

PASCAL ARRAYS: COUNTING GENERALISED CATALAN SETS

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ABSTRACT. The representation theory of algebras arising in the study of transfer matrices in statistical mechanics possesses a beautiful combinatorial structure. Guided by this we exhibit an interior structure to Catalan sequences of sets and many generalisations thereof. Certain of these coincide with well-known (but heretofore isolated) structures. The remainder are new. For example, we obtain a natural bijection between Temperley-Lieb diagrams and clusters of type A . We also show how the bijection between walks on the double Young graph and set partitions (which are counted by Bell numbers) fits into the same scheme.

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1. CATALAN SEQUENCES

1.1. Introduction.

The Catalan numbers form a sequence which begins

$$1, 1, 2, 5, 14, 42, \dots, C(n) = \frac{1}{n+1} \binom{2n}{n}, \dots,$$

They occur in a wide variety of distinct combinatorial contexts [3, 19, 38, 59, 60, 61], and have many generalisations [22, 26, 29, 47, 51]. In particular, $C(n)$ is equal to the number of non-crossing pair partitions of $2n$ objects [38].

The Temperley-Lieb algebra TL_n [64] has a natural basis given by such partitions [3, 42]. Consideration of non-crossing pair partitions from a Temperley-Lieb representation theoretic perspective reveals an interior structure to the Catalan integer sequence, and an enumeration of each set of non-crossing pair partitions. It is interesting to ask whether and how this structure also exists in the other combinatorial contexts mentioned above.

The first aim of this paper is to explain this structure and to give interesting examples. The second aim is to pass this structure wholesale into generalisations, both combinatorially and algebraically. The generalisations appropriate in this context can be considered to be classified by arbitrary rooted directed graphs (with the rooted semi-infinite chain $(A_\infty, 0)$ as the original).

Key motivating examples for our investigation were the Fomin-Zelevinsky cluster algebras [21]. The cluster algebras of finite type are classified by the Dynkin diagrams [22], and the number of clusters in a cluster algebra of type A_{n-1} is given by $C(n)$. We show in particular that such clusters possess the structure mentioned above (see Section 3.6). Numerical evidence suggests that clusters of type B can also be put into this framework. However, as it stands, clusters of type D do not fit into this picture, suggesting that the generalisation of the Catalan numbers we consider here (obtained by considering walks on rooted directed graphs) does not fully contain that arising for cluster algebras [23]. On the other hand, our generalisation brings other classical sequences such as the Bell numbers [39] into the same framework.

1.2. Representations and towers of algebras.

The structure in Catalan combinatorics we refer to above consists of two features of paths on graphs,

which we shall call *decomposition* and *edge maps*. These will be fully explained in Section 2. In order to do this, and to explain how they are connected to representations of algebras, we need first to recall some representation theory. In particular, we recall the formula for the dimension of a finite-dimensional algebra in terms of the dimensions of its simple and projective modules. We also discuss Bratteli diagrams for simple modules in a tower of algebras with a global limit, and their close relationship with paths on a related graph.

All our algebras will be finite-dimensional algebras with 1 over a field k (except where otherwise stated). Let $A \subset B$ be an identity-preserving injection of algebras, with left modules ${}_A M$ and ${}_B N$ respectively. Then the left-adjointness of induction to restriction [14] implies the Frobenius reciprocity:

$$(1.1) \quad \text{Hom}_B({}_B \text{Ind}_A M, {}_B N) \cong \text{Hom}_A({}_A M, {}_A \text{Res}_B N).$$

Consider the special case where $A = k$, $M = k$ and N is simple. Then ${}_B \text{Ind}_A k$ is isomorphic to the left regular module ${}_B B$. We also have that $\text{Hom}_B({}_B B, {}_B N)$ is generated by the maps from summands of ${}_B B$ with ${}_B N$ in their head — that is, the copies of the indecomposable projective module P_N covering ${}_B N$. By (1.1) we see that there are $\dim \text{Hom}(k, \text{Res} N) = \dim {}_B N$ summands P_N . Thus

$$(1.2) \quad \dim B = \sum_{\lambda} \dim P_{L_{\lambda}} \dim L_{\lambda}$$

where $\{L_{\lambda}\}_{\lambda}$ is the set of simple B -modules. In particular if B is semisimple then

$$(1.3) \quad \dim B = \sum_{\lambda} (\dim L_{\lambda})^2.$$

(Combinatorially, in case of the group algebra of the symmetric group, this is at the heart of the Robinson–Schensted correspondence [37, 58].) Thus if a combinatorial set can be equipped with the property of basis for B , it is a matter of representation theory to express its cardinality as a sum of squares. Of course there is no constructive procedure for equipping a set in this way, in general. The aim here is to present cases in which such an expression can be realised, via a bijection like the Robinson–Schensted correspondence (and more specifically via the Robinson–Schensted-like correspondence of [46]).

We have not yet used the fact that combinatorial sets often occur in *sequences*. The second aspect of representation theory which we want to employ is the relationship between the algebras in a *tower* of algebras [25, 12] giving these sequences.

Suppose that $\{A_n\}_{n=0}^{\infty} = A_{\bullet}$ is a tower of algebras (with identity-preserving inclusions). Let Λ_n be an index set for the simple A_n -modules, for each n . The *Bratteli diagram* of A_{\bullet} has vertices given by the simple modules of the A_n arranged in layers indexed by n . There are m arrows from the A_n -module $L_n(\mu)$ to the A_{n+1} -module $L_{n+1}(\lambda)$ whenever $L_n(\mu)$ appears with multiplicity m as a composition factor in ${}_{A_n} \text{Res}_{A_{n+1}} L_{n+1}(\lambda)$. (Bratteli diagrams appear in [6, 18] — a thorough discussion of the semisimple case may be found in [25].)

It is clear that the multiplicity of an A_{n-1} -module $L_{n-1}(\nu)$ as a composition factor in ${}_{A_{n-1}} \text{Res}_{A_{n+1}} L_{n+1}(\lambda)$ is given by the number of paths from $L_{n-1}(\nu)$ to $L_{n+1}(\lambda)$. If $A_0 \cong k$, so $\Lambda_0 = \{0\}$ for some element 0, it follows by induction on n that the dimension of an A_n -module $L_n(\lambda)$ is given by the number of paths from $L_0(0)$ to $L_n(\lambda)$ in the Bratteli diagram.

Suppose now, for a moment, that the following axioms hold:

(i) $\Lambda_n \leftrightarrow \Lambda_{n+2}$ for all n , so that the sequences of odd and even index have limits Λ' and Λ respectively. That is, for every simple module of A_n there is a simple module of A_{n+2} with the same label.

(ii) There is a graph G (subsequently referred to as the *Rollet graph*, cf. [46]) with vertices $\Lambda \sqcup \Lambda'$ such that for any $\lambda \in \Lambda_{n+1}$, $\mu \in \Lambda_n$, there is an arrow from $L_{n+1}(\lambda)$ to $L_n(\mu)$ in the Bratteli diagram if and only if there is an arrow from λ to μ in G , and in this case, the arrows have the same multiplicity, so that the multiplicity of $L_n(\mu)$ in ${}_{A_n} \text{Res}_{A_{n+1}} L_{n+1}(\lambda)$ is given by the number of arrows in G from μ to λ . (Note that there is a natural connection with the concept of *principal graph* as in [25, §4.1], from the context of finite dimensional von Neumann algebras.)

(iii) $A_0 = k$. (Set $\Lambda_0 = \{0\}$.)

An heuristic explanation for the significance of axioms of this kind in *statistical mechanics* and in *invariant theory* is given in Section 1.4; a more constructive and general axiom set is given in Section 4.1.

It follows that

Proposition 1.1. (i) *The walks on the Rollet graph G from 0 to λ of length n are a basis of $L_n(\lambda)$.*

(ii) *If A_n is semisimple, the walks on the Rollet graph G from 0 to 0 of length $2n$ are a basis for A_n itself.*
□

We remark that while we have a basis for $L_n(\lambda)$ in (i), the construction does not tell us the action of A_n on $L_n(\lambda)$.

The idea is that if A_n has an interesting combinatorial basis (in a sense to be elucidated in examples below) then such a representation-theoretic decomposition will be a way to understand it.

We do not restrict ourselves to consideration of simple modules as in the above discussion; our more general axiom set allows a much wider collection of examples. Particularly nice examples arise from towers of quasihereditary algebras (cf. [12]) where the modules considered are the standard modules over each algebra in the tower, and we will discuss some of these. It may also be the case that there are nice examples of towers of cellular algebras, with the modules taken to be the cell modules, although we do not consider that case in this article (other than quasihereditary examples). (Cellular algebras *per se* were introduced in [27].)

1.3. Structure of the paper.

The paper is structured as follows. In Section 2, we set up the appropriate graph-theoretic notation and define the notion of a Catalan (G, v_0) -sequence of sets for a rooted graph (G, v_0) . These are sequences of sets possessing the additional structure referred to above, in the sense that they arise from an underlying array of sets corresponding to walks on G starting at v_0 . In Section 3, we give a collection of examples of such sets for the rooted graph $(A_\infty, 0)$, including our motivating example of basis diagrams of the Temperley-Lieb algebra. Such examples will have cardinalities given by the sequence $(C(n))$ of Catalan numbers.

In Section 4 we consider towers of algebras, together with algebra modules at each level. We show that, provided a certain set of axioms holds, this gives rise to a Bratteli diagram and a Catalan (G, v_0) -sequence for a certain underlying rooted graph (G, v_0) . As a way of constructing examples of towers of algebras satisfying our axioms, we show that a set of axioms close to that in [12] is sufficient to ensure that our axioms hold. Examples arising in this way will always be quasihereditary, with the standard modules forming the set of modules at each layer. We discuss the example of the Temperley-Lieb algebra in this context.

In Section 5 we give our first example of a Catalan sequence corresponding to a rooted graph different from $(A_\infty, 0)$, i.e. blob diagrams. In this case the corresponding rooted graph is $(A_\infty^\infty, 0)$. In Section 6, we give a collection of natural examples of Catalan sequences corresponding to a wide collection of graphs — infinite rooted trees whose branching properties are governed by the distance to the root. We also show how these can arise from towers of algebras.

In Section 7, we consider examples for non-tree graphs. Our examples arise from partition algebras, Brauer algebras and Hecke algebras. The corresponding graphs in our framework include the double Young graph and the Young graph, respectively, showing that the underlying graph can be quite complex. The corresponding Catalan sequences have cardinalities given by the sequences of Bell numbers (counting partitions) and the sequence counting pair partitions.

In Section 8 we indicate how walks on a truncated graph can arise, by considering certain quotients of the ‘doubly critical’ blob algebra [47], and in Section 9 we give an example of a tower of algebras for the rooted graph $(D_\infty, 0)$ (see Figure 1(b)). These algebras first appeared in [29, §7].

1.4. Towers in statistical mechanics.

Towers of algebras A_\bullet of the kind described above occur ‘in nature’ as the transfer matrix algebras for sequences of lattice statistical mechanical models approaching the thermodynamic limit (indexed by lattice size — see [42] for details). In this setting different simple modules of A_n can be associated to different correlation functions, that is, to different observables on the system. Approaching the thermodynamic limit there has to be (on physical grounds) a stable notion of observables, essentially independent of lattice size. This implies a relationship between simple modules for different lattice sizes — i.e. between the simples responsible for the *same observable* in different algebras in the sequence. At the level of representation theory this is manifest, in practice, via functors embedding the category of modules for the smaller system into that of the larger. (There are several explicit examples in the literature — see [42, 45] and references therein.) This is the heuristic explanation for why we expect a non-empty set of examples satisfying axioms of the kind described above. That is, the inclusion $\Lambda_{n-2} \hookrightarrow \Lambda_n$ takes the label corresponding to a given observable in level $n-2$ to the label corresponding to the same observable in level n . We remark that the gap of two here is not forced: see Section 4.1 for consideration of a more general set up.

1.5. Geometrical considerations in representation theory.

The endowment of the index set for simple modules of an algebra with a geometrical structure (as in the weight space and weight lattice for Lie algebras) has been very useful in representation theory, *where it is possible*. For example the alcove geometry which describes the representation theory of quantum groups starts with an embedding of the index set in a space, on which a reflection group then acts. The number of instances of this *precise* abstract setting is manifestly limited [34]. Thus it is of interest to consider generalisations. A very mild generalisation of the classical use of alcove geometry [36] is discussed in [50], and another in [49], but there are many towers of algebras known for which, if geometry is to be applied, a more radical generalisation will be needed. Interpolating between the two examples above, it can be seen that while there may be no obvious metric space in which to embed a given index set, there may be a natural underlying graph. From a representation theory perspective this paper may be seen in part as an exploration of the uses of this idea.

2. PASCAL ARRAYS AND CATALAN SEQUENCES

2.1. Paths on graphs.

A *graph* G is defined to be a set of vertices $V = V_G$ together with a set of edges $E = E_G$ and a pair of maps i and f (initial and final) from E to V . Thus an element $e \in E$ is a directed edge from $i(e)$ to $f(e)$. If G is a graph then the *opposite graph* to G is the graph G^{op} obtained by interchanging the functions i and f .

We recall that a graph is said to be *loop-free* if it contains no edge e such that $i(e) = f(e)$; it is said to be *simply-laced* if it contains no pair e, e' of edges such that $i(e) = i(e')$ and $f(e) = f(e')$, and it is said to be *undirected* if for each edge e there is an edge e' such that $i(e') = f(e)$ and $f(e') = i(e)$.

Thus a simply-laced undirected graph G may be defined to be a set of vertices V together with a set of edges E consisting of subsets of V of cardinality 2.

A *rooted graph* (G, v_0) is a graph G together with a distinguished vertex v_0 . If v is a vertex in a graph, its adjacent vertices are defined to be those linked to v by an edge starting at v . We assume the *valency* of each vertex (the number of adjacent vertices) to be finite.

We use Dynkin diagram naming conventions for appropriate simply laced graphs — see, for example, Figures 1(a) and 1(b).

A *path* in graph G is defined to be a sequence $\mathbf{e} = (e_1, e_2, \dots, e_l)$ of edges of G such that $f(e_i) = i(e_{i+1})$ for $i = 1, 2, \dots, l$. The *length* of \mathbf{e} is defined to be the integer $l \geq 0$. For $v, w \in V$, let $P_G(n; v, w)$ denote the set of paths in G starting at v and ending at w with length n . If $v = v_0$ (the distinguished vertex) then

FIGURE 1. (a) The graph A_∞ .(b) The graph D_∞

we may omit it. We denote the cardinality of $P_G(n; v, w)$ by $N_{G,v}(n; w)$, and drop the subscripts G, v if they are not needed.

Definition 2.1. For fixed $v \in V$, we regard the collection of sets $P_G(n; v, w)$ as an array $Y_{G,v} = (Y_{G,v}(n; w))_{n \in \mathbb{N}, w \in V}$ indexed by the length n and the vertex w . We regard $(Y_{G,v}(n; w))_{w \in V}$ for fixed n as forming the n th layer of the array.

Example 2.2. Let G be the graph A_∞ or D_∞ — see Figure 1. We consider the rooted graph $(G, 0)$.

In this case the top of the array of paths is given in Figure 2 (respectively Figure 3).

Note that paths in A_∞ are displayed as finite graphs which are to be read from bottom to top; thus (for example) the graph in the bottom left of the picture corresponds to the path which visits the vertices 0, 1, 2, 1 and then 0 (in that order).

2.2. Arrays of paths and edge maps.

Let (G, v_0) be a rooted graph, with vertices V and edges E . For $e \in E$ let

$$\phi_e : Y_{G,v_0}(n-1; i(e)) \hookrightarrow Y_{G,v_0}(n; f(e))$$

be the natural embedding of the shorter path in the longer one, i.e. $\phi_e(e_1, e_2, \dots, e_l) = (e_1, e_2, \dots, e_l, e)$. We call these maps *edge maps*. Then by construction

Proposition 2.3.

$$(2.1) \quad Y_{G,v_0}(n; v) = \bigsqcup_{e \in E; f(e)=v} \phi_e(Y_{G,v_0}(n-1; i(e)))$$

And any total order on E passes to a total order on the union of sets in the n th layer of Y_{G,v_0} (i.e. $\cup_{v \in V} Y_{G,v_0}(n; v)$) via the lexicographic ordering of sequences on E . \square

Definition 2.4. An array of sets Y is said to be *equivalent* to $Y_{G,v_0} = (Y_{G,v_0}(n; w))_{n \in \mathbb{N}, w \in V}$, and called a *Pascal (G, v_0) -array*, if it has

- (1) the same index sets (V and \mathbb{N}),
- (2) pointwise bijections between the sets in layer $n = 0$ of Y and of Y_{G,v_0} , and
- (3) a set of inclusions corresponding to the edge maps ϕ_e which satisfy equalities corresponding to equation (2.1), i.e. for each edge e , a map

$$(\phi_e)' : Y(n-1; i(e)) \hookrightarrow Y(n; f(e))$$

satisfying:

$$(2.2) \quad Y(n; v) = \bigsqcup_{e \in E; f(e)=v} (\phi_e)'(Y(n-1; i(e))).$$

We will refer to the maps $(\phi_e)'$ in (3) as edge maps also.

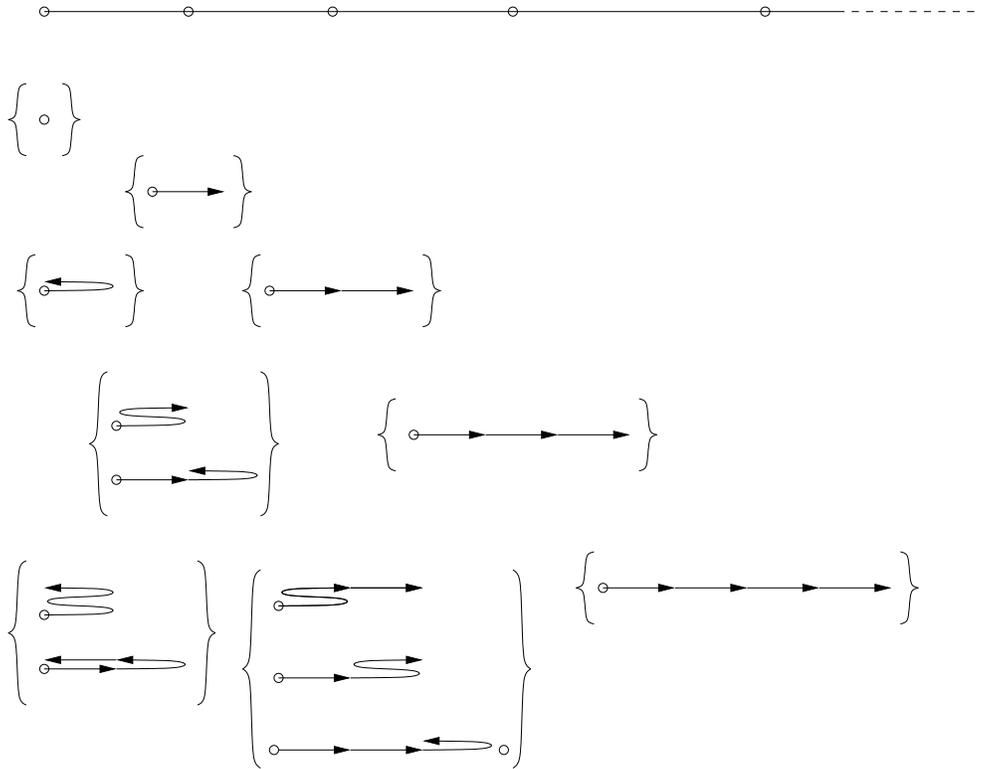


FIGURE 2. The array $Y_{A_\infty,0}$ of sets of paths from 0 on A_∞ , truncated below. The accompanying copy of the graph is positioned so as to associate vertices to the columns in the array. The column position in the array thus corresponds to the walk endpoint.

Proposition 2.5. *Let Y be an array of sets equivalent to Y_{G,v_0} . Then the entire array of sets is in pointwise explicit bijection with Y_{G,v_0} .*

Proof: This follows from requirement (3) of Definition 2.4 together with the fact (requirement (2)) that the sets in layer 0 all have cardinality 0 or 1. Each path in Y_{G,v_0} is given by a sequence of edge maps. Its image is given by the sequence of corresponding maps from (3). \square

Example: For any undirected graph G the array $\{P_{G^{op}}(n; w, v_0)\}_{n \in \mathbb{N}, w \in V}$ is equivalent to Y_{G,v_0} .

Suppose that Y is a Pascal (G, v_0) -array. Let $N(n; v)$ denote the cardinality of $Y(n; v)$ for all $n \in \mathbb{N}$ and $v \in V$. Note that necessarily we have $N(n; v) = |P_G(n; v_0, v)|$ for all $n \in \mathbb{N}$ and $v \in V$. We have an array of cardinalities corresponding to the array Y of sets satisfying:

$$(2.3) \quad \begin{aligned} N(0; v) &= 1 \\ N(n; v) &= \sum_{x \in G, e \in E} N(n-1; i(e)). \end{aligned}$$

Example: Let G be the graph A_∞ considered in the previous section. Then equations 2.3 become:

$$\begin{aligned} N(0; v) &= 1, \\ N(n; v) &= N(n-1; v-1) + N(n-1; v+1), \end{aligned}$$

where we define $N(n; -1) = 0$ for all $n \in \mathbb{N}$. The solution to this recurrence is the array given by the Catalan triangle, the top of which is shown in Figure 4; see for example [1, page 796].

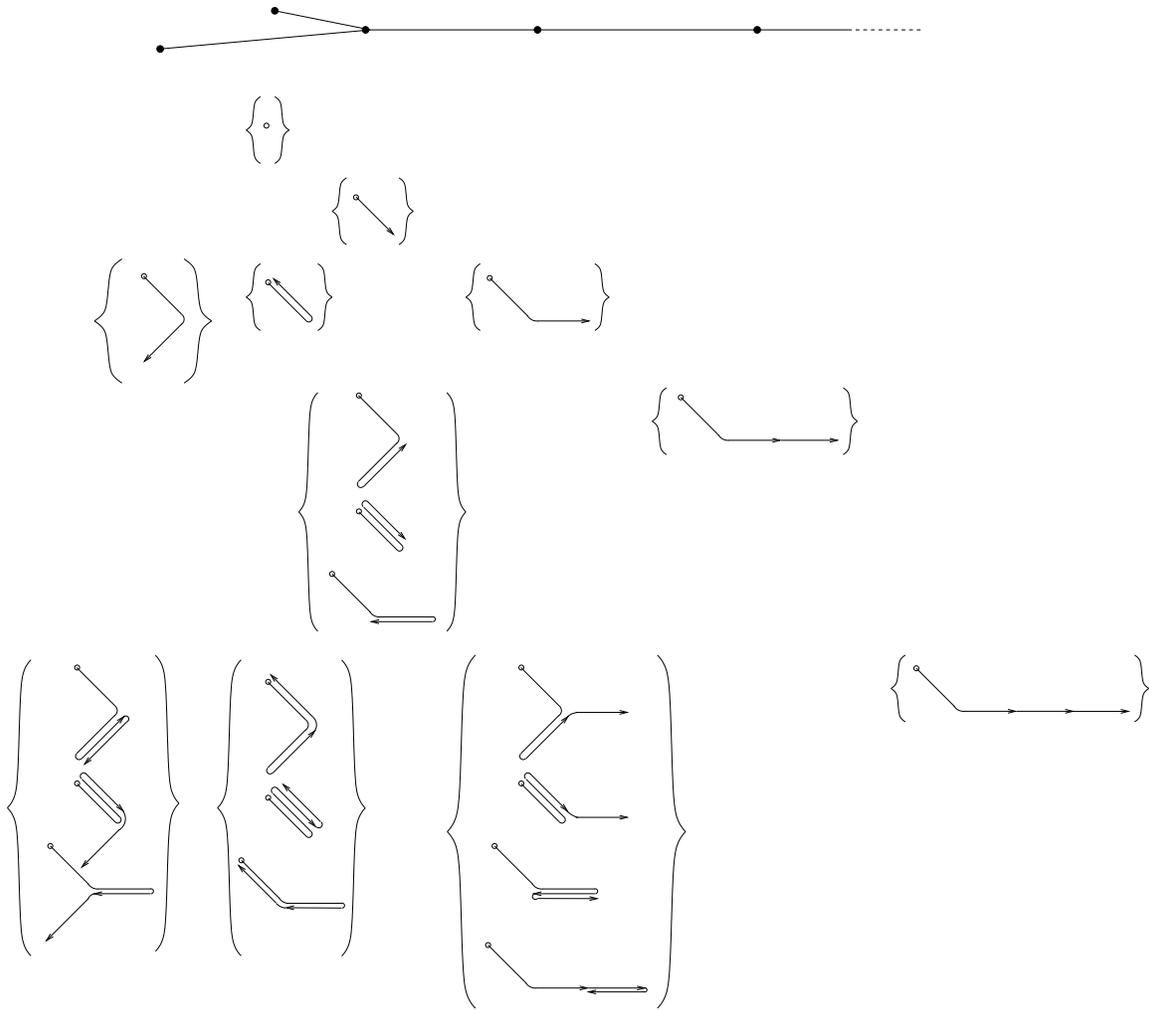


FIGURE 3. The Pascal array $Y_{D_\infty,0}$. As before we include a copy of the graph to label the columns.

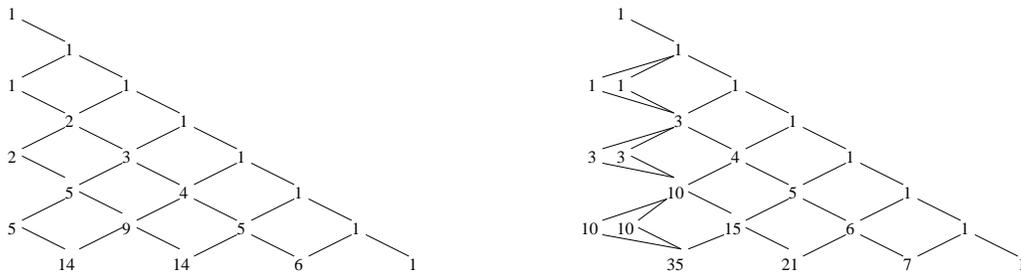


FIGURE 4. The Catalan triangle (cardinality array for Figure 2); and the cardinality array for Figure 3.

Remark 2.6. The above combinatorics can be encoded using the adjacency matrix $M(G) = (M_{v,w})_{v,w \in V}$ of the graph G , where $M_{v,w}$ is the number of edges e with $i(e) = v$ and $f(e) = w$. The v_0, w entry in $M(G)^n$ is equal to $N_{G,v_0}(n; w)$.

Remark 2.7. The entries in the Catalan triangle are also sometimes known as generalised Catalan numbers [54]. Note that, for $n - v$ even,

$$N(n, v) = \binom{n}{\frac{1}{2}(n-v)} - \binom{n}{\frac{1}{2}(n-v) - 1},$$

with the convention that $\binom{n}{-1} = 0$. (See e.g. [31, 54]).

2.3. Closed path decomposition and Catalan sequences.

Let $*$: $P_G(n; v, v') \times P_G(m; v', v'') \rightarrow P_G(n+m; v, v'')$ denote the composition of paths. Note that the triple $(V_G, \cup_n P_G(n; -, -), *)$, is a category with units given by paths of length zero (it is known as the *free category* on G). The following is immediate.

Proposition 2.8. *In general, we can express the set of paths from v_0 back to v_0 in $2n$ steps in the following way:*

$$(2.4) \quad P_G(2n; v_0, v_0) = *(\cup_{v \in G} P_G(n; v_0, v) \times P_G(n; v, v_0))$$

That is, $$ defines an bijection between from $\cup_{v \in G} P_G(n; v_0, v) \times P_G(n; v, v_0)$ to $P_G(2n; v_0, v_0)$. \square*

In this setting the inverse *decomposition* map to $*$ is obvious. In this paper we will be concerned with sets in bijection with $P_G(2n; v_0, v_0)$ (for various choices of (G, v_0)) for which the decomposition map is not so obvious (even the target set is unobvious), but which reveals important features of these sets (and associated algebras).

If G is undirected, it follows from Proposition 2.8 that

$$(2.5) \quad N_{G,v_0}(2n; v_0) = \sum_{x \in V} N_{G,v_0}(n; x)^2.$$

We call the sequence $\{N(2n; v_0)\}_n$ the *Catalan (G, v_0) -numbers*.

Definition 2.9. A *Catalan (G, v_0) -sequence* \mathcal{C} is any sequence of sets $(\mathcal{C}(n))_{n \in \mathbb{N}}$ in explicit natural bijection Φ with $\{\cup_{v \in G} Y(n; v) \times Y'(n; v)\}_{n \in \mathbb{N}}$ where Y is a Pascal (G, v_0) -array and Y' is a Pascal (G^{op}, v_0) -array. In this case we say that the Pascal array Y *underlies* the Catalan sequence (via Φ).

We remark that if G is undirected then Y' can be taken to be a Pascal (G, v_0) -array in the above: we make much use of this in the sequel. In this case, the sequence of cardinalities of such a sequence is given by the Catalan (G, v_0) -numbers.

We will see later that, beside the many well-known Catalan sequences (of ‘type-A’) this formalism brings many entirely distinct combinatorial sequences, such as the sets of partitions of sets of degree n , into the same framework.

Note that

Proposition 2.10. *The pointwise bijections from Proposition 2.5 together with the bijections Φ give an explicit pointwise bijection between any two Catalan (G, v_0) -sequences. \square*

The reason for this nomenclature is the following example.

Example: For $(G, v_0) = (A_\infty, 0)$ we obtain that

$$N_{A_\infty,0}(2n; v_0) = \sum_{v=2n, 2n-2, \dots, 1 \text{ or } 0} \left(\binom{n}{\frac{1}{2}(n-v)} - \binom{n}{\frac{1}{2}(n-v) - 1} \right)^2 = C(n).$$

That is, $\{P_{A_\infty,0}(2n; 0, 0)\}_n$ is the *ordinary* Catalan sequence. Equation 2.5 states that the n -th term is the sum of the squares in layer n of the triangle in Figure 4. This corresponds via equation 1.3 to a well-known result from representation theory in this case [42]. We will review this shortly (in Section 4.4) before going on to consider a number of generalisations.

3. EXAMPLES OF $(A_\infty, 0)$ -ARRAYS

In this section we recall some important Catalan sequences. By discovering decomposition maps for these (in the sense of Proposition 2.8), and edge maps for the target sets (in the sense of Proposition 2.3 and Definition 2.4), we give the corresponding Pascal G -arrays with G the rooted graph $(A_\infty, 0)$. Later we will give examples of the same kind for other rooted graphs.

We start with the core example from representation theory.

3.1. Temperley-Lieb diagrams.

Recall that a pair partition of $2n$ objects is a partition of the objects into parts each containing precisely two elements. One may draw a picture of such a pair partition by placing the objects on some horizontal line in the plane, and drawing lines between them in the plane below this horizontal. A pair partition is *non-crossing* if each pair may be connected by a line drawn on the lower half plane *simultaneously* with no lines crossing.

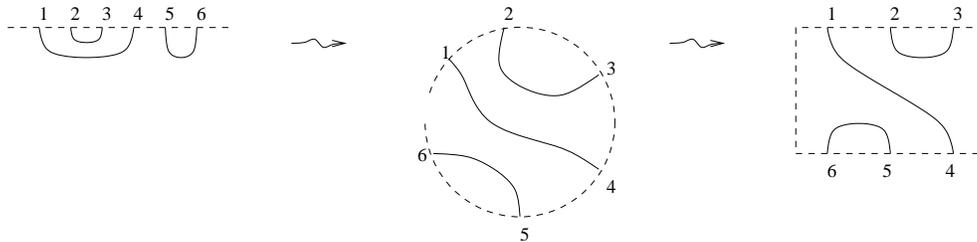
A famous Catalan sequence [60] has n^{th} term given by the set of non-crossing pair partitions of $2n$ objects. The $n = 3$ cases may be represented as follows.



The question here is: Is there an underlying Pascal G -array? This is the same as to ask: Is there a decomposition corresponding to the path decomposition and a set of edge maps? (The answer is yes. And in this case we should understand it in terms of representation theory.)

Note that the distinction between the embedding in the lower half plane and any other plane interval, such as a disk, is immaterial in this non-crossing pair partition construction. (The non-crossing property is preserved by conformal transformations, for example.) Let us think now of a representative non-crossing line realisation of such a pair partition as indeed being drawn on a disk.

Suppose we number the objects 1, 2, ... clockwise from some chosen point. The set $D(m, 2n - m)$ of *ordinary (Temperley-Lieb) diagrams* is simply a representative set of these realisations with a notional aggregation of the objects into ‘northern’ subset $\{1, 2, \dots, m\}$ of adjacent objects and a southern subset $\{m + 1, \dots, 2n\}$ of adjacent objects. Often the disk is also replaced by a rectangle, with the northern (resp. southern) subset lying on the northern (resp. southern) edge.



Note that for $n > 0$ (and $m + n$ even) there is a natural bijection

$$\phi^u : D(m, n) \rightarrow D(m + 1, n - 1)$$

via the common underlying set.

The *propagating* lines in a Temperley–Lieb diagram are the lines *between* the northern and southern subsets. Note that the number of these is *not* fixed by ϕ^u .

Let $D_l(m, n)$ denote the subset of $D(m, n)$ consisting of elements with l propagating lines.

The basic operations on Temperley–Lieb diagrams we need to consider are:

(1) Bra-ket decomposition: It is always possible to cut a diagram from east to west such that each propagating line is cut once, and no other line is cut. This process produces a well defined pair of half-diagrams, which itself may be thought of (in case $m = n$) as lying in $D(n, l) \times D(l, n)$ for some l . Indeed

$$(3.1) \quad D_l(n, n) \cong D_l(n, l) \times D_l(l, n)$$

where the map from left to right is given by cutting the propagating lines; the inverse map is to join the propagating lines in the obvious order. See Figure 5 for an example.

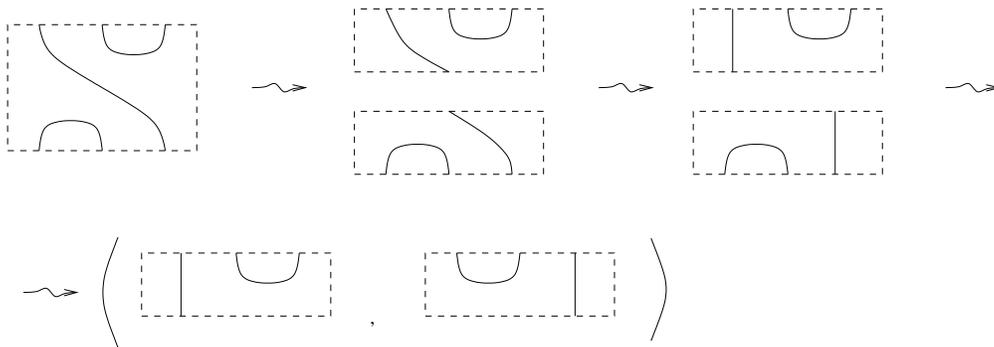
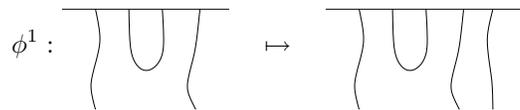


FIGURE 5. Splitting a $D(3, 3)$ diagram into two.

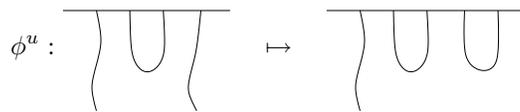
Remark. The reason for using Dirac’s bra-ket terminology will become apparent in Section 4.4.

(2) Edge maps:

(2a) Given an upper half diagram in $D_l(n, l)$ as above we may construct one in $D_{l+1}(n + 1, l + 1)$ by adding a propagating line on the right. We denote this map by ϕ^1 . For example:



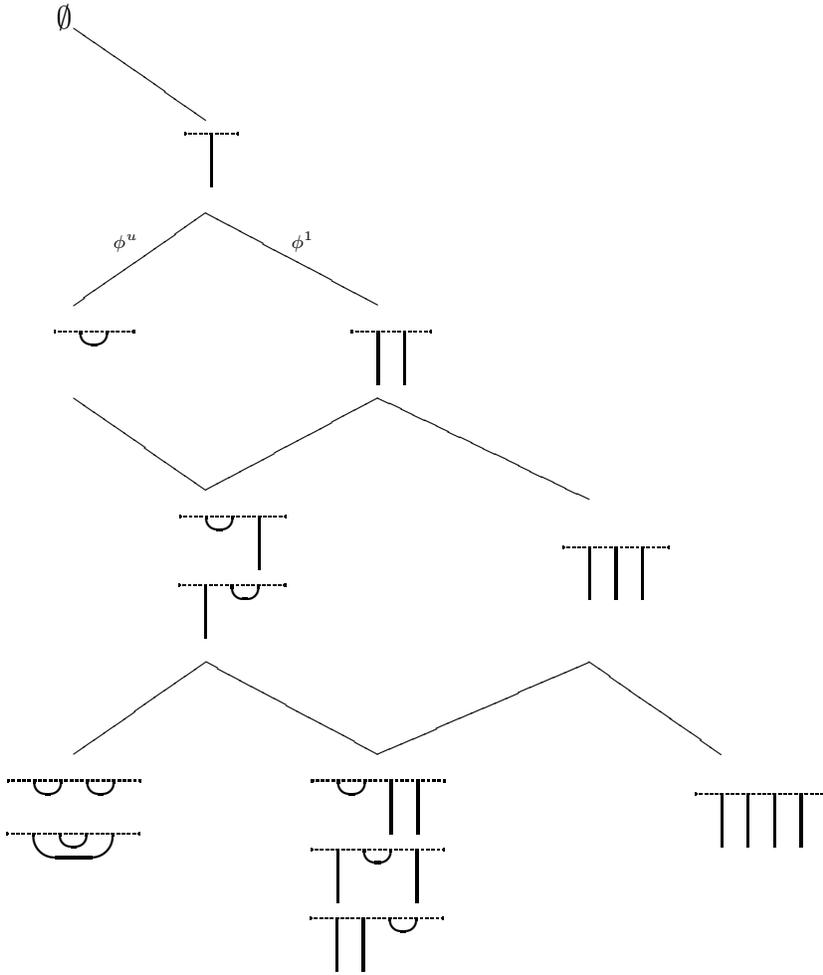
(2b) Given an upper half diagram in $D_l(n, l)$ with $l > 0$ we may construct another half diagram via the natural injection ϕ^u into $D_{l-1}(n + 1, l - 1)$, i.e. “bending over” the rightmost propagating line onto a new vertex on the horizontal line. For example:



Proposition 3.1. *The array of sets $Y_{TL} = ((D_l(n, l))_{l \in A_\infty})_n$ (see Figure 6) is a Pascal $(A_\infty, 0)$ -array with edge maps $(\phi_{i, i+1})'$ given by ϕ^1 and edge maps $(\phi_{i+1, i})'$ given by ϕ^u . This array underlies the sequence $(D(n, n))_n$ via equation (3.1).*

Proof: For the first part it suffices to note that

$$D_l(n, l) = \phi^u(D_{l+1}(n - 1, l + 1)) \sqcup \phi^1(D_{l-1}(n - 1, l - 1)).$$

FIGURE 6. The start of the Pascal array Y_{TL}

For the second part, note that an explicit construction (and *en passant* a counting) of the Catalan sets in the sequence $(D(n, n))_n$ follows from equation (3.1), and from the fact that $D(n, n) = \bigcup_l D_l(n, n)$. We obtain:

$$(3.2) \quad D(n, n) = \bigsqcup_l D_l(n, l) \times D_l(l, n).$$

□

Note that in Figure 6 we have drawn edges indicating when the edge maps are being applied (all such edges should be regarded as being oriented down the page). We sometimes use such edges for clarity; in such cases the brackets separating out the sets in the array are no longer necessary.

We shall see in Section 4.4 that $D(n, n)$ is a basis for the Temperley–Lieb algebra $TL_n(q)$ (independently of a choice of groundfield k and parameter q), and that $D_l(n, l)$ is a basis for a special kind of TL_n -module denoted $\Delta_n(l)$ (i.e. a standard module, see e.g. [15]).

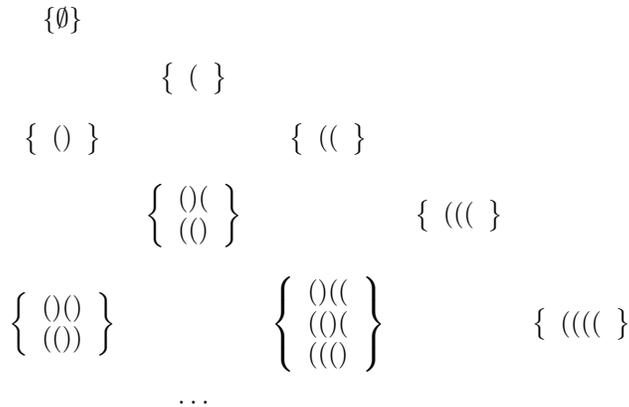


FIGURE 7. Pascal array Y_b . Here \emptyset denotes the empty sequence.

On the one hand (3.2) is (via the array equivalence) a special case of equation (2.4), and on the other hand it implies that

$$(3.3) \quad \dim(TL_n) = \sum_{\lambda} (\dim(\Delta_n(l)))^2.$$

Since for generic $q \in k = \mathbb{C}$, $\Delta_n(l)$ is isomorphic to the corresponding simple module $L_n(l)$, and TL_n is semisimple (see [42]), this provides a example of the deep algebraic result in equation 1.3.

3.2. Parentheses.

Consider the set $\mathcal{C}_b(n)$ consisting of matched bracket sequences containing n pairs of brackets, all of the same type. For example:

$$\mathcal{C}_b(3) = \{()()(), (())(), ()(()), ((())), (((()))\}.$$

Let us represent a walk in $Y_{A_\infty,0}$ (an element in one of the sets in Figure 2) by a sequence constructed from the symbols $\{1, 2\}$, where a 1 means step along an edge from i to $i + 1$ (some i) and a 2 means step from $i + 1$ to i . Thus all paths start with a 1, and the running total of 2's in any truncation never exceeds the running total of 1's (we also call this a *standard* sequence [62]). If we simply replace 1 with “(” and 2 with “)” we see that these sequences pass to properly nested, but not necessarily closed bracket sequences. That is, no close-bracket appears which does not close an open-bracket. Write $Y_b(n; l)$ for the set of sequences of n individual brackets in which there are l more open-brackets than close-brackets. The Pascal array corresponding to \mathcal{C}_b comes from the following operations:

- (1) Decomposition: An element $x \in \mathcal{C}_b(n)$ has $2n$ individual brackets. Break this sequence into two parts, each of n brackets. The left-hand sequence is of the form of an element of $Y_b(n, l)$ for some l . The right-hand sequence is not of such form, but the reverse sequence is.
- (2) Edge maps: For $i \geq 0$, the map $(\phi_{i,i+1})'$ is given by adding an open bracket and the map $(\phi_{i+1,i})'$ is given by adding a close bracket.

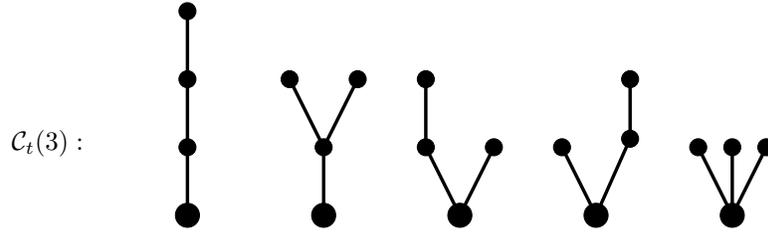
Thus we have:

Proposition 3.2. *The array Y_b (Figure 7) is a Pascal $(A_\infty, 0)$ -array (cf. Figure 2) underlying the sequence \mathcal{C}_b . \square*

3.3. Rooted planar trees.

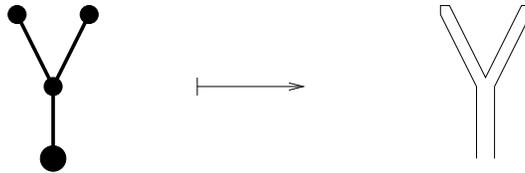
Another famous Catalan sequence [60] has n^{th} set the set $\mathcal{C}_t(n)$ of rooted planar trees with n edges. For

example, $\mathcal{C}_t(3)$ is given as follows:



More precisely, a *concrete* rooted planar tree is a tree (a connected acyclic graph of vertices with at most single edges between them), together with an embedding in the upper half-plane that takes the root vertex (and no other vertex) to the half-plane boundary, and has straight edges and no edge crossings. (Note that every tree has such an embedding.) Two such embedded trees are *equivalent* if they are related by an isotopy (note that this excludes moves which pass one branch over another). A rooted planar tree is an equivalence class of such embedded trees.

Algorithmically, two such embedded trees are distinct if they are distinguished by the following procedure. Consider the planar figure constructed by drawing around the outside of the embedded tree, starting from the root and drawing (say) anticlockwise (cf. the contour traversals in [10]). On completion this figure follows the shape of the tree, except that each edge is replaced by an outward/return pair of edges:



This figure may be called a *boundary tree*. Drawing clockwise produces the same boundary tree up to orientation. Traversing the boundary tree anticlockwise, each step is either moving away from or towards the root. Thus the embedded tree defines a sequence on any pair of symbols representing these two step types. (Our example could be the sequence 112122, say.)

Proposition 3.3. *Two such trees are distinct if their sequences are distinct.* \square

(Remark: choosing the symbols (and) gives a bijection between $\mathcal{C}_t(n)$ and $\mathcal{C}_b(n)$.)

Next we show that the boundary tree realisation leads us to a decomposition analogous to 3.1(1) and 3.2(1).

Every rooted planar tree has an embedding in which the root lies at the origin $(0,0)$, some sequence of vertices and edges starting at the root lie on the positive y -axis (and no others), and all vertices lie in the closure of the positive quadrant.

The edges and vertices lying on the y -axis are called the *trunk* in such an embedding. Each vertex v of the trunk defines a tree with root v induced by the child vertices of v obtained by moving along edges not in the trunk; we refer to these as the *branches* of the tree in such an embedding. The intersection of the boundary tree of such a tree with the positive quadrant is called a (*right*) *half-tree*. Note that, of the outward and return pair of boundary tree edges associated to an edge in the trunk, only the outward edge is retained in the right half-tree (while both are retained for edges not in the trunk). There is a corresponding notion of left half-trees.

The set of all right half-trees derived from trees with n edges in total and l edges in the trunk is denoted $Y_t(n; l) = Y_t^R(n; l)$. The sets $Y_t^L(n; l)$ of left half-trees are defined similarly.

The Pascal $(A_\infty, 0)$ -array corresponding to \mathcal{C}_t may be constructed as follows (cf. Figure 8).

(1) Bra-ket decomposition: Consider an n -edge planar tree drawn as described above. Note that each boundary tree contains $2n$ edges. We cut such a boundary tree figure into two halves by cutting after the n^{th} edge:



Lemma 3.4. *This defines a map*

$$\mathcal{C}_t(n) \rightarrow \bigsqcup_{l=0}^n Y_t^L(n;l) \times Y_t^R(n;l).$$

(2) Edge maps:

Let ϕ_t^+ denote the map from $Y_t(n;l)$ to $Y_t(n+1;l+1)$ which adds a single edge to the trunk of a half-tree and leaves the branches unchanged. Let ϕ_t^- denote the map from $Y_t(n;l)$ to $Y_t(n+1;l-1)$ which adds an extra edge to the last edge of the trunk (and rotates it clockwise off the y -axis, together with its branch, without going through another edge). The pair of edges thus created is regarded as part of the uppermost branch of the new half-tree. (See Figure 8).

A short argument shows that ϕ_t^+ and ϕ_t^- are both injective and that

$$Y_t(n;l) = \phi_t^+(Y_t(n-1;l-1)) \bigsqcup \phi_t^-(Y_t(n-1;l+1))$$

Proof of Lemma 3.4: If we cut the boundary tree of a tree T in $\mathcal{C}_t(n)$ in half as described above, the right-hand half lies in $Y_t(n;l)$ for some l (regarding edges in the boundary tree as being incident with the same vertices as the corresponding vertices in the tree itself). This is because a step in the path to a higher layer corresponds to ϕ_t^+ and a step in the path to a lower layer corresponds to ϕ_t^- . Similarly, the left-hand half is the reflection of a tree in $Y_t(n;l')$ in its trunk, for some l' . Since the edges of multiplicity one in each half come from the same set of edges of T , we must have $l = l'$. \square

Given a pair of half-trees in $Y_t(n;l)$, we can splice the first with the reflection of the second in its trunk to obtain a tree with all edges of multiplicity two which is the boundary tree of a tree in $\mathcal{C}_t(n)$. We denote this map by s . It is clear that this operation is inverse to the bra-ket extraction. We have confirmed that:

Proposition 3.5. *The array $Y_t(n;l)$ is a Pascal $(A_\infty, 0)$ -array underlying \mathcal{C}_t . \square*

It follows from Proposition 2.5 via a comparison of the edge maps in the Pascal $(A_\infty, 0)$ -arrays Y_t and Y_b that there is a bijection between $Y_t(n;l)$ and $Y_b(n;l)$ obtained by writing an open-bracket (respectively, close-bracket) for each step away from (respectively, towards) the root, as one traverses a half-tree anticlockwise from its root passing along each edge exactly once.

A comparison of the procedures for bra-ket extraction for $\mathcal{C}_t(n)$ and $\mathcal{C}_b(n)$ tells us that there is a bijection between $\mathcal{C}_t(n)$ and $\mathcal{C}_b(n)$ given by the same rule applied to the entirety of the boundary tree of a tree.

A similar argument shows there is a bijection between $D(n, n)$ and $\mathcal{C}_t(n)$ obtained by drawing a vertex in each connected component of the complement of a diagram in $D(n, n)$ in its bounding rectangle and an edge connecting any two vertices in components separated by a single line. Again the vertex in the region touching the western edge of the rectangle is taken to be the root.

3.4. Interval orders.

(For a general introduction to interval orders see [19].)

Consider the set χ_n of n -tuples of closed intervals of length 1 on the real line with distinct end-points. It will be convenient for each interval I to be considered as a function $I : [0, 1] \rightarrow \mathbb{R}$. We may label (totally

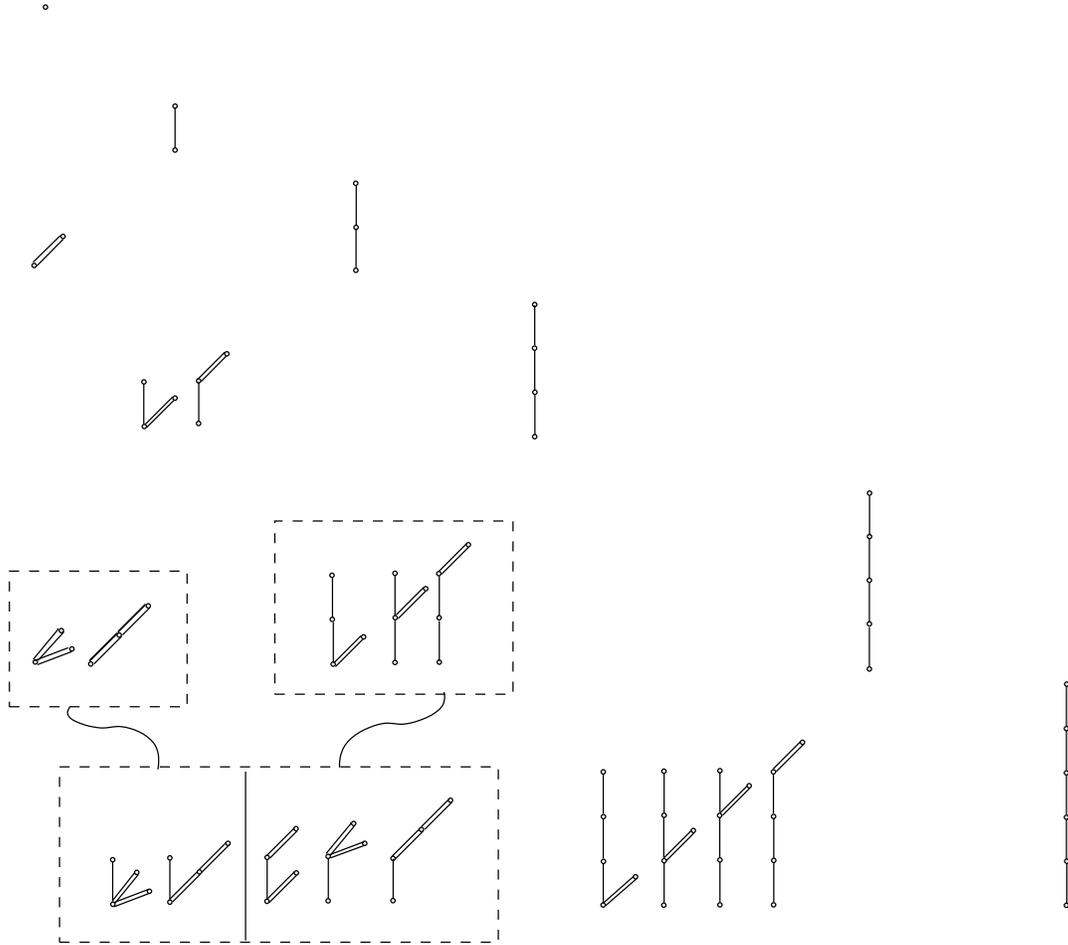


FIGURE 8. The Pascal $(A_\infty, 0)$ -array Y_t , underlying C_t . Note that the precise angle of edges is irrelevant.

order) the intervals in a tuple in χ_n by starting point: $I^1(0) < I^2(0) < \dots < I^n(0)$. We may partially order the intervals in a tuple in χ_n by stipulating that $I^i < I^j$ if $I^i(1) < I^j(0)$. The *unit interval orders* of degree n are the nonisomorphic partial orders of $\{I^1, I^2, \dots, I^n\}$ that can arise in this way, denoted by $\mathcal{C}_i(n)$. We remark that our restriction that the end-points are distinct does not change this set as a small enough displacement of an intervals will not change the isomorphism class.

We have [68] that $|\mathcal{C}_i(n)| = C(n)$. (See also [19, p98]). For example,

$$\mathcal{C}_i(3) = \{(I < J < K), (I < J, K), (I, J < K), (I < K), ()\}$$

Suppose that $0 \leq l \leq n$ and $n - 2l$ is even. Consider the set $\chi_n[l]$ of tuples of l distinct points and $(n - l)/2$ unit intervals such that the points and end-points of intervals never coincide and there is never a point a and an interval I such that $a < I(0)$.

Partially order the elements in a tuple by ordering the intervals as above. In addition, if I is an interval and a a point, define $I < a$ if $I(1) < a$. Thus two points are always incomparable, and a point can never be less than an interval. Let $Y_i(n; l)$ denote the set of distinct partial orders arising in this way. Each such partial order comes with a distinguished subset (given by the points). We call such labelled partial orders *unit interval-point orders*.

(1) Bra-ket decomposition: Given a unit interval order of degree n , we consider the lowest n end-points. If both end-points of an interval lie in this set we keep the interval, otherwise we just take the endpoint that does lie in the set. Note that it is not possible for such a point a to be such that $a < I(0)$ for one of the intervals I kept in the above procedure, since then a would be the start of an interval whose size must be greater than 1. The corresponding labelled partial order is thus an element of $Y_i(n; l)$ for some l . The negations of the remaining end points give rise to a second element of $Y_i(n; l)$ in the same way. It can be shown that this procedure is well-defined; the proof involves a description of the different interval representations of an interval order.

(2) Edge maps.

Let ϕ_i^+ denote the map from $Y_i(n; l)$ to $Y_i(n+1; l+1)$ given by adding a new labelled element greater in the ordering than all unlabelled elements and incomparable with all other labelled elements. Since this corresponds to adding a new point to a representative of the unit interval-point order greater than all the intervals in it, it is clear that the new labelled partial order is again a unit interval-point order.

Define the *height* of a labelled element to be the number of (necessarily unlabelled) elements less than it. Let ϕ_i^- denote the map from $Y_i(n; l)$ to $Y_i(n+1; l-1)$ given by removing the label from any labelled point of minimal height. It can be checked that the new labelled partial order is indeed an element of $Y_i(n+1; l-1)$ and is independent of the choice of labelled point.

We remark that two unit interval-point orders X_1 and X_2 with the same number of labelled points and orderings $<_1$ and $<_2$ can be combined to make a new order. The labelled points in X_1, X_2 can be ordered in increasing height. The new order has set $X_1 \cup X_2$, where we identify the labelled points in X_1 following this order with the labelled points in X_2 ordered in decreasing height.

We then set $x < y$ in the new order if either:

- (a) $x, y \in X_1$ and $x <_1 y$;
- (b) $x, y \in X_2$ and $y <_2 x$; or
- (c) $x \in X_1, y \in X_2$, and x, y are both unlabelled.

It can be seen that this is a unit interval order using a result of Roberts [57] (see [4] for a short proof) that states that a finite partial order is an interval order if and only if it has an interval representation without nested intervals.

The following proposition can be shown via an analysis of interval representations of interval orders.

Proposition 3.6. *The sets of unit interval-point orders $Y_i(n; l)$ described above form an $(A_\infty, 0)$ -array of sets underlying the sequence of sets $\mathcal{C}_i(n)$. \square*

The start of the array is displayed in Figure 9. Labelled points are shown as filled-in circles.

It follows from Proposition 2.5 via a comparison of the edge maps in the Pascal A_∞ -arrays Y_i and Y_b that there is a bijection between $Y_i(n; l)$ and $Y_b(n; l)$ obtained by writing an open-bracket (respectively, close-bracket) for each point or lower end-point of an interval (respectively, upper end-point of an interval) reading along the real line.

A comparison of the procedures for bra-ket extraction for $\mathcal{C}_i(n)$ and $\mathcal{C}_b(n)$ tells us that there is a bijection between $\mathcal{C}_i(n)$ and $\mathcal{C}_b(n)$ given by the same rule applied to an interval order (where the point case does not arise).

3.5. Noncrossing partitions.

Let $n \in \mathbb{N}$. A *noncrossing partition* of n is a partition of n such that, if the numbers $1, 2, \dots, n$ are equally spaced (in an anticlockwise order) around a circle in the plane then the convex hulls of the parts do not intersect [38] (see [52] for a recent survey). There is a bijection with noncrossing pair partitions of $2n$ obtained by drawing around the outside of the boundaries of the convex hulls (see [42, §6.2.1], or Figure 10 for an example). Our usual set of constructions follows from Section 3.1 via this bijection.

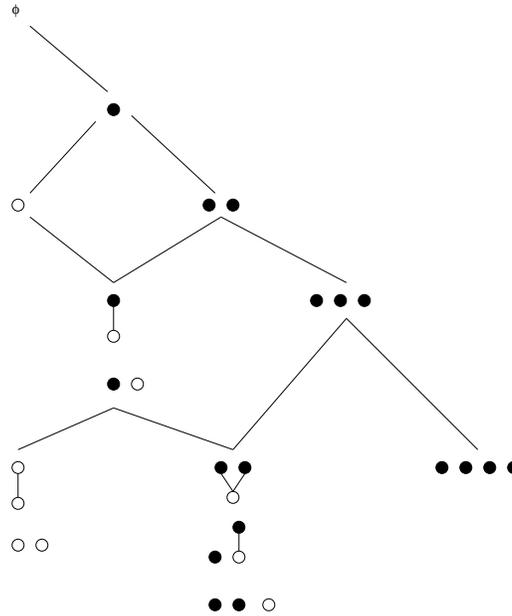


FIGURE 9. The start of the array of unit interval-point orders; a filled-in circle denotes a distinguished element, corresponding to a point in a representation. Empty circles correspond to intervals.

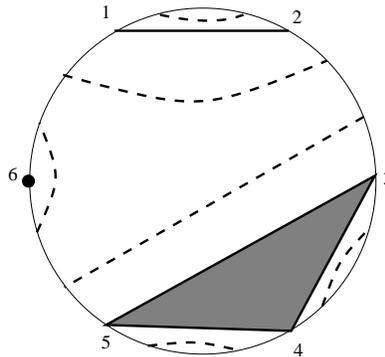


FIGURE 10. Example of correspondence between noncrossing pair partitions and noncrossing partitions. The dotted lines represent the noncrossing pair partition corresponding to the noncrossing partition $\{\{1, 2\}, \{3, 4, 5\}, \{6\}\}$.

3.6. Clusters.

In order to study the (dual) canonical basis of a quantized enveloping algebra of a simple Lie algebra, Fomin and Zelevinsky [21] have introduced the notion of a *cluster algebra*. Cluster algebras of finite type have been classified [22], and are parametrized (in families) by the Dynkin diagrams. The generators of a cluster algebra (apart from the coefficients) are organised into distinguished subsets all of the same size, known as *clusters*. In the finite type case, the generators can be parametrized by the *almost positive* roots $\Phi_{\geq -1}$ (i.e. the positive roots together with the negative simple roots) in the root system Φ of the corresponding Dynkin diagram, and the clusters can be described using root system combinatorics; the subsets of $\Phi_{\geq -1}$ corresponding to clusters are known as *root clusters*.

Root clusters can be regarded as a special case of the Γ -clusters of [40] associated to each orientation Γ of the Dynkin diagram; here Γ is taken to be the alternating orientation, in which each vertex is a sink or source.

Since Γ -clusters are known to form a sequence of sets with Catalan cardinalities in type A_n (this can be seen from [23, 3.8] and [8, 4.3,4.5]) it is natural to ask whether they fit into the framework of this paper (as we have already mentioned in the introduction). In this section, we show that in the simplest case, i.e. root clusters associated to the linear orientation of the Dynkin diagram of type A_n , this can be done. We describe a Pascal array for clusters that was constructed using the bijections given in Reading [56] between Coxeter-sortable elements and clusters and between Coxeter-sortable elements and noncrossing partitions. See Section 3.5 for information concerning the Pascal array underlying the sequence of sets of noncrossing partitions.

Reading [56] gives an elementary description of Γ -clusters which we use here. (Note that Reading generalises Γ -clusters to include the non-crystallographic case also). In Reading's approach, the clusters depend on a choice of Coxeter element c in the Weyl group W of Φ , and are therefore known as c -clusters. This is a natural generalisation of the initial combinatorics of clusters given in [23]; see also [22]. We regard c -clusters as subsets of $\Phi_{\geq -1}$ as in Fomin-Zelevinsky's work.

Let s_1, s_2, \dots, s_n be the simple reflections in W corresponding to the simple roots $\alpha_1, \alpha_2, \dots, \alpha_n$ in the root system Φ of type A_n . Note that

$$\Phi = \{\pm(\alpha_i + \alpha_{i+1} + \dots + \alpha_j) : 1 \leq i \leq j \leq n\}.$$

For positive roots $\alpha = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$ and $\alpha' = \alpha_{i'} + \alpha_{i'+1} + \dots + \alpha_{j'}$ we write $\alpha \subseteq \alpha'$ if $i \leq i' \leq j' \leq j$.

For each $i = 1, 2, \dots, n$ let σ_i be the involution of $\Phi_{\geq -1}$ given by the formula:

$$\sigma_i(\alpha) := \begin{cases} -\alpha, & \text{if } \alpha = \pm\alpha_i, \\ \alpha, & \text{if } \alpha = -\alpha_j, j \neq i, \\ s_i(\alpha), & \text{otherwise.} \end{cases}$$

Define a family of binary relations $\|_c$, for c a Coxeter element, on the set $\Phi_{\geq -1}$, unique satisfying:

(I) For any $-\alpha_i$ and any $\alpha \in \Phi^+$, $-\alpha_i \|_c \alpha$ if and only if α_i does not occur in the expansion of α in terms of the simple roots.

(II) For any $\alpha, \beta \in \Phi_{\geq -1}$ and any initial letter s (equiv. final letter) of c , $\alpha \|_c \beta$ if and only if $\sigma(s)(\alpha) \|_{scs} \sigma(s)(\beta)$.

Then a c -cluster is a maximal subset of $\Phi_{\geq -1}$ of elements all pairwise c -compatible in this sense. Note that it follows from repeated application of (II) that $\alpha \|_c \beta$ if and only if $c(\alpha) \|_c c(\beta)$ for all $\alpha, \beta \in \Phi_{\geq -1}$. In type A_n , we fix the Coxeter element $c = s_n s_{n-1} \dots s_1$, and in the sequel we will drop the c from c -cluster.

We set $\mathcal{C}_c(n)$ to be the set of clusters of type A_{n-1} .

Fix $n \in \mathbb{N}$ and $l \in \mathbb{N}$ such that $n - l$ is even. By a *tagged* cluster we mean a cluster together with a tag associated to each negative root (known as *root tags*) together with an additional (*global*) tag, which can be $+$ or empty. If a negative root is tagged, we write a bar over it; otherwise we leave it unadorned.

If n and l are odd, let $Y_c(n; l)$ denote the set of tagged clusters of type $A_{(n-1)/2}$ with $(l-1)/2$ root tags and no global tag. If n and l are even, let $Y_c(n; l)$ denote the set of tagged clusters of type $A_{n/2-1}$ with $l/2$ tags (one of which can be a global tag).

1. Bra-ket extraction

Suppose first that n is odd and let X be a cluster of type A_{n-1} . Let $r = (n-1)/2$. For each $k \in \{r+1, \dots, n-1\}$ consider the sets

$$X_k := \{\alpha \in X : \alpha > 0, \alpha = \alpha_i + \dots + \alpha_k, 1 \leq i \leq r\},$$

$$Y_k := \{\alpha \in X : \alpha > 0, \alpha = \alpha_i + \dots + \alpha_k\},$$

$$I_k := \{i \in \{1, 2, \dots, r\} : \alpha_i + \dots + \alpha_k \in X_k\}.$$

Suppose first that $|X_k| = 1$. Then $X_k = \{\alpha_i + \cdots + \alpha_k\}$ for some $i \in [1, r]$. If $|X_t| > 1$ for some $t \in [r+1, k-1]$, choose a maximal such t ; otherwise let $t = r$. Then the root $\alpha_i + \cdots + \alpha_k \in X$ is replaced by $\alpha_{t+1} + \cdots + \alpha_k$.

Suppose secondly that $|X_k| > 1$. Then the roots in $Y_k \subseteq X$ are replaced by the roots $-\alpha_k$ and the roots $-\alpha_{i-1}$ where $i \in I_k$ is not the minimum element of I_k . The roots $-\alpha_{j-1}$ where j is the second smallest element of I_k and $-\alpha_k$ are both tagged. Note that as $|X_k| > 1$ we must have $j \leq r$.

Let X' be the new set of roots so obtained and let

$$\begin{aligned} C &= \{\alpha : \alpha \in X', \alpha \subseteq \alpha_1 + \cdots + \alpha_r\} \\ D &= \{\sigma(\alpha) : \alpha \in X', \alpha \subseteq \alpha_{r+1} + \cdots + \alpha_{n-1}\}, \end{aligned}$$

where σ is the map taking α_i to α_{n-i} , extended linearly.

Then it can be shown that C and D are both clusters of type A_r with the same number of tags, i.e. elements of $Y_c(r; l)$ for some l .

Now suppose that n is even and let X be a cluster of type A_{n-1} . Let $r = n/2 - 1$. For each $k \in \{r+2, \dots, n-1\}$ consider the sets

$$\begin{aligned} X_k &:= \{\alpha \in X : \alpha > 0, \alpha = \alpha_i + \cdots + \alpha_k, 1 \leq i \leq r+1\}, \\ Y_k &:= \{\alpha \in X : \alpha > 0, \alpha = \alpha_i + \cdots + \alpha_k\}, \\ I_k &:= \{i \in \{1, 2, \dots, r+1\} : \alpha_i + \cdots + \alpha_k \in X_k\}. \end{aligned}$$

We also define:

$$\begin{aligned} X_{r+1} &:= \{\alpha \in X : \alpha > 0, \alpha = \alpha_i + \cdots + \alpha_k, 1 \leq i \leq r\}, \\ Y_{r+1} &:= \{\alpha \in X : \alpha > 0, \alpha = \alpha_i + \cdots + \alpha_k\}, \\ I_{r+1} &:= \{i \in \{1, 2, \dots, r\} : \alpha_i + \cdots + \alpha_k \in X_k\}. \end{aligned}$$

If $k > r+1$ and $I_k \subseteq [r+2, n-1]$, or if $k = r+1$ and $I_k = \{r+1\}$, then the elements of Y_k are left alone. Otherwise, if $|X_k| = 1$ then $X_k = \{\alpha_i + \cdots + \alpha_k\}$ for some $i \in [1, r+1]$ for $k > r+1$ or $i \in [1, r]$ for $k = r$. If $k > r+1$ and $|X_t| > 1$ for some $t \in [r+2, k-1]$ then choose a maximal such t ; otherwise let $t = r+1$, except that if $k = r+1$ then let $t = r$.

Then the root $\alpha_i + \cdots + \alpha_k \in C$ is replaced by $\alpha_{t+1} + \cdots + \alpha_k$.

If $|X_k| > 1$ then the roots in $Y_k \subseteq X$ are replaced by the root $-\alpha_k$ and the roots $-\alpha_{i-1}$ where $i \in I_k$ is not the minimum element of I_k . The roots $-\alpha_{j-1}$ where j is the second smallest element of I_k and $-\alpha_k$ are both tagged.

Let X' be the new set of roots so obtained and let

$$\begin{aligned} C &= \{\alpha : \alpha \in X', \alpha \subseteq \alpha_1 + \cdots + \alpha_r\} \\ D &= \{\sigma(\alpha) : \alpha \in X', \alpha \subseteq \alpha_{r+2} + \cdots + \alpha_{n-1}\}, \end{aligned}$$

where σ is the map taking α_i to α_{n-i} , extended linearly.

If X' contains α_r then a global tag is added to each of C and D . If X' contains $-\alpha_{r+1}$ then neither C nor D gets a global tag. If X' contains $-\overline{\alpha_{r+1}}$ then either C or D gets a global tag in such a way as to ensure that they have the same number of tags.

Then C and D are both clusters of type A_r with the same number of tags, so they both lie in $Y_c(n; l)$ for some l .

2. Edge maps:

Given a set S of (possibly tagged) almost positive roots, we need a function L_S from the set $\{1, 2, \dots, n\}$ to itself defined as follows. Let $i \in \{1, 2, \dots, n\}$. Then $L_S(i)$ is defined to be $j+1$ where $j < i$ is maximal such that $-\alpha_j \in S$ (possibly tagged). If no such j exists then $L_S(i) := 1$.

We now describe the maps ϕ_c^\pm on a tagged cluster C in $Y_c(n; l)$. The map ϕ_c^+ is given by adding a global tag if n is odd. If n is even it is given by adding a new negative simple root $-\alpha_{n/2}$ which is tagged if there was a global tag already present (in which case the global tag is removed).

The map ϕ_c^- is given by the identity map if n is odd (and $l \geq 1$). If n is even it is given in the following way. Note that C is a tagged cluster of type A_r where $r = n/2 - 1$, and we are supposing that $l \geq 1$ so C has at least one tag.

Suppose first that C has no global tag. Let k be maximal such that $-\overline{\alpha}_k \in C$. (Note that such a k must exist as C must have at least one tag and has no global tag). Then every root of the form $-\alpha_l$ with $l \geq k$ is replaced by $\alpha_{l+1} + \cdots + \alpha_{r+1}$. The extra root $\alpha_{L_C(k)} + \cdots + \alpha_{r+1}$ is added to the set C .

Suppose secondly that C does have a global tag. Then the extra root $\alpha_{L_C(r)} + \cdots + \alpha_{r+1}$ is added to the set C . In both cases, the global tag is removed.

The start of the array Y_c is shown in Figure 11, and an example of the bra-ket extraction is shown in Figure 12.

The following Theorem holds. The proof is nontrivial, but will be given in a separate publication [41] as it does not fit our purposes here.

Theorem 3.7. *The array $Y_c(n; l)$ of tagged clusters is a Pascal $(A_\infty, 0)$ -array underlying the sequence $\mathcal{C}_c(n)$ of root clusters.*

3.7. Remarks.

Note that we have *not* given a procedure for deriving a Pascal array from a Catalan sequence. Rather we have developed one in each case by some strategy.

Our main strategy is to use ideas from representation theory. Next we look explicitly at the representation theory side of things.

4. PASCAL ARRAYS ARISING FROM ALGEBRAS

4.1. Towers of algebras associated to rooted graphs.

Our objective in this section is to describe a set of axioms on a tower of algebras which is enough to associate a Pascal array to the tower (in the way suggested in section 3.1). The aim, of course, is to find an axiom set weak enough that concrete examples exist.

Let k be a field and A be a k -algebra. Let Δ be a set of A -modules. A filtration of an A -module M is called a Δ -filtration if it has sections in Δ . An A -module M is said to be Δ -good provided it has a Δ -filtration and the multiplicity of $L \in \Delta$ in any Δ -filtration of M is independent of the choice of filtration. We denote this multiplicity by $[M : L]$.

(This definition is motivated by the definition of Δ -good modules over a quasihereditary algebra).

Let k be a field. Suppose we are given a tower A_\bullet of k -algebras A_0, A_1, A_2, \dots . We consider the following axioms:

(N1) For all $n \in \mathbb{N}$, there is an algebra embedding $A_n \hookrightarrow A_{n+1}$.

(N2) For all $n \in \mathbb{N}$, there is a set Λ_n , with an A_n -module $\Delta_n(\lambda)$ for each $\lambda \in \Lambda_n$, such that if $\lambda \neq \mu$ then $\Delta_n(\lambda) \not\cong \Delta_n(\mu)$.

(N3) There is a fixed $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, $\Lambda_{n-N} \subseteq \Lambda_n$. Furthermore, whenever $\Lambda_n \cap \Lambda_m \neq \emptyset$, $n \equiv m \pmod N$.

By (N1), we have a restriction functor ${}_{A_{n-1}}\text{res}_{A_n} : A_n\text{-mod} \rightarrow A_{n-1}\text{-mod}$ for all $n \geq 1$. For given n , set $\Delta = \{\Delta_n(\lambda) \mid \lambda \in \Lambda_n\}$.

(N4) For all $n \geq 1$ and all $\lambda \in \Lambda_n$, ${}_{A_{n-1}}\text{res}_{A_n}(\Delta_n(\lambda))$ is a Δ -good A_{n-1} -module.

A_3 -cluster	C	D
$-\alpha_1, -\alpha_2, -\alpha_3$	$-\alpha_1$	$-\alpha_1$
$-\alpha_1, -\alpha_2, \alpha_3$	$-\alpha_1$	α_1
$\alpha_1, -\alpha_2, -\alpha_3$	α_1	$-\alpha_1$
$\alpha_1, -\alpha_2, \alpha_3$	α_1	α_1
$-\alpha_1, \alpha_2, -\alpha_3$	$\alpha_1, +$	$-\alpha_1, +$
$-\alpha_1, \alpha_2, \alpha_2 + \alpha_3$	$-\alpha_1, +$	$\alpha_1, +$
$-\alpha_1, \alpha_2 + \alpha_3, \alpha_3$	$-\alpha_1, +$	$-\overline{\alpha_1}$
$\alpha_1, \alpha_1 + \alpha_2, -\alpha_3$	$\alpha_1, +$	$-\alpha_1, +$
$\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3$	$\alpha_1, +$	$\alpha_1, +$
$\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3$	$\alpha_1, +$	$-\overline{\alpha_1}$
$\alpha_1 + \alpha_2, \alpha_2, -\alpha_3$	$-\overline{\alpha_1}$	$-\alpha_1, +$
$\alpha_1 + \alpha_2, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3$	$-\overline{\alpha_1}$	$\alpha_1, +$
$\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3$	$-\overline{\alpha_1}$	$-\overline{\alpha_1}$
$\alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_2 + \alpha_3$	$-\overline{\alpha_1}, +$	$-\overline{\alpha_1}, +$

FIGURE 12. Bra-ket extraction for clusters of type A_3

(N5) The cardinality of Λ_0 is 1. We denote its unique element by 0. The module $\Delta_0(0)$ is a one-dimensional simple A_0 -module.

(N6) For all $n \in \mathbb{N}$, $n \geq N$, there is an exact functor $F_n : A_n\text{-mod} \rightarrow A_{n-N}\text{-mod}$.

(N7) For all $\lambda \in \Lambda_n$, $F_n(\Delta_n(\lambda))$ is isomorphic to $\Delta_{n-N}(\lambda)$ if $\lambda \in \Lambda_{n-N}$, and otherwise is zero.

(N8) For all $n \in \mathbb{N}$, $n \geq N + 1$, and all $\lambda \in \Lambda_n$, we have that

$$(F_{n-1} \circ A_{n-1} \text{res}_{A_n})(\Delta_n(\lambda)) \cong (A_{n-N-1} \text{res}_{A_{n-N}} \circ F_n)(\Delta_n(\lambda)),$$

provided $F_n(\Delta_n(\lambda)) \neq 0$.

The Δ -Bratteli diagram of A_\bullet is defined to be the array $(\Delta_n(\lambda))_{n,\lambda}$ of modules. We draw m arrows from $\Delta_n(\lambda)$ to $\Delta_{n+1}(\mu)$ whenever $\Delta_n(\lambda)$ appears as a factor in $A_n \text{res}_{A_{n+1}}(\Delta_{n+1}(\mu))$ with multiplicity $m > 0$.

(N9) There is a graph G with vertices (indexed by) $\Lambda := \cup_{n \in \mathbb{N}} \Lambda_n$ with the following property. For every arrow α from λ to μ in G there is a corresponding arrow α_n in the Bratteli diagram from $\Delta_n(\lambda)$ to $\Delta_{n+1}(\mu)$ for any n such that $\lambda \in \Lambda_n$. These arrows are all distinct and, as λ, μ vary over Λ , they exhaust the arrows in the Bratteli diagram.

Note that $(G, 0)$ is a rooted graph.

If A_\bullet satisfies axioms (N1)-(N9) then we say that A_\bullet is a $(G, 0)$ -tower of algebras.

We shall see later that, for any tower of algebras satisfying (N1)-(N8), such a graph G exists, so that (N9) is automatic for the right choice of rooted graph $(G, 0)$.

There is a notable class of examples of (N1-4) with $N = 2$ given by Jones' Basic Construction [25, §2.4]. (Indeed F_n can be related to the conditional expectation [45], but we will not elaborate here.)

Lemma 4.1. *Suppose we have a tower A_\bullet of algebras satisfying axioms (N1)-(N8) as above. Let $\lambda \in \Lambda_{n-N}$, $\mu \in \Lambda_{n-N+1}$. Suppose that there are $m > 0$ arrows in the Bratteli diagram from $\Delta_{n-N}(\lambda)$ to $\Delta_{n-N+1}(\mu)$. Then there are $m > 0$ arrows in the Bratteli diagram from $\Delta_n(\lambda)$ to $\Delta_{n+1}(\mu)$.*

Proof: By assumption, $F_n(\Delta_n(\lambda))$ appears exactly $m > 0$ times as a quotient in any Δ -filtration of $A_{n-N} \text{res}_{A_{n-N+1}} F_{n+1}(\Delta_{n+1}(\mu))$. Suppose that $A_n \text{res}_{A_{n+1}}(\Delta_{n+1}(\mu))$ has a filtration with quotients $\Delta_n(\nu)$ for $\nu \in X$ for some index set X , each occurring with multiplicity $m_\nu > 0$. Then, since F_n is exact, $F_n A_n \text{res}_{A_{n+1}}(\Delta_{n+1}(\mu))$ has a filtration with quotients $F_n \Delta_n(\nu)$ for $\nu \in X'$, each with multiplicity m_ν , where $X' \subseteq X$ is the subset of those $\nu \in X$ such that $F_n \Delta_n(\nu) \neq 0$. Hence $A_{n-N} \text{res}_{A_{n-N+1}} F_{n+1}(\Delta_{n+1}(\mu))$

has such a filtration (by axiom (N8)). It follows that $F_n \Delta_n(\lambda) \cong F_n \Delta_n(\nu)$ for some $\nu \in X' \subseteq X$, and $m = m_\mu$. Hence $\Delta_n(\lambda) \cong \Delta_n(\nu)$ by axiom (N7). Hence $\mu = \nu$, by axiom (N2). So $\Delta_n(\lambda)$ appears in a filtration of ${}_{A_n} \text{res}_{A_{n+1}} \Delta_{n+1}(\mu)$, with multiplicity m_μ , as required. \square

The next lemma shows that there is a lot of uniformity in the Bratteli diagram of a tower of algebras satisfying (N1)-(N8).

Lemma 4.2. *Suppose that we have a tower A_\bullet of algebras satisfying axioms (N1)-(N8). Suppose that there are $n_1, n_2 \in \mathbb{N}$ such that $\lambda \in \Lambda_{n_1} \cap \Lambda_{n_2}$. Suppose further that there are m arrows from $\Delta_{n_1}(\lambda)$ to $\Delta_{n_1+1}(\mu)$ in the Bratteli diagram. Then there are m arrows from $\Delta_{n_2}(\lambda)$ to $\Delta_{n_2+1}(\mu)$ in the Bratteli diagram.*

Proof: For simplicity, we drop the subscripts from the functors F_n and ${}_{A_{n-1}} \text{res}_{A_n}$. Since $\lambda \in \Lambda_{n_1} \cap \Lambda_{n_2}$, we have that n_1 is congruent to n_2 modulo N by axiom (N3).

Case (I): We first suppose that $n_1 = n_2 + rN$ for some integer $r > 0$, so that $F^r(\Delta_{n_1}(\lambda)) \cong \Delta_{n_2}(\lambda)$. We note that $F^r(\Delta_{n_1+1}(\mu))$ is either zero or isomorphic to $\Delta_{n_2+1}(\mu)$, by axiom (N7). But $\Delta_{n_1}(\lambda)$ appears in a Δ -filtration of $\text{res} \Delta_{n_1+1}(\lambda)$ and $F^r(\Delta_{n_1}(\lambda)) = \Delta_{n_2}(\lambda) \neq 0$, so, since F is exact, $F^r(\Delta_{n_1+1}(\mu)) \cong \Delta_{n_2+1}(\mu)$ is not zero.

Hence $M := F^r \text{res}(\Delta_{n_1+1}(\mu)) \cong \text{res} F^r(\Delta_{n_1+1}(\mu)) = \Delta_{n_2+1}(\mu)$, the isomorphism by axiom (N8). By the exactness of F , this module has $F^r(\Delta_{n_1}(\lambda)) \cong \Delta_{n_2}(\lambda)$ as a quotient in a Δ -filtration; it appears with multiplicity m , so there are m arrows from $\Delta_{n_2}(\lambda)$ to $\Delta_{n_2+1}(\mu)$ in the Bratteli diagram as required.

Case (II): We now suppose that $n_1 = n_2 - rN$ for some $r > 0$, so that $F^r \Delta_{n_2}(\lambda) \cong \Delta_{n_1}(\lambda)$. Since $\mu \in \Lambda_{n_1+1}$, we know that $F^r \Delta_{n_2+1}(\mu) \cong \Delta_{n_1+1}(\mu)$. Furthermore, there are m arrows in the Bratteli diagram from $\Delta_{n_1}(\lambda)$ to $\Delta_{n_1+1}(\mu)$, i.e. from $F^r(\Delta_{n_2}(\lambda))$ to $F^r(\Delta_{n_2+1}(\mu))$, and that both of these are non-zero. Repeated application of Lemma 4.1 gives m arrows in the Bratteli diagram from $\Delta_{n_2}(\lambda)$ to $\Delta_{n_2+1}(\mu)$ as required. \square

Definition 4.3. Let G be the graph with vertices $\Lambda := \cup_{n \in \mathbb{N}} \Lambda_n$. Suppose that $\lambda \in \Lambda_n$ and $\mu \in \Lambda_{n+1}$. Then let G have m arrows from λ to μ , where m is the number of arrows in the Bratteli diagram from $\Delta_n(\lambda)$ to $\Delta_{n+1}(\mu)$ (note that this well-defined by Lemma 4.2). We choose a correspondence $\alpha \rightarrow \alpha_n$ between the arrows from λ to μ in G and the arrows in the Bratteli diagram from $\Delta_n(\lambda)$ to $\Delta_{n+1}(\mu)$ in the Bratteli diagram.

We have the following:

Corollary 4.4. *Suppose that A_\bullet is a tower of algebras satisfying axioms (N1)-(N8). Let G be the graph defined as above. Then A_\bullet is a $(G, 0)$ -tower of algebras.*

Proof: This is immediate from the definition of G . \square

Definition 4.5. Define the array Y_A of sets by setting $Y_A(n; \lambda)$ to be the set of paths in the Bratteli diagram from $\Delta_0(0)$ to $\Delta_n(\lambda)$, for $n \in \mathbb{N}$ and λ a vertex of G such that $\lambda \in \Lambda_n$.

Theorem 4.6. *Let $(G, 0)$ be a rooted graph, A_0, A_1, \dots a $(G, 0)$ -tower of algebras and Y_A the corresponding array of sets defined above. Then the array Y_A is equivalent to $Y_{G,0}$. In other words, Y_A is a Pascal G -array (see Definition 2.4).*

Proof: It is clear that Y_A has the same index sets (Λ and \mathbb{N}) as $Y = Y_{G,0}$. By definition, for $\lambda = 0 \in \Lambda_0$, there are $m > 0$ arrows from 0 to μ in G if and only if there are $m > 0$ arrows from $\Delta_0(0)$ to $\Delta_1(\mu)$ in the Bratteli diagram. It follows that there are pointwise bijections between the sets in layer $n = 1$ of Y_A and $Y_{G,0}$.

Suppose that there is an arrow α from λ to μ in G . Define $(\phi_\alpha)' : A(n; \lambda) \rightarrow A(n+1, \mu)$ composing a path from $\Delta_0(0)$ to $\Delta_n(\lambda)$ in the Bratteli diagram with the arrow α_n in the Bratteli diagram from $\Delta_n(\lambda)$ to $\Delta_{n+1}(\mu)$. Since the resulting path together with the arrow α_n determine the original path to $\Delta_{n-1}(\lambda)$, it is clear that $(\phi_\alpha)'$ is injective.

Any path π in the Bratteli diagram from $\Delta_0(0)$ to $\Delta_n(\mu)$ for some $\mu \in \Lambda_n$ must have visited some module $\Delta_{n-1}(\lambda)$ immediately before finishing at $\Delta_n(\mu)$. Let π' be the path π with the final arrow truncated, finishing at $\Delta_{n-1}(\lambda)$. The last arrow in π , which goes from $\Delta_{n-1}(\lambda)$ to $\Delta_n(\mu)$, must be of the form α_n where α is an arrow from λ to μ in G , and it is clear that $(\phi_\alpha)'(\pi') = \pi$.

Finally, suppose that $(\phi_{\alpha'})'(\pi'') = \pi$ for some path π'' in the Bratteli diagram from $\Delta_0(0)$ to $\Delta_{n-1}(\nu)$ and some arrow α' from ν to μ in G . Then the last arrow in π must be α'_n , so $\alpha'_n = \alpha_n$, so $\alpha = \alpha'$. Then $\pi'' = \pi'$ since $(\phi_\alpha)'$ is injective. The result is proved. \square

Corollary 4.7. *The array of sets Y_A is in pointwise explicit bijection with the array $Y_{G,0}$, i.e. there is a bijection between the set of oriented paths in the Bratteli diagram from $\Delta_0(0)$ to $\Delta_n(\lambda)$ and the set of paths in G from 0 to λ . \square*

We have the following:

Proposition 4.8. *Let $(G, 0)$ be a rooted graph, A_\bullet a $(G, 0)$ -tower of algebras and Y_A the corresponding array of sets defined above. Then there is a one-to-one correspondence between $Y_{G,0}(n; \lambda)$ and a basis for $\Delta_n(\lambda)$.*

Proof: The elements of $Y_{G,0}(n; \lambda)$ are in one-to-one correspondance with $Y_A(n; \lambda)$ by Corollary 4.7, i.e. with paths in the Bratteli diagram from $\Delta_0(0)$ to $\Delta_n(\lambda)$, so we have to show that such paths correspond to a basis for $\Delta_n(\lambda)$. We do this by induction on n ; since $\Delta_0(0)$ is one-dimensional (axiom N5), it holds for $n = 0$, so suppose that it holds for $n - 1$. The result for $\Delta_n(\lambda)$ then follows from the fact that it holds for $n - 1$, and thus for each section in a Δ -composition series for ${}_{A_{n-1}}\text{res}_{A_n}\Delta_n(\lambda)$, noting that each such section appears the same number of times as the number of arrows in the Bratteli diagram from that section to $\Delta_n(\lambda)$. \square

As we shall see, the G -tower axiom set is satisfied by many interesting examples. However it is not claimed that it is unique in meeting the stated objective. In melding the combinatorial array and representation theoretic aspects of the problem there are various possibilities as to how one might impose a suitable structure on the representation theory in a single layer. One could choose to require semisimplicity (which would bring us close to the ‘classical’ towers of [25]); or cellularity [27]; or tabularity [30]; or quasiheredity [15]. Here we have a setup which works (at least) with quasiheredity, as we shall see in Section 4.3. This is simply because the tower structure we need is already partly integrated with quasiheredity, in [12].

4.2. Undirected case.

Recall (2.5) that if G is undirected with vertex v_0 then

$$N_{G,v_0}(2n; v_0) = \sum_{x \in V} N_{G,v_0}(n; x)^2.$$

A path of length $2n$ from v_0 to v_0 can be regarded as two paths of length n from v_0 to some vertex l . Let $(C(n))_{n \in \mathbb{N}}$ be a Catalan (G, v_0) -sequence. Then by definition there is a bijection between $C(n)$ and the set of such pairs of paths, and therefore with $P_G(2n; v_0, v_0)$, and thus with the set $Y(2n; v_0)$ for any Pascal (G, v_0) -array Y : thus the Catalan (G, v_0) -sequence appears in the Pascal array itself.

Suppose that A_\bullet , together with modules $\Delta_n(\mu)$, is a (G, v_0) -tower of algebras. Then for each $\mu \in \Lambda_n$, in the restriction of $\Delta_{2n}(v_0)$ to A_n , the multiplicity of any module $\Delta_n(\mu)$ in a Δ -filtration is equal to the number of paths from v_0 to μ , i.e. $\dim(\Delta_n(\mu))$. That is, the multiplicity of any module $\Delta_n(\mu)$ is equal to its own dimension.

This situation is particularly nice if each A_n is semisimple and the modules $\Delta_n(\mu)$ are the simple A_n -modules. We see that the module $\Delta_{2n}(v_0)$ is isomorphic to the regular module ${}_{A_n}A_n$. Better still is the case in which the modules can be defined freely over some small ground ring, with extensions over fields which include both semisimple and non-semisimple cases (cf. [35]). The semisimple cases then ensure the above combinatoric for the non-semisimple cases. This is exemplified in the case of the tower TL_\bullet (see Sections 3.1,4.4).

Let us consider this in a little more detail. Let R be a ring. Suppose that we have a sequence of finite rank R -algebras $\{A_n^R\}_n$, such that each A_n^R is a free R -module, with basis B_n . For each homomorphism ϕ from R to a field k which makes k an R -algebra we get a k -algebra by $A_n^\phi = k \otimes_R A_n^R$. Suppose further that there is a filtration of the left regular module RB_n of A_n^R by submodules, each of which is a free R -module, and that there is a set of left modules $\Delta^R(\lambda)$ with basis, such that every section in the filtration is isomorphic to some $\Delta^R(\lambda)$; and that these pass to a complete set of simple modules in some semisimple algebra A_n^ϕ . (This might underly a cellularity or a quasiheredity [27, 15] say.) Then the rank of A_n^R is the sum of squares of ranks of the $\Delta^R(\lambda)$ s. Passing to any A_n^ϕ that is not semisimple we have a sum of squares formula like (1.3) for the Δ s, and a distinct formula (1.2) for the simples and projectives giving the same sum. We shall give examples in Section 8.

4.3. Towers of recollement.

We next explain the link between the systems of algebras described above and the towers of recollement considered in [12]. We firstly recall the set-up from [12]. Let k be an algebraically closed field, A a finite dimensional algebra over k and e an idempotent in A . We then have the *localisation* functor $F : A\text{-mod} \rightarrow eAe\text{-mod}$ taking an A -module M to eM , and the *globalisation* functor $G : eAe\text{-mod} \rightarrow A\text{-mod}$ taking an eAe -module N to $Ae \otimes_{eAe} N$. We have:

Theorem 4.9. (*GREEN* [28])

Let $\{L(\lambda), \lambda \in \Lambda\}$ be a full set of simple A -modules, and set $\Lambda^e = \{\lambda \in \Lambda : eL(\lambda) \neq 0\}$. Then $\{eL(\lambda) : \lambda \in \Lambda^e\}$ is a full set of simple eAe -modules. The remaining simple modules $L(\lambda)$ (with $\lambda \in \Lambda \setminus \Lambda^e$) are a full set of simple A/AeA -modules. \square

Suppose we have a family of finite dimensional algebras over k , $\mathcal{A} = A_0, A_1, A_2, \dots$, with idempotents $e_n \in A_n$. Let Λ_n denote the full set of simple A_n modules, for each n , and let Λ^n denote the full set of simple $A_n/A_n e_n A_n$ -modules. In [12] the following axioms are considered.

(A1) For each $n \geq 2$, there is an isomorphism $\Phi_n : A_{n-2} \rightarrow e_n A_n e_n$.

We remark that by Theorem 4.9, we have

$$\Lambda_n = \Lambda^n \sqcup \Lambda_{n-2}.$$

For $m, n \in \mathbb{N}$ with $m - n$ even, we set $\Lambda_m^n = \Lambda^n$ regarded as a subset of Λ_m if $m \geq n$ and $\Lambda_m^n = \emptyset$ otherwise. Setting $e_{n,0} = 1 \in A_n$, for $1 \leq i \leq n/2$ we define new idempotents in A_n by setting $e_{n,i} = \Phi_n(e_{n-2,i-1})$. We also set $A_{n,i} = A_n/(A_n e_{n,i+1} A_n)$.

(A2) (i) The algebra $A_n/A_n e_n A_n$ is semisimple.

(ii) For each $n \geq 0$ and $0 \leq i \leq n/2$, the surjective multiplication map $A_n e_{n,i} \otimes_{e_n A_n e_n} e_{n,i} A_n \rightarrow A_n e_{n,i} A_n$ is bijective.

It is remarked in [12] that axioms (A1) and (A2) are enough to obtain that each A_i is a quasihereditary algebra. Let $\Delta_n(\lambda)$, $\lambda \in \Lambda_n$, denote the standard modules for A_n .

(A3) For each $n \geq 0$, the algebra A_n can be identified with a subalgebra of A_{n+1} .

This enables us to define the restriction functor $\text{res}_n : A_n\text{-mod} \rightarrow A_{n-1}\text{-mod}$ and the induction functor $\text{ind}_n : A_n\text{-mod} \rightarrow A_{n+1}\text{-mod}$ given by $\text{ind}_n(M) = A_{n+1} \otimes_{A_n} M$.

(A4) For all $n \geq 1$, we have that $A_n e_n \cong A_{n-1}$ as a left A_{n-1} -, right A_{n-2} - bimodule.

Here we will also consider the following opposite version of (A4):

(A4') For all $N \geq 1$, we have that $e_n A_n \cong A_{n-1}$ as a right A_{n-1} -, left A_{n-2} -bimodule.

We divide the next axiom into two parts, in order to better relate it to (N1)–(N8):

(A5) (a) For each $\lambda \in \Lambda_n^m$, we have that $\text{res}(\Delta_n(\lambda))$ has a Δ -filtration.

(b) Furthermore,

$$\text{supp}(\text{res}(\Delta_n(\lambda))) \subseteq \Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}.$$

This implies that

$$\text{supp}(\text{ind}(\Delta_n(\lambda))) \subseteq \Lambda_{n+1}^{m-1} \sqcup \Lambda_{n+1}^{m+1}.$$

(A6) For each $\lambda \in \Lambda_n^m$ there exists $\mu \in \Lambda_{n+1}^{m-1}$ such that $\lambda \in \text{supp}(\text{res}(\Delta_{n+1}(\mu)))$.

A tower of algebras of satisfying (A1),(A2),(A3),(A4),(A5) and (A6) is called a *tower of recollement* in [12]. We can now relate $(G, 0)$ -towers of algebras with towers of recollement.

Proposition 4.10. *Suppose that k is an algebraically closed field, and A_0, A_1, A_2, \dots is a sequence of k -algebras satisfying axioms (A1),(A2),(A3),(A4),(A4'), (A5)(a) and (N5). Then A_0, A_1, A_2, \dots is a $(G, 0)$ -tower of algebras for some graph G .*

Proof: By (A3), (N1) holds. By (A1) and (A2) we know that each A_i is quasihereditary, and we can take the standard modules $\Delta_n(\lambda)$ for $\lambda \in \Lambda_n$ for (N2). By the comment after (A1), we see that $\Lambda_{n-2} \subseteq \Lambda_n$ for all $n \geq 2$. It is clear from (A1) and Green's theorem, Theorem 4.9, that, by appropriate choice of the Λ^n , we can ensure that the Λ_n are chosen so that $\Lambda_n \cap \Lambda_m \neq \emptyset$ implies that $n - m$ is even, so (N3) holds. The axiom (N4) follows from axiom (A5)(a). For (N6), we take F_n to be the localisation functor taking an A_n -module M to $e_n M$; note that this is an exact functor. The axiom (N7) follows from axioms (A1) and (A2) (as stated in [12, p3], this follows, for example, from [17, A1]). For (N8), we have:

$$\begin{aligned} F_{n+1} \text{ind}_n M &= e_{n+1}(A_{n+1} \otimes_{A_n} M) \\ &= e_{n+1} A_{n+1} \otimes_{A_n} M \\ &\cong A_n \otimes_{A_n} M = A_{n-1} \text{res}_{A_n} M \end{aligned}$$

as A_{n-1} -modules. If $F_n \Delta_n(\lambda) \neq 0$, then

$$\begin{aligned} F_{n-1} A_{n-1} \text{res}_{A_n} \Delta_n(\lambda) &\cong F_{n-1} A_{n-1} \text{res}_{A_n} G_{n-2} \Delta_{n-2}(\lambda) \\ &\cong F_{n-1} \text{ind}_{n-2} \Delta_{n-2}(\lambda) \\ &\cong A_{n-3} \text{res}_{A_{n-2}} \Delta_{n-2}(\lambda) \\ &\cong A_{n-3} \text{res}_{A_{n-2}} F_n \Delta_n(\lambda), \end{aligned}$$

using the fact that $G_{n-2} \Delta_{n-2}(\lambda) \cong \Delta_n(\lambda)$ which relies on (A1) and (A2), and the fact that

$$A_{n-1} \text{res}_{A_n} G_{n-2}(\Delta_{n-2}(\lambda)) \cong \text{ind}_{n-2}(\Delta_{n-2}(\lambda)),$$

which relies on (A1), (A2), (A3) and (A4) (see [12]). If we define the rooted graph $(G, 0)$ as in Definition 4.3, we are done (see Corollary 4.4). \square

4.4. The Temperley-Lieb algebra.

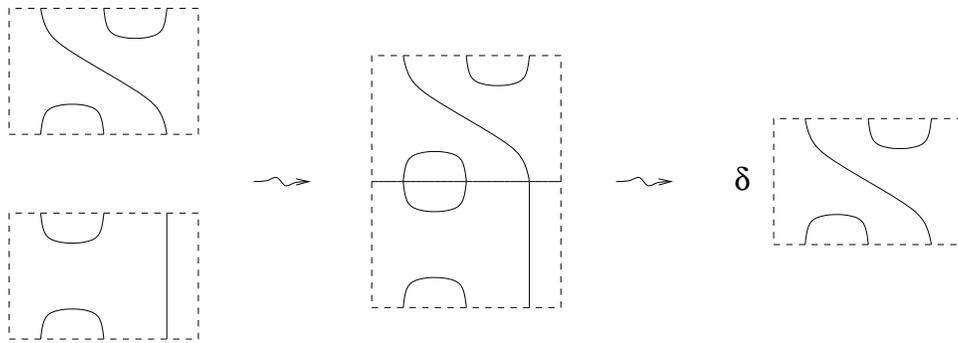
For each $\delta \in k$, the Temperley-Lieb algebras, $TL_n = TL_n(\delta)$, $n = 0, 1, 2, \dots$ form a tower of recollement TL_\bullet (provided $\delta \in k \setminus \{0\}$; see for example [12, 1.2]).

These algebras may be defined as follows (see [42], and cf. [64] for the original definition by presentation). Suppose that $\delta \in k$. The Temperley-Lieb algebra TL_n over k has the set $D(n, n)$ of diagrams as basis. To multiply two basis elements, the diagrams are combined, with the southern edge of the first meeting the northern edge of the second. If this process creates any loops (not connected to a vertex), they are removed and the resulting basis element is multiplied by δ for each one. (We remark that the order in which such loops are removed does not affect the result, which is required for this to be well-defined; see [2]). See Figure 13 for an example.

Note that the number of propagating lines cannot increase in composition. This means that the set $D_l(n, n)$ is a basis for a TL_n -module, where the action is algebra multiplication, modulo diagrams with fewer propagating lines. Now recall that each diagram $D \in D(n, n)$ can be expressed in the bra-ket decomposition (3.1). Write

$$D = |D\rangle \langle D|$$

for this.

FIGURE 13. Example of multiplication in TL_3 .

Lemma 4.11. [42] *If $DD' = \delta^r D''$ and $D'' \in D_l(n, n)$ then $D'' = |D\rangle \langle D'|$ (i.e. it is independent of $|D'\rangle$, $\langle D|$). \square*

It follows that the module $kD_l(n, n)$ above decomposes as a sum of isomorphic left submodules:

$$kD_l(n, n) \cong \bigoplus_{D \in D_l(l, n)} k|D_l(n, l)\rangle \langle D|$$

In particular, in each of these submodules the right-hand part of the component ($|D\rangle$) is irrelevant for the action of the Temperley-Lieb algebra. Thus $D_l(n, l)$ can be considered as a basis for a left TL_n -module. Varying l over the possible numbers of propagating lines, these will turn out below to give the set of standard modules, so we will denote these modules $\Delta_n(l)$.

By Lemma 4.11 there is a well-defined inner product $\langle D||D'\rangle$ on $\Delta_n(l) = kD_l(n, l)$ given by

$$(4.1) \quad \langle D||D'\rangle = \begin{cases} \delta^r & \text{in the case in Lemma 4.11 with } D'' \in D_l(n, n) \\ 0 & \text{if } D'' \notin D_l(n, n) \end{cases}$$

(see [42] for details).

Next we consider the tower structure. For TL_n we have the indexing set

$$\Lambda_n = \{n, n-2, n-4, \dots, 0 \text{ or } 1\}.$$

By [12, 1.2] we know that axioms (A1) to (A5)(a) hold. An argument similar to that for axiom (A4) shows that (A4') holds in this case also, and it is clear that (N5) holds. By Proposition 4.10 it follows that the Temperley-Lieb algebras (with the above set of modules) form a $(G, 0)$ -tower of algebras for some rooted graph $(G, 0)$. Since we have the short exact sequences

$$0 \rightarrow \Delta_{n-1}(l-1) \rightarrow \text{res}\Delta_n(l) \rightarrow \Delta_{n-1}(l+1) \rightarrow 0$$

for $0 \leq l < n$ and $\text{res}\Delta_n(n) \cong \Delta_{n-1}(n-1)$ (see [42, 11] for details), it follows from the definition of G (see Definition 4.3) that $(G, 0) = (A_\infty, 0)$, and hence that the array associated to the tower of Temperley-Lieb algebras (see Definition 4.5), is a Pascal $(A_\infty, 0)$ -array of sets and thus equivalent to the array Y_{TL} considered in Section 3.1. We see that the basis of $\Delta_n(\lambda)$ provided by Proposition 4.8 can be explicitly parametrized by the half diagrams $D_l(n, l)$, recovering the above description of the modules $\Delta_n(l)$.

Remark: if all the inner products (4.1) are non-degenerate (which will depend on δ), the algebra has a semisimple multimatrix structure as in (1.3). This inner product structure is the original reason for the bra-ket nomenclature.

Remark 4.12. Given a rooted directed graph satisfying suitable axioms, [25, p52] describes a tower of algebras with that graph as its Bratteli diagram (due to V. S. Sunder [63] and A. Ocneanu [55]). It follows

from this construction that, given any rooted directed graph $(G, 0)$, there is a tower of algebras whose Bratteli diagram is the directed cover of $(G, 0)$ (see Definition 6.1).

5. A FIRST GENERALISATION: $(A_\infty, 0)$ AND THE BLOB ALGEBRA

A *blob diagram* is a Temperley–Lieb diagram with the additional possibility of decorating any arc exposed to the western end of the diagram with a single blob [47, 49]. We denote the set of blob diagrams with n vertices on the north edge and m on the south edge by $D^b(n, m)'$. For example, let e denote the diagram in $D^b(n, n)'$ with n propagating lines and the westernmost line decorated with a blob. For $l \geq 0$ we write $D_l^b(n, m)'$ for the subset of blob diagrams with l propagating lines, one of which (if $l > 0$) is decorated; and write $D_{-l}^b(n, m)'$ for the subset with l propagating lines, all undecorated.

For $\delta, \delta' \in k$, the sequence of blob algebras [47] $b_\bullet = \{b_n(\delta, \delta')\}_{n=0,1,2,\dots}$ is a sequence of k -algebras defined as follows. A diagram basis of b_n is $D^b(n, n)'$. If the process of composition as in the Temperley–Lieb case creates loops then these are again removed: each undecorated loop is replaced with the scalar δ ; each decorated loop is replaced with the scalar δ' . Any arc with two blobs on it is replaced with the same arc with a single blob. It will be evident that this composition is again associative and unital.

We may retain the bra-ket construction as in (1) from Section 3.1, with the proviso that when starting with a propagating arc with a single blob on it, this is first replaced with two blobs; one each is then distributed to the bra and ket parts of the diagram. The half diagrams obtained from $D^b(n, n)'$ with l propagating lines thus lie in $D_{\pm l}^b(n, l)'$. The constructions in (2) in Section 3.1 go through unchanged.

As in [47], for symmetry we introduce a generator $f = 1 - e$. As a shorthand in the diagram calculus we may represent f as a propagating line carrying a square box. Two boxes on an arc can thus again be replaced by one. Now let $D^b(n, m)$ be a set consisting of the same tangles as $D^b(n, m)'$, except that every arc exposed to the western end of the rectangle carries either a blob or a square. Then another basis of b_n is $D^b(n, n)$. For $l \in \{n, n-2, n-4, \dots, 2-n, -n\}$, we denote by $D_l^b(n, l)$ the set of half diagrams in $D^b(n, l)$ with $|l|$ propagating lines including one propagating line with a blob if $l > 0$, respectively one propagating line with a square if $l < 0$.

The sequence b_\bullet is known to form a tower of recollement [12] for suitable δ, δ' .

In this case the index set for simples of b_n is given by $\Lambda_n = \{n, n-2, n-4, \dots, 2-n, -n\}$. Let $l \in \Lambda_n$. A basis for the standard module $\Delta_n^b(l)$ is given by the set $D^b(n, l)$. See [47, Proposition 2].

Proposition 5.1. *The array of sets $((D_l^b(n, l))_{l \in \Lambda_n})_n$ is a Pascal $(A_\infty, 0)$ -array. (a) If $i > 0$, the edge maps $(\phi_{i, i+1})'$ are given by ϕ^1 and the edge maps $(\phi_{i, i-1})'$ are given by ϕ^u (see Section 3.1 for the definitions).*

(b) If $i = 0$, the edge maps $(\phi_{i, i+1})'$ are given by ϕ^1 together with a blob on the new propagating line, and the edge maps $(\phi_{i, i-1})'$ are given by ϕ^1 together with a square on the new propagating line.

(c) If $i < 0$, the edge maps $(\phi_{i, i+1})'$ are given by ϕ^u and the edge maps $(\phi_{i, i-1})'$ are given by ϕ^1 . This array underlies the basis sequence $(D^b(n, n))_n$.

Proof: It suffices to note that

(a) If $l > 0$ then

$$D_l^b(n, l) = \phi^u(D_{l+1}^b(n-1, l+1)) \bigsqcup \phi^1(D_{l-1}^b(n-1, l-1)).$$

(b) If $l = 0$ then

$$D_l^b(n, 0) = \phi^u(D_{+1}^b(n-1, 1)) \bigsqcup \phi^u(D_{-1}^b(n-1, 1)).$$

(c) If $l < 0$ then

$$D_l^b(n, l) = \phi^u(D_{l-1}^b(n-1, l-1)) \bigsqcup \phi^1(D_{l+1}^b(n-1, l+1)).$$

□

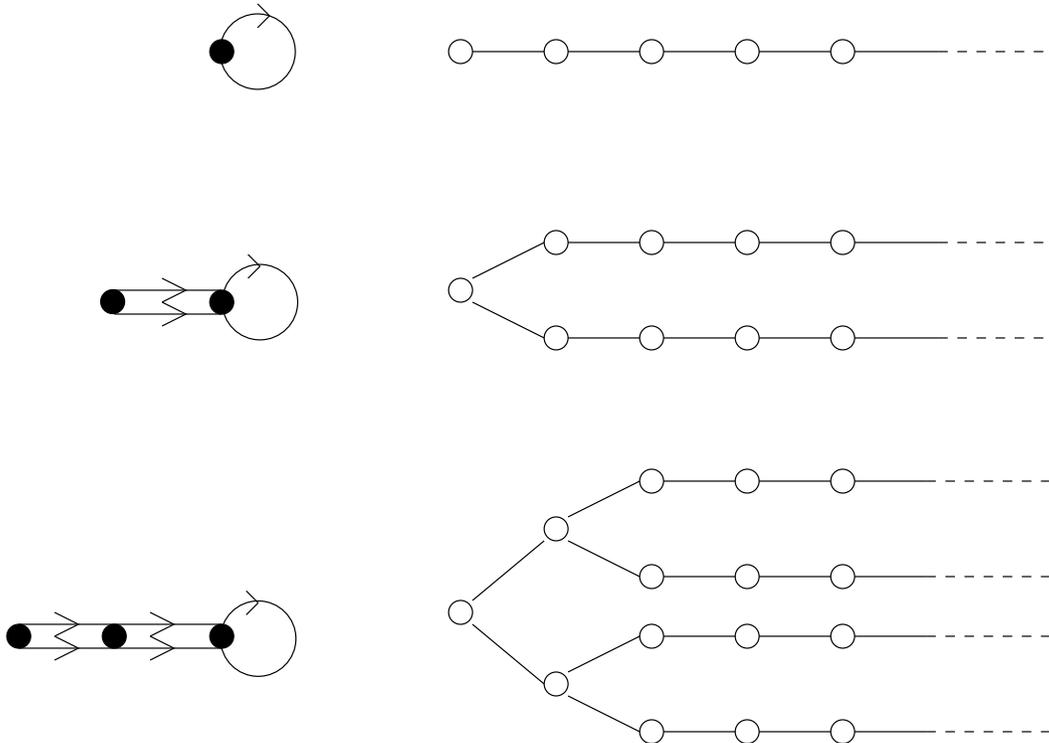


FIGURE 15. Some graphs and the underlying unoriented graphs of their directed covers. The graphs on the right are the rooted graphs $\mathcal{A}(1) = A_\infty$, $\mathcal{A}(2, 1) = A_\infty^\infty$ and $\mathcal{A}(2, 2, 1)$. These are the underlying unoriented graphs of the directed covers of the corresponding graphs on the left.

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is finite and the last entry λ_k is positive or zero then $\Gamma(\lambda)$ is the graph with vertices $0, 1, \dots, k - 1$, and λ_i directed edges from $i - 1$ to i for $i = 1, \dots, k - 1$, and λ_k directed edges from $k - 1$ to $k - 1$. (See the left hand side of Figure 15 for some examples.)

The *rooted λ -ary tree* $\mathcal{A}(\lambda)$ is the underlying unoriented graph of the directed cover $\mathcal{G}(\Gamma(\lambda))$ of $\Gamma(\lambda)$.

Note that $\mathcal{A}(\lambda)$ is a tree with a root and one (undirected) edge from the root to each of λ_1 children; each child has this single edge to the parent, then one (undirected) edge from it to each of λ_2 of its own children; and so on. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is finite and the last entry λ_k in λ is positive then the sequence of multiplicities of children in this construction of $\mathcal{A}(\lambda)$ is obtained by treating λ_k as infinitely repeating (i.e. $\lambda = (1)$ is equivalent to $\lambda = (1, 1, 1, \dots)$). On the other hand, if the last entry in λ is $\lambda_k = 0$ then the graph $\mathcal{A}(\lambda)$ terminates at the k -th generation of children.

Examples: $\mathcal{A}((1)) = A_\infty$, $\mathcal{A}((2, 1)) = A_\infty^\infty$, and $\mathcal{A}((2, 2, 1))$ is the last graph illustrated in Figure 15.

By ordering the children of each vertex, we can label the vertices of $\mathcal{A}(\lambda)$ in the j -th generation (any j) by sequences of the form (d_1, d_2, \dots, d_j) , where $1 \leq d_i \leq \lambda_i$ for all i .

Example 6.3. *The vertex labels for vertices in the j -th generation in $\mathcal{A}(2, 1)$ are of the form $(1, 1, 1, \dots, 1)$ or $(2, 1, 1, \dots, 1)$, corresponding to the two branches.*

Remark: Our definition here was partly motivated by the discussion in [53] of graphs with *finitely many cone types* (introduced in [9]). The graphs $\mathcal{A}(\lambda_1, \lambda_2, \dots, \lambda_k)$ all have finitely many cone types.

We now consider $Y_{(\mathcal{A}(\lambda),0)}$ and various equivalent arrays, in case $\lambda = (d^k, 1)$. The Catalan $(A_\infty, 0)$ -sequences in Sections 3.1, 3.2 and 3.3 have one (or more) generalisation corresponding to each of the graphs generalising $(A_\infty, 0) = (\mathcal{A}((1)), 0)$ here. In particular they all have sequences of algebras associated to them. We give a description of the combinatorial objects below, as well as a brief indication of the corresponding algebras.

6.1. Parentheses.

The set $\mathcal{C}_b(n)$ of properly nested bracket sequences (Section 3.2) can be generalised, for example, by allowing d different kinds of bracket pairs. If the choice for each pair of brackets is free then the set of matched bracket sequences containing $2n$ brackets simply has degree $d^n C(n)$ (choose a basic matched bracket sequence, then choose the type of bracket for each pair).

On the other hand suppose that only the brackets in the outermost layer of nesting can be chosen freely, so that the ‘internal’ brackets must be of the same type as their external brackets. We have, for example, with $d = 2$ and $n = 2$:

$$\{((), ()(), [](), ()[], [][], [[]])\}.$$

Write $\{\mathcal{C}_{b,(2,1)}(n)\}_n$ for the sequence of such sets. As with the blob diagrams from Section 5, the sequence $\{\mathcal{C}_{b,(2,1)}(n)\}_n$ has an underlying Pascal $Y_{(\mathcal{A}(2,1),0)}$ array (i.e. Pascal’s triangle itself). This can be seen by constructing the Pascal array, which we denote $Y_{b,(2,1)}$. Noting Example 6.3, we may more simply label the vertices of $\mathcal{A}(2, 1) (= A_\infty)$ from \mathbb{Z} , so that it has a positive branch and a negative branch (cf. the picture in Figure 15). For $l \in \mathbb{Z}$ then, the set $Y_{b,(2,1)}(n; l)$ is given by the set of sequences of n brackets with $|l|$ more open-brackets than close-brackets with the additional property that once a bracket is opened (from a choice of two types), any subsequent brackets must be of the same type, until it is closed, or the end of the sequence is reached. The sign of l is (say) $+$ if the open brackets are round, and $-$ if they are square.

(1) Decomposition: Evidently separating a sequence into two n -component subsequences gives

$$\mathcal{C}_{b,(2,1)}(n) \rightarrow \cup_l Y_{b,(2,1)}(n; l) \times Y_{b,(2,1)}^o(n; l)$$

(where $Y_{b,(2,1)}^o(n; l)$ is the set of opposite sequences).

(2) Edge maps: For $i \geq 0$, the map $(\phi_{i,i+1})'$ is given by adding a round open-bracket and the map $(\phi_{i+1,i})'$ is given by adding a round close-bracket. For $i \leq 0$, the map $(\phi_{i,i-1})'$ is given by adding a square open-bracket and the map $(\phi_{i-1,i})'$ is given by adding a square close-bracket.

Altogether the array begins as in Figure 16. That is, it is a form of the ordinary Pascal triangle, which is the Pascal array for $(A_\infty, 0)$ folded over (purely to match the way A_∞ is drawn in Figure 15) onto its right-hand side. The corresponding Catalan sequence is of course $1, 2, 6, 20, \dots$

More generally, given $\lambda = (\lambda_1, \lambda_2, \dots)$ as at the start of this section, consider the case where there are λ_1 possible types of brackets for the first layer of nesting, λ_2 for the second, and so on. For $\lambda = (d^k, 1)$ this corresponds to the case in which only the brackets in the outermost k layers of nesting can be chosen freely; beyond that point all the brackets must be of the same type as those in the k th layer of nesting. Write $\mathcal{C}_{b,\lambda}$ for the Catalan sequence, and $Y_{b,\lambda}$ for the Pascal array in this case.

Proposition 6.4. *The Pascal array $Y_{b,\lambda}$ is equivalent to $Y_{(\mathcal{A}(\lambda),0)}$.*

Proof: (1) For a vertex (d_1, d_2, \dots, d_j) of $\mathcal{A}(\lambda)$, the set $Y_{b,\lambda}(n; d_1, d_2, \dots, d_j)$ is given by the set of sequences of n brackets (with the above restriction on types of brackets) with j unmatched open brackets of types d_1, d_2, \dots, d_j in order from left to right.

(2) The edge maps corresponding to moving along an edge away from the root are given by adding an open-bracket corresponding to the branch taken, and the edge map corresponding to moving along an edge back towards the root is given by adding a close-bracket of the same type as the open bracket added on going out. \square

For example, in the case $d = 2$, $\lambda = (2, 2, 1)$, the Pascal array corresponds to the last graph shown explicitly in the list in Figure 15. The array of cardinalities begins as in Figure 17, so that the Catalan numbers are

Remark 6.5. We remark that if we have the rule that after certain sequences of open-brackets which have not been closed, we are not allowed to make the bracket nest any deeper, we obtain a Pascal array corresponding to a truncated version of the λ -ary tree, in which the corresponding branches of the tree have been removed. We will return to this point in Section 8.

Remark 6.6. We note that there is bijection between the matched bracket sequences of length $2n$ in $C_{b,(d,d,d,\dots)}$ discussed at the beginning of this section and words of length $2n$ in the usual generators of the Cuntz algebra which evaluate to 1 [51, §2.2]. The above provides an underlying Pascal array for the sequence of sets of such words. In this case the bra-ket extraction amounts to writing such a word as a concatenation of two words, expressions for each of a pair of elements whose product is the identity.

6.2. Contour algebra diagrams.

A Temperley–Lieb diagram can be thought of as a partition of the rectangular interval of the plane bounded by its frame. A line segment in a Temperley–Lieb diagram is 0 -covered if it forms part of the boundary of the part of the rectangular interval containing the western end of the frame of the diagram. A line segment in a Temperley–Lieb diagram is k -covered if it forms part of the boundary of a part of the rectangular interval also bounded by an $(k-1)$ -covered line and it is not k' -covered for any $k' < k$.

Definition 6.7. A level- k d -colour contour algebra diagram is a Temperley–Lieb diagram in which a line may be decorated (with up to $d-1$ ‘blobs’) if it is k' -covered for some $k' < k$. These diagrams may be constructed analogously to the Temperley–Lieb diagrams (which are the level-0 1-colour case).

Proposition 6.8. *The Pascal array for the sequence of sets $\mathcal{C}_{k,d}(n)$ of level- k d -colour contour diagrams with total $2n$ vertices is equivalent to that of the rooted graph $(\mathcal{A}(d^k, 1), 0)$.*

Proof: A vertex l of $\mathcal{A}(d^k, 1)$ is determined by a sequence (d_1, d_2, \dots, d_j) of integers where $1 \leq d_i \leq d$ for $i \leq k$ and $d_i = 1$ for $i > k$. The d_i determine which branches are taken in a path to l . So for $n \in \mathbb{N}$, we define $Y_{TL,(d^k,1)}(n; d_1, d_2, \dots, d_j)$ to be the set of d -colour level- k half-diagrams with $2n$ vertices, j propagating lines and $d_i - 1$ blobs on the i th propagating line from the left of the diagram for $i = 1, 2, \dots, j$. The edge map for stepping away from the root at a branch point adds a propagating line to the right of the diagram, with the appropriate number of blobs; at a non-branch point it adds an undecorated propagating line. The edge map for stepping towards the root bends over the rightmost propagating line in the diagram.

If a diagram in $Y_{TL,(d^k,1)}(n; d_1, d_2, \dots, d_j)$ has a propagating line on the right, it is clearly in the image of an edge map for the edge from $(n; d_1, d_2, \dots, d_{j-1})$. The only alternative is that its rightmost arc γ is non-propagating, and then it is in the image of the edge map for the edge from $(d_1, d_2, \dots, d_j, d_{j+1})$, where $d_{j+1} - 1$ is the number of blobs decorating γ , noting the fact that γ must be j -covered, so that $d_{j+1} = 1$ if $j \geq k$. Since the edge maps above are clearly injective, we see that $Y_{TL,(d^k,1)}(n; d_1, d_2, \dots, d_j)$ is the disjoint union of the images of the appropriate edge maps of the edges incident with $(n; d_1, d_2, \dots, d_j)$. Thus the sets of diagrams $Y_{TL,(d^k,1)}(n; d_1, d_2, \dots, d_j)$ form a Pascal array of type $(\mathcal{A}(d^k, 1))$ as required.

Finally, we note for the bra-ket construction that cutting a level k d -colour contour algebra diagram into two parts horizontally gives two diagrams in $Y_{TL,(d^k,1)}(n; d_1, d_2, \dots, d_j)$, where $d_1 - 1, d_2 - 1, \dots, d_j - 1$ are the numbers of blobs decorating the propagating lines counting from the western end of the diagram (so that a cut propagating line keeps the same number of blobs on each of the parts). Any two such diagrams can be stitched back together in an obvious way, and it is clear that these two operations are inverse to each other. Thus the set of level k , d -colour contour algebra diagrams on $2n$ vertices is equal to the union of the sets of stitchings of pairs of diagrams from the sets $Y_{TL,(d^k,1)}(n; d_1, d_2, \dots, d_j)$ over all possible vertices (d_1, d_2, \dots, d_j) and we are done. \square

The example corresponding to the rooted graph $(\mathcal{A}(2, 2, 2, 1), 0)$ has Pascal G -array as illustrated in Figure 18 (cf. [47] and Figure 14 for the corresponding figure for $(\mathcal{A}(2, 1), 0)$). We remark that the combinatorics here can also be generalised to arbitrary λ -ary graphs in the same way as the parentheses considered in the last section, by allowing λ_i colours on arcs which are $(i-1)$ -covered.

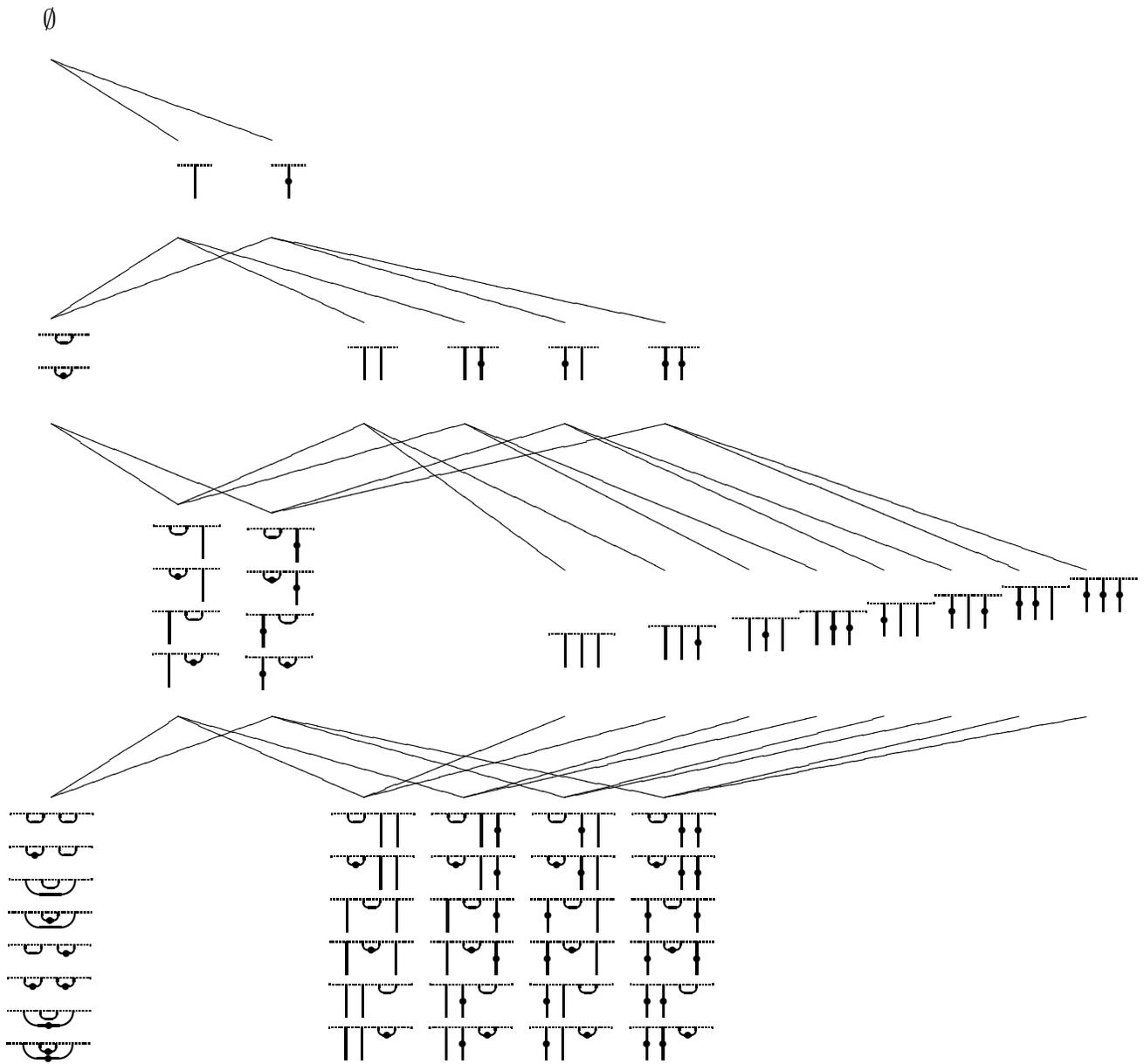


FIGURE 18. Pascal array for level-3 2-colour contour diagrams. The figure has been truncated on the right (in the last row only) and below.

Just as these decorated diagrams generalise the Temperley–Lieb diagrams, so there is a generalisation of diagram composition, giving us a generalised Temperley–Lieb algebra with these diagrams as basis. It will be evident that juxtaposition of these decorated diagrams in a manner analogous to Temperley–Lieb diagram juxtaposition gives another decorated diagram, with the underlying diagram (ignoring blobs) being given as in Section 4.4. The new features are that decorated lines of the composite may have too many blobs, and that loops may also be decorated, as in Section 5. Thus we need rules to reduce such

pseudodiagrams into linear combinations of legal diagrams. In fact contour diagrams can be seen as a basis for contour algebras in a number of different ways, and correspondingly different sets of rules arise. In our case it will be sufficient to proceed as follows. Let $(\delta^0, \delta^1, \dots, \delta^{d-1})$ be a d -tuple of elements of the field k . Then the contour algebra $c_n(\delta^0, \delta^1, \dots, \delta^{d-1})$ arises from one of the following reductions. Either

(a) Cyclotomic reduction: If a line in a pseudodiagram has $d' > d - 1$ blobs on it then reduce modulo d ;
Or

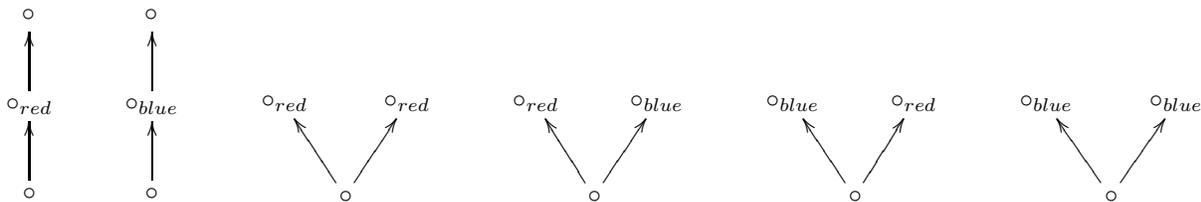
(b) Blob reduction: When a line segment with a run of d' blobs on it gets concatenated with a line with $d'' \geq d'$ blobs on it then discard the run of d' blobs.

Then, if a loop has l blobs on it, replace it with the scalar factor δ^l .

The case $d = 2, k = 1$ with the blob reduction is the blob algebra (briefly reviewed in Section 5). For other cases see [12].

6.3. Forests with special spinneys.

Next we consider rooted planar trees in which some subset of the vertices at distance one from the root (i.e. in layer 1) are designated as a marked subset. Alternatively we can think of this as a generalisation in which the vertices in layer 1 are *coloured*, from a choice of two colours. The consequence is that, for example, there are now six distinct trees with $n = 2$ edges:



For l a vertex of A_∞ , we define $Y_{t,(2,1)}(n; l)$ to be like the set of half-trees in $Y_t(n; |l|)$, generalised to have coloured vertices in layer 1. For $l \neq 0$, the vertex of the trunk in layer 1 is coloured by the sign of l . For $l = 0$, the trunk has only one vertex, so no vertex in layer 1, and there is no restriction on the colouring. The edge maps for moving away from the root are given by adding an edge to the trunk (labelled by the appropriate colour if the added vertex is in layer 1); for moving back towards the root they are given by adding an extra edge to the top edge of the trunk (these two edges then become part of the uppermost branch of the tree). In this way we obtain a Pascal array of type A_∞ — see Figure 19. The rules for splicing the trees together are the same as for the uncoloured trees; a vertex in layer 1 of the trunk inherits the same colouring as the colours of the corresponding vertices in the trunks of the half-trees.

It is clear that this set-up can be generalised to more colours and also to allowing colours in other layers. For a vertex (d_1, d_2, \dots, d_j) of $\mathcal{A}(\lambda)$, we define $Y_{t,\lambda}(d_1, d_2, \dots, d_j)$ to be the set of half-trees from $Y_t(n; j)$ except that the vertices in the first j layers of the trunk have colours d_1, d_2, \dots, d_j moving from bottom to top. The edge maps are given by adding a vertex (and corresponding edge) of the appropriate colour to the trunk (moving away from the root) or doubling the top edge of the trunk (moving towards the root). In this way we obtain a Pascal array of type $\mathcal{A}(\lambda)$, and we see that the set of rooted planar trees with the vertices in layer i coloured by λ_i colours is a Catalan sequence of sets of type $\mathcal{A}(\lambda)$.

Remark: There is a bijection between $Y_{t,\lambda}(d_1, d_2, \dots, d_j)$ and the set $Y_{b,\lambda}(d_1, d_2, \dots, d_j)$ considered in Section 6.1 given by following the boundary tree of a half-tree anticlockwise from the root; the colours correspond to the types of bracket. This induces a bijection on the level of the trees themselves.

An advantage in considering trees is that it helps to determine the generating functions for the corresponding sequences of generalised Catalan numbers (see Appendix A).

6.4. A generalisation to locally-finite rooted trees.

We remark that the combinatorial constructions considered above can be generalised by replacing the trees $\mathcal{A}(\lambda)$ with arbitrary locally-finite rooted trees, i.e. trees in which each vertex has finite valency.

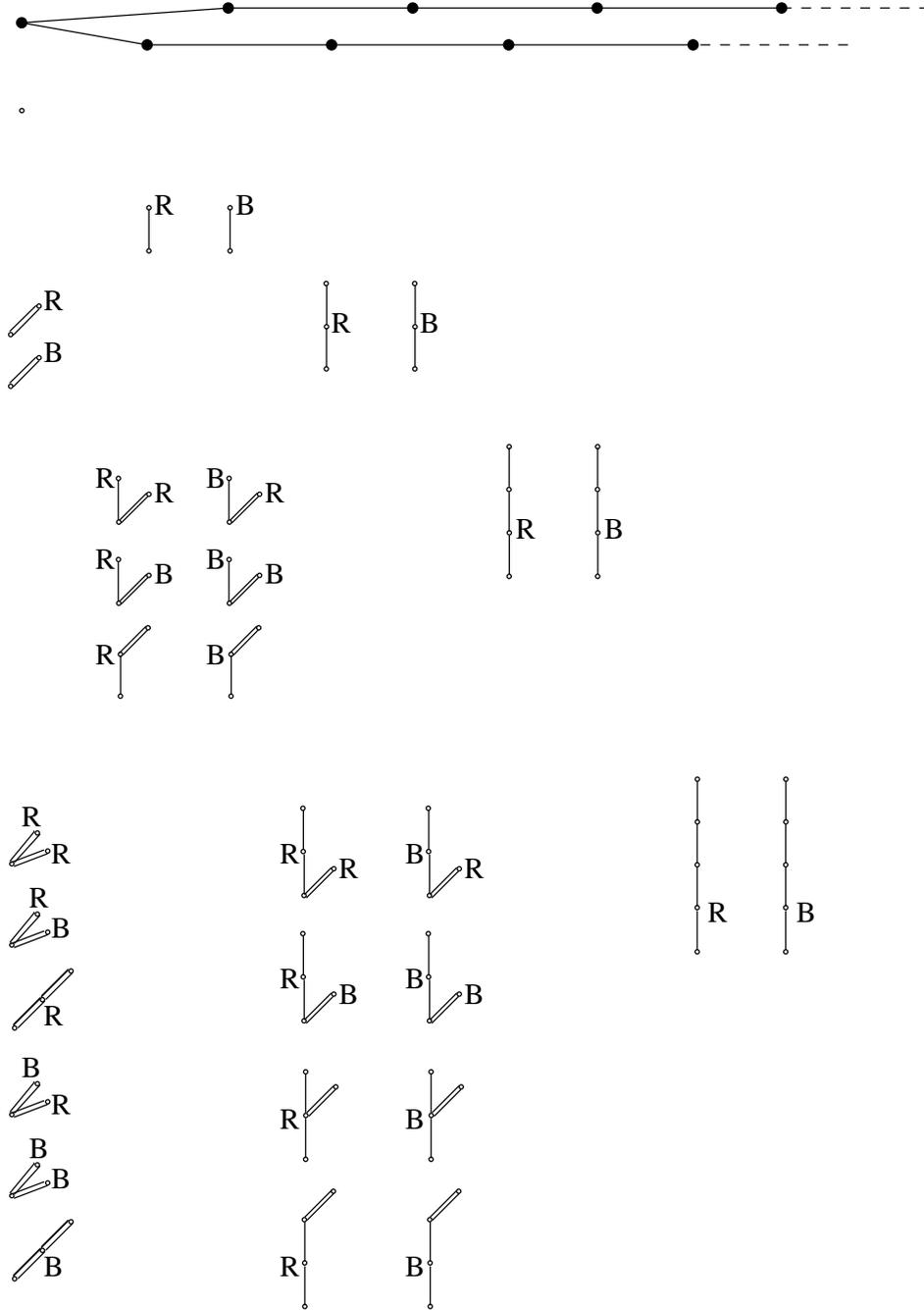


FIGURE 19. The Pascal $(A_{\infty}^{\infty}, 0)$ -array $Y_{t,(2,1)}$ of half-trees with coloured vertices in layer 1.

7. PASCAL ARRAY EXAMPLES: OTHER ROOTED GRAPHS

In our examples to this point each graph G has been a tree. The purpose of this section is to show that the notion of Pascal G -arrays underlying generalised Catalan sequences extends usefully beyond those cases in which G is a tree.

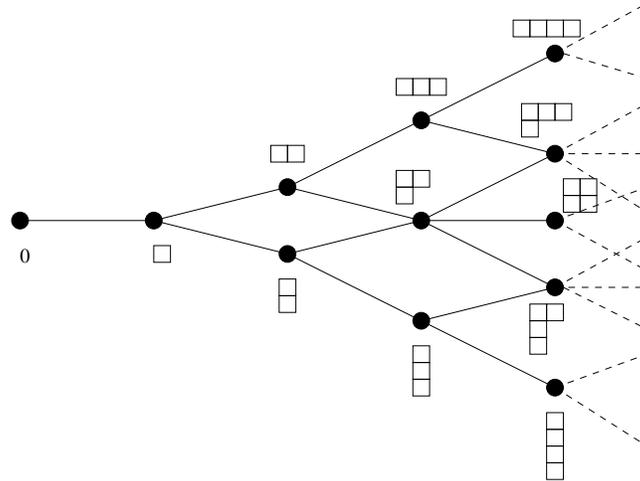


FIGURE 20. The beginning of the Young graph in Young diagram labelling.

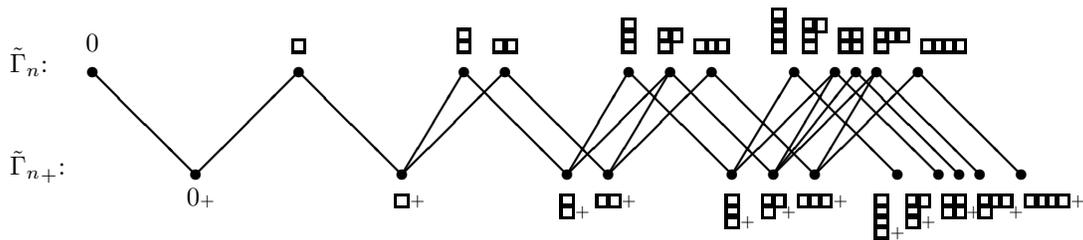


FIGURE 21. The beginning of the double Young graph in Young diagram labelling.

The ultimate guide and source for all our examples here is representation theory. But this is simply so that we can select graphs whose arrays we know in advance to have wider combinatorial interest.

It is appropriate to begin by introducing the graphs that we use.

Definition 7.1. The *Young graph* \mathcal{Y} is defined as follows. The vertex set is the set Γ of integer partitions, that is, the set of integer sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_i \geq \lambda_{i+1}$ for all i and all but finitely many entries zero. There is an edge from λ to λ' if $\lambda - \lambda'$ is of the form $(0, 0, \dots, \pm 1, 0, 0, \dots)$.

See Figure 20 for an illustration, with integer partitions displayed as Young diagrams.

Remarks:

- (1) The Young graph is infinite, undirected, simple and loop-free.
- (2) There is an inclusion of symmetric groups $I : S_n \hookrightarrow S_{n+1}$ given by $I(p)(i) = p(i)$ for $i \leq n$ and $I(p)(n+1) = n+1$. The Young graph is the Bratteli diagram for the sequence of symmetric group algebras $\mathbb{C} \subseteq \mathbb{C}S_1 \subset \mathbb{C}S_2 \subset \dots$ under I , with each $\nu \in \Gamma$ labelling an irreducible $S_{|\nu|}$ -module V_ν in the usual way [35, 65].

Definition 7.2. Let Γ^+ denote another distinct copy of Γ . The *double Young graph* \mathcal{Y}^+ is the simple undirected graph with vertex set $\Gamma \cup \Gamma^+$. For $\lambda, \mu \in \Gamma$, there is an edge between λ and ν^+ if $\lambda - \nu$ is of the form $(0, 0, \dots, 1, 0, 0, \dots)$.

It is clear that the graph \mathcal{Y}^+ is bipartite, with decomposition $\Gamma \cup \Gamma^+$. It is illustrated in Figure 21.

7.1. Bell numbers and the partition algebras.

The Bell number $b(n)$ is the number of ways to partition a set of n elements. The sequence begins:

$$1, 1, 2, 5, 15, \dots$$

For a set S denote by $E(S)$ the set of partitions of S . Define $\underline{n} = \{1, 2, \dots, n\}$ and $\underline{n}' = \{1', 2', \dots, n'\}$. Consider the sequence of sets $\mathcal{C}_p(0), \mathcal{C}_p(0+), \mathcal{C}_p(1), \mathcal{C}_p(1+), \dots$ defined as follows. Each $\mathcal{C}_p(n)$ is the set $E(\underline{n} \cup \underline{n}')$ of partitions of $\underline{n} \cup \underline{n}'$; and $\mathcal{C}_p(n+)$ is the set of partitions of $\underline{n+1} \cup \underline{(n+1)'}$ in which $n+1$ and $(n+1)'$ always lie in the same part (thus $\mathcal{C}_p(n+) \subset \mathcal{C}_p(n+1)$); we will also consider as canonical the inclusion of $\mathcal{C}_p(n)$ in $\mathcal{C}_p(n+)$ by $p \mapsto p \cup \{\{n+1, n+1'\}\}$. This sequence can be considered to be an example of a *Bell sequence*. The question, as before, is whether there is an underlying Pascal array.

Following [46], we shall define an array $\{Y_p(n; \nu)\}_{n; \nu}$ of sets equivalent to $Y_{\mathcal{Y}^+, 0}$. Here ν is of the form λ or $\lambda+$ for an integer partition λ of l where $l \leq n$. Remarkably, it will turn out that the corresponding Catalan sequence is the Bell sequence above.

Definition 7.3. We define a *half-partition* of \underline{n} to be an ordered partition of a partition of this set into two parts. The elements of the second part are known as propagating components.

For $n \in \mathbb{N}$ and an integer partition λ of $l \in \mathbb{N}$, we define $Y_p(2n; \lambda)$ to be the set of pairs (σ, T) where σ is a half-partition of \underline{n} with l propagating components and T is a standard tableau filling of the Young diagram corresponding to λ . We define $Y_p(2n+1; \lambda+)$ to be the set of pairs (σ, T) where σ is a half-partition of $\underline{n+1}$ with $l+1 \leq n+1$ propagating components including a part which has $n+1$ as a member and T is a filling of the Young diagram of λ as in the even case.

(1) Bra-ket decomposition.

Given a partition of $\underline{n} \cup \underline{n}'$ in $\mathcal{C}_p(n)$, we take the set of parts contained entirely within \underline{n} as the first component of a half-partition. For the second part we consider the intersections of the other parts of our original partition with \underline{n} (excluding the empty set). In this way we obtain (say) $l \leq n$ propagating components. We totally order these by the lowest number they contain. Repeating the same procedure with \underline{n}' we obtain a pair of half-partitions. The parts of the original partition match the l (ordered) propagating components of the first half-partition with those of the second half-partition and thus define an element of the symmetric group of degree l . Applying the Robinson-Schensted correspondence we obtain a pair of tableaux of the same shape λ (an integer partition of l) and associate the first to the first half-partition and the second to the second half-partition. In this way we obtain a pair of elements of $Y_p(2n; \lambda)$. (See also [32].)

Given a partition of $\underline{n+1} \cup \underline{(n+1)'}$ in $\mathcal{C}_p(n+)$, we follow the same procedure as above, to obtain a pair of half-partitions with $l+1$ propagating components in $Y_p(2n+1; \lambda)$, where λ is an integer partition of l .

For an example, take the partition

$$\{\{1, 4, 4'\}, \{2, 3, 1'\}, \{5, 6, 2', 5'\}, \{3', 6'\}\}$$

of $\underline{6} \cup \underline{6}'$. We obtain the two half-partitions $(\phi, \{\{1, 4\}, \{2, 3\}, \{5, 6\}\})$ and $(\{\{3, 6\}\}, \{\{1\}, \{2, 5\}, \{4\}\})$, each with 3 propagating components. The permutation in S_3 is given by $1 \rightarrow 3, 2 \rightarrow 1$ and $3 \rightarrow 2$. The half-partitions, together with their corresponding tableaux (from the Robinson-Schensted correspondence applied to this permutation) are shown in Figure 22.

(2) Edge maps.

If λ is an integer partition of $l \leq n$, we define $(\phi_{(2n; \lambda), (2n+1, \lambda+)})'$ to be the map that takes a pair (σ, T) to the half-partition σ with an extra propagating part $\{n+1\}$ and the same tableau T .

Let ν be an integer partition such that $\lambda - \nu$ is of the form $(0, 0, \dots, 1, 0, 0, \dots)$. Before defining $(\phi_{(2n; \lambda), (2n+1, \nu+)})'$ we need some notation. Let V_ν denote the corresponding irreducible S_{l-1} -module. Then ${}_{S_l} \text{Ind}_{S_{l-1}} V_\nu = \mathbb{C} S_l \otimes_{\mathbb{C} S_{l-1}} V_\nu$ has a composition series of irreducible S_{l-1} -modules $V_{\lambda'}$ where λ' varies over the integer partitions such that $\lambda' - \nu$ is of the form $(0, 0, \dots, 1, 0, 0, \dots)$ (thus including λ). It has a basis consisting of the elements $\pi_\kappa \otimes U$ where U is (a basis element corresponding to) a standard tableau filling of the Young

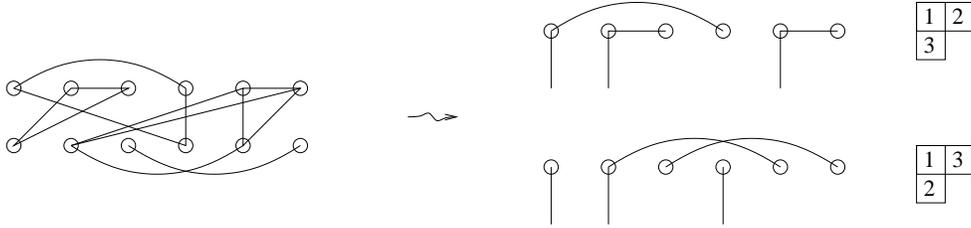


FIGURE 22. Example of bra-ket decomposition for partitions.

diagram corresponding to ν and

$$\pi_k := \begin{pmatrix} 1 & 2 & \cdots & l-1-k & l-k & l+1-k & \cdots & l \\ 1 & 2 & \cdots & l-1-k & l & l-k & \cdots & l-1 \end{pmatrix},$$

for $k = 1, 2, \dots, l$; note that $\pi_1, \pi_2, \dots, \pi_l$ is a system of coset representatives for S_{l-1} as a subgroup of S_l .

We choose a bijection between this basis for ${}_S \text{Ind}_{S_{l-1}} V_\nu$ and a basis compatible with a composition series for this module as S_l -module. Such a bijection, on restriction, gives rise to a bijection between a subset of the set of elements $\pi_k \otimes U$ as above and standard tableau fillings of a Young diagram of shape λ . This defines a map $(\phi_{(2n;\lambda), (2n+1, \nu+)})'$ in the following way. Given a pair (σ, T) in $\mathcal{C}_p(n)$, let $\pi_k \otimes U$ be the corresponding element under this bijection. Then $(\phi_{(2n;\lambda), (2n+1, \nu+)})'$ takes (σ, T) to the half partition σ transformed into a half partition σ' of $\underline{n+1}$ by adding $n+1$ to the k th propagating part of σ , together with the tableau U .

For λ an integer partition of l , we define $(\phi_{(2n+1; \lambda+), (2n+2, \lambda)})'$ to be the map which transforms the propagating component containing $n+1$ into a non-propagating component and leaves T unchanged. Let μ be an integer partition such that $\mu - \lambda$ is of the form $(0, 0, \dots, 1, 0, 0, \dots)$. We define $(\phi_{(2n+1; \lambda+), (2n+2, \mu)})'$ to be the map which leaves σ unchanged and labels the new box in μ with $l+1$.

We have:

Proposition 7.4. *The sets $Y_p(n; \lambda)$ described above form a $(\mathcal{Y}^+, 0)$ -array of sets with corresponding Catalan sequence of sets given by $\mathcal{C}_p(0) \subseteq \mathcal{C}_p(0+) \subseteq \mathcal{C}_p(1) \subseteq \mathcal{C}_p(1+) \subseteq \dots$. \square*

The start of the array is displayed in Figure 23. Note that propagating lines (indicating propagating components) have always been drawn at the right hand end of the part. The sizes of some of the sets in the array are displayed in Figure 24.

As a consequence of the above and Proposition 2.5 we have:

Proposition 7.5. [46] *Let v_0 denote the empty integer partition, a vertex of the double Young graph, \mathcal{Y}^+ . Then the sequence $\{N_{\mathcal{Y}^+}(2n; v_0)\}_n$ is the Bell numbers.*

Just as for $(A_\infty, 0)$, the above array also arises in an algebraic context, which we now describe.

The *partition algebra* P_n has \mathbb{C} -basis $\mathcal{C}_p(n)$. Let $\delta \in \mathbb{C}$. The multiplication is defined as follows (see [44]). For $T \subseteq S$ sets, and $p \in E(S)$, let $p|_T$ denote the *restriction* of p to T , obtained by removing all the elements of $S \setminus T$ from each part of p . For x a relation, let \bar{x} denote its transitive closure.

Let $\underline{n}'' := \{1'', 2'', \dots, n''\}$. For a partition p of $\underline{n} \cup \underline{n}'$, let p' denote the partition of $\underline{n}' \cup \underline{n}''$ obtained by replacing each $i \in \underline{n}$ with i' and each $i' \in \underline{n}'$ with i'' in every part of p . For a partition p of $\underline{n} \cup \underline{n}''$, let p' denote the partition of $\underline{n} \cup \underline{n}'$ obtained by replacing each $i'' \in \underline{n}''$ with i' and leaving each $i \in \underline{n}$ alone.

Let p, q be partitions in $\mathcal{C}_p(n)$, regarded as equivalence relations. The product of p and q in the partition algebra is then defined to be δ^k times $(\overline{p \cup q'}|_{\underline{n} \cup \underline{n}''})'$, where k is the number of parts of the transitive closure $\overline{p \cup q'}$ contained entirely in \underline{n}' . For an example, see Figure 25. The parts contributing to the exponent of δ in the product are indicated by filled-in circles.

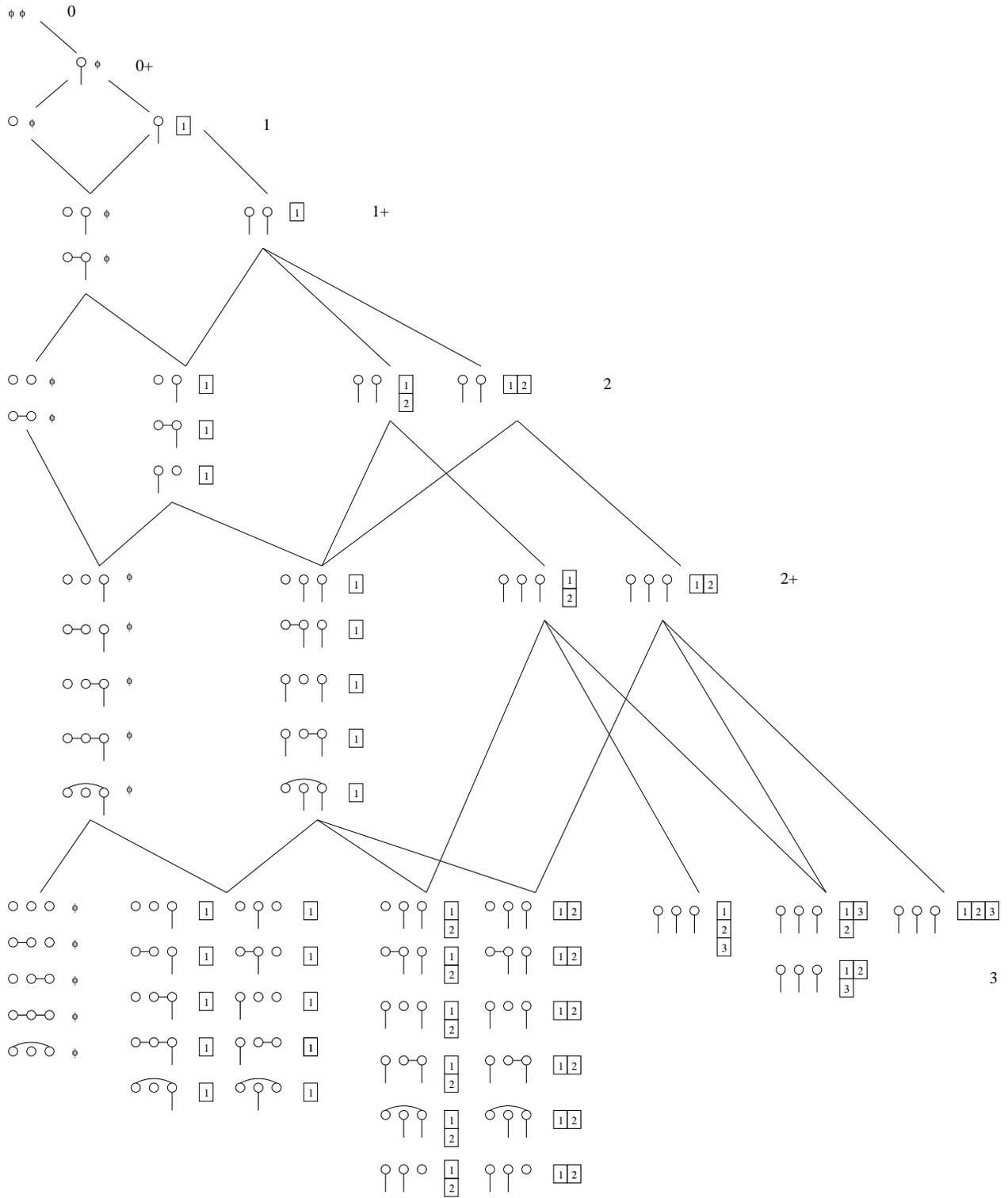


FIGURE 23. The start of the array of half-partitions.

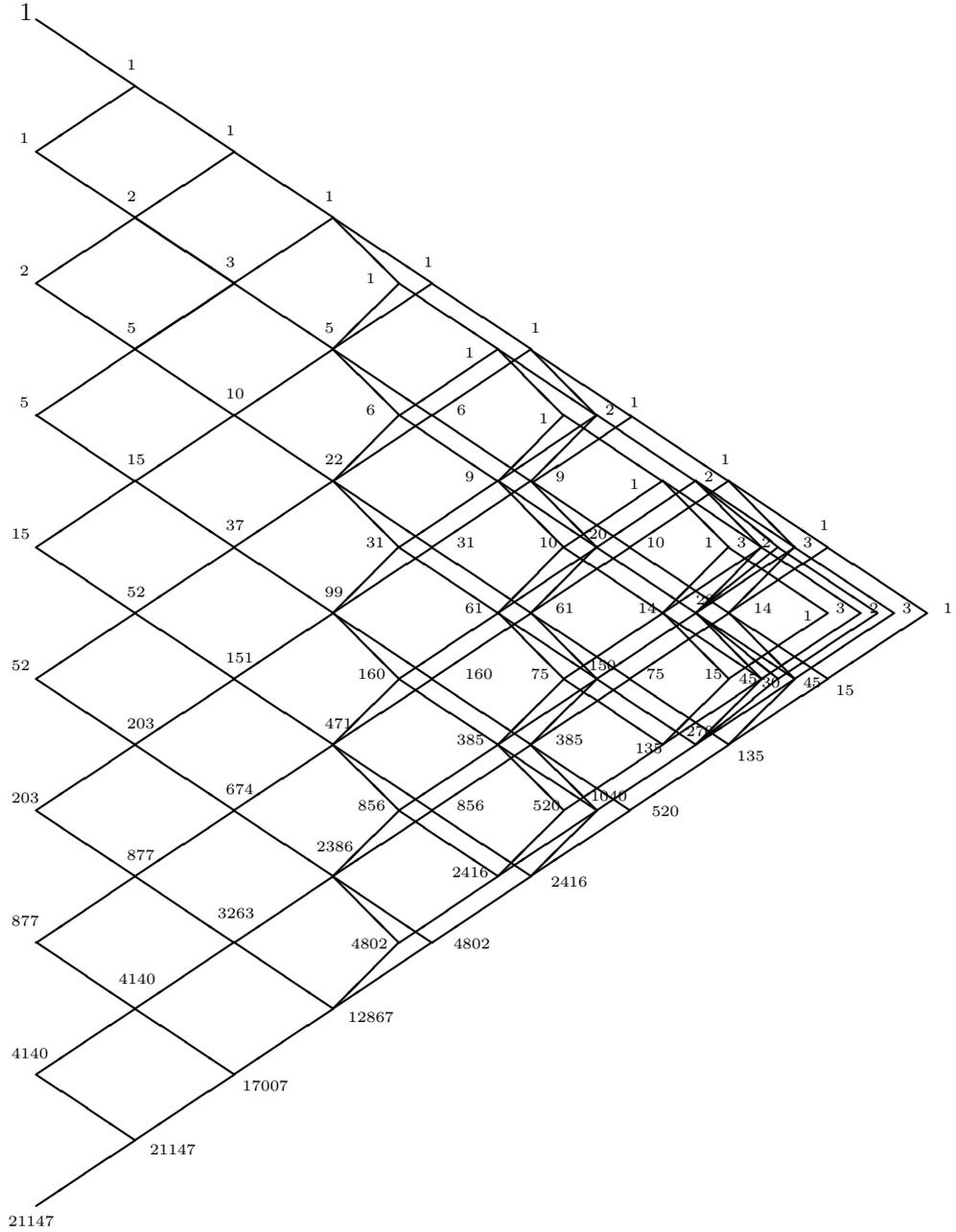


FIGURE 24. Counting walks on the double Young graph. That is, part of the ‘Bratteli’ diagram for $P_0 \subset P_{0+} \subset P_1 \subset P_{1+} \subset P_2 \cdots$ (to be precise, all restrictions of standard module $\mathcal{S}_{(0)}(9)$), in which each \mathcal{S}_λ is represented by its dimension.

There is also a related algebra P_n^+ , defined as the subalgebra of P_{n+1} with basis $\mathcal{C}_p(n+) \subseteq \mathcal{C}_p(n+1)$; see [45]. Following [45], we have the following.

Proposition 7.6. *The tower of algebras $P_0 \subseteq P_0^+ \subseteq P_1 \subseteq P_1^+ \subseteq \cdots$ is a $(\mathcal{Y}^+, 0)$ -tower of algebras (recall that \mathcal{Y}^+ is the double Young graph defined above).*

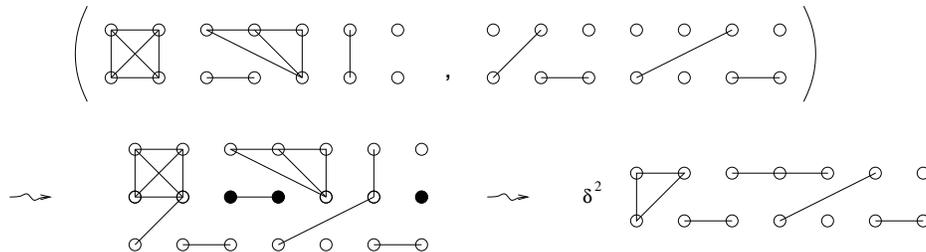


FIGURE 25. Multiplication in the partition algebra.

Proof: We explain why the axioms (N1) to (N9) hold. Axiom (N1) is clear from the definition. For $l = 0, 1, 2, \dots, n$, consider the ideal I_l of P_n generated (and indeed spanned) by partitions with at most l propagating components (i.e. parts containing both unprimed and primed numbers).

For each l , the quotient I_l/I_{l-1} is a left P_n -module with basis given by partitions with exactly l propagating components. By the definition of multiplication in P_n , it is clear that the left P_n action on such basis partitions does not depend on the parts of the partition entirely contained in \underline{n}' , and cannot change such parts. It follows that I_l/I_{l-1} is the direct sum of isomorphic copies of a module $M_n(l)$ indexed by the half-partitions of \underline{n}' with l propagating components. By fixing such a half partition σ we obtain a copy of $M_n(l)$ which has basis given by partitions of $\underline{n} \cup \underline{n}'$ whose ket component contains the half-partition σ . Via the description of the bra-ket decomposition we can see that these correspond to pairs consisting of a half-partition of \underline{n} together with an element of the symmetric group S_l .

We embed S_l into S_n as permutations of the smallest elements of each of the l propagating components of σ . Since S_n is naturally embedded in P_n (with each permutation ρ regarded as a pair partition with parts of the form $\{i, \rho(i)\}$ for $i \in \underline{n}$), we obtain an embedding of S_l into P_n . With this embedding, the right action of P_n on I_l/I_{l-1} restricts to a right action of S_l on $M_n(l)$. On the level of the basis pairs consisting of a half partition and an element of S_l , described above, this is just the regular action of S_l on the second element of the pair, so $M_n(l)$ becomes a free right S_l -module.

Let F be the exact functor from left S_l -modules to left P_n -modules taking an S_l -module V to $M_n(l) \otimes_{S_l} V$. For λ an integer partition of l , we define the module $\Delta_n(\lambda)$ to be $F(V_\lambda) = M_n(l) \otimes_{S_l} V_\lambda$, where, as usual, V_λ denotes the irreducible S_l -module indexed by λ . Then it is clear that a basis for $\Delta_n(\lambda)$ is given by the elements of $Y_p(2n, \lambda)$.

For P_n^+ , we consider, for each l , the left ideal I_l generated by partitions with at most $l + 1$ propagating components. The quotient I_l/I_{l-1} is a P_n^+ -module which can be written as a direct sum of copies of a module $M_{n+}(l+)$ indexed by half-partitions of \underline{n}' with $l + 1$ propagating components including one containing $(n + 1)'$. Each module $M_{n+}(l+)$ has basis given by pairs consisting of a half partition of $\underline{n+1}$ with $l + 1$ propagating components including one which contains $n + 1$, and an element of S_l . It is naturally a right S_l -module (where S_l acts regularly on the second element of the pair). For λ an integer partition of l , we define the module $\Delta_n(\lambda)$ to be the tensor product $M_{n+}(l+) \otimes_{S_l} V_\lambda$. A basis for $\Delta_n(\lambda)$ is given by the elements of $Y_p(2n + 1, \lambda+)$.

Let e_n be the idempotent given by $1/\delta$ times the partition $\{\{1, 1'\}, \{2, 2'\}, \dots, \{n - 1, (n - 1)'\}, \{n\}, \{n'\}\}$, and let F_n be the functor taking a module M to $e_n M$ as in [45]. Acting e_n on the basis elements for $\Delta_n(\lambda)$ gives all basis elements containing a part $\{n\}$ (noting that e_n kills any basis element which has a propagating component whose intersection with \underline{n} is $\{n\}$). These span a subspace isomorphic to $\Delta_{n-1}(\lambda)$ over P_{n-1} . Hence $F_n \Delta_n(\lambda) = \Delta_{n-1}(\lambda)$. If we define e_{n+} to be the idempotent given by $1/\delta$ times the partition $\{\{1, 1'\}, \{2, 2'\}, \dots, \{n - 1, (n - 1)'\}, \{n\}, \{n'\}, \{n + 1, (n + 1)'\}\}$ and $F_{n+} M = e_{n+} M$ we obtain similarly that $F_{n+} \Delta_n(\lambda+) = \Delta_{(n-1)+}(\lambda+)$ if $l \leq n - 1$ and is zero otherwise. It is clear that F_n and F_{n+}

are both exact. In this way, (N3), (N6) and (N7) follow with $N = 2$; the labelling is designed for this to work.

It follows easily by considering the action of elements of P_n with various numbers of propagating components, that modules $\Delta_n(\lambda)$ with λ having different numbers of propagating components cannot be isomorphic. Under the natural embedding of the symmetric group S_n into P_n the module $\Delta_n(\lambda)$ becomes the S_n -module V_λ if λ is a partition of n . It follows that no two modules $\Delta_n(\lambda), \Delta_n(\mu)$ such that $\lambda \neq \mu$ and λ and μ each have n propagating components can be isomorphic. Suppose that λ and μ are distinct and have $k < n$ propagating components and $\Delta_n(\lambda) \cong \Delta_n(\mu)$. Then by applying $F_k F_{k+1} \cdots F_n$, we obtain $\Delta_k(\lambda) \cong \Delta_k(\mu)$, a contradiction to the above. A similar argument applies to the P_n^+ -modules $\Delta_{n+}(\lambda+)$. Thus axiom (N2) holds.

To see axiom (N4), we first consider restricting from P_n to $P_{(n-1)+}$. It is clear that the basis elements of $\Delta_n(\lambda)$ with a propagating component containing $\{n\}$ form a submodule isomorphic over $P_{(n-1)+}$ to $M_{(n-1)+}(l-1) \otimes_{S_{l-1}} V_\lambda$ and thus to a module with a series whose sections are isomorphic to the modules $\Delta_{n+}(\nu)$ where $\lambda - \nu$ is of the form $(0, 0, \dots, 1, 0, 0, \dots)$ (where λ is a partition of l). The remaining basis elements of $\Delta_n(\lambda)$ correspond to a basis for the quotient which is isomorphic to $\Delta_{(n-1)+}(\lambda)$ over $P_{(n-1)+}$.

Next, restricting from P_n^+ to P_n , we see that the basis elements of $\Delta_{n+}(\lambda+)$ with a propagating component of the form $\{n+1\}$ form a submodule (over P_n) isomorphic to $\Delta_n(\lambda)$. The remaining basis elements of $\Delta_{n+}(\lambda+)$ form a basis for the quotient, isomorphic to $M_n(l+) \otimes_{\mathbb{C}S_{l+1} \otimes_{\mathbb{C}S_l}} V_\lambda$. Such a basis element (σ, T) has a non-trivial part containing $\{n+1\}$ in its half-partition and corresponds to $(\sigma, \pi_k \otimes T)$ where σ is now interpreted as a P_n half-partition by removing $n+1$ and k is 1 plus the number of propagating components containing parts between $n+1$ and the largest element not equal to $n+1$ of the part containing $n+1$. Thus the quotient is isomorphic to a module with a series with sections isomorphic to the modules $\Delta_n(\mu)$ where $\mu - \lambda$ is of the form $(0, 0, \dots, 1, \dots, 0, 0)$. Axiom (N4) is shown, and (N9) also, with corresponding rooted graph $(\mathcal{Y}^+, 0)$, the double Young graph.

Axiom (N5) is trivial from the definitions, and axiom (N8) follows from a standard diagram algebra argument. \square

7.2. Pair partitions.

A subset of the set of partitions of \underline{n} is the set $j(n)$ of partitions into pairs, known as *pair partitions*. Obviously this set is empty unless n is even, and we have $|j(2m)| = \frac{(2m)!}{2^m m!} = (2m-1)(2m-3)\dots(1)$. Let $\mathcal{C}_{br}(n)$ denote the set of pair partitions of $\underline{n} \cup \underline{n}'$. Define a half pair partition of \underline{n} to be a partition of \underline{n} into pairs and singletons (known as propagating components). For λ an integer partition of l where $n-l$ is even, let $Y_{br}(n; \lambda)$ denote the set of pairs (σ, T) where σ is a half pair partition of \underline{n} with l singletons and T is a standard tableau filling of the Young diagram corresponding to λ .

(1) Bra-ket decomposition.

Given a partition of $\underline{n} \cup \underline{n}'$ we take the set of parts contained entirely within \underline{n} as the pairs of a half pair partition; the remaining elements are designated as singletons. This gives a half pair partition; the same procedure is followed for the parts contained in \underline{n}' and in this way we obtain a pair of half pair partitions. The correspondence between the propagating components of the two partitions determines an element of the symmetric group S_l and thus, via the Robinson-Schensted correspondence, a pair of standard tableaux of the same shape. The first is associated with the first half pair partition, and the second with the second half pair partition. In this way we obtain a pair of elements of $Y_{br}(n; \lambda)$ for some integer partition λ .

(2) Edge maps.

If λ is an integer partition of $l \leq n$, let ν be an integer partition such that $\mu - \lambda$ is of the form $(0, 0, \dots, 1, 0, 0, \dots)$. We define $(\phi_{(n;\lambda), (n+1, \mu)})'$ to be the map that takes a pair (σ, T) to the half partition with an extra propagating component $\{n+1\}$ and the tableau of shape μ obtained from T by adding a new box labelled $l+1$. Let ν be an integer partition such that $\lambda - \nu$ is of the form $(0, 0, \dots, 1, 0, 0, \dots)$. The map $(\phi_{(n;\lambda), (n+1, \nu)})'$ is defined entirely analogously to the map $(\phi_{(2n;\lambda), (2n+1, \nu+)})'$ for the partition algebra case considered above.

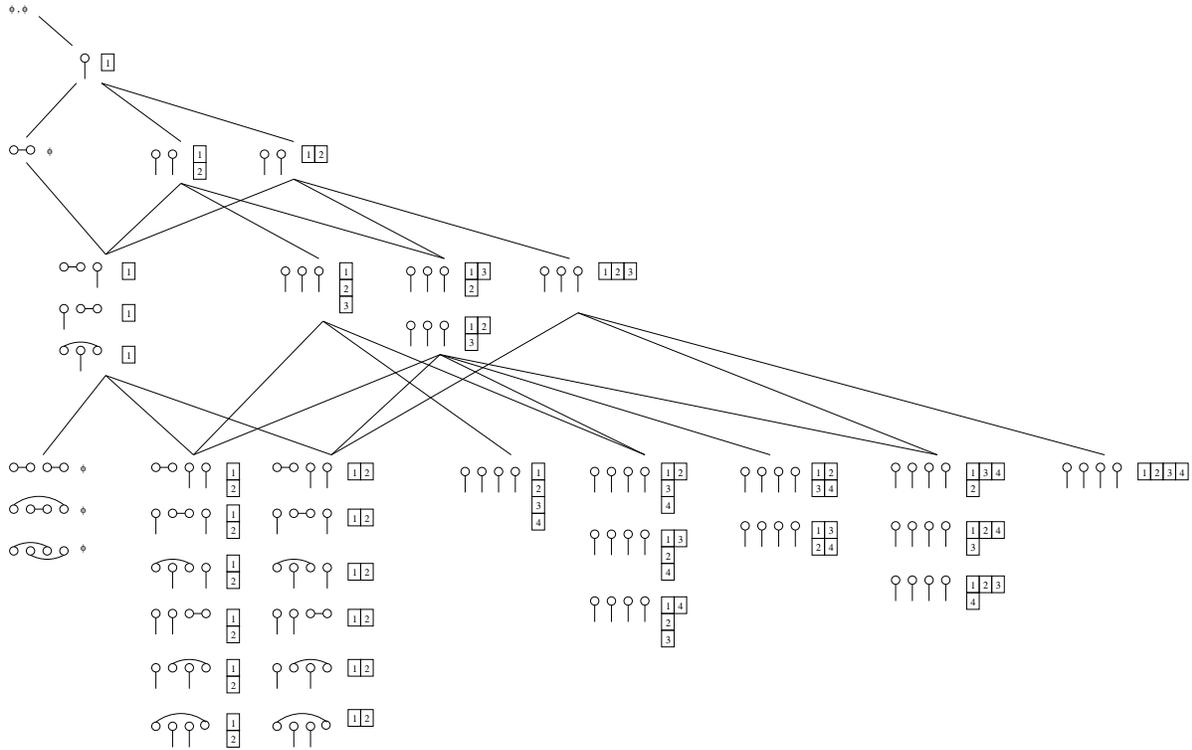


FIGURE 26. The start of the array of half pair partitions.

We have:

Proposition 7.7. *The sets $Y_{br}(n; \lambda)$ described above form a $(\mathcal{Y}, 0)$ -array of sets with corresponding Catalan sequence of sets given by $\mathcal{C}_{br}(n)$. \square*

The start of the array is displayed in Figure 26. The sizes of some of the sets in the array are displayed in Figure 27. As a consequence of the above and Proposition 2.5 we have:

Proposition 7.8. *Let \mathcal{Y} be the Young graph with distinguished vertex 0 given by the empty integer partition. Then the sequence $\{N_{\mathcal{Y}}(2n; 0)\}_n$ is the sequence $1, 1, 3, 15, 105, \dots, (2m - 1)(2m - 3)\dots(1), \dots$*

Proof: This is analogous to Proposition 7.5 (cf. [46]), using the Brauer algebra instead of the partition algebra. \square

We remark that the above array also arises in an algebraic context. The Brauer algebra $J_n(\delta)$ has \mathbb{C} -basis $\mathcal{C}_{br}(n)$ with multiplication defined as for P_n (see also [66]). Each basis element is usually envisaged as a partition of an arrangement of two rows of m vertices into pairs. The following is implicit in [7] (see also [13]). The proof is analogous to that for the partition algebra case considered above.

Proposition 7.9. *The tower of algebras $J_0(\delta) \subseteq J_1(\delta) \subseteq \dots$ is a $(\mathcal{Y}, 0)$ -tower of algebras, where \mathcal{Y} is the Young graph with distinguished vertex 0 given by the empty partition. \square*

Remarks:

Both the Brauer and partition algebras have interesting non-semisimple specialisations, associated to their roles in invariant theory. We briefly review this setting, and consider the combinatorics of the non-semisimple cases, in Section 8.

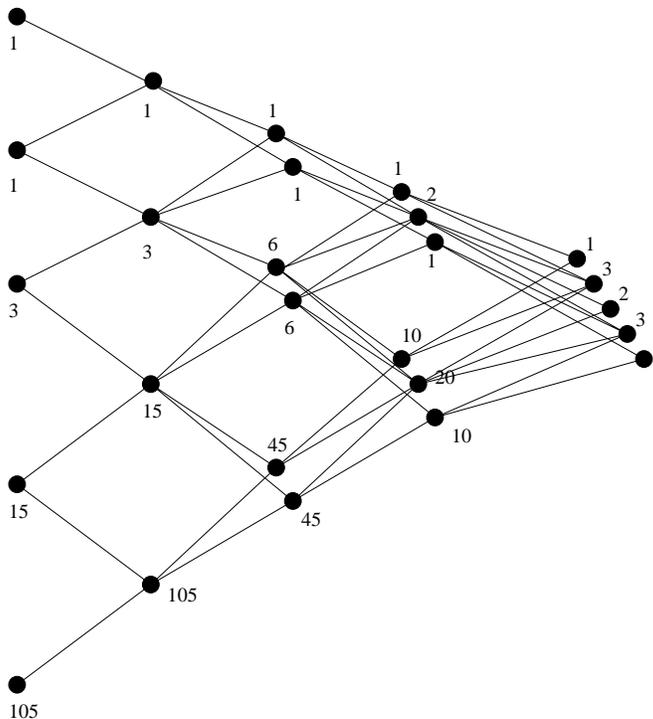


FIGURE 27. Counting walks on the Young graph. That is, part of the ‘Bratteli’ diagram for the Brauer algebras $J_0(\delta) \subset J_1(\delta) \subset J_2(\delta) \subset J_3(\delta) \subset J_4(\delta) \cdots$ (to be precise, all restrictions of the standard module $\mathcal{S}_{(0)}(4)$), in which each \mathcal{S}_λ is represented by its dimension.

7.3. Weight lattice graphs.

Here we briefly touch on cases in which N (from Section 4.1) is different from 2 (and the graph G is directed). The objective here is only to demonstrate that examples exist. We reserve a detailed exposition for a later work.

Our $(A_\infty, 0)$ graph may be considered as the dominant part of the $\Delta = A_1$ weight lattice [5], with the root at the boundary of the dominant region, in the following sense.

For $N \in \mathbb{N}$, let $P(A_{N-1})$ denote the integral weight lattice of type A_{N-1} , with the dominant integral weights denoted by $P^+(A_{N-1})$. Let e_1, e_2, \dots, e_N be the standard basis for \mathbb{R}^N , and let V be the subspace consisting of elements whose coordinates in this basis sum to zero. Then $P_{A_{N-1}}$ can be realised in V as the integral linear combinations of the fundamental weights ω_i , where $\omega_i := e_1 + e_2 + \dots + e_i$, for $i = 1, 2, \dots, N - 1$. The dominant weights are the nonnegative integral combinations of the ω_i . Note also that the dominant weights are in one-to-one correspondance with the N -row Young diagrams, which we denote by Λ^N .

Denote by $\mathcal{L}(A_{N-1})$ the graph with vertices given by $P(A_{N-1})$ and an arrow from λ to μ whenever $\lambda - \mu = e_i$ for some i . Let $\mathcal{L}^+(A_{N-1})$ denote the induced subgraph on the vertices $P^+(A_{N-1})$. Each graph has distinguished vertex given by the zero weight.

In the $\Delta = A_1$ case this simply says that each adjacent pair of vertices has an edge between them in each direction (i.e. the graph is undirected), and we thus see that $(\mathcal{L}(A_1), 0)$ is isomorphic to $(A_\infty, 0)$ and $(\mathcal{L}^+(A_1), 0)$ is isomorphic to $(A_\infty, 0)$. The case A_{N-1} is N -partite, rather than bipartite as all our previous examples are. We shall return to this point shortly.

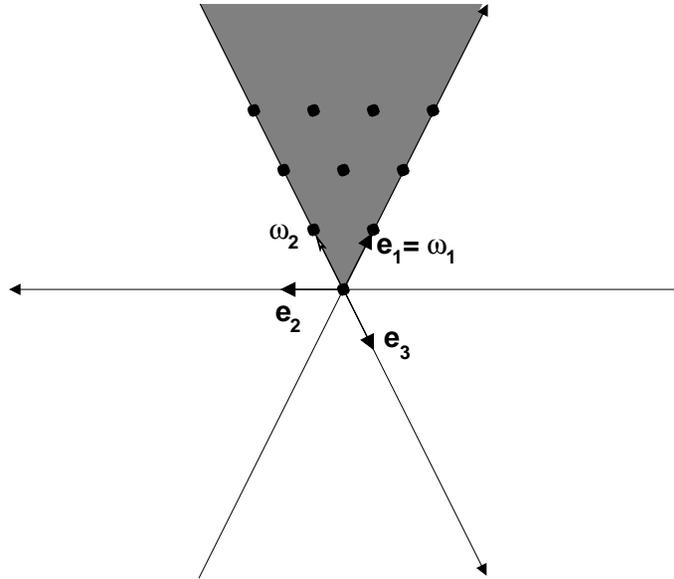


FIGURE 28. The A_2 dominant integral weight lattice.

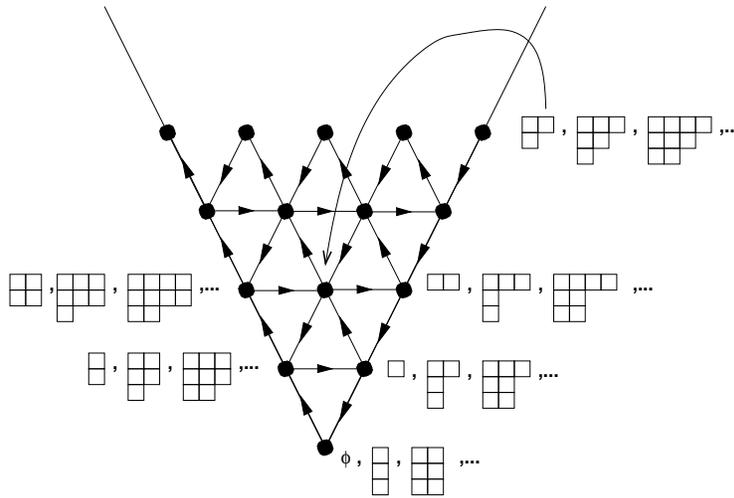


FIGURE 29. The graph $\mathcal{L}^+(A_2)$.

These cases have an extensive literature (see [50] for examples).

The case for the $\Delta = A_2$ weight lattice is illustrated in Figures 28 and 29. In this case, it will be convenient to label the vertices as linear combinations of fundamental weights, so that $d_1\omega_1 + d_2\omega_2$ is denoted (d_1, d_2) . Thus the root is $(0, 0)$; the unique vertex one step from the root is $(0, 1)$; the two vertices one step on are $(1, 0)$ and $(0, 2)$. In the third layer of the Pascal array there are sets labelled by the vertices $(0, 0)$, $(1, 1)$ and $(0, 3)$. The set $Y_{\mathcal{L}^+(A_2)}(3; (1, 1))$ has two paths in it, and the others considered above have one. (Since the layers of the Pascal array are now effectively two-dimensional we will not attempt to draw it.)

One way to put this in an algebraic context is to consider the action of the group algebra of the symmetric group S_n on the tensor space $(V_N)^{\otimes n}$, where V_N is the k -vector space of dimension N . Let us say for definiteness that $k = \mathbb{C}$, and we consider $N = 3$. The former condition says that irreducible representations

$L_\lambda(n)$ of S_n may be indexed by the set Λ_n of integer partitions of n , or equivalently by Young diagrams with n boxes. For $n < N + 1$ the action of S_n on $(V_N)^{\otimes n}$ is faithful, and otherwise the annihilator of this action is generated by the S_{N+1} antisymmetriser $e_{(1^{N+1})} \in S_{N+1} \subseteq S_n$. This means that only irreducible representations indexed by the set Λ_n^N of Young diagrams with fewer than $N + 1$ rows (and exactly n boxes) appear in the decomposition of tensor space. The dimensions of these representations are given by the hook law (which counts the number of standard Young tableaux — see e.g. [33]). The point here is that they are also given, in case $N = 3$, by

$$\dim L_\lambda(n) = |Y_{\mathcal{L}^+(A_2)}(n; (\lambda_1^f - \lambda_2^f, \lambda_2^f))|$$

where $\lambda_i^f = \lambda_i - \lambda_3$. For example, our $n = 3$ layer as given above shows that $\dim L_{(2,1,0)}(3) = 2$. To see this let H_n^N denote the quotient of kS_n by its annihilator on tensor space. Then there is an isomorphism $H_n^N \cong e_{(1^N)} H_{n+N}^N e_{(1^N)}$ [42]. This enables us to define a functor

$$\begin{aligned} F : H_{n+N}^N\text{-mod} &\rightarrow H_n^N\text{-mod} \\ M &\mapsto e_{(1^N)} M \end{aligned}$$

(where the action on $e_{(1^N)} M$ is via the isomorphism). This takes the simple module with label $(\lambda_1, \lambda_2, \dots, \lambda_N) \in \Lambda_n^N$ to one with label $(\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_N - 1)$ (or to zero if $\lambda_N = 0$). This says that the sequence of algebras $(H_n^N)_n$ has a global limit with irreducible modules indexed by the set Λ^{N-1} of $N - 1$ -row Young diagrams — which are in bijection with the set $P^+(A_{N-1})$ of dominant integral A_{N-1} -weights. This brings us to an example of the setup in Section 4.1 with N different from 2. Let us write λ^f for the image of $\lambda \in \Lambda_n^N$ in Λ^{N-1} (i.e. $\lambda_i^f = \lambda_i - \lambda_N$). Thus for each $\mu \in \Lambda^{N-1}$ such that $|\mu| \leq n$ and $|\mu| \equiv n$ modulo N we have an irreducible representation $L[\mu](n)$ of H_n^N given by $L[\lambda^f](n) = L_\lambda$. In Figure 29, we have labelled each $\mu \in \Lambda^{N-1}$ with the set of tableaux corresponding to all λ arising in this way. We have (see, for example, [24, 7.3, Cor. 3])

$$\text{Res}_{S_{n-1}} L[\mu](n) = \bigoplus_{\nu} L[\nu](n-1)$$

where the sum is over weights which have an edge incoming from μ in $\mathcal{L}^+(A_{N-1})$. Now cf. Section 4.1.

As with our previous examples there are a number of other combinatorial sets related to sequences, Young diagrams and so on, which are also counted in this Pascal array $Y_{\mathcal{L}^+(A_2)}$. Sadly a diagram algebra style basis for our quotient algebra, analogous to the Temperley–Lieb diagrams, is *not* known however. (We will return to this in another work.)

Remark 7.10. In the manner of this section we may also embed the Young graph into $\mathcal{L}^+(A_\infty)$, and hence give a conventional geometric setting for the index set for standard P_n -modules [48].

8. TRUNCATION, ALTERNATIVE ROOTS, AND NON-SEMISIMPLICITY

The Stirling numbers of the second kind count set partitions with certain extra properties [39, 46]. It is interesting both from the combinatorial and representation theoretic point of view to ask if walks with special properties on the double Young graph can be put in correspondence with these special partitions, as a refinement of the Bell sequence analysis above. The answer is that the appropriate walks are those which are restricted to a certain subgraph — a *truncation*. This truncation then also has a role in the representation theory of the partition algebra for non-semisimple values of the parameters. This case was first treated in [46]. Here we begin with a much simpler example of such a truncation, which also illustrates the combinatorial and representation theoretic effects of ‘moving’ the root of the graph.

8.1. Temperley–Lieb and blob cases.

In this section we reparameterise $TL_n(\delta)$ by $\delta = q + q^{-1}$. When q is a (primitive l -th) root of unity the tower $(TL_n(q + q^{-1}))_{n \in \mathbb{N}}$ from Section 4.4 is not semisimple, and the standard modules are not all simple. It is interesting to ask what variant of our combinatoric can describe the simple modules in these cases. Here we recall the case over the complex field. In this case the only parameter is l . A convenient

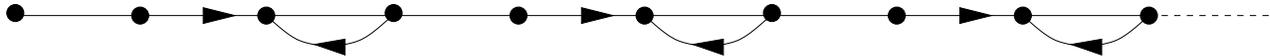


FIGURE 30. Rollet for $l = 3$. Undirected edges have arrows in both directions.

geometrical starting point is to embed $(A_\infty, 0)$ in $(A_\infty^\infty, 0)$ in the obvious way. One then views TL_\bullet as a quotient of the blob algebra b_\bullet (Section 5) by the relation

$$e = 0$$

(For such a non-zero quotient we need $\delta' = 0$). Only one standard module in each layer respects this quotient, but the consistency requirement $\delta' = 0$ is not a semisimple specialisation, and, appropriately arranged, the simple modules corresponding to the vertices of $(A_\infty, 0) \hookrightarrow (A_\infty^\infty, 0)$ do respect the quotient.

Proposition 8.1. [49] *Fix $\delta' = 0$. Walks on $(A_\infty^\infty, 0)$ from 0 to λ of length n which never touch vertex -1 are a basis for the δ -generically simple module $L^{b_n}(\lambda) = \text{Head}(\Delta^{b_n}(\lambda))$. \square*

Of course this restriction on walks precisely corresponds to the embedding of $(A_\infty, 0)$ in $(A_\infty^\infty, 0)$ and we have that these modules are the simple modules of the $TL_\bullet(q + q^{-1})$ quotient. Indeed these bases make sense for any q , and give the standard modules of $TL_\bullet(q + q^{-1})$.

Next we consider $l \in \mathbb{N}$.

Proposition 8.2. [42] *Let q be a primitive l -th root of unity, and let $\lambda \in \mathbb{Z}$. Let $m \in \mathbb{Z}$ be minimal such that $\lambda < ml - 1$. Walks on $(A_\infty^\infty, 0)$ from 0 to $\lambda \geq 0$ of length n which never touch vertex -1 and never touch vertex $ml - 1$ unless they subsequently touch $(m - 1)l - 1$ are a basis for the simple module $L^{TL_n}(\lambda)$ (note that this last restriction is vacuous if $\lambda = ml - 1$ for some m). \square*

Example: Suppose $\lambda < l - 1$. Since the walks in the proposition never touch -1 they may never touch $l - 1$, so we have the set of walks on the truncated graph $(A_l, 0) \hookrightarrow (A_\infty, 0)$.

Another natural question at this point, then, is: Is there a Rollet graph *unrestricted* walks on which encode the restriction of simple modules over the Temperley-Lieb algebra? To answer this, note the following alternative statement of Prop 8.2: For $r = -1, -2, \dots, l - 2$,

$$\text{Res}L(ml + r) = \begin{cases} L(ml + r + 1) + L(ml + r - 1) & r \neq -1, l - 2 \\ L(ml + r - 1) & r = l - 2 \\ 2L(ml + r + 1) + L(ml + r - 1) & r = -1 \end{cases}$$

(where $L(s)$ is regarded as zero for negative s). This gives us the Rollet graph in Figure 30.

This in turn raises another natural question. The new Rollet graph is similar to that for standard modules — i.e. it is obtained from A_∞ by borrowing one edge and giving it to the next pair of vertices, at three step intervals. We have equation (1.2), showing that there is a kind of combinatorial duality between the sets of simple and projective modules. So the question is: Can we construct a ‘dual’ graph, walks on which give bases for projectives? This is a question of a different nature, since unsurprisingly

Proposition 8.3. *Not every Temperley–Lieb projective module restricts to a module with a projective filtration. \square*

(And induction of Temperley–Lieb modules is a bit of an art.) It is possible to give projective decompositions of restricted projective modules up to some ‘boundary effects’ (rather like Temperley–Lieb tilting modules [16, 43]), that is, to adding in one or two not-quite projective modules in finitely many cases. In the interests of brevity we will return to this interesting question in a separate paper.

Having given some examples of truncation and non-semisimplicity, now now turn to the algebra of shifted roots.

Set $[n] = q^{n-1} + q^{n-3} + \dots + q^{1-n}$. Returning to the blob algebra, with $\delta = [2]$ as above it is convenient to reparameterise $\delta' = [m+1]/[m]$. The blob algebra b_n is then semisimple (over \mathbb{C}) unless m is integral (see [47, 49]). Note that $\delta' = 0$ is the case $m = -1$. For each b_n note that there are two standard modules of dimension one, that is $\Delta^b(n)$ and $\Delta^b(-n)$. Since the algebras are generically semisimple we may associate a primitive central idempotent $E(n)$ (resp. $E(-n)$) to each of these modules, which is well defined for q and m indeterminate (but which is ‘singular’ in certain specialisations). For example, the idempotent for $\Delta(-1)$ is $E(-1) = e$. It is always possible to find a scalar multiple $E'(-n)$ of $E(-n)$ which is well defined on specialising m to an integer l_0 (say) (leaving q indeterminate). We may then define a quotient tower of b_\bullet by $E'(-n) \equiv 0$ (for fixed n), analogous to the Temperley–Lieb quotient (the case $n = 1$). We have

Proposition 8.4. [49] *Fix $\delta' = [l_0 + 1]/[l_0]$, where $l_0 < 0$ is integral. Walks on $(A_\infty, 0)$ from 0 to λ of length n which never touch vertex l_0 are a basis for the δ -generically simple module $L^{b_n}(\lambda) = \text{Head}(\Delta^{b_n}(\lambda))$. These are the simple modules of the quotient by $E'(l_0) = 0$. \square*

Details of the associated representation theory are given in [49]. The point for our purposes is simply this. The tower of quotient algebras obtained by quotienting out by $E'(l_0)$ (for $l_0 < 0$ integral) is a G -tower with $G = (A_\infty, -l_0 - 1)$. To see this we simply note that the truncation is at l_0 rather than at -1 . This is a shift of $l_0 + 1$, which may be seen as a different embedding of A_∞ in A_∞ , shifting the root by $l_0 + 1$.

8.2. Brauer case.

For $\delta \in \mathbb{N}$ let V_δ be the (complex) defining module for the action of the general linear group $GL(\delta)$. Then the natural action of $GL(\delta)$ on $V_\delta^{\otimes m}$ is in Schur–Weyl duality with an action of the symmetric group S_m permuting the tensor factors [66]. Restricting to the action of the orthogonal group $O(\delta) \subset GL(\delta)$, then the Brauer algebra $J_m(\delta)$ has a natural action on $V_\delta^{\otimes m}$ extending the S_m action, which commutes with the $O(\delta)$ action. This was the context in which the Brauer algebra was originally introduced. Restricting further to the action of $S_\delta \subset O(\delta)$ permuting standard ordered basis elements in each V_δ , the dual action extends to one of the partition algebra.

These specialisations of δ are non-semisimple cases of the Brauer algebra, so it is natural to ask if there are ‘truncations’ of the set of walks on the Young graph which enumerate the bases of simple modules in these cases. In general this seems to be a very hard problem (see [13]). A simpler (but closely related) problem is to restrict attention to the simple modules which appear in the tensor space module.

The simplest case is $\delta = 1$. Here only the (one-dimensional) simple module in the head of the spine standard module (with $\lambda = 0$) survives. The truncation is to walks on the full subgraph containing the vertices for S_0 and S_1 . There is only one such walk in each layer, so this truncation works almost trivially. Subsequent cases are less straightforward to describe, and we reserve details on them for a separate work.

9. A $(D_\infty, 0)$ TOWER OF ALGEBRAS

9.1. A blob algebra in type D .

The defining Pascal D -array is sketched in Figure 3. More precisely, this is the Pascal array for the rooted graph $(D_\infty, 0)$ where D_∞ is as in Figure 1.

To obtain such an array in an algebraic setting we consider the algebra d_n introduced by Richard Green in [29, §7]. A diagram basis of d_n is given by the Temperley–Lieb diagrams with $2n$ vertices (divided into north and south vertices as in the Temperley–Lieb case). As in the blob algebra case, there is the additional possibility of decorating any arc exposed to the western end of the diagram. There is an additional restriction: there must be an even number of blobs in total.

Multiplication is given by concatenation of diagrams in the usual way; in addition, any undecorated loop is replaced with the scalar δ , any loop with an odd number of blobs is zero and any pair of blobs on the same arc can be removed; see Figure 31. We assume in the sequel that $\delta \neq 0$.

We denote the set of d_n diagrams with n vertices on the north edge and m on the south edge by $D^d(n, m)$. Thus a basis for d_n is $D^d(n, n)$. We still have the bra-ket construction as in Section 5 above, with the

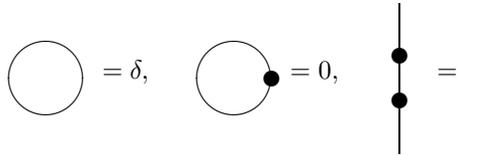


FIGURE 31. The rules for simplifying diagrams for the algebra d_n .

proviso that if the two diagrams produced have at least one undecorated propagating line and an odd number of blobs in total, the westernmost propagating line is decorated with a blob on each of the half diagrams (and any pair of blobs on a single arc produced in this way is removed). We denote the half diagrams obtained from $D^d(n, n)$ with l propagating lines by $D_l^d(n, l)'$; note that if $l > 0$ then these half diagrams always have an even number of blobs. The constructions in (2) and (3) in the previous section go through unchanged.

See Figure 32 for an illustration of the Pascal array in this case.

For $l \in \{n, n-2, \dots, 2 \text{ or } 1\}$, we define $D_l^d(n, l) = D_l^d(n, l)'$. If n is even, we define $D_0^d(n, 0)$ to be the set of half diagrams in $D^d(n, 0)'$ with an even number of blobs, and $D_{0'}^d(n, 0')$ to be the set of half diagrams in $D^d(n, 0)'$ with an odd number of blobs.

Proposition 9.1. *The array of sets $((D_l^d(n, l))_{l \in D_\infty})_n$ is a Pascal D^∞ -sequence. (a) If $l \neq 0'$, the edge maps $(\phi_{i, i+1})'$ are given by ϕ^1 and the edge maps $(\phi_{i+1, i})'$ are given by ϕ^u .*

(b) *The edge maps $(\phi_{0', 1})'$ are given by $\phi^{1'}$, i.e. ϕ^1 modified by decorating the new propagating edge with a blob.*

(c) *The edge maps $(\phi_{1, 0'})'$ are given by $\phi^{u'}$, i.e. ϕ^u modified by decorating the propagating edge that is “bent over” (to form a new vertex) with an extra blob (so if a blob is already there, it is removed by the relation involving two blobs).*

Proof: It suffices to note that

(a) If $l \in \{n, n-2, \dots, 3 \text{ or } 2\}$ then

$$D_l^d(n, l) = \phi^u(D_{l+1}^d(n-1, l+1)) \cup \phi^1(D_{l-1}^d(n-1, l-1)).$$

(b) If $l = 1$ then

$$D_1^d(n, 1) = \phi^u(D_2^d(n-1, 2)) \cup \phi^1(D_0^d(n-1, 0)) \cup \phi^{1'}(D_{0'}(n-1, 0')).$$

(c) If $l = 0$ then

$$D_0^d(n, 0) = \phi^u(D_1^d(n-1, 1)).$$

(d) If $l = 0'$ then

$$D_{0'}^d(n, 0) = \phi^{u'}(D_1^d(n-1, 1)).$$

□

It can be shown that the axioms (A1) to (A4), (A4'), (A5)(a) and (N5) hold. This has the consequence that the axioms (N1) to (N8) all hold for the sequence of algebras d_1, d_2, \dots (note that we start our numbering at 1 rather than 0). We remark that a consequence of this is that the d_n are all quasihereditary. In this case the standard module $\Delta_n(l)$ has a basis parametrized by $D_l^d(n, l)$.

By Proposition 4.10 it follows that the algebras d_\bullet (with the standard modules) form a $(G, 0)$ -tower of algebras for some rooted graph $(G, 0)$. Analysis of the restriction of the standard modules for d_n to d_{n-1} shows that $(G, 0) = (D_\infty, 0)$ (see Figure 1), and hence that the array associated to the tower d_\bullet is a Pascal $(D_\infty, 0)$ -array of sets and thus equivalent to the array of sets $((D_l^d(n, l))_{l \in D_\infty})_n$ shown in Figure 32. We thus have:

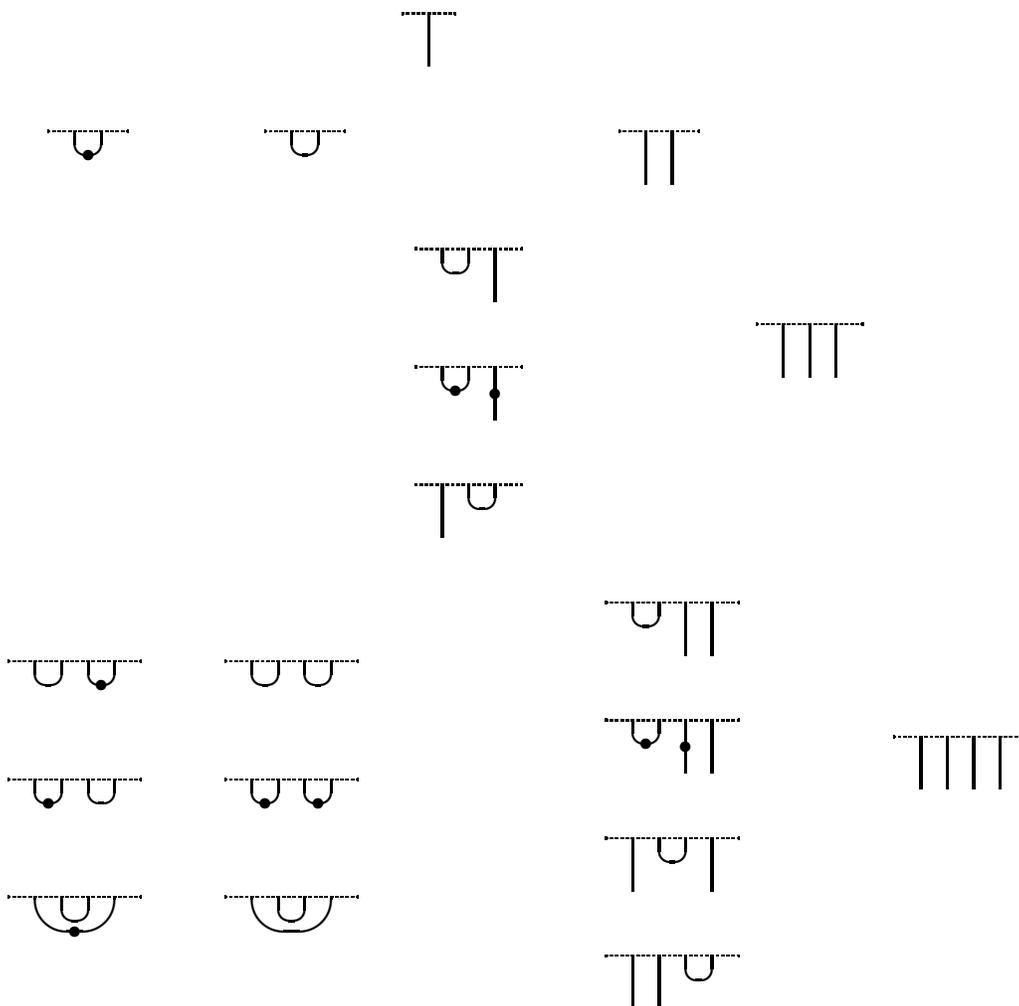


FIGURE 32. D -type blob ‘bra’ diagrams. These sets have the property that if combined with their corresponding ‘ket’ set they produce diagrams with an even number of blobs.

Theorem 9.2. *The tower of algebras d_\bullet , together with their standard modules, is a $(D_\infty, 0)$ -tower of algebras.*

We see that the basis of $\Delta_n(\lambda)$ provided by Proposition 4.8 can be explicitly parametrized by the half diagrams $D_l^d(n, l)$, recovering the above description of the modules $\Delta_n(l)$.

10. FINAL REMARKS

Recall that the Robinson–Schensted correspondence can be regarded as giving a bijection between pairs of paths (of length n from the empty diagram to the same vertex in the Young graph) and permutations of n elements. The Young graph is a oriented graph with a natural \mathbb{Z} -grading. Furthermore, if g_1, g_2 are vertices then the set of vertices g such that $g \rightarrow g_1$ and $g \rightarrow g_2$ has the same cardinality (which must be 0 or 1) as the set of vertices g such that $g_1 \rightarrow g$ and $g_2 \rightarrow g$. In addition, for any vertex g , the number

of vertices h such that $g \rightarrow h$ is equal to 1 more than the number of vertices k such that $k \rightarrow g$. A graph with all of these properties is known as a Y -graph.

Fomin [20] has shown that a generalised Robinson-Schensted correspondence can be defined on such a graph G provided it has a unique source 0, giving a bijection between pairs of paths of length n from 0 to the same vertex and permutations of n elements. The graph G could be, for example, the Young graph or the Young-Fibonacci graph considered by Fomin. In this case the Catalan sequence of sets given by the symmetric groups has underlying Pascal $(G, 0)$ -array $Y_{G,0}$, and [20, K7] can be regarded as giving a natural construction of the bra-ket extraction and its inverse. The result [20, Theorem C] does not obviously fit into the general framework considered here, and suggests the consideration of more general (sequences of) paths.

APPENDIX A. GENERATING FUNCTIONS FOR CATALAN NUMBERS

Let $H_0(x) = \sum_{n \geq 0} C(n)x^n$ be the ordinary generating function for the Catalan numbers. Since the Catalan numbers count rooted planar trees with n vertices, we see that the constant term is 1, since there is a unique tree with no edges. Every other tree has at least one edge. Having drawn this edge the remainder of every tree which contains it as a subtree can be decomposed as two trees: one growing from the end of this edge, and one growing from the root (on the right, say). Thus we have, solving for H_0 (e.g. [3]):

$$H_0 = 1 + xH_0^2 = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

We can think of each such tree with $n + 1$ vertices as a forest of rooted trees with n vertices in total embedded in the upper half-plane, each with its root lying on the boundary. We do this by simply moving the boundary up to the line of vertices one-removed from the original root and then discarding the root. Note that this does not work for the 1-vertex tree, since this procedure would give us the empty tree, which is not rooted.

Now suppose in the forest picture that there are actually two different types of tree in the forest. They look the same as trees, but a forest with two pines is different from a forest with a pine and an oak, and so on. Let H_1 denote the generating function for such two-species forests. As the two-species forests are counted by the $(\mathcal{A}_{\infty,0})$ -, or $(\mathcal{A}(2,1),0)$ -Catalan numbers, H_1 is the generating function for these.

We note that such a forest can be regarded as a tree of one of the two species together with a remaining forest. By an analogous argument to the above we have that

$$H_1 = 1 + 2xH_0H_1 = \frac{1}{\sqrt{1 - 4x}}$$

We have proved:

Proposition A.1. *The sequence $\{N_{(\mathcal{A}_{\infty,0})}(2n; v_0)\}_n$ is the sequence generated by H_1 . \square*

Viewed in isolation, this is nothing but a well known result for the Pascal triangle. But we can continue. Similarly the $(\mathcal{A}(2,2,1),0)$ Catalan numbers are given by

$$H_2 = 1 + 2xH_1H_2 = \frac{1}{1 - \frac{2x}{\sqrt{1-4x}}}$$

and so on. Thus the $(\mathcal{A}(2^l,1),0)$, Catalan numbers are given by

$$H_l = 1 + 2xH_{l-1}H_l$$

Let us write H^λ for the generating function for the $(\mathcal{A}(\lambda),0)$ -Catalan numbers. We have:

Proposition A.2. *Let G be the graph $\mathcal{A}(\lambda)$. Then the generating function H^λ for the sequence $\{N_G(2n; v_0)\}_n$ satisfies the relation*

$$H^\lambda = 1 + \lambda_1 x H^{\lambda'} H^\lambda$$

where $\lambda' = (\lambda_2, \lambda_3, \dots)$ and $H^0 = H_0$. \square

Finally, recall that the exponential generating function for the Bell numbers is

$$\sum_{n \geq 0} \frac{b(n)x^n}{n!} = \exp(\exp(x) - 1) = 1 + x + \frac{2x^2}{2} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \dots$$

(see for example [67]), and that the Rollet graph for the Bell sequence considered in Section 7.1 was not a tree. It is intriguing to speculate on the relationship between the generating function of a Catalan sequence of sets and the properties of the Rollet graph of a Pascal array for the sequence. For example, it may be that ordinary functions correspond to the case where the Rollet graph is a tree, with exponential generating functions corresponding to the case where the Rollet graph is not a tree.

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