A GEOMETRIC DESCRIPTION OF $m$-CLUSTER CATEGORIES

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Abstract. We show that the $m$-cluster category of type $A_{n-1}$ is equivalent to a certain geometrically-defined category of diagonals of a regular $nm + 2$-gon. This generalises a result of Caldero, Chapoton and Schiffler for $m = 1$. The approach uses the theory of translation quivers and their corresponding mesh categories. We also introduce the notion of the $m$th power of a translation quiver and show how it can be used to realise the $m$-cluster category in terms of the cluster category.

Introduction

Let $n, m \in \mathbb{N}$ and let $\Pi$ be a regular $nm + 2$-sided polygon. We show that a category $\mathcal{C}_{A_{n-1}}^m$ of diagonals can be associated to $\Pi$ in a natural way. The objects of $\mathcal{C}_{A_{n-1}}^m$ are the diagonals in $\Pi$ which divide $\Pi$ into two polygons whose numbers of sides are congruent to 2 modulo $m$, as considered in [PS]. A quiver $\Gamma_{A_{n-1}}^m$ can be defined on the set of such diagonals, with arrows given by a simple geometrical rule. It is shown that this quiver is a stable translation quiver in the sense of Riedtmann [Rie] with translation $\tau$ given by a certain rotation of the polygon. For a field $k$, the category $\mathcal{C}_{A_{n-1}}^m$ is defined as the mesh category associated to $(\Gamma_{A_{n-1}}^m, \tau)$.

Let $Q$ be a Dynkin quiver of type $A_{n-1}$, and let $D^b(kQ)$ denote the bounded derived category of finite dimensional $kQ$-modules. Let $\tau$ denote the Auslander-Reiten translate of $D^b(kQ)$, and let $S$ denote the shift. These are both autoequivalences of $D^b(kQ)$. Our main result is that $\mathcal{C}_{A_{n-1}}^m$ is equivalent to the quotient of $D^b(kQ)$ by the autoequivalence $\tau^{-1}S^m$. We thus obtain a geometric description of this category in terms of $\Pi$.

The $m$-cluster category $D^b(kQ)/\tau^{-1}S^m$ associated to $kQ$ was introduced in [Kel] and has also been studied by Thomas [Tho], Wrasen [Wra] and Zhu [Zhu]. It is a generalisation of the cluster category defined in [CCS1] (for type $A$) and [BMRRT] (general hereditary case). Keller has shown that it is Calabi-Yau of dimension $m + 1$ [Kel]. We remark that such Calabi-Yau categories have also been studied in [KR].

Our definition is motivated by and is a generalisation of the construction of the cluster category in type $A$ given in [CCS1], where a category of diagonals of a polygon is introduced. The authors show that this category is equivalent to the cluster category associated to $kQ$. This can be regarded as the case $m = 1$ here. The aim of the current paper is to generalise the construction of [CCS1] to the diagonals arising in the $m$-divisible polygon dissections considered in [PS]. Note that Tzanaki [Tza] has also studied such diagonals. We also remark that


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a connection between the \( m \)-cluster category associated to \( kQ \) and the diagonals considered here was given in [Tho].

We further show that if \((\Gamma, \tau)\) is any stable translation quiver then the quiver \( \Gamma^m \)
with the same vertices but with arrows given by sectional paths in \( \Gamma \) of length \( m \) is again a stable translation quiver with translation given by \( \tau^m \). If \((\Gamma, \tau)\) is taken to be the Auslander-Reiten quiver of the cluster category of a Dynkin quiver of type \( A_{nm-1} \), we show that \( \Gamma^m \) contains \( \Gamma^m_{A_{nm-1}} \) as a connected component; it follows that the \( m \)-cluster category is a full subcategory of the additive category generated by the mesh category of \( \Gamma^m \).

Since \( \Gamma \) is known to have a geometric construction [CCS1], our definition provides a geometric construction for the additive category generated by the mesh category of any connected component of \( \Gamma^m \). We give an example to show that this provides a geometric construction for quotients of \( D^b(kQ) \) other than the \( m \)-cluster category.

1. Notation and Definitions

In [Tza], E. Tzanaki studied an abstract simplicial complex obtained by dividing a polygon into smaller polygons.

We recall the definition of an abstract simplicial complex. Let \( X \) be a finite set and \( \Delta \subseteq \mathcal{P}(X) \) a collection of subsets. Assume that \( \Delta \) is closed under taking subsets (i.e. if \( A \in \Delta \) and \( B \subseteq A \) then \( B \in \Delta \)). Then \( \Delta \) is an abstract simplicial complex on the ground set \( X \). The vertices of \( \Delta \) are the single element subsets of \( \Delta \) (i.e. \( \{ A \} \in \Delta \)). The faces of \( \Delta \) are the elements of \( \Delta \), the facets are the maximal among those (i.e. the \( A \in \Delta \) such that if \( A \subseteq B \) and \( B \in \Delta \) then \( A = B \)). The dimension of a face \( A \) is equal to \( |A| - 1 \) (where \( |A| \) is the cardinality of \( A \)). The complex is said to be pure of dimension \( d \) if all its facets have dimension \( d \).

Let \( \Pi \) be an \( nm + 2 \)-gon, \( m, n \in \mathbb{N} \), with vertices numbered clockwise from 1 to \( nm + 2 \). We regard all operations on vertices of \( \Pi \) modulo \( nm + 2 \). A diagonal \( D \) is denoted by the pair \((i, j)\) (or simply by the pair \( ij \) if \( 1 \leq i, j \leq 9 \)). Thus \( (i, j) \) is the same as \((j, i)\). We call a diagonal \( D \) in \( \Pi \) an \( m \)-diagonal if \( D \) divides \( \Pi \) into an \((mj + 2)\)-gon and an \((m(n - j) + 2)\)-gon where \( j = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor \). Then Tzanaki defines the abstract simplicial complex \( \Delta = \Delta^m_{A_{nm-1}} \) on the \( m \)-diagonals of \( \Pi \) as follows.

The vertices of \( \Delta \) are the \( m \)-diagonals. The faces of \( \Delta^m_{A_{nm-1}} \) are the sets of \( m \)-diagonals which pairwise do not cross. They are called \( m \)-divisible dissections (of \( \Pi \)). Then the facets are the maximal collections of such \( m \)-diagonals. Each facet contains exactly \( n - 1 \) elements, so the complex \( \Delta^m_{A_{nm-1}} \) is pure of dimension \( n - 2 \).

The case \( m = 1 \) is the complex whose facets are triangulations of an \( n + 2 \)-gon.

2. A Stable Translation Quiver of Diagonals

To \( \Delta = \Delta^m_{A_{nm-1}} \) we associate a category along the lines of [CCS1]. As a first step, we associate to the simplicial complex a quiver, called \( \Gamma^m_{A_{nm-1}} \). The vertices of the quiver are the \( m \)-diagonals in the defining polygon \( \Pi \), i.e. the vertices of \( \Delta^m_{A_{nm-1}} \).

The arrows of \( \Gamma^m_{A_{nm-1}} \) are obtained in the following way:

Let \( D, D' \) be \( m \)-diagonals with a common vertex \( i \) of \( \Pi \). Let \( j \) and \( j' \) be the other endpoints of \( D \), respectively \( D' \). The points \( i, j, j' \) divide the boundary of the polygon \( \Pi \) into three arcs, linking \( i \) to \( j \), \( j \) to \( j' \) and \( j' \) to \( i \). (We usually refer to a part of the boundary connecting one vertex to another as an arc.) If \( D, D' \) and the arc from \( j \) to \( j' \) form an \( m + 2 \)-gon in \( \Pi \) and if furthermore, \( D \) can be rotated clockwise to \( D' \) about the common endpoint \( i \), we draw an arrow from \( D \) to \( D' \) in \( \Gamma^m_{A_{nm-1}} \). (By this we mean that \( D \) can be rotated clockwise to the line through \( D' \).)
Note that if \( D, D' \) are vertices of the quiver \( \Gamma_m^{A_{n-1}} \) then there is at most one arrow between them.

Examples 2.4 and 2.5 below illustrate this construction.

We then define an automorphism \( \tau_m \) of the quiver: let \( \tau_m : \Gamma_m^{A_{n-1}} \rightarrow \Gamma_m^{A_{n-1}} \) be the map given by \( D \mapsto D' \) if \( D' \) is obtained from \( D \) by an anticlockwise rotation through \( \frac{2\pi}{nm+2} \) about the centre of the polygon. Clearly, \( \tau_m \) is a bijective map and a morphism of quivers.

**Figure 1.** The translation \( \tau_m, \tau_m(14) = 92 \), where \( n = 4, m = 2 \)

**Definition 2.1.** (1) A translation quiver is a pair \( (\Gamma, \tau) \) where \( \Gamma \) is a locally finite quiver and \( \tau : \Gamma_0' \rightarrow \Gamma_0 \) is an injective map defined on a subset \( \Gamma_0' \) of the vertices of \( \Gamma \) such that for any \( X \in \Gamma_0, Y \in \Gamma_0' \), the number of arrows from \( X \) to \( Y \) is the same as the number of arrows from \( \tau(Y) \) to \( X \). The vertices in \( \Gamma_0 \setminus \Gamma_0' \) are called projective. If \( \Gamma_0' = \Gamma_0 \) and \( \tau \) is bijective, \( (\Gamma, \tau) \) is called a stable translation quiver.

(2) A stable translation quiver is said to be connected if it is not a disjoint union of two non-empty stable subquivers.

**Proposition 2.2.** The pair \( (\Gamma = \Gamma_m^{A_{n-1}}, \tau_m) \) is a stable translation quiver.

**Proof.** By definition, \( \tau_m \) is a bijective map from \( \Gamma \) to \( \Gamma \), and \( \Gamma \) is a finite quiver. We have to check that the number of arrows from \( D \) to \( D' \) in \( \Gamma \) is the same as the number of arrows from \( \tau_m D' \) to \( D \). Since there is at most one arrow from one vertex to another, we only have to see that there is an arrow \( D \rightarrow D' \) if and only if there is an arrow \( \tau_m D' \rightarrow D \).

Assume that there is an arrow \( D \rightarrow D' \), let \( i \) be the common vertex of \( D \) and \( D' \) in the polygon, \( D = (i, j), D' = (i, j + m) \). Then \( \tau_m D' = (i - m, j) \). In particular, \( j \) is the common vertex of \( D \) and \( \tau_m D' \). Furthermore, we obtain \( D \) from \( \tau_m D' \) by a clockwise rotation about \( j \) and these two \( m \)-diagonals form an \( m + 2 \)-gon together with an arc from \( i - m \) to \( i \), hence there is an arrow \( \tau_m D' \rightarrow D \). See Figure 2.

The converse follows with the same reasoning. \( \square \)

**Proposition 2.3.** \( (\Gamma, \tau_m) \) is a connected stable translation quiver.

**Proof.** Note that every vertex of \( \Pi \) is incident with some element of any given \( \tau_m \)-orbit of \( m \)-diagonals: any \( m \)-diagonal is of the form \( (i, i + km + 1) \) and

\[
\tau_m^{k-n}(i, i + km + 1) = (i + (n - k)m, i + nm + 1) = (i + (n - k)m, i - 1).
\]
Let $\Pi$ be a 6-gon. The rotation group given by rotation about the centre of $\Pi$ through $k \times \frac{2\pi}{3}$ degrees ($k = 1, \ldots, 5$) acts on the facets of $\Delta^m_{\Lambda_{n-1}}$. There are four orbits, $\mathcal{O}_{13,14,15}$ of size 6, $\mathcal{O}_{13,14,46}$ and $\mathcal{O}_{13,36,46}$ of size 3 and $\mathcal{O}_{13,15,35}$ with two elements, making a total of 14 elements.

The vertices of the quiver $\Gamma_{A_3}$ are the nine 1-diagonals $\{13, 14, 15, 24, 25, 26, 35, 36, 46\}$ and we draw the quiver as follows:

![Quiver Diagram](image)

**Example 2.5.** Let $m = 2$ and $n = 4$, i.e. $\Pi$ is a 10-gon. The rotation group is generated by the rotation about the centre of $\Pi$ through $k \times \frac{2\pi}{3}$ degrees ($k = 1, \ldots, 9$) and acts on the facets of $\Delta^2_{A_3}$. The orbits are $\mathcal{O}_{14,16,18}$, $\mathcal{O}_{14,18,47}$, $\mathcal{O}_{18,38,47}$ and $\mathcal{O}_{47,38,39}$ of size 10, and $\mathcal{O}_{14,16,69}$, $\mathcal{O}_{14,49,69}$ and $\mathcal{O}_{29,38,47}$ of size 5, making a total of 55 elements. The vertices of $\Gamma^2_{A_3}$ are the fifteen 2-diagonals $\{14, 16, 18, 25, 27, 29, 36, 38, (3, 10), 47, 49, 58,(5, 10), 69, (7, 10)\}$.
and the quiver is:

![Diagram](image)

3. m-Cluster Categories

Let $G$ be a simply-laced Dynkin diagram with vertices $I$. Let $Q$ be a quiver with underlying graph $G$, and let $k$ be an algebraically-closed field. Let $kQ$ be the corresponding path algebra. Let $D^b(kQ)$ denote the bounded derived category of finitely generated $kQ$-modules, with shift denoted by $S$, and Auslander-Reiten translate given by $\tau$. It is known that $D^b(kQ)$ is triangulated, Krull-Schmidt and has almost-split triangles (see [Hap]). Let $ZQ$ be the stable translation quiver associated to $Q$, with vertices $(n, i)$ for $n \in \mathbb{Z}$ and $i$ a vertex of $Q$. For every arrow $\alpha : i \to j$ in $Q$ there are arrows $(n, i) \to (n, j)$ and $(n, j) \to (n + 1, i)$ in $ZQ$, for all $n \in \mathbb{Z}$. Together with the translation $\tau$, taking $(n, i)$ to $(n - 1, i)$, $ZQ$ is a stable translation quiver. We note that $ZQ$ is independent of the orientation of $Q$ and can thus be denoted $ZG$.

We recall the notion of the mesh category of a stable translation quiver with no multiple arrows (the mesh category is defined for a general translation quiver but we shall not need that here). Recall that for a quiver $\Gamma$, $k(\Gamma)$ denotes the path category on $\Gamma$, with morphisms given by arbitrary $k$-linear combinations of paths.

**Definition 3.1.** Let $(\Gamma, \tau)$ be a stable translation quiver with no multiple arrows. Let $Y$ be a vertex of $\Gamma$ and let $X_1, \ldots, X_k$ be all the vertices with arrows to $Y$, denoted $\alpha_i : X_i \to Y$. Let $\beta_i : \tau(Y) \to X_i$ be the corresponding arrows from $\tau(Y)$ to $X_i$ ($i = 1, \ldots, k$). Then the mesh ending at $Y$ is defined to be the quiver consisting of the vertices $Y, \tau(Y), X_1, \ldots, X_k$ and the arrows $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $\beta_1, \beta_2, \ldots, \beta_k$. The mesh relation at $Y$ is defined to be

$$m_Y := \sum_{i=1}^k \beta_i \alpha_i \in \text{Hom}_{k(\Gamma)}(\tau(Y), Y)$$

Let $J_m$ be the ideal in $k(\Gamma)$ generated by the mesh relations $m_Y$ where $Y$ runs over all vertices of $\Gamma$.

Then the mesh category of $\Gamma$ is defined as the quotient $k(\Gamma)/J_m$.

For an additive category $\mathcal{E}$, denote by $\text{ind} \mathcal{E}$ the full subcategory of indecomposable objects. Happel [Hap] has shown that $\text{ind} D^b(kQ)$ is equivalent to the mesh category of $ZQ$, from which it follows that it is independent of the orientation of $Q$. Its Auslander-Reiten quiver is $ZG$.

For $m \in \mathbb{N}$, we denote by $C^m_G$ the $m$-cluster category associated to the Dynkin diagram $G$, so

$$C^m_G = \frac{D^b(kQ)}{F_m},$$

where $Q$ is any orientation of $G$ and $F_m$ is the autoequivalence $\tau^{-1} \circ S^m$ of $D^b(kQ)$. This was introduced by Keller [Kel] and has been studied by Thomas [Tho], Walsen [Wra] and Zhu [Zhu]. It is known that $C^m_G$ is triangulated [Kel], Krull-Schmidt and has almost split triangles [BMRRT, 1.2.1.3]. Let $\varphi_m$ denote the automorphism of $ZG$ induced by the autoequivalence $F_m$. The Auslander-Reiten quiver of $C^m_G$ is the quotient $ZG/\varphi_m$, and $\text{ind} C^m_G$ is equivalent to the mesh category of $ZG/\varphi_m$. 


4. Coloured almost positive roots

Our main aim in the next two sections is to show that, if $G$ is of type $A_{n-1}$, then $\text{ind} C^m_G$ is equivalent to the mesh category $\mathcal{D}^m_{A_{n-1}}$ of the stable translation quiver $\Gamma^m_{A_{n-1}}$ defined in the previous section. From the previous section we can see that it is enough to show that, as translation quivers, $ZG/\varphi_m$ is isomorphic to $\Gamma^m_{A_{n-1}}$. In this section, we recall the discussion of $m$-diagonals and $m$-coloured almost positive roots in Fomin-Reading [FR].

4.1. $m$-coloured almost positive roots and $m$-diagonals. For $\Phi$ a root system, with positive roots $\Phi^+$ and simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, let $\Phi^m_{\geq -1}$ denote the set of $m$-coloured almost positive roots (see [FR]). An element of $\Phi^m_{\geq -1}$ is either a $m$-coloured positive root $\alpha^k$ where $\alpha \in \Phi^+$ and $k \in \{1, 2, \ldots, m\}$ or a negative simple root $-\alpha_i$ for some $i$ which we regard as having colour 1 for convenience (it is thus also denoted $-\alpha_i^1$). Fomin-Reading [FR] show that there is a one-to-one correspondence between $m$-diagonals of the regular $nm + 2$-gon $\Pi$ and $\Phi^m_{\geq -1}$ when $\Phi$ is of type $A_{n-1}$. We now recall this correspondence.

Recall that $R_m$ denotes the anticlockwise rotation of $\Pi$ taking vertex $i$ to vertex $i - 1$ for $i \geq 2$, and vertex 1 to vertex $nm + 2$. For $1 \leq i \leq \frac{n}{2}$, the negative simple root $-\alpha_{2i-1}$ corresponds to the diagonal $((i-1)m+1, (n-i)m+2)$. For $1 \leq i \leq \frac{n+1}{2}$, the negative simple root $-\alpha_{2i}$ corresponds to the diagonal $(im+1, (n-i)m+2)$. Together, these diagonals form what is known as the $m$-snake, cf. Figure 3. For $1 \leq i \leq j \leq n$, there are exactly $m$ $m$-diagonals intersecting the diagonals labelled $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ and no other diagonals labelled with negative simple roots. These diagonals are of the form $D, R^1_m(D), \ldots, R^{m-1}_m(D)$ for some diagonal $D$, and $\alpha^k$ corresponds to $R^{k-1}_m(D)$ for $k = 1, 2, \ldots, m$, where $\alpha$ denotes the positive root $\alpha_i + \cdots + \alpha_j$. For an $m$-coloured almost positive root $\beta^k$, we denote the corresponding diagonal by $D(\beta^k)$.

It is clear that, for $1 \leq i \leq \frac{n}{2}$, the coloured root $\alpha^i_{2i-1}$ corresponds to the diagonal $(im+1, (n+1-i)n+2)$. Also, the diagonals $D(-\alpha_i)$, for $i$ even, together with $D(\alpha^1_j)$, for $j$ odd, form a ‘zig-zag’ dissection of $\Pi$ which we call the opposite $m$-snake, cf. Figure 3.

![Figure 3. m-snake and opposite m-snake for $n = 6, m = 2$](image)

Let $I = I^+ \cup I^-$ be a decomposition of the vertices $I$ of $G$ so that there are no arrows between vertices in $I^+$ or between vertices in $I^-$; such a decomposition
Lemma 5.2. For every second statement, since \( C \) the set of \( m \) and therefore that \( \psi \) deduce from this that The first statement follows immediately from Lemmas 4.1 and 4.2. We can

\[ \tau_\varepsilon(\beta) = \begin{cases} \alpha & \text{if } \beta = -\alpha_i, \text{ for } i \in I^- \\ (\prod_{i \in I^-} s_i)(\beta) & \text{otherwise.} \end{cases} \]

Then, for \( \beta^k \in \Phi^m_{\geq -1} \), we have

\[ R_m(\beta^k) = \begin{cases} \beta^{k+1} & \text{if } \alpha \in \Phi^+ \text{ and } k < m, \\ (\tau - \tau_+)(\beta^k) & \text{otherwise.} \end{cases} \]

Lemma 4.1 (FOMIN-READING). For all \( \beta^k \in \Phi^m_{\geq -1} \), we have: \( D(R_m(\beta^k)) = R_mD(\beta^k) \).

**Proof.** See the discussion in [FR, 4.1]. \( \square \)

4.2. Indecomposable objects in the \( m \)-cluster category and \( m \)-diagonals. Let \( Q_{\text{alt}} \) denote the orientation of \( G \) obtained by orienting every arrow to go from a vertex in \( I^+ \) to a vertex in \( I^- \), so that the vertices in \( I^+ \) are sources and the vertices in \( I^- \) are sinks.

For a positive root \( \alpha \), let \( V(\alpha) \) denote the corresponding \( kQ_{\text{alt}} \)-module, regarded as an indecomposable object in \( D(kQ_{\text{alt}}) \). Then it is clear from the definition that the indecomposable objects in \( C^m_G \) are the objects \( S^{k-1}V(\beta) \) for \( k = 1, 2, \ldots, m \) and \( \alpha \in \Phi^+ \) and \( s^{-1}I_i \) for \( I \) an indecomposable injective \( kQ_{\text{alt}} \)-module corresponding to the vertex \( i \in I \) (all regarded as objects in the \( m \)-cluster category). Following Thomas [Tho] or Zhu [Zhu], we define \( V(\alpha^k) \) to be \( S^{k-1}V(\beta) \) for \( k = 1, 2, \ldots, m \), \( \alpha \in \Phi^+ \), and \( V(-\alpha) = s^{-1}I_i \) for \( i \in I \).

We have:

Lemma 4.2 (THOMAS,ZHU). For all \( \beta^k \in \Phi^m_{\geq -1} \), \( V(R_m(\beta^k)) \cong SV(\beta^k) \), where \( S \) denotes the autoequivalence of \( C^m_G \) induced by the shift on \( D(kQ) \).

**Proof.** See [Tho, Lemma 2] or [Zhu, 3.8]. \( \square \)

5. An isomorphism of stable translation quivers

From the previous two sections, we see that in type \( A_{n-1} \), we have a bijection \( D \) from \( \Phi^m_{\geq -1} \) to the set of \( m \)-diagonals of \( \Pi \) and a bijection \( V \) from \( \Phi^m_{\geq -1} \) to the objects of \( \text{ind}C^m_{A_{n-1}} \) up to isomorphism, i.e. to the vertices of the Auslander-Reiten quiver of \( C^m_{A_{n-1}} \). Composing the inverse of \( D \) with \( V \) we obtain a bijection \( \psi \) from the set of \( m \)-diagonals of \( \Pi \) to \( \text{ind}C^m_{A_{n-1}} \).

Lemma 5.1. For every \( m \)-diagonal \( D \) of \( \Pi \), we have that \( \psi(R_m(D)) \cong S\psi(D) \), and therefore that \( \psi(\tau_m(D)) \cong \tau(\psi(D)) \).

**Proof.** The first statement follows immediately from Lemmas 4.1 and 4.2. We can deduce from this that \( \psi(\tau_m(D)) = \psi(R^m_m(D)) = S^m\psi(D) \) and thus obtain the second statement, since \( S^m \) coincides with \( \tau \) on every indecomposable object of \( C^m_{A_{n-1}} \) by the definition of this category. \( \square \)

It remains to show that \( \psi \) and \( \psi^{-1} \) are morphisms of quivers.

Lemma 5.2. For \( 1 \leq i \leq \frac{n-1}{2} \), there is an arrow in \( \Gamma^m_{A_{n-1}} \) from \( D(-\alpha_{2i-1}) \) to \( D(-\alpha_{2i}) \).
For $1 \leq i \leq \frac{n-1}{2}$, there is an arrow in $\Gamma_{A_{n-1}}^m$ from $D(-\alpha_{2i+1})$ to $D(-\alpha_{2i})$.

For $1 \leq i \leq \frac{n}{2}$, there is an arrow in $\Gamma_{A_{n-1}}^m$ from $D(-\alpha_{2i})$ to $D(\alpha_{2i-1})$.

For $1 \leq i \leq \frac{n+2}{2}$, there is an arrow in $\Gamma_{A_{n-1}}^m$ from $D(-\alpha_{2i+1})$ to $D(\alpha_{2i+1})$.

These are the only arrows amongst the diagonals $D(-\alpha_i)$ and $D(\alpha_i)$, for $1 \leq i, j \leq n-1$, with $j$ odd, in $\Gamma_{A_{n-1}}^m$.

Proof. We firstly note that, for $1 \leq i \leq \frac{n-1}{2}$, the diagonals corresponding to the negative simple roots $-\alpha_{2i-1}$ and $-\alpha_{2i}$, together with an arc of the boundary containing vertices $(i-1)m+1, \ldots, im+1$, bound an $m+2$-gon. The other vertex is numbered $(n-i)m+2$. Furthermore, $D(-\alpha_{2i-1})$ can be rotated clockwise about the common end point $(n-i)m+2$ to $D(-\alpha_{2i})$, so there is an arrow in $\Gamma_{A_{n-1}}^m$ from $D(-\alpha_{2i-1})$ to $D(-\alpha_{2i})$.

Similarly, for $1 \leq i \leq \frac{n+2}{2}$, the diagonals corresponding to the negative simple roots $-\alpha_{2i+1}$ and $-\alpha_{2i+2}$, together with an arc of the boundary containing vertices $(n-i)m+1, \ldots, (n-i)m+2$, bound an $m+2$-gon (with the other vertex being numbered $im+1$), and $D(-\alpha_{2i+1})$ can be rotated clockwise about the common end point $im+1$ to $D(-\alpha_{2i+2})$, so there is an arrow in $\Gamma_{A_{n-1}}^m$ from $D(-\alpha_{2i+1})$ to $D(-\alpha_{2i+2})$.

We have observed that, for $1 \leq i \leq \frac{n}{2}$, the coloured root $\alpha_{2i-1}^1$ corresponds to the diagonal $(im+1, (n+1-i)m+2)$. Consideration of the $m+2$-gon with vertices $(n-i)m+2, \ldots, (n+1-i)m+2$ and $im+1$ shows that there is an arrow from $D(-\alpha_{2i})$ to $D(\alpha_{2i-1}^1)$. For $1 \leq i \leq \frac{n+2}{2}$, consideration of the $m+2$-gon with vertices $im+1, \ldots, (i+1)m+1$ and $(n-i)m+2$ shows that there is an arrow from $D(-\alpha_{2i+1})$ to $D(\alpha_{2i+1}^1)$.

The statement that these are the only arrows amongst the diagonals considered is clear. □

The following follows from the well-known structure of the Auslander-Reiten quiver of $D^b(kQ)$.

Lemma 5.3. For $1 \leq i \leq \frac{n+1}{2}$, there is an arrow in the Auslander-Reiten quiver of $C_{A_{n-1}}^m$ from $I_{2i-1}[-1]$ to $I_{2i}[-1]$.

For $1 \leq i \leq \frac{n-2}{2}$, there is an arrow from $I_{2i+1}[-1]$ to $I_{2i}[-1]$.

For $1 \leq i \leq \frac{n}{2}$, there is an arrow from $I_{2i}[-1]$ to $P_{2i-1}$.

For $1 \leq i \leq \frac{n+2}{2}$, there is an arrow from $I_{2i-1}[-1]$ to $P_{2i+1}$.

These are the only arrows amongst the vertices $I_i[-1]$ and $P_j$ for $1 \leq i, j \leq n-1$, with $j$ odd, in the Auslander-Reiten quiver of $C_{A_{n-1}}^m$.

Proposition 5.4. The map $\psi$ from $m$-diagonals in $\Pi$ to indecomposable objects in $C_{A_{n-1}}^m$ is an isomorphism of quivers.

Proof. Suppose that $D, E$ are $m$-diagonals in $\Pi$ and that there is an arrow from $D$ to $E$. Write $D = D(\beta^k)$ and $E = D(\gamma^l)$ for coloured roots $\beta^k$ and $\gamma^l$. Then $V := \psi(D) = V(\beta^k)$ and $W := \psi(E) = V(\gamma^l)$ are corresponding vertices in the Auslander-Reiten quiver of $C_{A_{n-1}}^m$. Since there is an arrow from $D$ to $E$, there is an $m+2$-gon bounded by $D$ and $E$ and an arc of the boundary of $\Pi$.

Since $D$ is an $m$-diagonal, on the side of $D$ not in the $m+2$-gon, there is a $dm+2$-gon bounded by $D$ and an arc of the boundary of $\Pi$ for some $d \geq 1$. Similarly, since $E$ is an $m$-diagonal, on the side of $E$ not in the $m+2$-gon, there is an $em+2$-gon bounded by $D$ and an arc of the boundary of $\Pi$, for some $e \geq 1$. It is clear that each of these polygons can be dissected by an $m$-snake such that, together with $D$ and $E$, we obtain a ‘zig-zag’ dissection $\chi$ of $\Pi$. Let $v$ be one of its endpoints. The other endpoint of the diagonal containing $v$ must be $v-m-1$ or $v+m+1$ (modulo $nm+2$).
In the first case, we have that for some $t \in \mathbb{Z}$, $R^t_m(v) = 1$ and $R^t_m$ applied to $\chi$ is the $m$-snake. In the second case, we have that, for some $t \in \mathbb{Z}$, $R^t_m(v) = nm + 2$ and $R^t_m$ applied to $\chi$ is the opposite $m$-snake. It follows from Lemma 5.3 that there is an arrow from $R^t_m(V)$ to $R^t_m(W)$ in the Auslander-Reiten quiver of $C^m_{\mathcal{A}_{n-1}}$, and hence from $V$ to $W$.

Conversely, suppose that $V, W$ are vertices of the Auslander-Reiten quiver of $C^m_{\mathcal{A}_{n-1}}$, and that there is an arrow from $V$ to $W$. We can write $V = V(\beta^l)$ and $W = V(\gamma^l)$ for coloured roots $\beta^k$ and $\gamma^l$. Let $D := \psi^{-1}(V) = D(\beta^k)$ and let $E := \psi^{-1}(W) = D(\gamma^l)$. It is clear that $\tau^u(V) \cong I_{t|-1}$ for some $i$ and some $u$.

By Lemma 5.3, we must have that either $\tau^u(W) \cong I_{t+1|-1}$ or $\tau^u(W) \cong P_{t \pm 1}$. In the latter case we must have that $i$ is even. Note that $S^{um}(V) \cong \tau^u(V)$ and $S^{um}(W) \cong \tau^u(W)$. It follows from Lemmas 5.1 and 5.2 that there is an arrow from $R^u_m(D)$ to $R^u_m(E)$ in $\Gamma^m_{\mathcal{A}_{n-1}}$, and thus from $D$ to $E$.

It follows that $\psi$ is an isomorphism of quivers. 

**Proposition 5.5.** There is an isomorphism $\psi$ of translation quivers between the stable translation quiver $\Gamma^m_{\mathcal{A}_{n-1}}$ of $m$-diagonals and the Auslander-Reiten quiver of the $m$-cluster category $C^m_{\mathcal{A}_{n-1}}$.

**Proof.** This now follows immediately from Proposition 5.4 and Lemma 5.1. 

We therefore have our main result.

**Theorem 5.6.** The $m$-cluster category $C^m_{\mathcal{A}_{n-1}}$ is equivalent to the additive category generated by the mesh category of the stable translation quiver $\Gamma^m_{\mathcal{A}_{n-1}}$ of $m$-diagonals.

We remark that a connection between the $m$-cluster category and the $m$-diagonals has been given in [Tho]. In particular, Thomas gives an interpretation of Ext-groups in the $m$-cluster category in terms of crossings of diagonals. However, Thomas does not give a construction of the $m$-cluster category using diagonals.

6. The $m$-th Power of a Translation Quiver

In this section we define a new category in natural way in which the $m$-cluster category $C^m_{\mathcal{A}_{n-1}}$ will appear as a full subcategory. We start with a translation quiver $\Gamma$ and define its $m$-th power.

Let $\Gamma$ be a translation quiver with translation $\tau$.

Let $\Gamma^m$ be the quiver whose objects are the same as the objects of $\Gamma$ and whose arrows are the sectional paths of length $m$. A path $(x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_{m-1} \rightarrow x_m = y)$ in $\Gamma$ is said to be sectional if $\tau x_{i+1} \neq x_{i-1}$ for $i = 1, \ldots, m-1$ (for which $\tau x_{i+1}$ is defined) (cf. [Rin]). Let $\tau^m$ be the $m$-th power of the translation, i.e. $\tau^m = \tau \circ \tau \circ \cdots \circ \tau$ ($m$ times). Note that the domain of definition of $\tau^m$ is a subset of the domain of definition of $\Gamma_0'$ of $\tau$.

Recall that a translation quiver is said to be hereditary (see [Rin]) if:

- for any non-projective vertex $z$, there is an arrow from some vertex $z' \rightarrow z$;
- there is no (oriented) cyclic path of length at least one containing projective vertices, and
- If $y$ is a projective vertex and there is an arrow $x \rightarrow y$, then $x$ is projective.

The last condition is what we need to ensure that $(\Gamma^m, \tau^m)$ is again a translation quiver.

**Theorem 6.1.** Let $(\Gamma, \tau)$ be a translation quiver such that if $y$ is a projective vertex and there is an arrow $x \rightarrow y$, then $x$ is projective. Then $(\Gamma^m, \tau^m)$ is a translation quiver.
Proof. We prove the following statement by induction on $m$:

Suppose that there is a sectional path

$$x = x_0 \to x_1 \to \cdots \to x_m = y$$

in $\Gamma$ and $\tau^m y$ is defined. Then $\tau^i x_i$ is defined for $i = 0, 1, \ldots, m$ and there is a sectional path

$$\tau^m y = \tau^m x_m \to \tau^{m-1} x_{m-1} \to \cdots \to \tau x_1 \to x = x_0$$

in $\Gamma$. Furthermore, if the multiplicities of arrows between consecutive vertices in the first path are $k_1, k_2, \ldots, k_m$, the multiplicities of arrows between consecutive vertices in the second path are $k_m, k_{m-1}, \ldots, k_1$.

This is clearly true for $m = 1$, since $\Gamma$ is a translation quiver. Suppose it is true for $m - 1$, and that

$$x = x_0 \to x_1 \to \cdots \to x_m = y$$

is a sectional path in $\Gamma$. Since $\tau^{m-1} x_m$ is defined, we can apply induction to the section path:

$$x_1 \to x_2 \to \cdots \to x_m$$

to obtain that $\tau^{i-1} x_i$ is defined for $i = 1, 2, \ldots, m$ and that there is a sectional path

$$\tau^{m-1} x_m \to \tau^{m-2} x_{m-1} \to \cdots \to x_1$$

in $\Gamma$, with multiplicities $k_2, k_3, \ldots, k_m$. As $\tau^m x_m$ is defined, $\tau^{m-1} x_m$ is not projective, and it follows that $\tau^{i-1} x_i$ is not projective for $i = 1, 2, \ldots, m$ by our assumption. Therefore $\tau^i x_i$ is defined for $i = 1, 2, \ldots, m$. For $i = 2, 3, \ldots, m$, there are $k_1$ arrows from $\tau^{i-1} x_i$ to $\tau^{i-2} x_{i-1}$. Therefore there are $k_i$ arrows from $\tau^{i-1} x_{i-1}$ to $\tau^{i-1} x_i$. Thus there are $k_i$ arrows from $\tau^i x_i$ to $\tau^{i-1} x_{i-1}$. As there are $k_1$ arrows from $x_0$ to $x_1$, there are $k_1$ arrows from $\tau x_1$ to $x_0$. If $\tau(\tau^i x_i) = \tau^{i+2} x_{i+2}$ for some $i$ then $x_i = \tau x_{i+2}$, contradicting the fact that $x_0 \to x_1 \to \cdots \to y$ is sectional. It follows that

$$\tau^m x_m \to \tau^{m-1} x_{m-1} \to \cdots \to x_0 = x$$

is a sectional path with multiplicities of arrows $k_1, k_2, \ldots, k_m$ as required.

It follows that the number of sectional paths with sequence of vertices $x_0, x_1, \ldots, x_m$ is less than or equal to the number of sectional paths with sequence of vertices $\tau^m y = \tau^m x_m, \tau^{m-1} x_{m-1}, \ldots, \tau x_1, x_0 = x$.

Suppose that

$$x = x_0' \to x_1' \to \cdots \to x_m' = y$$

is a sectional path from $x$ to $y$ with a different sequence of vertices. Then $x_i \neq x_i'$ for some $i, 0 < i < m$. It follows that $\tau^i x_i \neq \tau^i x_i'$ and thus that the sectional path from $\tau^m y$ to $x$ provided by the above argument is also on a different sequence of vertices. Thus, applying the above argument to every sectional path of length $m$ from $x$ to $y$, we obtain an injection from the set of sectional paths of length $m$ from $x$ to $y$ to the set of sectional paths of length $m$ from $\tau^m y$ to $x$.

A similar argument shows that whenever there is a sectional path

$$\tau^m y = y_0 \to y_1 \to \cdots \to y_m = x$$

in $\Gamma$ with multiplicities $l_1, l_2, \ldots, l_m$, then $\tau^{i-m} y_i$ is defined for all $i$ and there is a sectional path

$$x \to \tau^{-1} y_{m-1} \to \cdots \to \tau^{-1} y_1 \to \tau^m y = y_0$$

in $\Gamma$ with multiplicities $l_m, l_{m-1}, \ldots, l_1$ and as above we obtain an injection from the set of sectional paths of length $m$ from $\tau^m y$ to $x$ to the set of sectional paths of length $m$ from $x$ to $y$.

Since $\Gamma$ is locally finite, the number of sectional paths of fixed length between two vertices is finite. It follows that the number of sectional paths of length $m$ from
A GEOMETRIC DESCRIPTION OF $m$-CLUSTER CATEGORIES

We consider the construction of Section 6 in the case where $\Gamma$ is the quiver given by the diagonals of an $N$-gon $\Pi$, i.e. $\Gamma = \Gamma_{A_{N-3}}$ as in Section 2. Here, we fix $m = 1$, i.e. the vertices of the quiver are the usual diagonals of $\Pi$ and there is an arrow from $D$ to $D'$ if $D, D'$ have a common endpoint $i$ so that $D, D'$ together with the arc from $j$ to $j'$ between the other endpoints form a triangle and $D$ is rotated to $D'$ by a clockwise rotation about $i$. We will call this rotation $\rho_i$. Furthermore, we have introduced an automorphism $\tau_1$ of $\Gamma$: $\tau_1$ sends $D$ to $D'$ if $D$ can be rotated to

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$x$ to $y$ is the same as the number of sectional paths of length $m$ from $\tau^m y$ to $x$. Hence $(\Gamma^m, \tau^m)$ is a translation quiver.

We remark that the square of the translation quiver below, which does not satisfy the additional assumption of the theorem, is not a translation quiver:

\[
\begin{array}{ccc}
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xleftarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xleftarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\end{array}
\]

**Corollary 6.2.** (1) Let $(\Gamma, \tau)$ be a hereditary translation quiver. Then $(\Gamma^m, \tau^m)$ is a translation quiver.

(2) Let $(\Gamma, \tau)$ be a stable translation quiver. Then $(\Gamma^m, \tau^m)$ is a stable translation quiver.

**Proof.** Part (1) is immediate from Theorem 6.1 and the definition of a hereditary translation quiver. For (2), note that if $(\Gamma, \tau)$ is stable, no vertex is projective, so $(\Gamma^m, \tau^m)$ is a translation quiver by Theorem 6.1. Since $\tau$ is defined on all vertices of $\Gamma$, so is $\tau^m$.

We remark that the $m$th power of a hereditary translation quiver need not be hereditary: there can be non-projective vertices $z$ without any vertex $z'$ such that $z' \rightarrow z$. For example, consider the hereditary translation quiver below. It is clear that its square in the above sense has no arrows, but does have non-projective vertices.

\[
\begin{array}{ccc}
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\circ & \xrightarrow{\circ} & \circ \\
\end{array}
\]

However, we do have the following:

**Proposition 6.3.** Let $(\Gamma, \tau)$ be a translation quiver such that for any arrow $x \rightarrow y$ in $\Gamma$, $x$ is projective whenever $y$ is projective. Then the translation quiver $(\Gamma^m, \tau^m)$ has the same property.

**Proof.** We know by Theorem 6.1 that $(\Gamma^m, \tau^m)$ is a translation quiver. Suppose that

\[x_0 = x \rightarrow x_1 \rightarrow \cdots \rightarrow x_m = y\]

is a sectional path in $\Gamma$ and that $\tau^m x$ is defined, i.e. $x$ is not projective in $(\Gamma, \tau)$. Then $\tau x$ is defined, so $x$ is not projective in $(\Gamma, \tau)$. Hence $x_1, x_2, \ldots, x_m$ are not projective in $(\Gamma, \tau)$. Since there are arrows $x_{i-1} \rightarrow x_i$ for $i = 1, 2, \ldots, m$, there are arrows $\tau x_i \rightarrow x_{i-1}$ and therefore arrows $\tau x_{i-1} \rightarrow \tau x_i$ for $i = 1, 2, \ldots, m$. Repeating this argument we see that $\tau^m x_i$ is defined for all $i$. In particular, $\tau^m x_m$ is defined, so $y = x_m$ is not projective in $(\Gamma^m, \tau^m)$ and we are done.

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7. THE $m$-CLUSTER CATEGORY IN TERMS OF $m$TH POWERS

We consider the construction of Section 6 in the case where $\Gamma$ is the quiver given by the diagonals of an $N$-gon $\Pi$, i.e. $\Gamma = \Gamma_{A_{N-3}}$ as in Section 2. Here, we fix $m = 1$, i.e. the vertices of the quiver are the usual diagonals of $\Pi$ and there is an arrow from $D$ to $D'$ if $D, D'$ have a common endpoint $i$ so that $D, D'$ together with the arc from $j$ to $j'$ between the other endpoints form a triangle and $D$ is rotated to $D'$ by a clockwise rotation about $i$. We will call this rotation $\rho_i$. Furthermore, we have introduced an automorphism $\tau_1$ of $\Gamma$: $\tau_1$ sends $D$ to $D'$ if $D$ can be rotated to
$D'$ by an anticlockwise rotation about the centre of the polygon through $\frac{2\pi}{N}$. Then $\Gamma = \Gamma_{A_{N-3}}^1$ is a stable translation quiver (cf. Proposition 2.2).

The geometric interpretation of a sectional path of length $m$ from $D$ to $D'$ is given by the map $\rho^m$: $\rho^m_i$ sends the diagonal $D$ to $D'$ if $D$, $D'$ have a common endpoint $i$ and form an $m+2$-gon together with the arc between the other endpoints $j, j'$ and if $D$ can be rotated to $D'$ with a clockwise rotation about the common endpoint.

Furthermore, the $m$-th power $\tau^m_1$ of the translation $\tau_1$ corresponds to an anticlockwise rotation through $\frac{2m\pi}{N}$ about the centre of the polygon. From that one obtains:

**Proposition 7.1.** 1) The quiver $(\Gamma_{A_{N-3}}^1)^m$ contains a translation quiver of $m$-diagonals if and only if $N = nm + 2$ for some $n$.

2) $\Gamma_{A_{n-3}}^m$ is a connected component of $(\Gamma_{A_{n-3}}^1)^m$.

**Proof.** 1) Note that if $N \neq nm + 2$ (for some $n$) then $\Gamma_{A_{N-3}}^m$ contains no $m$-diagonals.

So assume that $N = nm + 2$ for some $n$. Let $\Gamma := \Gamma_{A_{N-3}}^1 = \Gamma_{A_{n-3}}^m$. We have to show that $\Gamma^m$ contains $Q := \Gamma_{A_{n-3}}^{m-1}$. Recall that the vertices of the quiver $\Gamma^m$ are the diagonals of an $nm + 2$-gon and that $Q$ is the quiver whose vertices are the $m$-diagonals of an $nm + 2$-gon. So the vertices of $Q$ are vertices of $\Gamma^m$.

We claim that the arrows between those vertices are the same for $Q$ and for $\Gamma^m$. In other words, we claim that there is a sectional path of length $m$ between $D$ and $D'$ if and only if $D$ can be rotated clockwise to $D'$ about a common endpoint and $D$ and $D'$ together with the arc joining the other endpoints bound an $(m + 2)$-gon.

Let $D \rightarrow D'$ be an arrow in $\Gamma^m$, where $D$ is the diagonal $(i, j)$ from $i$ to $j$. Without loss of generality, let $i < j$. The arrow $D \rightarrow D'$ in $\Gamma^m$ corresponds to a sectional path of length $m$ in $\Gamma$, $D \rightarrow D_1 \rightarrow \cdots \rightarrow D_{m-1} \rightarrow D_m = D'$. We describe such sectional paths. The first arrow is either $D = (i, j) \rightarrow (i, j + 1)$ or $(i, j) \rightarrow (i-1, j)$, i.e. $D_1 = (i, j+1)$ or $D_1 = (i-1, j)$ (vertices taken mod $N$). In the first case, one then gets an arrow $D_2 = (i, j + 1) \rightarrow (i+1, j + 1)$ or $D_1 = (i, j + 2)$. Now $\tau(i + 1, j + 1) = (i, j)$ and since the path is sectional, we get that $D_2$ can only be the diagonal $(i, j + 2)$.

\[
\begin{align*}
(i, j) & \rightarrow (i, j + 1) \\
(i, j + 1) & \rightarrow (i + 1, j + 1) \\
(i - 1, j) & \rightarrow (i - 2, j) \\
\end{align*}
\]

Repeatedly using the above argument, we see that the sectional path has to be of the form

$$D = D_0 = (i, j) \rightarrow (i, j + 1) \rightarrow (i, j + 2) \rightarrow \cdots \rightarrow (i, j + m) = D_m = D'$$

where all vertices are taken mod $N$.

Similarly, if $D_1 = (i - 1, j)$ then $D_2 = (i - 2, j)$ and so on, $D_m = (i - m, j)$ (mod $N$).

In particular, in the first case, the arrow $D \rightarrow D'$ corresponds to a rotation $\rho^m_i$ about the common endpoint $i$ of $D, D'$. In the second case, the arrow $D \rightarrow D'$ corresponds to $\rho^m_{i-1}$. In each case $D, D'$ and an arc between them bound an $(m + 2)$-gon, so there is an arrow from $D$ to $D'$ in $Q$.

Since it is clear that every arrow in $Q$ arises in this way, we see that the arrows between the vertices of $Q$ and of the corresponding subquiver of $\Gamma^m$ are the same.
2) We know by Proposition 2.3 that $Q = \Gamma_{A_{n-1}}^m$ is a connected stable translation quiver. If there is an arrow $D \to D'$ in $\Gamma^m$ where $D$ is an $m$-diagonal then $D'$ is an $m$-diagonal. Similarly, $r_i^m(D)$ is also an $m$-diagonal. □

**Theorem 7.2.** The $m$-cluster category $C_{A_{n-1}}^m$ is a full subcategory of the additive category generated by the mesh category of $(\Gamma_{A_{n+1}}^1)^m$.

**Proof.** This is a consequence of Proposition 7.1 and Theorem 5.6 □

**Remark 7.3.** Even if $\Gamma$ is a connected quiver, $\Gamma^m$ need not be connected. As an example we consider the quiver $\Gamma = \Gamma_{A_5}^1$ and its second power $(\Gamma_{A_5}^1)^2$ pictured in Figures 4 and 5. The connected components of $(\Gamma_{A_5}^1)^2$ are $\Gamma_{A_2}^2$ and two copies of a translation quiver whose mesh category is equivalent to $\text{ind}D^b(A_3)/[1]$ (where $D^b(A_3)$ denotes the derived category of a Dynkin quiver of type $A_3$). We thus obtain a geometric construction of a quotient of $D^b(A_3)$ which is not an $m$-cluster category.

**Figure 4.** The quiver $\Gamma_{A_5}^1$

**Figure 5.** The three components of $(\Gamma_{A_5}^1)^2$

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