p, X)$ is surjective since $\operatorname{Ext}^1_H(U_p, X^*) \simeq D \operatorname{Hom}_H(\tau^{-1} X^*, U_p) = 0$.

We claim that $\operatorname{Hom}_H(U_r, B) \to \operatorname{Hom}_H(U_r, X)$ is also surjective. For this note that by the AR-structure of the tube, there is an indecomposable direct summand B_0 in B such that the restriction $B_0 \to X$ is surjective. Let U'_r be a summand in U_r such that $\operatorname{Hom}_H(U'_r, X) \neq 0$. By assumption $\operatorname{Hom}_H(U'_r, \tau X) = 0$, since $\operatorname{Hom}_H(U'_r, \tau X) \simeq D \operatorname{Ext}^1(X, U'_r)$. Hence any map $U'_r \to X$ is either an epimorphism or a monomorphism. Since U'_r is not in $\mathcal{W}_Z$, it is also not in $\mathcal{W}_X$, and it follows that any non-zero map $U'_r \to X$ is an epimorphism, and hence factors through $B_0 \to X$, and the claim follows. Hence $B \to X$ is a minimal right $\operatorname{add}(U \amalg Y)$-approximation. This completes the proof of (a). $\qquad\square$

Lemma 5.2. *Let $\mathcal{T}$ be a tube such that T has a summand M, lying in $\mathcal{T}$. Assume q.l. M is maximal among the direct summands of T in $\mathcal{T}$. By Lemma 4.2, we have that $\operatorname{add} \tau T \cap \mathcal{W}_{\tau M} = \operatorname{add} \tau T'$ for a tilting object $\tau T'$ in $\mathcal{W}_{\tau M}$. Let $T = T' \amalg T''$. Then we have:*

    (a) *All tilting objects in $\mathcal{W}_{\tau M}$ are complements of $\tau T''$.*

    (b) *All objects in $\mathcal{W}_{\tau M}$ have a T-denominator.*

Proof. Note that there is a hereditary algebra H', with $\mathcal{C}_{H'} = \mathcal{C}_H$, such that $\tau T''$ as a H'-module has only regular and preprojective direct summands (see Remark 3.1). Assume q.l. $M \leq t$, and that the rank of $\mathcal{T}$ is $t + 1$. Let $U = \tau T' = \tau N_1 \amalg \cdots \amalg \tau N_{t-1} \amalg \tau M$ be the tilting object in $\mathcal{W}_{\tau M}$.

Using Proposition 3.3 and Lemma 3.4, we have that all tilting objects in $\mathcal{W}_{\tau M}$ can be reached from U by a finite number of exchanges, given by exchange sequences in $\mathcal{W}_{\tau M}$. Using Lemma 5.1 these exchange sequences are also exchange sequences in $\operatorname{mod} H'$ and hence in $\mathcal{C}_{H'} = \mathcal{C}_H$. This shows (a). For (b) it suffices to show that each such exchange pair is compatible with T. Consider the exchange triangle

$$X' \to \bar{X} \to X'' \to . \tag{8}$$

By Proposition 1.7, the pair (X', X'') is compatible with all summands in T which are $\mathcal{C}$-bricks. It is also compatible with any regular summand T_j of T with $\mathcal{T}_{T_j} \neq \mathcal{T}$, since $\operatorname{Hom}(T_j, \ )$ vanishes on all terms of the sequence. By

Lemma 4.1(a) we only need to consider compatibility with summands of T which lie in $\mathcal{T}$ and have quasilength t. By Lemma 4.1(b), T has at most one such summand. Since M is assumed to have maximal quasilength amongst indecomposable direct summands of T in $\mathcal{T}$, if T has such a summand, it must be M. But, since the exchange triangle (8) lies inside $\mathcal{W}_{\tau M}$, we see that $\mathrm{Hom}(M, \ )$ vanishes when applied to (8). This finishes the proof of (b). $\hfill\square$

Lemma 5.3. *Let X be an exceptional regular indecomposable object of $\mathcal{C}$ which is a $\mathcal{C}$-brick.*

(a) *An exchange pair (X, Z) is compatible with any regular object M for which either M is a $\mathcal{C}$-brick, or $\mathcal{T}_M \neq \mathcal{T}_X$, or $X \in \mathcal{W}_{\tau M'}$, where $M' \to M$ is an irreducible monomorphism.*

(b) *There is an exchange triangle of the form $Y \to Q \amalg X' \to X \to$ where $X' \to X$ is an irreducible monomorphism in case q.l. $X > 1$ and $X' = 0$ otherwise, with the property that Y and Q are transjective.*

Proof. (a) If M is a $\mathcal{C}$-brick then this holds by Proposition 1.7. For the other cases note that $\mathrm{Hom}(M, X) = 0 = \mathrm{Hom}(M, \tau^{-1}X)$, and hence when $\mathrm{Hom}(M, \ )$ is applied to the exchange triangle $Z \to Q' \to X \to$, one obtains a short exact sequence.

For (b), let $X_1, \ldots, X_t = X$ be regular exceptional indecomposable modules such that there is an irreducible monomorphism $X_i \to X_{i+1}$ for all i. Let Q be the preprojective complement of $X_1 \amalg \cdots \amalg X_t$ provided by Lemma 4.5, and let $\overline{U} = Q \amalg X_1 \amalg \cdots \amalg X_{t-1}$. Consider the minimal right add $\overline{U}$-approximation $U' \to X$ (as H-module). Since Q generates X, so does $\overline{U}$, so the approximation is surjective, and we have a short exact sequence $0 \to Y \to U' \to X \to 0$ and thus an induced approximation triangle, $Y \to U' \to X \to$ in $\mathcal{C}$. Since all maps from X_i to X (with $1 \leq i \leq t - 1$) factor through a non-zero map $X_{t-1} \to X$ (taking $X_0 = 0$), $X' = X_{t-1}$ is the only regular summand of U' and the other summands are preprojective. Since no non-zero map $X_{t-1} \to X$ is surjective, while $U' \to X$ is surjective, U' must have a preprojective summand, and it follows that Y is preprojective. $\hfill\square$

We now deal with the transjective objects.

Proposition 5.4. *All transjective objects have a T-denominator.*

Proof. By Proposition 3.8 it is sufficient to show that that there is one transjective cluster-tilting object all of whose indecomposable direct summands have T-denominators. Without loss of generality we can assume that T has at least one indecomposable direct summand which is not a $\mathcal{C}$-brick.

Assume $T = Q \amalg R$, where Q is transjective and R is regular. Then, using Lemma 4.2, there are indecomposable summands $M_1, \ldots, M_z$ of R such that each summand of R lies in one of the wings $\mathcal{W}_{M_i}$. We choose a minimal such set of summands. Since $\mathrm{Ext}^1_{\mathcal{C}}(M_i, A) \neq 0$ for any object A whose wing overlaps $\mathcal{W}_{M_i}$, any two of the $\mathcal{W}_{M_i}$ must be either equal or disjoint.

By definition, all summands of τT have T-denominators. By Lemma 5.2, we can, for each i, replace the summands of τT in $\mathcal{W}_{\tau M_i}$ with the indecomposable objects in the tube of M_i which have a monomorphism to τM_i.

We obtain a new cluster-tilting object $U = (\amalg_{i=1}^{z} \tau M_i) \amalg U'$ all of whose indecomposable direct summands have T-denominators.

Suppose M_1 has quasilength t and let $N_1, N_2, \ldots, N_t = M_1$ be the indecomposable objects in $\mathcal{T}_{M_1}$ with monomorphisms to M_1, where $\mathrm{q.\,l.}(N_i) = i$ for all i. Then we can write $U = (\amalg_{i=1}^{t} \tau N_i) \amalg Y$. We claim that, via a sequence of exchanges, the τN_i can be replaced by transjective summands Q_i which have T-denominators. When repeating this for $M_1, M_2, \ldots, M_z$, we will end up with a transjective cluster-tilting object having T-denominators as required.

We exchange τM_1 with a complement $(\tau M_1)^*$, via the exchange triangles:

$$(\tau M_1)^* \to B \to \tau M_1 \to$$

$$\tau M_1 \to B' \to (\tau M_1)^* \to .$$

Claim: The object $(\tau M_1)^*$ is transjective.

If $(\tau M_1)^*$ is not induced by an H-module, it is induced by the shift of a projective module, and we are done. So we can assume that $(\tau M_1)^*$ is induced by a module. Then one of these two exchange triangles must arise from a short exact sequence of modules.

If it is the first, then clearly $\mathrm{Hom}_H(X, \tau M_1) = 0$ for any regular summand X of U not in $\mathcal{T}_{M_1}$. But if X lies in $\mathcal{T}_{M_1}$ and not in $\mathcal{W}_{\tau M_1}$, again $\mathrm{Hom}_H(X, \tau M_1) = 0$ since the wings $\mathcal{W}_{\tau M_i}$ do not overlap (and $\mathrm{q.\,l.}(M_i)$ is less than the rank of its tube for all i). Let $N_0 = 0$. Since τN_{t-1} does not generate $\tau N_t = \tau M_1$, it follows that B has a nonzero preprojective summand, and hence that $(\tau M_1)^*$ is preprojective.

If it is the second, then clearly $\mathrm{Hom}_H(\tau M_1, X) = 0$ for any regular summand X of U not in $\mathcal{T}_{M_1}$. But if X lies in $\mathcal{T}_{M_1}$ and not in $\mathcal{W}_{\tau M_1}$, again $\mathrm{Hom}_H(\tau M_1, X) = 0$ since the wings $\mathcal{W}_{\tau M_i}$ do not overlap. Since $\mathrm{Hom}_H(\tau M_1, \tau N_j) = 0$ for all j, it follows that B' has a nonzero preinjective summand, and hence that $(\tau M_1)^*$ is preinjective.

Hence, in either case, $(\tau M_1)^*$ is transjective. We next show that $(\tau M_1)^*$ has a T-denominator, by considering two cases:

CASE I: We assume first that $\mathrm{End}(M_1) = k$, i.e. M_1 is a $\mathcal{C}$-brick.

Every summand of T in $\mathcal{T} = \mathcal{T}_{M_1}$ is a $\mathcal{C}$-brick (by the choice of the M_i), so by Lemma 5.3(a) we obtain that the exchange pair $(\tau M_1, (\tau M_1)^*)$ is compatible with all summands of T, and hence that $(\tau M_1)^*$ has a T-denominator by Proposition 1.6. We then repeat this procedure for $\tau N_{t-1}, \ldots, \tau N_1$.

CASE II: $\mathrm{End}(M_1) \neq k$, i.e. M_1 is not a $\mathcal{C}$-brick. Arguing as above, we see that we can exchange τM_1 with a transjective object $(\tau M_1)^*$. M_1 is compatible with the exchange pair $(\tau M_1, (\tau M_1)^*)$ by definition. The other direct summands in T are either $\mathcal{C}$-bricks, or they are in other tubes. In both cases they are compatible with $(\tau M_1, (\tau M_1)^*)$. Hence T is compatible with that exchange pair. So $(\tau M_1)^*$ has a T-denominator by Proposition 1.6. We can then exchange the other summands $\tau N_{t-1}, \ldots, \tau N_1$ with transjectives by Lemma 5.3(b). By the last assertion of Lemma 5.3(a), each exchange pair is compatible with M_1. As in case I, they are also compatible with the other direct summands of T. Hence, we obtain a transjective cluster-tilting object having a T-denominator, and we are done. □

Lemma 5.5. *Let $\mathcal{T}$ be a tube such that each direct summand of T lying in $\mathcal{T}$ is a $\mathcal{C}$-brick, or such that $\mathcal{T}$ has no summands in T. Then each exceptional indecomposable object in $\mathcal{T}$ has a T-denominator.*

Proof. Let X be an exceptional indecomposable object in $\mathcal{T}$. We prove the Lemma by induction on the quasilength of X.

If q. l. $X = 1$, then by Lemma 5.3(b) there is an exchange triangle $Y \to Q \to X \to$ with Q and Y transjective. By Proposition 1.7, we need only show that (Y, X) is compatible with any regular non $\mathcal{C}$-brick summand M of T. But this follows from Lemma 5.3.

Now assume that any exceptional indecomposable object Y of quasilength less than t has a T-denominator. We want to show that the result also holds for the exceptional indecomposable X with q. l. $X = t$. For this we use Lemma 5.3. $\square$

It now remains to deal with the exceptional objects which are in $\overline{\mathcal{W}_{\tau M}}$ for a non $\mathcal{C}$-brick summand M of T. Here $\overline{\mathcal{W}_{\tau M}}$ denotes the complement of the wing $\mathcal{W}_{\tau M}$ inside the tube $\mathcal{T}_M$ (i.e. the indecomposable objects in $\mathcal{T}_M$ which are not in $\mathcal{W}_{\tau M}$). For this the following lemma is crucial.

Lemma 5.6. *For each indecomposable exceptional object X^* in $\overline{\mathcal{W}_{\tau M}}$, there are exchange triangles*
$$X^* \to B \to X \to$$
and
$$X \to B' \to X^* \to$$
such that

   (i)
$$\max(\dim \operatorname{Hom}(M, B), \dim \operatorname{Hom}(M, B'))$$
$$= \dim \operatorname{Hom}(M, X^*) + \dim \operatorname{Hom}(M, X) - 1$$

  (ii) *The object X and all indecomposable summands of the objects B and B' have T-denominators.*
 (iii) *The object X is induced by a preprojective module.*

Proof. By Lemma 4.6(a), there is an object N in $\mathcal{W}_{X^*}$ such that $N \amalg X^*$ is a tilting object in the wing $\mathcal{W}_{X^*}$ and all direct summands of N are in $\mathcal{W}_{\tau M}$.

By Lemma 4.6(b), we have that $N \amalg X^*$ has a preprojective complement Q in $\operatorname{mod} H$, such that Q generates X^*. Let $R = Q \amalg N$ and let $B' \to X^*$ (respectively, $X^* \to B$) be the minimal right, (respectively, minimal left) add R-approximations of X^*. We claim that the induced exchange triangles satisfy (i), (ii) and (iii).

Consider the exchange triangle
$$X \to B' \to X^* \to .$$

Since Q generates X^* in $\operatorname{mod} H$, it is clear that this triangle is induced by a short exact sequence in $\operatorname{mod} H$, and hence X is induced by a preprojective module (showing (iii)), since $X \to B'$ is nonzero and B' must have a preprojective summand as N doesn't generate X^*.

Apply $\operatorname{Hom}(M, \ )$ to obtain the long exact sequence
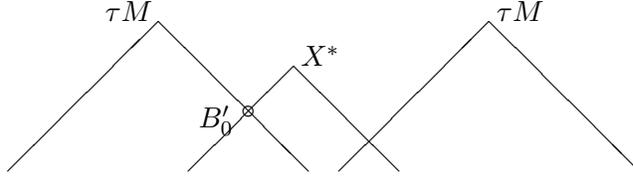$$(M, \tau^{-1}X^*) \to (M, X) \to (M, B') \to (M, X^*) \to (M, \tau X).$$

FIGURE 2. A summand B_0 of N with $\mathrm{Hom}_H(\tau M, B_0) \neq 0$:
see proof of Lemma 5.6.

We claim that every H-map $\tau M \to X^*$ factors through B'. This follows
from the configuration of M, N and X^* in the Auslander-Reiten quiver of the
tube, noting that $\mathrm{Hom}_H(\tau M, N) \neq 0$ if and only if $\mathrm{Hom}(\tau M, X^*) \neq 0$ (if and
only if $N \neq 0$). Figure 2 displays this case, when N has a summand B_0' (oc-
curring in B') with $\mathrm{Hom}_H(\tau M, B_0') \neq 0$; compare with Figure 1. By [BMR2,
Lemma 5.1] it follows that $(M, X) \to (M, B')$ is a monomorphism.

We claim that $\dim \mathrm{coker}((M, B') \to (M, X^*)) = 1$. By Lemma 4.1, we
have that $\dim \mathrm{Hom}_H(M, X^*) = 1$ and it is clear that an H-map $M \to X^*$
will not factor through B', since N is in $\mathcal{W}_{\tau M}$, and hence $\mathrm{Hom}_H(M, N) = 0$,
by Lemma 4.1.

By Lemma 4.1 the space of F-maps $M \to X^*$ is also one-dimensional.
We claim that such F-maps will factor through B'. For this we consider
two possible cases: the object X is either induced by a projective H-module
P or not. First assume that X is non-projective. Since the composition of
two F-maps is 0, it is clear that all F-maps $M \to X^*$ will factor through
$B' \to X^*$. Hence the claim follows in this case. Now consider the case where
X^* is projective. Then the composition $M \to \tau^{-1}X^*[1] \to \tau^{-1}(P_i[1])[1]$ is
clearly zero, so the claim follows in this case.

We next want to show that when $\mathrm{Hom}(M, \ )$ is applied to the second
exchange triangle

$$X^* \to B \to X \to,$$

we do not obtain an exact sequence. The map $X^* \to B$ decomposes into
$X^* \to Q_0 \amalg N_0$, with Q_0 preprojective and N_0 in $\mathcal{W}_{\tau M}$. Hence $X^* \to Q_0$
is an F-map. There is a non-zero F-map $M \to X^*$ and the composition
$M \to X^* \to B$ will be zero since $M \to X^* \to Q_0$ is the composition of two
F-maps and $\mathrm{Hom}(M, N_0) = 0$, since $M \in \mathcal{W}_{\tau M}$.

Hence we obtain (i), and (ii) follows using Lemmas 3.6 and 5.2(b), using
the fact that X and all indecomposable summands of B and B' are either
transjective or in $\mathcal{W}_{\tau M}$, and noting that, by Lemma 4.1(a), M has maximal
quasilength amongst the direct summands of T in $\mathcal{T}_M$. $\square$

The proof of the following is an adaptation of parts the proof of [BMRT,
Prop. 3.1]. It completes the proof of our main result, Theorem A.

Proposition 5.7. *Let $\mathcal{T}$ be a tube such that T has a non $\mathcal{C}$-brick summand
M, lying in $\mathcal{T}$. Then each object in $\overline{\mathcal{W}_{\tau M}}$ has a diminished T-denominator.*

Proof. Let X^* be an indecomposable object in $\overline{\mathcal{W}_{\tau M}}$. By Lemma 5.6 there
is an indecomposable object X and exchange triangles

$$X^* \to B \to X \to$$

and
$$X \to B' \to X^* \to$$
such that Lemma 5.6 holds.

We have [BMR1] that

$$x_X x_{X^*} = x_B + x_{B'}. \tag{9}$$

Assume $M = T_l$. We need to discuss two different cases.

CASE I: Suppose that neither X nor X^* is isomorphic to τT_i for any i. Let $B = B_0 \amalg B_1$, where no summand of B_0 is of the form τT_i for any i, and B_1 is in $\operatorname{add} \tau T$. Similarly, write $B' = B_0' \amalg B_1'$. Let t_{B_0} denote $\prod t_Y$ where the product is over all indecomposable direct summands Y of B_0; similarly write $t_{B_0'}$. Note that t_Y is defined in Definition 1.2.

We then have $x_B = \frac{f_{B_0} y_{B_1}}{t_{B_0}}$ and $x_{B'} = \frac{f_{B_0'} y_{B_1'}}{t_{B_0'}}$.

Let $m = \frac{\operatorname{lcm}(t_{B_0}, t_{B_0'})}{t_{B_0}}$ and $m' = \frac{\operatorname{lcm}(t_{B_0}, t_{B_0'})}{t_{B_0'}}$. We then have:

$$(x_{X^*}) = \frac{x_B + x_{B'}}{x_X} = \frac{(f_{B_0} m y_{B_1} + f_{B_0'} m' y_{B_1'})/f_X}{\operatorname{lcm}(t_B, t_{B'})/t_X},$$

using that $t_B = t_{B_0}$ since $\operatorname{Hom}_{\mathcal{C}}(T_i, \tau T_j) = 0$ for all i, j, and similarly $t_{B'} = t_{B_0'}$.

Since $M = T_l$ is a summand in T, we have by Lemma 5.6 that

$$\max(\dim \operatorname{Hom}(T_l, B), \dim \operatorname{Hom}(T_l, B'))$$
$$= \dim \operatorname{Hom}(T_l, X^*) + \dim \operatorname{Hom}(T_l, X) - 1.$$

For any other summand of T, say T_i with $i \neq l$, we have that T_i is compatible with (X, X^*), and hence

$$\max(\dim \operatorname{Hom}(T_i, B), \dim \operatorname{Hom}(T_i, B'))$$
$$= \dim \operatorname{Hom}(T_i, X^*) + \dim \operatorname{Hom}(T_i, X).$$

We thus obtain:

$$\begin{aligned} t_X t_{X^*} &= \prod y_i^{\dim \operatorname{Hom}_{\mathcal{C}}(T_i, X) + \dim \operatorname{Hom}_{\mathcal{C}}(T_i, X^*)} \\ &= y_l \cdot \prod y_i^{\max(\dim \operatorname{Hom}_{\mathcal{C}}(T_i, B), \dim \operatorname{Hom}_{\mathcal{C}}(T_i, B'))} = y_l \cdot \operatorname{lcm}(t_B, t_{B'}). \end{aligned}$$

Hence

$$x_{X^*} = \frac{(f_{B_0} m y_{B_1} + f_{B_0'} m' y_{B_1'})/f_X}{t_{X^*}/y_l}. \tag{10}$$

We have that m and m' are coprime, by definition of least common multiple. Since B and B' have no common direct summands [BMR1, 6.1], y_{B_1} and $y_{B_1'}$ are coprime. Suppose that m and $y_{B_1'}$ had a common factor y_i. Then we would have a summand Z of B_0' such that $\operatorname{Hom}_{\mathcal{C}}(T_i, Z) \neq 0$, and τT_i was a summand of B'. But then

$$\operatorname{Ext}_{\mathcal{C}}^1(Z, \tau T_i) \simeq D \operatorname{Hom}_{\mathcal{C}}(\tau T_i, \tau Z) \simeq D \operatorname{Hom}_{\mathcal{C}}(T_i, Z) \neq 0.$$

This contradicts the fact that B' is the direct sum of summands of a cluster-tilting object. Therefore m and $y_{B_1'}$ are coprime, and similarly m' and y_{B_1} are coprime. It follows that $m y_{B_1}$ and $m' y_{B_1'}$ are coprime.

Since all indecomposable summands of B and B' have T-denominators, it follows (see Definition 1.2) that $f_{B_0}(e_i) > 0$ and $f_{B'_0}(e_i) > 0$ for each $i \in \{1, 2, \ldots, n\}$. It is clear that $(my_{B_1})(e_i) \geq 0$ and $(m'y_{B'_1})(e_i) \geq 0$. Using that my_{B_1} and $m'y_{B'_1}$ are coprime, it follows that these two numbers cannot simultaneously be zero, so $(f_B my_{B_1} + f_{B'} m'y_{B'_1})(e_i) > 0$. Hence $f_B my_{B_1} + f_{B'} m'y_{B'_1}$ satisfies the positivity condition. By assumption, f_X also satisfies the positivity condition.

By the Laurent phenomenon [FZ1, 3.1], x_{X^*} is a Laurent polynomial in $y_1, y_2, \ldots, y_n$. Clearly t_{X^*}/y_l is also a Laurent polynomial. Hence $u = (f_B my_{B_1} + f_{B'} m'y_{B'_1})/f_X = \frac{x_{X^*} t_{X^*}}{y_l}$ is also a Laurent polynomial. Since u is defined at e_i for all i, it must be a polynomial. By the above, u satisfies the positivity condition.

We have that y_l divides $t_{X^*} = \prod y_i^{\dim \operatorname{Hom}_\mathcal{C}(T_i, X^*)}$, since $\dim \operatorname{Hom}_\mathcal{C}(T_l, X^*) = 2$. Hence we get that t_{X^*}/y_l is a monomial. This finishes the proof in Case (I).

CASE II: Assume that $X \simeq \tau T_i$ for some i. Note that $i \neq l$, since X and hence T_i is transjective, while T_l is regular.

Since $\operatorname{Ext}^1_\mathcal{C}(T_r, T_s) = 0$ for all r, s, we have that $X^* \not\simeq \tau T_j$ for any j.

Using Proposition 1.6 and Lemma 5.6, we have

$$\dim \operatorname{Hom}_\mathcal{C}(T_j, X) + \dim \operatorname{Hom}_\mathcal{C}(T_j, X^*)$$
$$= \max(\dim \operatorname{Hom}_\mathcal{C}(T_j, B), \dim \operatorname{Hom}_\mathcal{C}(T_j, B')) + \epsilon_j,$$

where

$$\epsilon_j = \begin{cases} 1 & \text{if } j = i \text{ or } j = l \\ 0 & \text{otherwise} \end{cases}.$$

As in Case (I), but using that $x_X = y_i$ (as $X = \tau T_i$), we obtain the expression

$$x_{X^*} = \frac{(f_{B_0} my_{B_1} + f_{B'_0} m'y_{B'_1})}{\operatorname{lcm}(t_B, t_{B'})y_i}.$$

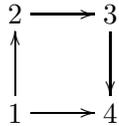Using $\operatorname{lcm}(t_B, t_{B'}) = t_X t_{X^*} y_i^{-1} y_l^{-1}$, we get

$$x_{X^*} = \frac{(f_{B_0} my_{B_1} + f_{B'_0} m'y_{B'_1})}{t_{X^*}/y_l}.$$

As in Case (I), we get that the numerator satisfies positivity and is a polynomial, and that $t_{X^*} y_l^{-1}$ is a monomial. The proof is complete. $\square$
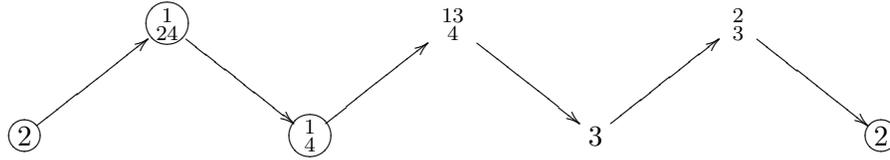
6. An example

We give a small example illustrating the main theorem.
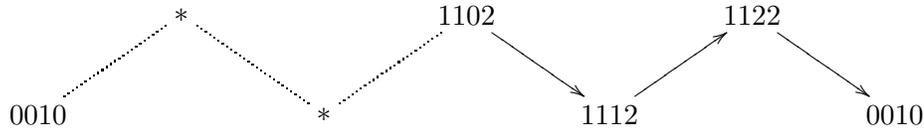
Let Q be the extended Dynkin quiver

and let $H = kQ$ be the path algebra. Then H is a tame hereditary algebra where the AR-quiver has one exceptional tube $\mathcal{T}$, which is of rank 3. The

(exceptional part of) the AR-quiver of $\mathcal{T}$ is as follows, where the composition factors (in radical layers) of indecomposable modules are given.

$$
\begin{array}{ccccc}
& \overset{1}{\underset{24}{\bigcirc}} & & \overset{13}{4} & & \overset{2}{3} \\
\nearrow & & \searrow \nearrow & & \searrow \nearrow & & \searrow \\
\overset{}{\underline{2}} & & \overset{1}{\underset{4}{\bigcirc}} & & 3 & & \underline{2}
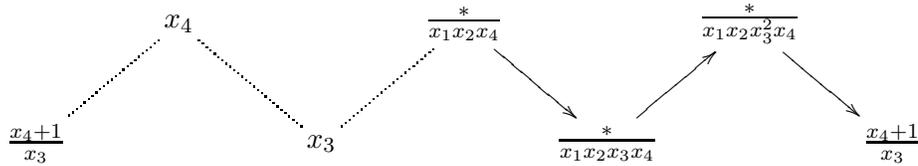\end{array}
$$

Let $P_i = He_i$ denote the indecomposable projective H-modules. Let $T = T_1 \amalg T_2 \amalg T_3 \amalg T_4 = \tau^{-1}P_4 \amalg \tau^{-1}P_1 \amalg 3 \amalg {}^{13}_{4}$. It is easily verified that this is a cluster-tilting object. The encircled modules in the above figure are those which are in $\mathcal{W}_{\tau T_4}$.

For each exceptional object Y in the tube $\mathcal{T}$, we give the dimension vector of $\mathrm{Hom}_{\mathcal{C}}(T, Y)$ as a $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$-module. Note that $\mathrm{Hom}_{\mathcal{C}}(T, \tau T_3) = 0 = \mathrm{Hom}_{\mathcal{C}}(T, \tau T_4)$.

$$
\begin{array}{ccccccc}
& * & & 1102 & & 1122 & \\
\nearrow & & \searrow \nearrow & & \searrow \nearrow & & \searrow \\
0010 & & * & & 1112 & & 0010
\end{array}
$$

We consider the initial seed $\{x_1, \ldots, x_4\}$ where $x_i = x_{\tau T_i}$. We give the corresponding cluster variables x_Y (with most of the numerators skipped).

$$
\begin{array}{ccccccc}
& x_4 & & \frac{*}{x_1 x_2 x_4} & & \frac{*}{x_1 x_2 x_3^2 x_4} & \\
\nearrow & & \searrow \nearrow & & \searrow \nearrow & & \searrow \\
\frac{x_4+1}{x_3} & & x_3 & & \frac{*}{x_1 x_2 x_3 x_4} & & \frac{x_4+1}{x_3}
\end{array}
$$

We observe that the denominators of these cluster variables can be computed from the dimension vectors of the corresponding modules over $\mathrm{End}_{\mathcal{C}}(T)$, as described by our main Theorem.

References

[AHK] L. Angeleri Hügel, D. Happel and H. Krause, *Handbook of tilting theory*, London Mathematical Society Lecture Note Series **332**. Cambridge University Press, Cambridge, 2007.

[ABS] I. Assem, T. Brüstle and Ralf Schiffler, *Cluster-tilted algebras as trivial extensions*, Bull. London Math. Soc. **40** (2008) 151-162.

[ARS] M. Auslander, I. Reiten, S. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, **36**. Cambridge University Press, Cambridge, 1997.

[ASS] I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1: Techniques of representation theory*, London Mathematical Society Student Texts, **65**. Cambridge University Press, Cambridge, 2006.

[BKL] M. Barot, D. Kussin, H. Lenzing, *The cluster category of a canonical algebra*, preprint arXiv:0801.4540v1 [math.RT], 2008.

[Bo] K. Bongartz, *Tilted algebras*, Representations of algebras (Puebla, 1980), pp. 26–38, Lecture Notes in Math., **903**, Springer, Berlin-New York, 1981.

[BCKMRT] A. B. Buan, P. Caldero, B. Keller, R. J. Marsh, I. Reiten and G. Todorov, *Appendix to: Clusters and seeds in acyclic cluster algebras*, Proc. Amer. Math. Soc. **135** (2007), no. 10, 3059–3060.

[BM] A. B. Buan and R. J. Marsh. *Cluster-tilting theory*, In: Trends in Representation Theory of Algebras and Related Topics, Workshop August 11-14, 2004, Queretaro, Mexico. Editors J. A. de la Pena and R. Bautista, Contemporary Mathematics **406** (2006), 1-30.

[BMR1] A. B. Buan, R. J. Marsh and I. Reiten, *Cluster mutation via quiver representations*, Comm. Math. Helv. **83** (2008), no.1, 143–177.

[BMR2] A. B. Buan, R. J. Marsh and I. Reiten, *Denominators of cluster variables*, preprint arxiv:0710.4335v1 [math.RT], 2007.

[BMRRT] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics **204** (2) (2006), 572–618.

[BMRT] A. B. Buan, R. J. Marsh, I. Reiten and G. Todorov, *Clusters and seeds in acyclic cluster algebras*, Proc. Amer. Math. Soc. **135** (2007), no. 10, 3049–3060.

[CC] P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Commentarii Mathematici Helvetici, **81**, (2006), 595-616.

[CCS1] P. Caldero, F. Chapoton and R. Schiffler *Quivers with relations arising from clusters (A_n case)*, Transactions of the American Mathematical Society **358** (2006), 1347–1364.

[CCS2] P. Caldero, F. Chapoton and R. Schiffler *Quivers with relations and cluster-tilted algebras*, Algebras and Representation Theory, 9, No. 4 , (2006), 359–376.

[CK1] P. Caldero and B. Keller, *From triangulated categories to cluster algebras*, Inv. Math. **172** (2008), 169-211.

[CK2] P. Caldero and B. Keller, *From triangulated categories to cluster algebras II*, Ann. Sci. Ecole Norm. Sup, 4eme serie, **39**, (2006), 983–1009.

[CZ] P. Caldero and A. Zelevinsky, *Laurent expansions in cluster algebras via quiver representations*, Moscow Math. J. **6** (2006), No. 3, 411-429.

[DWZ] H. Derksen, J. Weyman and A. Zelevinsky, *Quivers with potentials and their representations II: Applications to cluster algebras.*, preprint arXiv:0904.0676v1 [math.RA], 2009.

[D] G. Dupont, *Cluster multiplication in stable tubes via generalized Chebyshev polynomials*, preprint arXiv:0801.3964 [math.RT], 2008. To appear in J. Alg. Appl.

[FZ1] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[FZ2] S. Fomin and A. Zelevinsky, *Cluster algebras: notes for the CDM-03 conference.* Current developments in mathematics, 2003, 1–34, Int. Press, Somerville, MA, 2003.

[FZ3] S. Fomin and A. Zelevinsky, *Cluster algebras IV: Coefficients*, Compositio Math. **143** (2007), 112–164.

[FK] C. Fu and B. Keller. *On cluster algebras with coefficients and 2-Calabi-Yau categories*, preprint arXiv:0710.3152v2 [math.RT], 2007. To appear in Trans. Amer. Math. Soc.

[GLS] C. Geiss, B. Leclerc and J. Schöer, *Preprojective algebras and cluster algebras*, Conference Proceedings, International Conference on Representations of Algebras (ICRA), Torun (2007), 31 pages.

[Ha] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, **119**. Cambridge University Press, Cambridge, 1988.

[HU] D. Happel and L. Unger, *Almost complete tilting modules*, Proc. Amer. Math. Soc. **107** (3) (1989), 603–610.

[Hu] A. Hubery, *Acyclic cluster algebras via Ringel-Hall algebras*, preprint.

[Kel1] B. Keller, *On triangulated orbit categories*, Documenta Math. 10 (2005), 551–581.

[Kel2] B. Keller, *Categorification of acyclic cluster algebras: an introduction*, preprint arXiv:0801.3103v1 [math.RT], 2008.

[M] G. Musiker. *A graph theoretic expansion formula for cluster algebras of classical type*, preprint arXiv:0710.3574v2 [math.CO], 2007. To appear in Ann. Comb.

[MP] G. Musiker and J. Propp, *Combinatorial Interpretations for Rank-Two Cluster Algebras of Affine Type*, Electron. J. Combin. **14** (2007), no. 1, Research Paper 15, 23 pp.

[Pal] Y. Palu, *Cluster characters for triangulated 2-Calabi–Yau categories*, Annales de l'institut Fourier, **58**, no. 6 (2008), p. 2221-2248.

[Par] M. Parsons, *On indecomposable modules over cluster-tilted algebras of type A*, PhD thesis, University of Leicester, UK, 2008.

[PA] M.I. Platzeck and M. Auslander, *Representation theory of hereditary Artin algebras*, Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976), pp. 389–424. Lecture Notes in Pure Appl. Math., V ol. 37, Dekker, New York, 1978

[Pr] J. Propp, *The combinatorics of frieze patterns and Markoff numbers*, preprint arXiv:math/0511633v3 [math.CO], 2005.

[RT] I. Reiten and G. Todorov, unpublished.

[RS] C. Riedtmann, A. Schofield, *On a simplicial complex associated with tilting modules*, Comm. Math. Helv. **66** (1991), no.1, 70–78.

[R] C. M. Ringel, *Tame algebras and integral quadratic forms*, Springer Lecture Notes in Mathematics **1099** (1984).

[Sc] R. Schiffler, *A cluster expansion formula (A_n case)*, Electron. J. Combin. **15** (2008), # R64 1.

[ST] R. Schiffler and H. Thomas, *On cluster algebras arising from unpunctured surfaces*, IMRN, Advance Access published on April 11, 2009; doi: doi:10.1093/imrn/rnp047.

[SZ] P. Sherman and A. Zelevinsky, *Positivity and canonical bases in rank 2 cluster algebras of finite and affine types*, Moscow Math. J. **4** (2004), No. 4, 947-974.

[St] H. Strauss, *On the perpendicular category of a partial tilting module*, J. Algebra **144** (1991), no. 1, 43–66.

[XX1] Jie Xiao and Fan Xu, Green's formula with $\mathbb{C}^*$-action and Caldero-Keller's formula for cluster algebras. preprint arXiv:0707.1175 [math.QA], 2007.

[XX2] Jie Xiao and Fan Xu, *A multiplication formula for algebras with 2-Calabi-Yau properties*, preprint arXiv:0804.2014 [math.RT], 2008.

[XX3] Jie Xiao and Fan Xu, *Green formula in Hall algebras and cluster algebras*, preprint arXiv:0804.2015 [math.RT], 2008.

[YZ] Shih-Wei Yang and A. Zelevinsky, *Cluster algebras of finite type via Coxeter elements and principal minors*, Transformation Groups **13** (2008), No. 3-4, 855-895.

[Ze] A. Zelevinsky, *Semicanonical basis generators of the cluster algebra of type $A_1^{(1)}$* Electon. J. Combin. **14** (2007), no. 1, Note 4, 5 pp. (electronic).

[Zh] Bin Zhu, *Preprojective cluster variables of acyclic cluster algebras*, Comm. Algebra **35** (2007) no.9, 2857–2871.

Institutt for matematiske fag, Norges teknisk-naturvitenskapelige universitet, N-7491 Trondheim, Norway

E-mail address: `aslakb@math.ntnu.no`

School of Mathematics, University of Leeds, Leeds, LS2 9JT, UK

E-mail address: `marsh@maths.leeds.ac.uk`