A GEOMETRIC DESCRIPTION OF m-CLUSTER CATEGORIES

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Abstract. We show that the m-cluster category of type A_{n-1} is equivalent to a certain geometrically-defined category of diagonals of a regular $nm+2$-gon. This generalises a result of Caldero, Chapoton and Schiffler for $m=1$. The approach uses the theory of translation quivers and their corresponding mesh categories. We also introduce the notion of the mth power of a translation quiver and show how it can be used to realise the m-cluster category in terms of the cluster category.

Introduction

Let $n, m \in \mathbb{N}$ and let Π be a regular $nm+2$-sided polygon. We show that a category $\mathcal{C}^m_{A_{n-1}}$ of diagonals can be associated to Π in a natural way. The objects of $\mathcal{C}^m_{A_{n-1}}$ are the diagonals in Π which divide Π into two polygons whose numbers of sides are congruent to 2 modulo m, as considered in [PS]. A quiver $\Gamma^m_{A_{n-1}}$ can be defined on the set of such diagonals, with arrows given by a simple geometrical rule. It is shown that this quiver is a stable translation quiver in the sense of Riedtmann [Rie] with translation τ given by a certain rotation of the polygon. For a field k, the category $\mathcal{C}^m_{A_{n-1}}$ is defined as the mesh category associated to $(\Gamma^m_{A_{n-1}}, \tau)$.

Let Q be a Dynkin quiver of type A_{n-1}, and let $D^b(kQ)$ denote the bounded derived category of finite dimensional kQ-modules. Let τ denote the Auslander-Reiten translate of $D^b(kQ)$, and let S denote the shift. These are both autoequivalences of $D^b(kQ)$. Our main result is that $\mathcal{C}^m_{A_{n-1}}$ is equivalent to the quotient of $D^b(kQ)$ by the autoequivalence $\tau^{-1}S^m$. We thus obtain a geometric description of this category in terms of Π.

The m-cluster category $D^b(kQ)/\tau^{-1}S^m$ associated to kQ was introduced in [Kel] and has also been studied by Thomas [Tho], Wralsen [Wra] and Zhu [Zhu]. It is a generalisation of the cluster category defined in [CCS1] (for type A) and [BMRRT] (general hereditary case). Keller has shown that it is Calabi-Yau of dimension $m+1$ [Kel]. We remark that such Calabi-Yau categories have also been studied in [KR].

Our definition is motivated by and is a generalisation of the construction of the cluster category in type A given in [CCS1], where a category of diagonals of a polygon is introduced. The authors show that this category is equivalent to the cluster category associated to kQ. This can be regarded as the case $m=1$ here. The aim of the current paper is to generalise the construction of [CCS1] to the diagonals arising in the m-divisible polygon dissections considered in [PS]. Note that Tzanaki [Tza] has also studied such diagonals. We also remark that

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a connection between the m-cluster category associated to kQ and the diagonals considered here was given in [Tho].

We further show that if (Γ, τ) is any stable translation quiver then the quiver Γ^m with the same vertices but with arrows given by sectional paths in Γ of length m is again a stable translation quiver with translation given by τ^m. If (Γ, τ) is taken to be the Auslander-Reiten quiver of the cluster category of a Dynkin quiver of type A_{nm-1}, we show that Γ^m contains $\Gamma^m_{A_{n-1}}$ as a connected component; it follows that the m-cluster category is a full subcategory of the additive category generated by the mesh category of Γ^m.

Since Γ is known to have a geometric construction [CCS1], our definition provides a geometric construction for the additive category generated by the mesh category of any connected component of Γ^m. We give an example to show that this provides a geometric construction for quotients of $D^b(kQ)$ other than the m-cluster category.

1. Notation and Definitions

In [Tza], E. Tzanaki studied an abstract simplicial complex obtained by dividing a polygon into smaller polygons.

We recall the definition of an abstract simplicial complex. Let X be a finite set and $\triangle \subseteq \mathcal{P}(X)$ a collection of subsets. Assume that $\triangle$ is closed under taking subsets (i.e. if $A \in \triangle$ and $B \subseteq A$ then $B \in \triangle$). Then $\triangle$ is an *abstract simplicial complex* on the ground set X. The vertices of S are the single element subsets of $\triangle$ (i.e. $\{A\} \in \triangle$). The faces are the elements of $\triangle$, the facets are the maximal among those (i.e. the $A \in \triangle$ such that if $A \subseteq B$ and $B \in \triangle$ then $A = B$). The dimension of a face A is equal to $|A| - 1$ (where $|A|$ is the cardinality of A). The complex is said to be *pure* of dimension d if all its facets have dimension d.

Let Π be an $nm + 2$-gon, $m, n \in \mathbb{N}$, with vertices numbered clockwise from 1 to $nm + 2$. We regard all operations on vertices of Π modulo $nm + 2$. A diagonal D is denoted by the pair (i, j) (or simply by the pair ij if $1 \leq i, j \leq 9$). Thus (i, j) is the same as (j, i). We call a diagonal D in Π an *m-diagonal* if D divides Π into an $(mj + 2)$-gon and an $(m(n - j) + 2)$-gon where $j = 1, \dots, \lceil \frac{n-1}{2} \rceil$. Then Tzanaki defines the abstract simplicial complex $\triangle = \triangle^m_{A_{n-1}}$ on the m-diagonals of Π as follows.

The vertices of $\triangle$ are the m-diagonals. The faces of $\triangle^m_{A_{n-1}}$ are the sets of m-diagonals which pairwise do not cross. They are called *m-divisible dissections (of Π)*. Then the facets are the maximal collections of such m-diagonals. Each facet contains exactly $n - 1$ elements, so the complex $\triangle^m_{A_{n-1}}$ is pure of dimension $n - 2$.

The case $m = 1$ is the complex whose facets are triangulations of an $n + 2$-gon.

2. A stable translation quiver of diagonals

To $\triangle = \triangle^m_{A_{n-1}}$ we associate a category along the lines of [CCS1]. As a first step, we associate to the simplicial complex a quiver, called $\Gamma^m_{A_{n-1}}$. The vertices of the quiver are the m-diagonals in the defining polygon Π, i.e. the vertices of $\triangle^m_{A_{n-1}}$.

The arrows of $\Gamma^m_{A_{n-1}}$ are obtained in the following way:

Let D, D' be m-diagonals with a common vertex i of Π. Let j and j' be the other endpoints of D, respectively D'. The points i, j, j' divide the boundary of the polygon Π into three arcs, linking i to j, j to j' and j' to i. (We usually refer to a part of the boundary connecting one vertex to another as an arc.) If D, D' and the arc from j to j' form an $m + 2$-gon in Π and if furthermore, D can be rotated clockwise to D' about the common endpoint i, we draw an arrow from D to D' in $\Gamma^m_{A_{n-1}}$. (By this we mean that D can be rotated clockwise to the line through D'.)

Note that if D, D' are vertices of the quiver $\Gamma^m_{A_{n-1}}$ then there is at most one arrow between them.

Examples 2.4 and 2.5 below illustrate this construction.

We then define an automorphism τ_m of the quiver: let $\tau_m : \Gamma^m_{A_{n-1}} \to \Gamma^m_{A_{n-1}}$ be the map given by $D \mapsto D'$ if D' is obtained from D by an anticlockwise rotation through $\frac{2m\pi}{nm+2}$ about the centre of the polygon. Clearly, τ_m is a bijective map and a morphism of quivers.

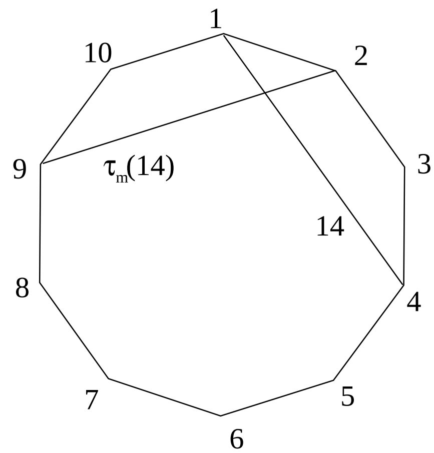


FIGURE 1. The translation τ_m, $\tau_m(14) = 92$, where $n = 4, m = 2$

Definition 2.1. *(1) A* translation quiver *is a pair* (Γ, τ) *where* Γ *is a locally finite quiver and* $\tau : \Gamma'_0 \to \Gamma_0$ *is an injective map defined on a subset* Γ'_0 *of the vertices of* Γ *such that for any* $X \in \Gamma_0$, $Y \in \Gamma'_0$, *the number of arrows from* X *to* Y *is the same as the number of arrows from* $\tau(Y)$ *to* X. *The vertices in* $\Gamma_0 \setminus \Gamma'_0$ *are called* projective. *If* $\Gamma'_0 = \Gamma_0$ *and* τ *is bijective,* (Γ, τ) *is called a* stable translation quiver.

(2) A stable translation quiver is said to be connected *if it is not a disjoint union of two non-empty stable subquivers.*

Proposition 2.2. *The pair* $(\Gamma = \Gamma^m_{A_{n-1}}, \tau_m)$ *is a stable translation quiver.*

Proof. By definition, τ_m is a bijective map from Γ to Γ, and Γ is a finite quiver. We have to check that the number of arrows from D to D' in Γ is the same as the number of arrows from $\tau_m D'$ to D. Since there is at most one arrow from one vertex to another, we only have to see that there is an arrow $D \to D'$ if and only if there is an arrow $\tau_m D' \to D$.

Assume that there is an arrow $D \to D'$, let i be the common vertex of D and D' in the polygon, $D = (i, j)$, $D' = (i, j + m)$. Then $\tau_m D' = (i - m, j)$. In particular, j is the common vertex of D and $\tau_m D'$. Furthermore, we obtain D from $\tau_m D'$ by a clockwise rotation about j and these two m-diagonals form an $m + 2$-gon together with an arc from $i - m$ to i, hence there is an arrow $\tau_m D' \to D$. See Figure 2.

The converse follows with the same reasoning. □

Proposition 2.3. (Γ, τ_m) *is a connected stable translation quiver.*

Proof. Note that every vertex of Π is incident with some element of any given τ_m-orbit of m-diagonals: any m-diagonal is of the form $(i, i + km + 1)$ and

$$\begin{aligned} \tau_m^{k-n}(i, i + km + 1) &= (i + (n - k)m, i + nm + 1) \\ &= (i + (n - k)m, i - 1). \end{aligned}$$

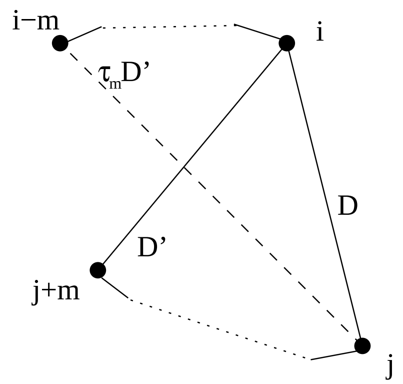


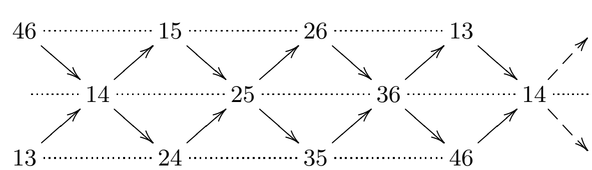
FIGURE 2. $D \to D' \iff \tau_m D' \to D$

Assume that Γ is the disjoint union of two non-empty stable subquivers. So there exist m-diagonals $D = (i,j)$ and $D' = (i',j')$ that cannot be connected by any path in Γ. After rotating D' using τ_m we can assume that $i = i'$. By assumption, $j' \neq j + rm$ for any r. Without loss of generality, $j < j'$. The diagonal D can be rotated clockwise about i to another m-diagonal $D'' = (i, j'')$ such that $j' = j'' + s$ with $0 < s < m$. Since D'' is an m-diagonal, the arc from i to j'' not including j' together with D'' bounds a $(um+2)$-gon for some u. But then the arc from i to j' including j'' together with the diagonal D' bound a $(um+2+s)$-gon where $um+2 < um+2+s < (u+1)m+2$. Hence D' cannot be an m-diagonal. $\square$

In the examples below we draw the quiver associated to the complex $\triangle^m_{A_{n-1}}$ in the standard way of Auslander-Reiten theory: the vertices and arrows are arranged so that the translation τ_m is a shift to the left. We indicate it by dotted lines.

Example 2.4. *Let $n = 4$, $m = 1$, i.e. Π is a 6-gon. The rotation group given by rotation about the centre of Π through $k \times \frac{\pi}{3}$ degrees ($k = 1, \dots, 5$) acts on the facets of $\triangle^1_{A_3}$. There are four orbits, $\mathcal{O}_{\{13,14,15\}}$ of size 6, $\mathcal{O}_{\{13,14,46\}}$ and $\mathcal{O}_{\{13,36,46\}}$ of size 3 and $\mathcal{O}_{\{13,15,35\}}$ with two elements, making a total of 14 elements.*

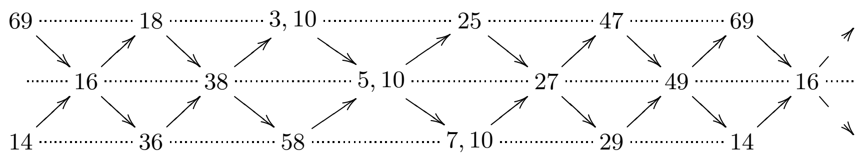
The vertices of the quiver $\Gamma^1_{A_3}$ are the nine 1-diagonals $\{13, 14, 15, 24, 25, 26, 35, 36, 46\}$ and we draw the quiver as follows:



Example 2.5. *Let $m = 2$ and $n = 4$, i.e. Π is a 10-gon. The rotation group is generated by the rotation about the centre of Π through $k \times \frac{2\pi}{5}$ degrees ($k = 1, \dots, 9$) and acts on the facets of $\triangle^2_{A_3}$. The orbits are $\mathcal{O}_{\{14,16,18\}}$, $\mathcal{O}_{\{14,18,47\}}$, $\mathcal{O}_{\{18,38,47\}}$ and $\mathcal{O}_{\{47,38,39\}}$ of size 10, and $\mathcal{O}_{\{14,16,69\}}$, $\mathcal{O}_{\{14,49,69\}}$ and $\mathcal{O}_{\{29,38,47\}}$ of size 5, making a total of 55 elements. The vertices of $\Gamma^2_{A_3}$ are the fifteen 2-diagonals*

$\{14, 16, 18, 25, 27, 29, 36, 38, (3,10), 47, 49, 58, (5,10), 69, (7,10)\}$

and the quiver is:



3. m-Cluster Categories

Let G be a simply-laced Dynkin diagram with vertices I. Let Q be a quiver with underlying graph G, and let k be an algebraically-closed field. Let kQ be the corresponding path algebra. Let $D^b(kQ)$ denote the bounded derived category of finitely generated kQ-modules, with shift denoted by S, and Auslander-Reiten translate given by τ. It is known that $D^b(kQ)$ is triangulated, Krull-Schmidt and has almost-split triangles (see [Hap]). Let $\mathbb{Z}Q$ be the stable translation quiver associated to Q, with vertices (n, i) for $n \in \mathbb{Z}$ and i a vertex of Q. For every arrow $\alpha : i \to j$ in Q there are arrows $(n, i) \to (n, j)$ and $(n, j) \to (n + 1, i)$ in $\mathbb{Z}Q$, for all $n \in \mathbb{Z}$. Together with the translation τ, taking (n, i) to $(n - 1, i)$, $\mathbb{Z}Q$ is a stable translation quiver. We note that $\mathbb{Z}Q$ is independent of the orientation of Q and can thus be denoted $\mathbb{Z}G$.

We recall the notion of the mesh category of a stable translation quiver with no multiple arrows (the mesh category is defined for a general translation quiver but we shall not need that here). Recall that for a quiver Γ, $k\langle\Gamma\rangle$ denotes the path category on Γ, with morphisms given by arbitrary k-linear combinations of paths.

Definition 3.1. *Let (Γ, τ) be a stable translation quiver with no multiple arrows. Let Y be a vertex of Γ and let $X_1, \ldots, X_k$ be all the vertices with arrows to Y, denoted $\alpha_i : X_i \to Y$. Let $\beta_i : \tau(Y) \to X_i$ be the corresponding arrows from $\tau(Y)$ to X_i $(i = 1, \ldots, k)$. Then the mesh ending at Y is defined to be the quiver consisting of the vertices $Y, \tau(Y), X_1, \ldots, X_k$ and the arrows $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $\beta_1, \beta_2, \ldots, \beta_k$. The mesh relation at Y is defined to be*

$$m_Y := \sum_{i=1}^{k} \beta_i \alpha_i \in \mathrm{Hom}_{k\langle\Gamma\rangle}(\tau(Y), Y)$$

Let J_m be the ideal in $k\langle\Gamma\rangle$ generated by the mesh relations m_Y where Y runs over all vertices of Γ.

Then the mesh category of Γ is defined as the quotient $k\langle\Gamma\rangle/J_m$.

For an additive category ε, denote by $\operatorname{ind}\varepsilon$ the full subcategory of indecomposable objects. Happel [Hap] has shown that $\operatorname{ind} D^b(kQ)$ is equivalent to the mesh category of $\mathbb{Z}Q$, from which it follows that it is independent of the orientation of Q. Its Auslander-Reiten quiver is $\mathbb{Z}G$.

For $m \in \mathbb{N}$, we denote by $\mathcal{C}_G^m$ the m-cluster category associated to the Dynkin diagram G, so

$$\mathcal{C}_G^m = \frac{D^b(kQ)}{F_m},$$

where Q is any orientation of G and F_m is the autoequivalence $\tau^{-1} \circ S^m$ of $D^b(kQ)$. This was introduced by Keller [Kel] and has been studied by Thomas [Tho], Wralsen [Wra] and Zhu [Zhu]. It is known that $\mathcal{C}_G^m$ is triangulated [Kel], Krull-Schmidt and has almost split triangles [BMRRT, 1.2,1.3]. Let φ_m denote the automorphism of $\mathbb{Z}G$ induced by the autoequivalence F_m. The Auslander-Reiten quiver of $\mathcal{C}_G^m$ is the quotient $\mathbb{Z}G/\varphi_m$, and $\operatorname{ind}\mathcal{C}_G^m$ is equivalent to the mesh category of $\mathbb{Z}G/\varphi_m$.

4. Coloured almost positive roots

Our main aim in the next two sections is to show that, if G is of type A_{n-1}, then $\mathrm{ind}\,\mathcal{C}_G^m$ is equivalent to the mesh category $\mathcal{D}^m_{A_{n-1}}$ of the stable translation quiver $\Gamma^m_{A_{n-1}}$ defined in the previous section. From the previous section we can see that it is enough to show that, as translation quivers, $\mathbb{Z}G/\varphi_m$ is isomorphic to $\Gamma^m_{A_{n-1}}$. In this section, we recall the discussion of m-diagonals and m-coloured almost positive roots in Fomin-Reading [FR].

4.1. **m-coloured almost positive roots and m-diagonals.** For Φ a root system, with positive roots Φ^+ and simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, let $\Phi^m_{\geq -1}$ denote the set of m-coloured almost positive roots (see [FR]). An element of $\Phi^m_{\geq -1}$ is either a m-coloured positive root α^k where $\alpha \in \Phi^+$ and $k \in \{1, 2, \ldots, m\}$ or a negative simple root $-\alpha_i$ for some i which we regard as having colour 1 for convenience (it is thus also denoted $-\alpha_i^1$). Fomin-Reading [FR] show that there is a one-to-one correspondence between m-diagonals of the regular $nm+2$-gon Π and $\Phi^m_{\geq -1}$ when Φ is of type A_{n-1}. We now recall this correspondence.

Recall that R_m denotes the anticlockwise rotation of Π taking vertex i to vertex $i-1$ for $i \geq 2$, and vertex 1 to vertex $nm+2$. For $1 \leq i \leq \frac{n}{2}$, the negative simple root -α_{2i-1} corresponds to the diagonal $((i-1)m+1, (n-i)m+2)$. For $1 \leq i \leq \frac{n-1}{2}$, the negative simple root $-\alpha_{2i}$ corresponds to the diagonal $(im+1, (n-i)m+2)$. Together, these diagonals form what is known as the *m-snake*, cf. Figure 3. For $1 \leq i \leq j \leq n$, there are exactly m m-diagonals intersecting the diagonals labelled $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ and no other diagonals labelled with negative simple roots. These diagonals are of the form $D, R_m^1(D), \ldots, R_m^{m-1}(D)$ for some diagonal D, and α^k corresponds to $R_m^{k-1}(D)$ for $k = 1, 2, \ldots, m$, where α denotes the positive root $\alpha_i + \cdots + \alpha_j$. For an m-coloured almost positive root β^k, we denote the corresponding diagonal by $D(\beta^k)$.

It is clear that, for $1 \leq i \leq \frac{n}{2}$, the coloured root α^1_{2i-1} corresponds to the diagonal $(im+1, (n+1-i)m+2)$. Also, the diagonals $D(-\alpha_i)$, for i even, together with $D(\alpha_j^1)$, for j odd, form a 'zig-zag' dissection of Π which we call the *opposite m-snake*, cf. Figure 3.

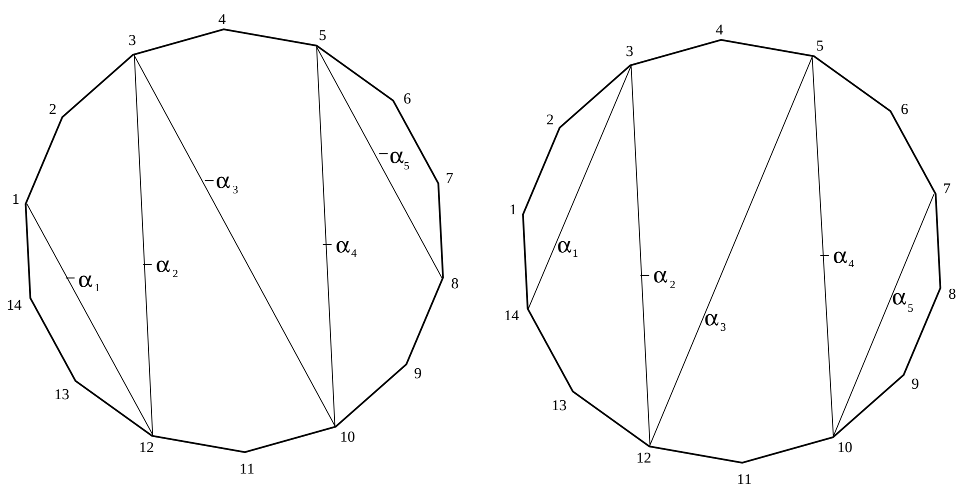


Figure 3. m-snake and opposite m-snake for $n = 6, m = 2$

Let $I = I^+ \cup I^-$ be a decomposition of the vertices I of G so that there are no arrows between vertices in I^+ or between vertices in I^-; such a decomposition exists because G is bipartite. For type A_{n-1}, we take I^+ to be the even-numbered vertices and I^- to be the odd-numbered vertices.

Let $R_m : \Phi^m_{\geq -1} \to \Phi^m_{\geq -1}$ be the bijection introduced by Fomin-Reading [FR, 2.3]. This is defined using the involutions [FZ2] $\tau_\pm : \Phi_{\geq -1} \to \Phi_{\geq -1}$ given by

$$\tau_\varepsilon(\beta) = \begin{cases} \alpha & \text{if } \beta = -\alpha_i, \text{ for } i \in I^{-\varepsilon} \\ \left(\prod_{i \in I^\varepsilon} s_i\right)(\beta) & \text{otherwise.} \end{cases}$$

Then, for $\beta^k \in \Phi^m_{\geq -1}$, we have

$$R_m(\beta^k) = \begin{cases} \beta^{k+1} & \text{if } \alpha \in \Phi^+ \text{ and } k < m, \\ ((\tau_-\tau_+)(\beta))^1 & \text{otherwise.} \end{cases}$$

Lemma 4.1 (FOMIN-READING). *For all $\beta^k \in \Phi^m_{\geq -1}$, we have: $D(R_m(\beta^k)) = R_m D(\beta^k)$.*

Proof. See the discussion in [FR, 4.1]. □

4.2. **Indecomposable objects in the m-cluster category and m-diagonals.** Let Q_{alt} denote the orientation of G obtained by orienting every arrow to go from a vertex in I^+ to a vertex in I^-, so that the vertices in I^+ are sources and the vertices in I^- are sinks.

For a positive root α, let $V(\alpha)$ denote the corresponding kQ_{alt}-module, regarded as an indecomposable object in $D^b(kQ_{alt})$. Then it is clear from the definition that the indecomposable objects in $\mathcal{C}^m_G$ are the objects $S^{k-1}V(\beta)$ for $k = 1, 2, \ldots, m$ and $\alpha \in \Phi^+$ and $S^{-1}I_i$ for I_i an indecomposable injective kQ_{alt}-module corresponding to the vertex $i \in I$ (all regarded as objects in the m-cluster category). Following Thomas [Tho] or Zhu [Zhu], we define $V(\alpha^k)$ to be $S^{k-1}V(\beta)$ for $k = 1, 2, \ldots, m$, $\alpha \in \Phi^+$, and $V(-\alpha_i) = S^{-1}I_i$ for $i \in I$.

We have:

Lemma 4.2 (THOMAS,ZHU). *For all $\beta^k \in \Phi^m_{\geq -1}$, $V(R_m\beta^k) \cong SV(\beta^k)$, where S denotes the autoequivalence of $\mathcal{C}^m_G$ induced by the shift on $D^b(kQ)$.*

Proof. See [Tho, Lemma 2] or [Zhu, 3.8]. □

5. An isomorphism of stable translation quivers

From the previous two sections, we see that in type A_{n-1}, we have a bijection D from $\Phi^m_{\geq -1}$ to the set of m-diagonals of Π and a bijection V from $\Phi^m_{\geq -1}$ to the objects of $\operatorname{ind} \mathcal{C}^m_{A_{n-1}}$ up to isomorphism, i.e. to the vertices of the Auslander-Reiten quiver of $\mathcal{C}^m_{A_{n-1}}$. Composing the inverse of D with V we obtain a bijection ψ from the set of m-diagonals of Π to $\operatorname{ind} \mathcal{C}^m_{A_{n-1}}$.

Lemma 5.1. *For every m-diagonal D of Π, we have that*

$$\psi(R_m(D)) \cong S\psi(D),$$

and therefore that

$$\psi(\tau_m(D)) \cong \tau(\psi(D)).$$

Proof. The first statement follows immediately from Lemmas 4.1 and 4.2. We can deduce from this that $\psi(\tau_m(D)) = \psi(R^m_m(D)) = S^m\psi(D)$ and thus obtain the second statement, since S^m coincides with τ on every indecomposable object of $\mathcal{C}^m_{A_{n-1}}$ by the definition of this category. □

It remains to show that ψ and ψ^{-1} are morphisms of quivers.

Lemma 5.2.
- *For $1 \leq i \leq \frac{n-1}{2}$, there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i-1})$ to $D(-\alpha_{2i})$.*

- *For $1 \leq i \leq \frac{n-1}{2}$, there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i+1})$ to $D(-\alpha_{2i})$.*
- *For $1 \leq i \leq \frac{n}{2}$, there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i})$ to $D(\alpha^1_{2i-1})$.*
- *For $1 \leq i \leq \frac{n-2}{2}$, there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i})$ to $D(\alpha^1_{2i+1})$.*

These are the only arrows amongst the diagonals $D(-\alpha_i)$ and $D(\alpha^1_j)$, for $1 \leq i, j \leq n-1$, with j odd, in $\Gamma^m_{A_{n-1}}$.

Proof. We firstly note that, for $1 \leq i \leq \frac{n-1}{2}$, the diagonals corresponding to the negative simple roots $-\alpha_{2i-1}$ and $-\alpha_{2i}$, together with an arc of the boundary containing vertices $(i-1)m+1, \ldots, im+1$, bound an $m+2$-gon. The other vertex is numbered $(n-i)m+2$. Furthermore, $D(-\alpha_{2i-1})$ can be rotated clockwise about the common end point $(n-i)m+2$ to $D(-\alpha_{2i})$, so there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i-1})$ to $D(-\alpha_{2i})$.

Similarly, for $1 \leq i \leq \frac{n-2}{2}$, the diagonals corresponding to the negative simple roots $-\alpha_{2i}$ and $-\alpha_{2i+1}$, together with an arc of the boundary containing vertices $(n-i-1)m+2, \ldots, (n-i)m+2$, bound an $m+2$-gon (with the other vertex being numbered $im+1$), and $D(-\alpha_{2i+1})$ can be rotated clockwise about the common end point $im+1$ to $D(-\alpha_{2i})$, so there is an arrow in $\Gamma^m_{A_{n-1}}$ from $D(-\alpha_{2i+1})$ to $D(-\alpha_{2i})$.

We have observed that, for $1 \leq i \leq \frac{n}{2}$, the coloured root α^1_{2i-1} corresponds to the diagonal $(im+1, (n+1-i)m+2)$. Consideration of the $m+2$-gon with vertices $(n-i)m+2, \ldots, (n+1-i)m+2$ and $im+1$ shows that there is an arrow from $D(-\alpha_{2i})$ to $D(\alpha^1_{2i-1})$. For $1 \leq i \leq \frac{n-2}{2}$, consideration of the $m+2$-gon with vertices $im+1, \ldots (i+1)m+1$ and $(n-i)m+2$ shows that there is an arrow from $D(-\alpha_{2i})$ to $D(\alpha^1_{2i+1})$.

The statement that these are the only arrows amongst the diagonals considered is clear. □

The following follows from the well-known structure of the Auslander-Reiten quiver of $D^b(kQ)$.

Lemma 5.3.
- *For $1 \leq i \leq \frac{n-1}{2}$, there is an arrow in the Auslander-Reiten quiver of $\mathcal{C}^m_{A_{n-1}}$ from $I_{2i-1}[-1]$ to $I_{2i}[-1]$.*
- *For $1 \leq i \leq \frac{n-2}{2}$, there is an arrow from $I_{2i+1}[-1]$ to $I_{2i}[-1]$.*
- *For $1 \leq i \leq \frac{n}{2}$, there is an arrow from $I_{2i}[-1]$ to P_{2i-1}.*
- *For $1 \leq i \leq \frac{n-2}{2}$ there is an arrow from $I_{2i}[-1]$ to P_{2i+1}.*

These are the only arrows amongst the vertices $I_i[-1]$ and P_j for $1 \leq i, j \leq n-1$, with j odd, in the Auslander-Reiten quiver of $\mathcal{C}^m_{A_{n-1}}$.

Proposition 5.4. *The map ψ from m-diagonals in Π to indecomposable objects in $\mathcal{C}^m_{A_{n-1}}$ is an isomorphism of quivers.*

Proof. Suppose that D, E are m-diagonals in Π and that there is an arrow from D to E. Write $D = D(\beta^k)$ and $E = D(\gamma^l)$ for coloured roots β^k and γ^l. Then $V := \psi(D) = V(\beta^k)$ and $W := \psi(E) = V(\gamma^l)$ are corresponding vertices in the Auslander-Reiten quiver of $\mathcal{C}^m_{A_{n-1}}$. Since there is an arrow from D to E, there is an $m+2$-gon bounded by D and E and an arc of the boundary of Π.

Since D is an m-diagonal, on the side of D not in the $m+2$-gon, there is a $dm+2$-gon bounded by D and an arc of the boundary of Π for some $d \geq 1$. Similarly, since E is an m-diagonal, on the side of E not in the $m+2$-gon, there is an $em+2$-gon bounded by D and an arc of the boundary of Π, for some $e \geq 1$. It is clear that each of these polygons can be dissected by an m-snake such that, together with D and E, we obtain a 'zig-zag' dissection χ of Π. Let v be one of its endpoints. The other endpoint of the diagonal containing v must be $v-m-1$ or $v+m+1$ (modulo $nm+2$).

In the first case, we have that for some $t \in \mathbb{Z}$, $R_m^t(v) = 1$ and R_m^t applied to χ is the m-snake. In the second case, we have that, for some $t \in \mathbb{Z}$, $R_m^t(v) = nm + 2$ and R_m^t applied to χ is the opposite m-snake. It follows from Lemma 5.3 that there is an arrow from $R_m^t(V)$ to $R_m^t(W)$ in the Auslander-Reiten quiver of $\mathcal{C}^m_{A_{n-1}}$, and hence from V to W.

Conversely, suppose that V, W are vertices of the Auslander-Reiten quiver of $\mathcal{C}^m_{A_{n-1}}$ and that there is an arrow from V to W. We can write $V = V(\beta^k)$ and $W = V(\gamma^l)$ for coloured roots β^k and γ^l. Let $D := \psi^{-1}(V) = D(\beta^k)$ and let $E := \psi^{-1}(W) = D(\gamma^l)$. It is clear that $\tau^u(V) \cong I_i[-1]$ for some i and some u. By Lemma 5.3, we must have that either $\tau^u(W) \cong I_{i\pm 1}[-1]$ or $\tau^u(W) \cong P_{i\pm 1}$. In the latter case we must have that i is even. Note that $S^{um}(V) \cong \tau^u(V)$ and $S^{um}(W) \cong \tau^u(W)$. It follows from Lemmas 5.1 and 5.2 that there is an arrow from $R_m^{um}(D)$ to $R_m^{um}(E)$ in $\Gamma^m_{A_{n-1}}$, and thus from D to E.

It follows that ψ is an isomorphism of quivers. □

Proposition 5.5. *There is an isomorphism ψ of translation quivers between the stable translation quiver $\Gamma^m_{A_{n-1}}$ of m-diagonals and the Auslander-Reiten quiver of the m-cluster category $\mathcal{C}^m_{A_{n-1}}$.*

Proof. This now follows immediately from Proposition 5.4 and Lemma 5.1. □

We therefore have our main result.

Theorem 5.6. *The m-cluster category $\mathcal{C}^m_{A_{n-1}}$ is equivalent to the additive category generated by the mesh category of the stable translation quiver $\Gamma^m_{A_{n-1}}$ of m-diagonals.*

We remark that a connection between the m-cluster category and the m-diagonals has been given in [Tho]. In particular, Thomas gives an interpretation of Ext-groups in the m-cluster category in terms of crossings of diagonals. However, Thomas does not give a construction of the m-cluster category using diagonals.

6. The m-th power of a translation quiver

In this section we define a new category in natural way in which the m-cluster category $\mathcal{C}^m_{A_{n-1}}$ will appear as a full subcategory. We start with a translation quiver Γ and define its m-th power.

Let Γ be a translation quiver with translation τ.

Let Γ^m be the quiver whose objects are the same as the objects of Γ and whose arrows are the sectional paths of length m. A path $(x = x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m = y)$ in Γ is said to be *sectional* if $\tau x_{i+1} \neq x_{i-1}$ for $i = 1, \ldots, m-1$ (for which τx_{i+1} is defined) (cf. [Rin]). Let τ^m be the m-th power of the translation, i.e. $\tau^m = \tau \circ \tau \circ \cdots \circ \tau$ (m times). Note that the domain of definition of τ^m is a subset of the domain of definition Γ_0' of τ.

Recall that a translation quiver is said to be *hereditary* (see [Rin]) if:

- for any non-projective vertex z, there is an arrow from some vertex z' to z;
- there is no (oriented) cyclic path of length at least one containing projective vertices, and
- If y is a projective vertex and there is an arrow $x \to y$, then x is projective.

The last condition is what we need to ensure that (Γ^m, τ^m) is again a translation quiver:

Theorem 6.1. *Let (Γ, τ) be a translation quiver such that if y is a projective vertex and there is an arrow $x \to y$, then x is projective. Then (Γ^m, τ^m) is a translation quiver.*

Proof. We prove the following statement by induction on m:
Suppose that there is a sectional path

$$x = x_0 \to x_1 \to \cdots \to x_m = y$$

in Γ and $\tau^m y$ is defined. Then $\tau^i x_i$ is defined for $i = 0, 1, \ldots, m$ and there is a sectional path

$$\tau^m y = \tau^m x_m \to \tau^{m-1} x_{m-1} \to \cdots \to \tau x_1 \to x = x_0$$

in Γ. Furthermore, if the multiplicities of arrows between consecutive vertices in the first path are $k_1, k_2, \ldots, k_m$, the multiplicities of arrows between consecutive vertices in the second path are $k_m, k_{m-1}, \ldots, k_1$.

This is clearly true for $m = 1$, since Γ is a translation quiver. Suppose it is true for $m - 1$, and that

$$x = x_0 \to x_1 \to \cdots \to x_m = y$$

is a sectional path in Γ. Since $\tau^{m-1} x_m$ is defined, we can apply induction to the section path:

$$x_1 \to x_2 \to \cdots \to x_m$$

to obtain that $\tau^{i-1} x_i$ is defined for $i = 1, 2, \ldots, m$ and that there is a sectional path

$$\tau^{m-1} x_m \to \tau^{m-2} x_{m-1} \to \cdots x_1$$

in Γ, with multiplicities $k_2, k_3, \ldots, k_m$. As $\tau^m x_m$ is defined, $\tau^{m-1} x_m$ is not projective, and it follows that $\tau^{i-1} x_i$ is not projective for $i = 1, 2, \ldots, m$ by our assumption. Therefore $\tau^i x_i$ is defined for $i = 1, 2, \ldots, m$. For $i = 2, 3, \ldots, m$, there are k_i arrows from $\tau^{i-1} x_i$ to $\tau^{i-2} x_{i-1}$. Therefore there are k_i arrows from $\tau^{i-1} x_{i-1}$ to $\tau^{i-1} x_i$. Thus there are k_i arrows from $\tau^i x_i$ to $\tau^{i-1} x_{i-1}$. As there are k_1 arrows from x_0 to x_1, there are k_1 arrows from τx_1 to x_0. If $\tau(\tau^i x_i) = \tau^{i+2} x_{i+2}$ for some i then $x_i = \tau x_{i+2}$, contradicting the fact that $x_0 \to x_1 \to \cdots \to y$ is sectional. It follows that

$$\tau^m x_m \to \tau^{m-1} x_{m-1} \to \cdots \to x_0 = x$$

is a sectional path with multiplicities of arrows $k_1, k_2, \ldots, k_m$ as required.

It follows that the number of sectional paths with sequence of vertices $x_0, x_1, \ldots, x_m$ is less than or equal to the number of sectional paths with sequence of vertices $\tau^m y = \tau^m x_m, \tau^{m-1} x_{m-1}, \ldots, \tau x_1, x_0 = x$.

Suppose that

$$x = x'_0 \to x'_1 \to \cdots \to x'_m = y$$

is a sectional path from x to y with a different sequence of vertices. Then $x_i \neq x'_i$ for some i, $0 < i < m$. It follows that $\tau^i x_i \neq \tau^i x'_i$ and thus that the sectional path from $\tau^m y$ to x provided by the above argument is also on a different sequence of vertices. Thus, applying the above argument to every sectional path of length m from x to y, we obtain an injection from the set of sectional paths of length m from x to y to the set of sectional paths of length m from $\tau^m y$ to x.

A similar argument shows that whenever there is a sectional path

$$\tau^m y = y_0 \to y_1 \to \cdots \to y_m = x$$

in Γ with multiplicities $l_1, l_2, \ldots, l_m$, then $\tau^{i-m} y_i$ is defined for all i and there is a sectional path

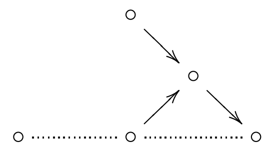
$$x \to \tau^{-1} y_{m-1} \to \cdots \to \tau^{m-1} y_1 \to \tau^m y = y_0$$

in Γ with multiplicities $l_m, l_{m-1}, \ldots, l_1$ and as above we obtain an injection from the set of sectional paths of length m from $\tau^m y$ to x to the set of sectional paths of length m from x to y.

Since Γ is locally finite, the number of sectional paths of fixed length between two vertices is finite. It follows that the number of sectional paths of length m from

x to y is the same as the number of sectional paths of length m from $\tau^m y$ to x. Hence (Γ^m, τ^m) is a translation quiver. □

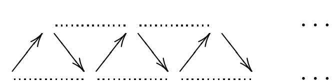
We remark that the square of the translation quiver below, which does not satisfy the additional assumption of the theorem, is not a translation quiver:



Corollary 6.2. *(1) Let (Γ, τ) be a hereditary translation quiver. Then (Γ^m, τ^m) is a translation quiver.*
(2) Let (Γ, τ) be a stable translation quiver. Then (Γ^m, τ^m) is a stable translation quiver.

Proof. Part (1) is immediate from Theorem 6.1 and the definition of a hereditary translation quiver. For (2), note that if (Γ, τ) is stable, no vertex is projective, so (Γ^m, τ^m) is a translation quiver by Theorem 6.1. Since τ is defined on all vertices of Γ, so is τ^m. □

We remark that the mth power of a hereditary translation quiver need not be hereditary: there can be non-projective vertices z without any vertex z' such that $z' \to z$. For example, consider the hereditary translation quiver below. It is clear that its square in the above sense has no arrows, but does have non-projective vertices.



However, we do have the following:

Proposition 6.3. *Let (Γ, τ) be a translation quiver such that for any arrow $x \to y$ in Γ, x is projective whenever y is projective. Then the translation quiver (Γ^m, τ^m) has the same property.*

Proof. We know by Theorem 6.1 that (Γ^m, τ^m) is a translation quiver. Suppose that

$$x_0 = x \to x_1 \to \cdots \to x_m = y$$

is a sectional path in Γ and that $\tau^m x$ is defined, i.e. x is not projective in (Γ^m, τ^m). Then τx is defined, so x is not projective in (Γ, τ). Hence $x_1, x_2, \ldots, x_m$ are not projective in (Γ, τ). Since there are arrows $x_{i-1} \to x_i$ for $i = 1, 2, \ldots, m$, there are arrows $\tau x_i \to x_{i-1}$ and therefore arrows $\tau x_{i-1} \to \tau x_i$ for $i = 1, 2, \ldots, m$. Repeating this argument we see that $\tau^m x_i$ is defined for all i. In particular, $\tau^m x_m$ is defined, so $y = x_m$ is not projective in (Γ^m, τ^m) and we are done. □

7. The m-cluster category in terms of mth powers

We consider the construction of Section 6 in the case where Γ is the quiver given by the diagonals of an N-gon Π, i.e. $\Gamma = \Gamma^1_{A_{N-3}}$ as in Section 2. Here, we fix $m = 1$, i.e. the vertices of the quiver are the usual diagonals of Π and there is an arrow from D to D' if D, D' have a common endpoint i so that D, D' together with the arc from j to j' between the other endpoints form a triangle and D is rotated to D' by a clockwise rotation about i. We will call this rotation ρ_i. Furthermore, we have introduced an automorphism τ_1 of Γ: τ_1 sends D to D' if D can be rotated to

D' by an anticlockwise rotation about the centre of the polygon through $\frac{2\pi}{N}$. Then $\Gamma = \Gamma^1_{A_{N-3}}$ is a stable translation quiver (cf. Proposition 2.2).

The geometric interpretation of a sectional path of length m from D to D' is given by the map ρ_i^m: ρ_i^m sends the diagonal D to D' if D, D' have a common endpoint i and form an $m{+}2$-gon together with the arc between the other endpoints j, j' and if D can be rotated to D' with a clockwise rotation about the common endpoint.

Furthermore, the m-th power τ_1^m of the translation τ_1 corresponds to a anti-clockwise rotation through $\frac{2m\pi}{N}$ about the centre of the polygon. From that one obtains:

Proposition 7.1. *1) The quiver $(\Gamma^1_{A_{N-3}})^m$ contains a translation quiver of m-diagonals if and only if $N = nm + 2$ for some n.*

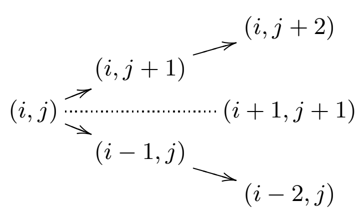
2) $\Gamma^m_{A_{n-1}}$ is a connected component of $(\Gamma^1_{A_{nm-1}})^m$.

Proof. 1) Note that if $N \neq nm{+}2$ (for some n) then $\Gamma^m_{A_{N-3}}$ contains no m-diagonals.

So assume that $N = nm + 2$ for some n. Let $\Gamma := \Gamma^1_{A_{N-3}} = \Gamma^1_{A_{nm-1}}$. We have to show that Γ^m contains $Q := \Gamma^m_{A_{n-1}}$. Recall that the vertices of the quiver Γ^m are the diagonals of an $nm + 2$-gon and that Q is the quiver whose vertices are the m-diagonals of an $nm + 2$-gon. So the vertices of Q are vertices of Γ^m.

We claim that the arrows between those vertices are the same for Q and for Γ^m. In other words, we claim that there is a sectional path of length m between D and D' if and only if D can be rotated clockwise to D' about a common endpoint and D and D' together with an arc joining the other endpoints bound an $(m + 2)$-gon.

Let $D \to D'$ be an arrow in Γ^m, where D is the diagonal (i, j) from i to j. Without loss of generality, let $i < j$. The arrow $D \to D'$ in Γ^m corresponds to a sectional path of length m in Γ, $D \to D_1 \to \cdots \to D_{m-1} \to D_m = D'$. We describe such sectional paths. The first arrow is either $D = (i, j) \to (i, j + 1)$ or $(i, j) \to (i{-}1, j)$, i.e. $D_1 = (i, j{+}1)$ or $D_1 = (i{-}1, j)$ (vertices taken mod N). In the first case, one then gets an arrow $D_1 = (i, j + 1) \to (i + 1, j + 1)$ or $D_1 \to (i, j + 2)$. Now $\tau(i + 1, j + 1) = (i, j)$ and since the path is sectional, we get that D_2 can only be the diagonal $(i, j + 2)$.



Repeatedly using the above argument, we see that the sectional path has to be of the form

$$D = D_0 = (i, j) \to (i, j + 1) \to (i, j + 2) \to \cdots \to (i, j + m) = D_m = D'$$

where all vertices are taken mod N.

Similarly, if $D_1 = (i - 1, j)$ then $D_2 = (i - 2, j)$ and so on, $D_m = (i - m, j)$ (mod N).

In particular, in the first case, the arrow $D \to D'$ corresponds to a rotation ρ_i^m about the common endpoint i of D, D'. In the second case, the arrow $D \to D'$ corresponds to ρ_j^m. In each case D, D' and an arc between them bound an $(m{+}2)$-gon, so there is an arrow from D to D' in Q.

Since it is clear that every arrow in Q arises in this way, we see that the arrows between the vertices of Q and of the corresponding subquiver of Γ^m are the same.

2) We know by Proposition 2.3 that $Q = \Gamma^m_{A_{n-1}}$ is a connected stable translation quiver. If there is an arrow $D \to D'$ in Γ^m where D is an m-diagonal then D' is an m-diagonal. Similarly, $\tau_1^m(D)$ is also an m-diagonal. □

Theorem 7.2. *The m-cluster category $\mathcal{C}^m_{A_{n-1}}$ is a full subcategory of the additive category generated by the mesh category of $(\Gamma^1_{A_{nm-1}})^m$.*

Proof. This is a consequence of Proposition 7.1 and Theorem 5.6 □

Remark 7.3. *Even if Γ is a connected quiver, Γ^m need not be connected. As an example we consider the quiver $\Gamma = \Gamma^1_{A_5}$ and its second power $(\Gamma^1_{A_5})^2$ pictured in Figures 4 and 5. The connected components of $(\Gamma^1_{A_5})^2$ are $\Gamma^2_{A_2}$ and two copies of a translation quiver whose mesh category is equivalent to* $\mathrm{ind} D^b(A_3)/[1]$ *(where $D^b(A_3)$ denotes the derived category of a Dynkin quiver of type A_3). We thus obtain a geometric construction of a quotient of $D^b(A_3)$ which is not an m-cluster category.*

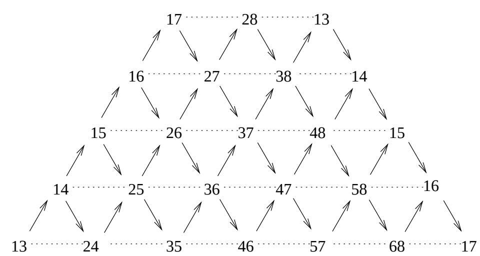


FIGURE 4. The quiver $\Gamma^1_{A_5}$

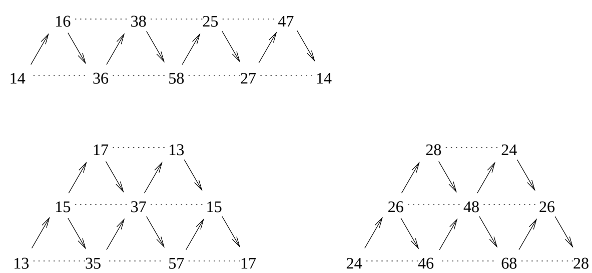


FIGURE 5. The three components of $(\Gamma^1_{A_5})^2$

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