Cluster-tilting theory

Aslak Bakke Buan and Robert Marsh

ABSTRACT. Cluster algebras were introduced by Fomin and Zelevinsky in order to understand the dual canonical basis of the quantized enveloping algebra of a quantum group and total positivity for algebraic groups. A cluster category is obtained by forming an appropriate quotient of the derived category of representations of a quiver. In this survey article, we describe the connections between cluster categories and cluster algebras, and we survey the representation-theoretic applications of cluster categories, in particular how they provide an extended version of classical tilting theory. We also describe a number of interesting new developments linking cluster algebras, cluster categories, representation theory and the canonical basis.

Introduction

Cluster categories are certain quotients of derived categories of hereditary algebras (or hereditary categories), and were introduced in [**BMRRT**]. A graphical description in type A_n in terms of triangulations of a polygon was given in [**CCS1**]. The aim was to model the cluster algebras introduced by Fomin and Zelevinsky [**FZ3**] using the representation theory of quivers. The definition in [**BMRRT**] arose from work by Marsh, Reineke and Zelevinsky [**MRZ**] linking cluster algebras and the representation theory of Dynkin quivers. This was a result of the combination of two approaches to the canonical basis [**Ka, Lu1, Lu3**] of a quantized enveloping algebra: via the representation theory of the preprojective algebra and via cluster algebras.

Cluster categories have led to new developments in the theory of the canonical basis and particularly its dual. They are providing insight into cluster algebras and their related combinatorics, and they have also been used to define a new kind of tilting theory, known as cluster-tilting theory, which generalises APR-tilting for hereditary algebras. A particular consequence is the definition of a new class of algebras closely related to hereditary algebras.

This survey article is organised as follows. In Section 1 we give an introduction to the theory of cluster algebras; in Section 2 we describe classical tilting theory

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for hereditary algebras and categories; in Section 3 we give motivation for and the definition of cluster categories; we also give an introduction to cluster-tilting theory and describe a graphical method for defining cluster categories in type A; in Section 4 we describe cluster-tilted algebras with particular emphasis on the case of finite representation type; in Section 5 we explain how cluster mutation can be described using cluster categories; in Section 6 we describe the links with the canonical basis of the quantized enveloping algebra of a simple Lie algebra; in Section 7 we describe a number of related developments in the field, including the links with Calabi-Yau categories.

1. Cluster algebras

Cluster algebras were first introduced in 2001 by Fomin and Zelevinsky [**FZ3**]. See [**FZ6, Ze2**] for survey articles, and also the lecture notes of Fomin and Reading [**FR**] for an introduction. Cluster algebras arose from the study of two related problems. The first problem is that of understanding the *canonical* [**Lu1, Lu3**], or *global crystal* [**Ka**], basis of a quantized enveloping algebra associated to a semisimple complex Lie algebra. It is expected that the positive part of the quantized enveloping algebra has a (quantum) cluster structure, with the cluster monomials forming part of the dual canonical basis. More details are given in Section 6.

The second problem is the description of totally positive matrices in a reductive algebraic group. An invertible matrix with real entries is said to be *totally positive* if all of its minors are positive. In fact total positivity was defined for all reductive groups by Lusztig [**Lu5**]. There are corresponding definitions of total positivity for related subgroups and varieties. To check total positivity for an upper uni-triangular matrix, only a certain collection of minors needs to be checked. The minimal sets of minors that need to be checked all have the same cardinality, and possess an intriguing property — often, when one minor is removed from a minimal set, there is a unique alternative minor which can be put in its place to obtain another such set. See [**BZ2, BFZ1**]. These two minors are related by a certain relation. This exchange, or *mutation*, was one of the motivations behind the definition of a cluster algebra; see [**FZ1, Ze1**]. In some cases there is no new minor to exchange; then other functions on the group arise. The algebraic and combinatorial (i.e. cluster) structure obtained in this way can be used to give a description of the entire coordinate ring of the algebraic group, or, in a similar fashion, one of its related varieties. For example, the coordinate rings of $SL_2(\mathbb{C})$ and $SL_3(\mathbb{C})$ are known to be cluster algebras [**FZ6**, Fig. 6], as are the coordinate rings of double Bruhat cells [**BFZ2**] and the homogeneous coordinate rings of the Grassmannians [**Sc**]; see also [**GSV1**].

Cluster algebras and their related combinatorics have, since then, been found to have many other connections and applications in algebra, geometry, combinatorics and physics: generalisations of the Stasheff polytopes (associahedra) to other Dynkin types [**CFZ, FZ2**]; solution [**FZ2**] of a conjecture of Zamolodchikov concerning Y-systems, a class of functional relations important in the theory of the thermodynamic Bethe ansatz; solution [**FZ4**] of various recurrence problems involving Laurent polynomials, including a conjecture of Gale and Robinson on the integrality of generalised Somos sequences; Poisson geometry [**GSV1**] and Teichmüller spaces [**GSV2**]; positive spaces and stacks [**FG**]; dual braid monoids [**Be**]; ad-nilpotent ideals of a Borel subalgebra of a simple Lie algebra [**P**], toric varieties

[**Ch**], as well as representation theory, e.g. [**BGZ, BMRRT, BMR1, BMR2, CC, CCS1, CCS2, GLS2, Ke, MRZ, Zh1, Zh2, Zh3**].

The general definition of a cluster algebra involves a base ring of coefficients. We remark that in the preprojective algebra model of [**GLS2**] the indecomposable projective modules can be regarded as playing the role of coefficients; see Section 6.5 for more details. However, for simplicity (and also because the coefficients do not appear in the cluster category), we give here the definition of a cluster algebra in which all of the coefficients are set equal to 1. Let $n \in \mathbb{N}$ and let $\mathbb{F} = \mathbb{Q}(u_1, u_2, \dots, u_n)$ be the field of rational functions in indeterminates $u_1, u_2, \dots u_n$. Let $\mathbf{x} \subset \mathbb{F}$ be a transcendence basis over $\mathbb{Q}$, and let $B = (b_{xy})_{x,y\in\mathbf{x}}$ be an $n \times n$ sign-skew-symmetric integer matrix with rows and columns indexed by the elements of $\mathbf{x}$. This means that for all $x, y \in \mathbf{x}$, $b_{xy} = 0$ if and only if $b_{yx} = 0$, that $b_{xy} > 0$ if and only if $b_{yx} < 0$, and that $b_{xx} = 0$. Such a pair $(\mathbf{x}, B)$ is called a *seed*. The cluster algebra [**FZ3, FZ5**] is a subring $\mathcal{A}(\mathbf{x}, B)$ of $\mathbb{F}$ associated to the seed $(\mathbf{x}, B)$. Given such a seed, and an element $z \in \mathbf{x}$, define a new element $z' \in \mathbb{F}$ via the *binary exchange relation*:

$$zz' = \prod_{x\in\mathbf{x}, b_{xz}>0} x^{b_{xz}} + \prod_{x\in\mathbf{x}, b_{xz}<0} x^{-b_{xz}}. \tag{1.1}$$

In such circumstances, we say that z, z' form an *exchange pair*. Let $\mathbf{x}' = \mathbf{x} \cup \{z'\} \setminus \{z\}$, a new transcendence basis of $\mathbb{F}$. Let B' be the *mutation* of the matrix B in direction z (as defined in [**FZ3**]). Then

$$b'_{xy} = \begin{cases} -b_{xy} & \text{if } x = z \text{ or } y = z, \\ b_{xy} + \frac{1}{2}(|b_{xz}|b_{zy} + b_{xz}|b_{zy}|) & \text{otherwise.} \end{cases}$$

The row and column labelled z in B are relabelled z' in B'. The pair $(\mathbf{x}', B')$ is called the *mutation* of the seed $\mathbf{x}$ in direction z. Let $\mathcal{S}$ be the set of seeds obtained by iterated mutation of $(\mathbf{x}, B)$. Then the set of *cluster variables* is, by definition, the union χ of the transcendence bases appearing in the seeds in $\mathcal{S}$, and the cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is the subring of $\mathbb{F}$ generated by χ. Up to isomorphism of cluster algebras, $\mathcal{A}(\mathbf{x}, B)$ does not depend on the initial choice $\mathbf{x}$ of transcendence basis, so can be denoted $\mathcal{A}_B$. The *exchange graph* of $\mathcal{A}_B$ has vertices corresponding to the clusters of $\mathcal{A}$ and edges corresponding to mutations.

If its matrix B is skew-symmetric, a seed $(\mathbf{x}, B)$ determines a quiver with vertices corresponding to its rows and columns, and b_{xy} arrows from vertex x to vertex y whenever $b_{xy} > 0$. If χ is finite, the cluster algebra $\mathcal{A}_B$ is said to be of finite type. In [**FZ5**], it is shown that, up to isomorphism, the cluster algebras of finite type can be classified by the Dynkin diagrams; they are precisely those for which there exists a seed whose corresponding quiver is of Dynkin type. There are other methods for determining if a cluster algebra has finite type: via subgraphs [**Se1**] or via Cartan matrices and related types of positive matrices [**BGZ**].

As might be expected, in the finite type case, there is a strong connection with the combinatorics of the corresponding root system of the simple Lie algebra [**FZ5**]. Let Δ be a simply-laced graph of Dynkin type, and let $\mathcal{A} = \mathcal{A}(\Delta)$ denote the corresponding cluster algebra. Let Φ denote the set of roots of the corresponding Lie algebra. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ denote the simple roots, and let Φ^+ denote the positive roots. Let

$$\Phi_{\geq -1} = \Phi^+ \cup \{-\alpha_1, -\alpha_2, \dots - \alpha_n\}$$

denote the set of *almost positive* roots. The cluster variables of $\mathcal{A}$ are in bijection with the almost positive roots, and we denote the cluster variable corresponding to $\alpha \in \Phi_{\geq -1}$ by x_α. Fomin and Zelevinsky associate a nonnegative integer $(\alpha||\beta)$, known as the *compatibility degree*, to each pair α, β of almost positive roots, which is defined in the following way. Let s_i be the Coxeter generator of the Weyl group of Φ corresponding to the simple root α_i, and let σ_i be the permutation of $\Phi_{\geq -1}$ defined as follows:

$$\sigma_i(\alpha) = \begin{cases} \alpha & \alpha = -\alpha_j,\ j \neq i, \\ s_i(\alpha) & \text{otherwise.} \end{cases}$$

Let $I = I^+ \sqcup I^-$ be a partition of the set of vertices I of Δ into completely disconnected subsets and define:

$$\tau_\pm = \prod_{i \in I^\pm} \sigma_i.$$

Then $(\ ||\ )$ is defined by setting $(-\alpha_i||\beta)$ to be the coefficient of α_i in β for $\beta \in \Phi^+$ and α_i any simple root, $(-\alpha_i|| - \alpha_j) = 0$ for all i, j, and by specifying that it is $\tau_\pm$-invariant. The clusters of $\mathcal{A}$ correspond to the maximal subsets of $\Phi_{\geq -1}$ which contain pairwise compatible elements; these subsets all have equal cardinality.

Example 1.1. We give an example to demonstrate these definitions. Let $n = 2$ and consider the skew-symmetric matrix $B := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $\mathcal{A} = \mathcal{A}_B$ denote the cluster algebra which is a subring of $\mathbb{Q}(u_1, u_2)$ with initial seed $(\mathbf{x}, B)$, where $\mathbf{x} = \{u_1, u_2\}$. Mutating at u_1, we obtain the new seed $(\mathbf{x}', B')$, where $\mathbf{x}' = (u_1', u_2)$, $u_1' = \frac{u_2+1}{u_1}$ and $B' = B^T$, the transpose of B. Note that $u_1 u_1' = u_2 + 1$ is the exchange relation in this case. Mutating this new seed at u_1' returns us to $(\mathbf{x}, B)$, since mutation is symmetric in this way. Thus at each stage we only have one new mutation that is possible; after doing this five times, we return again to $(\mathbf{x}, B)$, except that the entries in $\mathbf{x}$ have been reversed, and that there is a corresponding simultaneous permutation of the rows and columns of B. Since an ordering on the elements of a seed is not specified, this is the same as the original seed. It follows that the exchange graph is a pentagon. The result is shown in Figure 1. We note that the five cluster variables are:

$$u_1 = \frac{1}{u_1^{-1}}, u_2 = \frac{1}{u_2^{-1}}, \frac{u_2+1}{u_1}, \frac{u_1+1}{u_2}, \frac{u_1+u_2+1}{u_1 u_2},$$

where in each case the numerator is coprime to u_1 and u_2. Considering the exponents of u_1 and u_2 as coefficients of the simple roots α_1 and α_2 in the root system of type A_2, we can see the correspondence between the cluster variables and the almost positive roots. The almost positive roots corresponding to the above cluster variables are $-\alpha_1, -\alpha_2, \alpha_1, \alpha_2$ and $\alpha_1 + \alpha_2$ respectively. See Section 6 for more details of this phenomenon.

See Figure 2 for the exchange graph in type A_3. We note that the exchange graph can be seen as the 1-skeleton of the associahedron; see [**CFZ, FZ2**].

2. Tilting theory for hereditary algebras (and categories)

In this section we briefly review a few basic facts about tilting theory for hereditary algebras [**BB, HR**]. Let H be a hereditary finite-dimensional algebra over a field k. We can assume that H is basic, and thus H is (isomorphic to) the path

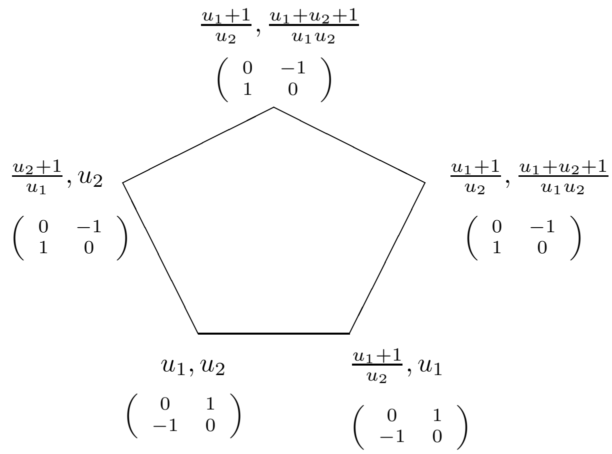


FIGURE 1. The seeds of the cluster algebra of type A_2.

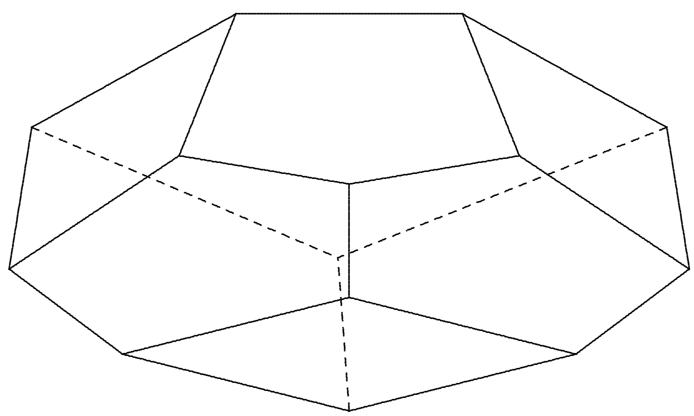


FIGURE 2. The exchange graph in type A_3.

algebra kQ for some finite quiver Q with no oriented cycles. Let $\operatorname{mod} H$ be the category of finite dimensional (left) modules over H. A module T in $\operatorname{mod} H$ is called *tilting* if $\operatorname{Ext}^1_H(T,T) = 0$, and if there is an exact sequence $0 \to H \to T_1 \to T_2 \to 0$, with T_1, T_2 in $\operatorname{add} T$. Here $\operatorname{add} T$ is the smallest additive subcategory of $\operatorname{mod} H$ containing T. We are usually only interested in the multiplicity-free tilting modules, so from now on we assume that any two indecomposable direct summands of a tilting module T are non-isomorphic. Then the number of indecomposable direct summands of T, denoted $\delta(T)$, is equal to the number n of vertices of Q. Bongartz showed [**Bo**] that in fact a module T with $\mathrm{Ext}^1_H(T,T) = 0$ is a tilting module if and only if $\delta(T) = n$.

For a hereditary algebra H and a tilting module T, there are associated torsion theories $(\mathcal{T},\mathcal{F})$ in $\operatorname{mod} H$, where $\mathcal{T} = \operatorname{Fac} T$, and $(\mathcal{X},\mathcal{Y})$ in $\operatorname{mod}\operatorname{End}_H(T)^{\mathrm{op}}$, such that there are equivalences of categories $\operatorname{Hom}_H(T,\ )\colon \mathcal{T} \to \mathcal{Y}$ and $\mathrm{Ext}^1_H(T,\ )\colon \mathcal{F} \to \mathcal{X}$; see [**BB**]. Here $\operatorname{Fac} T$ denotes the full subcategory of $\operatorname{mod} H$ with objects given by factors of modules in $\operatorname{add} T$. In addition there is an induced equivalence of derived categories (see [**Ha**])

$$\mathrm{RHom}_H(T,\ )\colon D^b(\operatorname{mod} H) \to D^b(\operatorname{mod}\operatorname{End}_H(T)^{\mathrm{op}}).$$

Let T be a tilting module in $\operatorname{mod} H$. Then $\Lambda := \operatorname{End}_H(T)^{\mathrm{op}}$ is called a tilted algebra.

This nice relationship between $\operatorname{mod} H$ and $\operatorname{mod}\Lambda$ is one of the main motivations for tilting theory.

2.1. APR-tilting. A special case of tilting is so-called APR-tilting [**APR**]. The notion of APR-tilting modules had its origin in BGP-reflection functors [**BGP**], occurring in the proof by Bernstein, Gelfand and Ponomarev of Gabriel's theorem [**G**].

Consider two quivers Q and Q', where Q' can be obtained from Q by reflection at a source or at a sink in Q. That is, let v be a source or a sink in Q, and let Q' be the quiver obtained from Q by reversing all arrows incident with v. Let v' denote the resulting sink or source in Q'.

One wants to compare the module categories of the path algebras $H = kQ$ and $H' = kQ'$. Assume v is a sink. Then there is a simple projective H-module S. Assume $H = P \amalg S$ as an H-module. Then one can show that $T = P \amalg \tau^{-1} S$ is a tilting module, and that $kQ' \simeq \operatorname{End}_H(T)^{\mathrm{op}}$. Using the results on torsion theories from the previous subsection, one obtains that $\operatorname{ind} H \setminus \{S\}$ is equivalent to $\operatorname{ind} H' \setminus \{S'\}$, where S' is the simple H'-module corresponding to v' and $\operatorname{ind} H$ denotes the full subcategory of $\operatorname{mod} H$ consisting of the indecomposable H-modules, similarly for H'.

2.2. Complements and the tilting graph. Another direction within tilting theory has been the study of combinatorial structures on the set of (basic) tilting modules. An H-module U with $\mathrm{Ext}^1_H(U,U) = 0$ can always be completed to a tilting module [**Bo**], i.e. there is a module X (called a complement) such that $U \amalg X$ is a tilting module. If $\delta(U) = n-1$, then U is called an almost complete tilting module, and then U has (up to isomorphism) at most two complements [**HU1**]. This implies that there is a natural combinatorial structure on the set of basic tilting modules. Riedtmann and Schofield [**RS**] assigned to the set of basic tilting modules a graph $\mathcal{G}_H$ with vertices corresponding to the (basic) tilting modules, and with an edge $T \to T'$, whenever $T = U \amalg X$ and $T' = U \amalg Y$ for some almost complete tilting module U. This graph is in general not connected, an easy example of this is the path algebra H of the Kronecker quiver. Here the graph $\mathcal{G}_H$ has two infinite components, both of the form

$$\cdot \text{———} \cdot \text{———} \cdot \text{———} \cdots$$

One component corresponds to preprojective tilting modules, and one to preinjective tilting modules.

2.3. Tilting for hereditary categories. A more general set-up for tilting theory is the following: Let $\mathcal{H}$ be a hereditary abelian category with finite dimensional Hom-spaces and Ext-spaces (over some field k). An object T is a tilting object if $\mathrm{Ext}^1_{\mathcal{H}}(T,T) = 0$, and T is maximal with respect to this property. We suppose that $\mathcal{H}$ has a tilting object T. The category $D^b(\mathcal{H})$ has Serre-duality (and hence almost split triangles). The endomorphism rings of tilting objects are called *quasi-tilted algebras*. These concepts were defined and studied in [**HRS**].

There is a combinatorial structure on the set of tilting objects for $\mathcal{H}$, as in the module case. Here, we note one interesting property. The following was shown by Happel and Unger [**HU2**], and is useful later (see Section 3.4).

THEOREM 2.1. *Assume the hereditary category $\mathcal{H}$ is not equivalent to a module category, but $\mathcal{H}$ is derived equivalent to a module category. Then the tilting graph $\mathcal{G}_{\mathcal{H}}$ is connected.*

3. Cluster categories

3.1. Motivation: Decorated Representations. In [**MRZ**] links between the theory of cluster algebras and tilting theory were first discovered. Let us here review some results from that paper.

As we saw in Section 1, the cluster variables of a cluster algebra of finite type are parametrized by the almost positive roots of the corresponding root system. Gabriel's Theorem states that the indecomposable representations of a quiver of Dynkin type are parametrized by the positive roots in the corresponding root system. This already suggests that there might be some representation-theoretic interpretation of cluster algebras. Such a connection was made in the paper [**MRZ**] in terms of *decorated representations* of a quiver of Dynkin type. The key idea in this paper was to model the combinatorics of the cluster algebra via the indecomposable representations of a modified version of the Dynkin quiver.

Let Q be a Dynkin quiver (with underlying unoriented graph given by a disjoint union of Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8), and let I be the set of vertices. Let k be an algebraically closed field. Let $\widetilde{Q}$ be the quiver Q with an extra vertex i^- for each $i \in I$, with no arrows incident with the new vertices. Then a *decorated representation* of Q is just a representation of $\widetilde{Q}$. Such a representation must be of the form $M = M^+ \oplus V$, where M is a representation of Q and V is an I-graded vector space. The signed dimension vector of such a representation is defined to be:

$$\underline{\mathrm{sdim}}(M) = \underline{\mathrm{dim}}(M^+) - \sum_{i \in I} (\dim V_i)\alpha_i,$$

where $\underline{\mathrm{dim}}(M^+)$ is the usual dimension vector of M^+ in the root lattice:

$$\underline{\mathrm{dim}}(M^+) = \sum_{i \in I} \dim(M_i^+)\alpha_i,$$

and M_i^+ is the vector space at vertex i in the representation M^+. The following follows from Gabriel's Theorem:

PROPOSITION 3.1. *The map $\underline{\mathrm{sdim}}$ induces a bijection between the set of indecomposable decorated representations of Q (up to isomorphism) and the set of almost positive roots $\Phi_{\geq -1}$.*

For $\alpha \in \Phi_{\geq -1}$, let $M(\alpha)$ denote the indecomposable representation of $\widetilde{Q}$ such that $\underline{\mathrm{sdim}}(M_\alpha) = \alpha$. MRZ define a bifunctor E_{kQ} on the category of decorated representations of Q in the following way. Let $M = M^+ \oplus V$ and $N = N^+ \oplus W$ be decorated representations of $\widetilde{Q}$. Then set

$$E_{kQ}(M,N) := \mathrm{Ext}^1_{kQ}(M^+,N^+) \oplus \mathrm{Ext}^1_{kQ}(N^+,M^+)^* \oplus \\ \oplus \mathrm{Hom}^I(M^+,W) \oplus \mathrm{Hom}^I(V,N^+),$$

where Hom^I denotes homomorphisms as I-graded vector spaces.

Let Q_{alt} be an alternating quiver of Dynkin type (i.e. so that each vertex is a source or a sink). Marsh, Reineke and Zelevinsky prove the following result:

Theorem 3.2. *Let* $\alpha \in \Phi^{-1}$. *Then* $(\alpha||\beta) = \dim E_{kQ_{\mathrm{alt}}}(M(\alpha), M(\beta))$.

It follows that the map $\underline{\mathrm{sdim}}$ induces a bijection between the maximal E_{kQ}-orthogonal collections of indecomposable decorated representations and the clusters of the corresponding finite type cluster algebra (regarded as sets of almost positive roots). The theorem is proved using the reflection functors of Bernstein, Gelfand and Ponomarev [**BGP**]. Restricting to the positive roots only, this gives a bijection between basic tilting modules over kQ and the set of *positive* clusters for the cluster algebra (i.e. those clusters consisting entirely of positive roots). This approach also provides a representation-theoretic interpretation of the *generalized associahedra* of [**CFZ, FZ2**].

The bifunctor E_{kQ} is a symmetrized version of the Ext^1_{kQ} bifunctor. In fact, E_{kQ} can be interpreted as the Ext^1 bifunctor in an appropriate triangulated quotient of the derived category of representations of kQ, known as the cluster category [**BMRRT**]. The appearance of tilting modules in [**MRZ**] suggests connections with tilting theory, and the (appropriately defined) "cluster-" tilting objects in the cluster category correspond to the clusters in the corresponding cluster algebra. The connections with tilting theory are in fact very strong, and the cluster category can be used to define a generalisation of APR-tilting [**APR**]; see [**BMR1**]. In the next sections of this article, we will give a definition of the cluster category and describe its main properties and the new tilting theory arising from it.

3.2. Definition and main properties. In this section cluster categories are defined, and the main properties are summarized. The results here are all from [**BMRRT**].

Let H be a hereditary algebra. We assume that H is basic and given as a path algebra kQ over a field k, where Q is a finite quiver with no oriented cycles. Let $\mathcal{D} = D^b(\mathrm{mod}\, H)$ be the bounded derived category of H. The objects of $\mathcal{D}$ are bounded complexes of modules, and there is a shift functor [1] which is the suspension functor in a triangulated structure on $\mathcal{D}$. It was shown by Happel [**Ha**] that $\mathcal{D}$ has Auslander-Reiten triangles, and that there is a auto-equivalence τ on $\mathcal{D}$, such that $D\mathrm{Ext}^1_{\mathcal{D}}(X,Y) \simeq \mathrm{Hom}_{\mathcal{D}}(Y,\tau X)$. Especially, the AR-sequences in $\mathrm{mod}\, H$ give rise to AR-triangles in $\mathcal{D}$, using the canonical embedding. For an indecomposable projective module P, we have $\tau P = I[-1]$, where I is an indecomposable with socle P/JP. Here J denotes the Jacobson radical.

Consider the auto-equivalence $F = \tau^{-1}[1]$. The cluster-category $\mathcal{C}_H$ is defined as the orbit category $\mathcal{D}/F$. The objects of $\mathcal{C}$ are the objects of $\mathcal{D}$, while maps in $\mathcal{C}$ are given by $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \amalg_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, F^iY)$. Let us list some of the basic properties of $\mathcal{C}$.

(i) $\mathcal{C}$ is a Krull-Schmidt category.
(ii) $\mathcal{C}$ is triangulated, and the canonical functor $\mathcal{D} \to \mathcal{C}$ is a triangle-functor (this is due to Keller [**Ke**]). We denote the suspension functor on $\mathcal{C}$ also by [1].
(iii) $\mathcal{C}$ has AR-triangles induced by the AR-structure on $\mathcal{D}$. We denote the induced AR-translate on $\mathcal{C}$ also by τ.
(iv) For any objects X, Y in $\mathcal{C}$, we have $D\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, \tau X)$.

An especially nice property is the following Ext-symmetry.

Proposition 3.3. *For X, Y in $\mathcal{C}$, we have* $\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \simeq D\mathrm{Ext}^1_{\mathcal{C}}(Y,X)$.

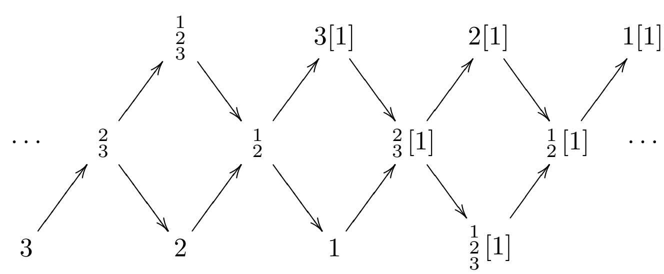
Given $D^b(H)$ for a hereditary algebra H, there is a canonical functor $\pi\colon D^b(H) \to \mathcal{C}_H$. It is clear from the definition of F that every indecomposable object in $\mathcal{C}$ is isomorphic to an object $\pi(X)$ for an object X in $\mathrm{mod}\, H \vee \mathcal{P}[1]$. That is, X is either in degree 0 or of the form P[1] for an indecomposable projective module P.

Note that for a hereditary abelian category $\mathcal{H}$ with tilting object and extra properties as mentioned in Section 2.3, we can also form the cluster category $\mathcal{C}_{\mathcal{H}} = D^b(\mathcal{H})/F$. Then $\mathcal{C}_{\mathcal{H}}$ is also triangulated, by Keller [**Ke**].

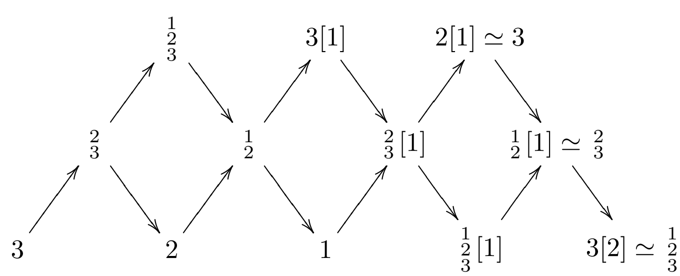
3.3. Example. Let us end with an example of a cluster category. We consider the cluster category of the path algebra $H = kQ$ where the quiver Q is A_3 with the linear orientation:

$$1 \longrightarrow 2 \longrightarrow 3$$

The AR-quiver of $D^b(H)$ is:



From this we can find the AR-quiver of $\mathcal{C}$



with identifications as indicated in the figure.

3.4. Tilting theory in cluster categories. One of the motivations in [**BMRRT**] for defining cluster categories was to study tilting theory in such categories. The set of tilting objects in $\mathcal{C}$ has a combinatorial structure which has strong links to cluster algebras. We will see that the tilting theory in $\mathcal{C}$ naturally extends tilting theory for $\operatorname{mod} H$.

An object T in $\mathcal{C}$ is called a *tilting object* if $\operatorname{Ext}^1_{\mathcal{C}}(T,T)=0$, and T is maximal with respect to this property. That is, $\operatorname{Ext}^1_{\mathcal{C}}(T \amalg X, T \amalg X)=0$ implies that X is in $\operatorname{add} T$, where $\operatorname{add} T$ is the smallest additive subcategory containing T. In fact, we only consider basic tilting objects, where basic means that if $T = \amalg_i T_i$, with T_i indecomposable, then $T_i \not\simeq T_j$ for $i \neq j$. The algebras obtained as $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, for a tilting object T, are called *cluster-tilted algebras*. Given $\mathcal{C} = \mathcal{C}_H$, there is a canonical map from $\operatorname{mod} H' \to \mathcal{C}$ for every hereditary algebra H' derived equivalent to H. These maps send tilting modules to tilting objects, and in fact every tilting object in $\mathcal{C}$ arises in this way.

Theorem 3.4. *Let T be a basic tilting object in $\mathcal{C}_H$, where H is a hereditary algebra with n simple modules. Then T has n indecomposable direct summands, and T is induced by a tilting module over a hereditary algebra H', where H' is derived equivalent to H. On the other hand, every tilting module T in* $\operatorname{mod} H$ *induces a tilting object in $\mathcal{C}_H$.*

Recall from Section 2 that we defined almost complete tilting modules $\bar{T}$ over a hereditary algebra H, and that such a module $\bar{T}$ has either one or two complements. An *almost complete tilting object* in $\mathcal{C}_H$ is an object $\bar{T}$ with $\operatorname{Ext}^1_{\mathcal{C}}(\bar{T},\bar{T})=0$ and such that there is an indecomposable object X (called a complement), such that $\bar{T} \amalg X$ is a tilting object. In the cluster category there are always exactly two complements.

Theorem 3.5. *Let H be a hereditary algebra and let $\bar{T}$ be an almost complete tilting object in $\mathcal{C}_H$. Then $\bar{T}$ has exactly two complements (up to isomorphism).*

Let us now fix an almost complete tilting object $\bar{T}$, and let M and M^* be the two (non-isomorphic) complements. For an indecomposable object X, consider the division ring $D_X = \operatorname{End}_{\mathcal{C}}(X)/\operatorname{rad}(X,X)$. Here $\operatorname{rad}(X,X)$ is the ideal of all non-isomorphisms. The following was also proved in [**BMRRT**].

Theorem 3.6. *If M and M^* are two complements of some almost complete tilting object, then* $\operatorname{Ext}^1_{\mathcal{C}}(M,M^*)$ *has dimension 1 over both D_M and D_{M^*}. On the other hand, if M, M^* are two indecomposable objects such that* $\operatorname{Ext}^1_{\mathcal{C}}(M,M^*)$ *has dimension 1 over both D_M and D_{M^*}, then there is an almost complete tilting object $\bar{T}$, such that M and M^* are the complements of $\bar{T}$.*

A pair of indecomposable objects M, M^* with the property in the Theorem is called an *exchange pair*.

There are corresponding triangles in $\mathcal{C}$, uniquely defined up to isomorphism, known as *connecting triangles*, having some interesting properties. There is one triangle $M^* \to B \to M \to$, and one triangle $M \to B' \to M^* \to$, with B, B' both in $\operatorname{add}(\bar{T})$. The proof of this result was motivated by similar results due to Happel and Unger in the algebra case, as referred to in Section 2. These triangles play an important role in Section 5.

The tilting graph $\mathcal{G}_{\mathcal{C}}$ of the cluster category is defined as in the module case. The vertices are the tilting objects, and the edges correspond to almost complete tilting objects which can be complemented to the tilting objects corresponding to the starting and ending vertices. The proof of the following depends on Theorem 2.1.

THEOREM 3.7. *The tilting graph $\mathcal{G}_{\mathcal{C}}$ is connected.*

3.5. Example. We consider the cluster category of kQ, when Q is the Dynkin diagram of type A_3 with any orientation. Then $\mathcal{C}$ has 14 tilting objects and the tilting graph is isomorphic to the exchange graph of type A_3 shown in Figure 2.

3.6. Links to cluster algebras. Given a finite quiver Q with no oriented cycles, we have seen that there is a corresponding (acyclic) cluster algebra $\mathcal{A}(Q)$. On the other hand, we can define the cluster category of Q as $\mathcal{C}_{\mathbb{C}Q}$. Let $H = \mathbb{C}Q$ denote the path algebra. It is known by work of Fomin and Zelevinsky [**FZ5**], that $\mathcal{A}(Q)$ is of finite type (i.e. there is only a finite number of cluster variables) if and only if Q is a mutation equivalent to a Dynkin quiver. It is also well known and easy to see that there are only a finite number of tilting modules in $\operatorname{mod} H$ (equivalently in $\mathcal{C}_H$) if and only if Q is Dynkin.

It is also clear from the above sections that there is some stronger analogy between the combinatorial definition of $\mathcal{A}(Q)$, and the combinatorics of the tilting graph $\mathcal{G}_{\mathbb{C}Q}$. We have seen that for $Q = A_3$, the exchange graph of the cluster category and the tilting graph are isomorphic. The explanation for this is the following.

THEOREM 3.8. *Let Q be a (simply-laced) Dynkin quiver, and let $\mathcal{A} = \mathcal{A}(Q)$ and $\mathcal{C} = \mathcal{C}_{\mathbb{C}Q}$ be the cluster algebra and the cluster category of Q, respectively. Then there is a 1–1 correspondence between the cluster variables of $\mathcal{A}$ and the indecomposable objects of $\mathcal{C}$. This correspondence induces a 1–1 correspondence between the clusters of $\mathcal{A}$ and the tilting objects of $\mathcal{C}$.*

This result is from [**BMRRT**], where it also was conjectured that this can be generalized. For an arbitrary finite, acyclic quiver Q, the conjectured correspondence is between the cluster variables and the exceptional objects of $\mathcal{C}$, where an indecomposable object X in $\mathcal{C}$ is called exceptional if $\operatorname{Ext}^1_{\mathcal{C}}(X, X) = 0$. The fact that the exchange graph of $\mathcal{C}$ is connected supports this conjecture. Theorem 3.2 from [**MRZ**] is applied in [**BMRRT**] to prove Theorem 3.8. Here we give a direct argument. For a positive root α, let M_α denote the kQ-module corresponding to α via Gabriel's Theorem. Let $M_{-\alpha_i}$ denote the object of $\mathcal{C}$ corresponding to $P_i[1]$ in the derived category. This gives a one-one correspondence between the indecomposable objects of $\mathcal{C}$ and the almost positive roots, since every $\tau^{-1}[1]$-orbit on $\operatorname{ind} D^b(kQ)$ contains an object M_α up to isomorphism.

The Theorem follows from the following Proposition.

PROPOSITION 3.9. *Let Q_{alt} be an alternating quiver of Dynkin type, and let α, β be almost positive roots. Then $(\alpha||\beta) = \dim \operatorname{Ext}^1_{\mathcal{C}}(M_\alpha, M_\beta)$.*

PROOF. The direct proof here was given by Mark Parsons. We have

$$\begin{aligned}\dim \operatorname{Ext}^1_{\mathcal{C}}(M_{-\alpha_i}, M_\alpha) &= \dim \operatorname{Ext}^1_{\mathcal{C}}(P_i[1], M)\\ &= \dim \operatorname{Ext}^1_{\mathcal{C}}(\tau^{-1}P_i[1], \tau^{-1}M)\\ &= \dim \operatorname{Ext}^1_{\mathcal{C}}(P_i, \tau^{-1}M)\\ &= \dim \operatorname{Ext}^1_{\mathcal{C}}(\tau^{-1}M, P_i)\\ &= \dim \operatorname{Hom}^1_{\mathcal{C}}(P_i, M)\\ &= \dim \operatorname{Hom}^1_{kQ_{alt}}(P_i, M)\\ &= (-\alpha_i||\alpha).\end{aligned}$$

using [**BMRRT**, 1.7d]. It is clear that $(-\alpha_i|| - \alpha_j) = \dim \operatorname{Ext}^1_{\mathcal{C}}(P_i[1], P_j[1]) = 0$ for all i, j. Let I_+ denote the source vertices of Q_{alt} and I_- the sink vertices of Q_{alt}. It can then be shown that for any almost positive root α, $M_{\tau_+\tau_-\alpha} \simeq \tau M_\alpha$. This holds for non-projective modules since then $\tau_+\tau_-$ coincides with the Coxeter element, and can be extended to the almost positive roots using a combinatorial argument on the root system. □

3.7. Cluster-tilted algebras from triangulations. In this section, we describe an alternative method for assigning finite-dimensional algebras to clusters, as developed by P. Caldero, F. Chapoton and R. Schiffler [**CCS1, CCS2**]. Suppose that $\mathcal{A}$ is a cluster algebra, C is a cluster of $\mathcal{A}$, and B is its exchange matrix. Thus B is a square sign-skew-symmetric matrix with integer entries. A quiver Q_C can be associated to C, with vertices corresponding to the rows or columns of B, i.e. corresponding to the elements of the cluster. If $x, y \in C$ then there are B_{xy} arrows from x to y if $B_{xy} > 0$, and otherwise there are no arrows from x to y (although if $B_{xy} < 0$ there will be arrows from y to x as $B_{yx} > 0$).

Let k be an algebraically-closed field. We now suppose that $\mathcal{A}$ is a cluster algebra of type A_n. In [**CCS1**], the authors impose relations on the path algebra kQ_C as follows. Suppose that $a : i \to j$ is an arrow in Q_C. As in [**BGZ**] we say that an oriented cycle in Q_C is *chordless* if the induced subgraph on the cycle is again an oriented cycle. A path from j to i is said to be a *shortest path* if, together with the arrow a, it forms a chordless cycle. A relation on kQ_C corresponding to a is defined as follows:

(1) If there is no shortest path in Q_C from j to i, there is no relation.
(2) If there is exactly one shortest path p from j to i, the relation is $p = 0$.
(3) If there are exactly two shortest paths p_1, p_2 from j to i, the relation is that $p_1 = p_2$.

Let A_C denote the quotient of Q_C by the ideal generated by these relations. In [**CCS2**], the following is proved:

THEOREM 3.10. *Let C be any cluster of a cluster algebra $\mathcal{A}$ of type A and let A_C be the algebra defined above. Let T_C be the corresponding cluster-tilting object in the cluster category $\mathcal{C}$ corresponding to $\mathcal{A}$. Then the cluster-tilted algebra* $\operatorname{End}_{\mathcal{C}}(T_C)^{op}$ *is isomorphic to A_C.*

The combinatorics of the cluster algebra of type A_n has a nice description in terms of triangulations of a regular $n + 3$ polygon $\mathcal{P} = \mathcal{P}_{n+3}$ [**FZ2**]. In [**CCS1**], motivated by this description, the authors give an interesting graphical interpretation of the category of finite-dimensional modules over A_C, which we shall now

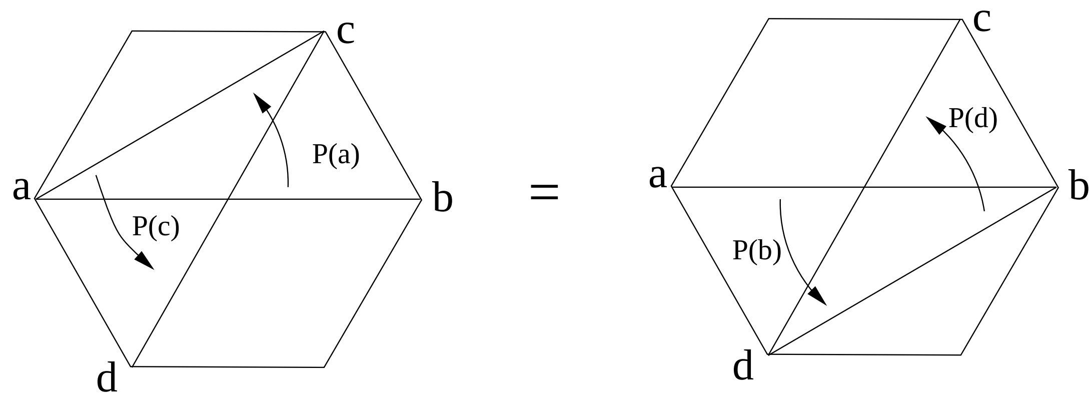


FIGURE 3. A mesh relation in type A_3: $P(c)P(a) = P(d)P(b)$.

describe. Fix $n \in \mathbb{N}$ and let Δ be a triangulation of $\mathcal{P}$. Let $\mathcal{C}(\Delta)$ be the additive category with objects given by formal direct sums of diagonals of the polygon (i.e. line segments joining non-adjacent vertices) which are *not* in Δ. Two diagonals are related by a *pivoting elementary move* if the associated diagonals share a vertex on the boundary (the pivot), the other vertices of the diagonals form a boundary edge of the polygon, and the rotation about the pivot is anticlockwise. See Figure 3 for some examples in type A_3. The pivoting elementary move is denoted by $P(v)$, where v is the pivot. There is, by definition, a corresponding morphism in $\mathcal{C}(\Delta)$, which we shall also denote by $P(v)$. The morphisms in $\mathcal{C}(\Delta)$ are generated by the pivoting elementary moves, modulo the *mesh relations*, which are defined as follows. For any pair α, β of diagonals not in Δ, the mesh relation is given by $P(c)P(a) = P(d)P(b)$, where a, b (respectively c, d) are the vertices of α (respectively, β) such that the product $P(c)P(a)$ of pivotal elementary moves takes α to β. Diagonal or boundary edges are allowed, with the convention that if one of the intermediate edges is a border edge or a diagonal in Δ, the corresponding term in the mesh relation is regarded as zero. See Figure 3 for an example of a mesh relation.

There is a quiver Q_Δ corresponding to any triangulation Δ, defined as follows. The vertices of Q_Δ are the midpoints of the diagonals in Δ. An arrow is drawn between two vertices i, j in Q_Δ if the corresponding diagonals bound a common triangle, with orientation $i \to j$ if the diagonal corresponding to j is obtained from the diagonal corresponding to i by rotating anticlockwise about their common vertex. Let A_Δ be the path algebra kQ_Δ modulo the relations (1), (2) and (3) given above. See Figure 4 for an example; in this case the relations are that the product of any two arrows in any triangle is zero. Using a mutation argument and [**FZ2**, §3.5], it is easy to see that there is a correspondence between the triangulations of $\mathcal{P}$ and the clusters of $\mathcal{A}$, such that $A_\Delta = A_C$ if C is the cluster of $\mathcal{A}$ corresponding to the triangulation Δ. Caldero, Chapoton and Schiffler prove the following result:

THEOREM 3.11. *Let Δ be a triangulation of $\mathcal{P}_{n+3}$ and A_Δ the algebra defined above. Then the category $\mathcal{C}(\Delta)$ and the category of finite-dimensional modules over A_Δ are equivalent.*

We remark that this gives in particular an interesting new way of thinking about the module category of a path algebra (in type A). In [**CCS1**] it is pointed

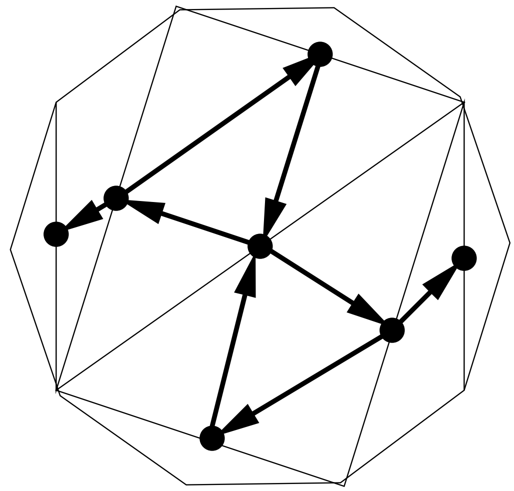


FIGURE 4. The quiver of a triangulation in type A_7.

out that the construction of the category $\mathcal{C}(\Delta)$ above can be extended to include all of the diagonals of $\mathcal{P}$ as indecomposable objects. The morphisms are still generated by the pivoting elementary moves modulo the mesh relations. The rule that terms in mesh relations corresponding to intermediate edges in the initial triangulation are regarded as zero no longer makes sense (as a fixed initial triangulation is no longer being chosen), and is therefore omitted. The authors show that the category $\widetilde{\mathcal{C}}$ defined in this way is equivalent to the cluster category $\mathcal{C}$ corresponding to $\mathcal{A}$.

In [**CCS1**], it is pointed out that two diagonals α and α, when regarded as indecomposable objects in $\widetilde{\mathcal{C}}$, satisfy $\mathrm{Ext}^1_{\widetilde{\mathcal{C}}}(\alpha, \alpha') = 0$ if and only if they do not cross. It follows that cluster-tilting objects in $\mathcal{C} \simeq \widetilde{\mathcal{C}}$ correspond to triangulations of $\mathcal{P}$, as indeed we should expect, since cluster-tilting objects in $\mathcal{C}$ are known to correspond to clusters. From this perspective, Theorem 4.1 (in the next Section) can be regarded as saying that the module category of a cluster-tilted algebra is obtained from the cluster category by forming the quotient by a fixed triangulation; the above construction makes this very clear for the type A case.

In [**CCS2**], the authors use the above results to prove the following Theorem:

THEOREM 3.12. *Let $C = \{x_1, x_2, \ldots, x_n\}$ be any cluster of $\mathcal{A}$, a cluster algebra of arbitrary simply-laced Dynkin type. Let $T = T_1 \amalg T_2 \amalg \cdots \amalg T_n$ be the corresponding cluster-tilting object in $\mathcal{C}$. Then there is a bijection between the indcomposable objects of $\mathcal{C}$ and the cluster variables of $\mathcal{A}$, denoted $M \mapsto x_M$, such that*

$$x_M = \frac{P(x_1, x_2, \ldots, x_n)}{\prod_{i=1}^n x_i^{d_i}},$$

where $d_i = \dim \mathrm{Ext}^1_{\mathcal{C}}(T_i, M)$.

In particular, this establishes a conjecture of Fomin and Zelevinsky [**FZ5**] in this case. In general, the results in this section should hold (in a suitably modified form) for arbitrary finite type cluster algebras. Some ideas in this direction are described in [**CCS1, CCS2**].

4. Cluster-tilted algebras and generalized APR-tilting

As mentioned in Section 2, the main idea of (hereditary) tilting theory is to compare the representation theory of a hereditary algebra H with the representation theory of a tilted algebra Λ, obtained as the endomorphism ring of a tilting module T in $\operatorname{mod} H$.

This motivates the investigation of algebras of the form $\Gamma = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ where T is a tilting object T in the cluster category $\mathcal{C}$. Such an algebra is called a *cluster-tilted algebra*. The results from this section are from [**BMR1**] unless otherwise stated.

4.1. Cluster-tilted algebras. Let T be a tilting object in $\mathcal{C} = \mathcal{C}_H$ for some hereditary algebra H, and let $\Gamma = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ be the corresponding cluster-tilted algebra.

The functor $G = \operatorname{Hom}_{\mathcal{C}}(T, \ )\colon \mathcal{C} \to \operatorname{mod} \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ is full and dense. This is Proposition 2.1 in [**BMR1**]. It is clearly not faithful, since $\operatorname{Hom}_{\mathcal{C}}(T, \tau T) = \operatorname{Ext}^1_{\mathcal{C}}(T, T) = 0$. So there is an induced functor $\bar{G}\colon \mathcal{C} / \operatorname{add}(\tau T) \to \operatorname{mod} \Gamma$. The main result of [**BMR1**] is the following.

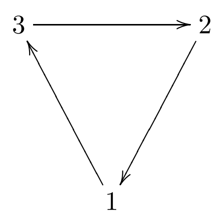
Theorem 4.1. *The functor* $\bar{G}\colon \mathcal{C} / \operatorname{add}(\tau T) \to \operatorname{mod} \Gamma$ *is an equivalence.*

Especially, this shows that the cardinality of the set of indecomposable modules is the same for Γ and H.

There is an even stronger link between $\operatorname{mod} \Gamma$ and $\operatorname{mod} H$. In fact the AR-quiver of Γ can easily be obtained from the AR-quiver of H, since we have the following.

Proposition 4.2. *The almost split sequences in* $\mathcal{C} / \operatorname{add}(\tau T)$ *are induced by the almost split triangles in* $\mathcal{C}$.

To illustrate this, let us revisit the example of Section 3.2. Let T be the tilting module $T = T_1 \amalg T_2 \amalg T_3 = S_3 \amalg P_1 \amalg S_1$, let $\Lambda = \operatorname{End}_H(T)^{\mathrm{op}}$ be the corresponding tilted algebra and $\Gamma = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ the cluster-tilted algebra. Then Γ is the path algebra of the quiver



with relations $J^2 = 0$. We get the AR-quiver of Γ by deleting the vertices corresponding to τT in the AR-quiver of $\mathcal{C}$. See Figure 5.

4.2. Generalized APR-tilting. Let $H = kQ$ be a path algebra. Assume v is a sink in Q. Let Q' be the quiver obtained by reversing the arrows starting in v. Recall that this can be interpreted as APR-tilting, namely Q' will be the quiver of the algebra H' obtained as $\operatorname{End}_H(T)^{\mathrm{op}}$, where $T = P \amalg \tau^{-1} S$, S is the simple projective module corresponding to v, and $H = P \amalg S$. By [**APR**], in this situation $\operatorname{ind} H \setminus \{S\}$ is equivalent to $\operatorname{ind} H' \setminus \{S'\}$, for the simple H'-module S' corresponding to the source v' in Q'. There is a similar result for a source v in Q.

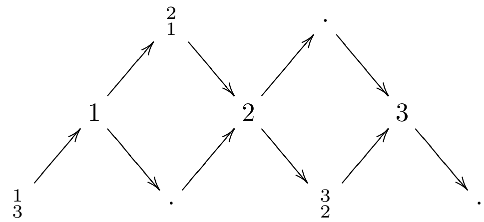


Figure 5. Obtaining the AR-quiver of Γ from the AR-quiver of $\mathcal{C}$.

We shall now see that Theorem 4.1 gives a generalization of this result to arbitrary vertices v. Let $\bar{T}$ be an almost complete tilting object with complements M, M^* and connecting triangles $M^* \to B \to M \to$ and $M \to B' \to M^* \to$. Let $T = \bar{T} \amalg M$, and let $T' = \bar{T} \amalg M^*$. The corresponding cluster-tilted algebras are $\Gamma = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ and $\Gamma' = \operatorname{End}_{\mathcal{C}}(T')^{\mathrm{op}}$. Let S_M denote the simple top of the projective Γ-module $\operatorname{Hom}_{\mathcal{C}}(T, M)$, and let S_{M^*} denote the simple top of the projective Γ'-module $\operatorname{Hom}_{\mathcal{C}}(T', M^*)$. There is an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(T, B) \to \operatorname{Hom}_{\mathcal{C}}(T, M) \to \operatorname{Hom}_{\mathcal{C}}(T, \tau M^*) \to 0$$

and $\operatorname{Hom}_{\mathcal{C}}(T, \tau M^*)$ is a simple Γ-module. It follows that $S_M \simeq \operatorname{Hom}_{\mathcal{C}}(T, \tau M^*)$, and similarly one can show that $S_{M^*} \simeq \operatorname{Hom}_{\mathcal{C}}(T', \tau M)$, Using the equivalences given by Theorem 4.1, one obtains the following.

Theorem 4.3. *There is an equivalence* $\operatorname{mod} \Gamma / \operatorname{add} S_M \to \operatorname{mod} \Gamma' / \operatorname{add} S_{M^*}$.

What we call *generalized APR-tilting* is the special case when M is an indecomposable projective H-module. Assume H decomposes as $H = P \amalg M$. Then P is an almost complete tilting object, and we denote as usual the second complement by M^*, and $\Gamma' = \operatorname{End}_{\mathcal{C}}(P \amalg M^*)^{\mathrm{op}}$. Theorem 4.3 now gives $\operatorname{mod} H / \operatorname{add} S_M \simeq \operatorname{mod} \Gamma' / \operatorname{add} S_{M^*}$. In fact, our example in the previous subsection illustrates this. The algebra Γ can be regarded as "cluster-tilted from H" at the vertex 2. Note that (classical) APR-tilting is the special case where M is simple.

4.3. Cluster-tilted algebras of finite representation type. In this section we discuss the results of [**BMR3**]. Here, assume the field k is algebraically closed. Then an algebra Γ over k is (up to Morita equivalence) given as a path algebra modulo relations. One of the main results of [**BMR3**] is that a cluster-tilted algebras of finite representation type is (up to isomorphism) determined by its quiver. It is an open question if this holds more generally.

As by-product of this one gets some precise information both about the quivers and the relations for such algebras.

Note that it follows from Theorem 4.1 that the cluster-tilted algebras of finite (representation) type are exactly those of the form $\operatorname{End}_{\mathcal{C}_H}(T)^{\mathrm{op}}$ for some hereditary algebra H of finite type.

In [**BMR3**] the following is shown.

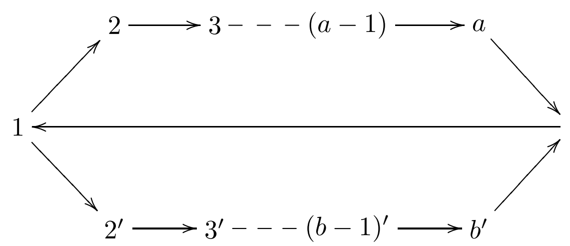
Theorem 4.4. *Let* $\Gamma = kQ/I$ *be a cluster-tilted algebra of finite representation type, and let* i, j *be vertices of* Q.

(a) *If there is an arrow $i \to j$ there are at most two paths from j to i that are non-zero in Γ.*
(b) *If there is an arrow $i \to j$ in Q then the sum of all paths from j to i is zero in Γ.*
(c) *If there is a minimal relation involving path(s) from j to i then there is an arrow $i \to j$.*
(d) *A minimal relation is either a zero-relation, or of the form $\rho + \mu = 0$, where ρ and μ are two non-zero, disconnected paths.*

Here two paths from j to i are called disconnected if the full subquiver generated by the paths contains no additional arrows, except possibly an arrow $i \to j$. This has the following consequence.

COROLLARY 4.5. *A cluster-tilted algebra is determined by its quiver, up to isomorphism.*

The minimal relations are either of the form $\rho = 0$ for some path, or of the form $\rho + \mu = 0$, for two disconnected paths from j to i. In the latter case one can show that the subquiver generated by these paths is of the form



Denote such a quiver by $G(a, b)$, where a is the length of ρ and b is the length of μ There are restrictions of the length of the paths. In [**BMR3**] it is shown that $G(a, b)$ is the quiver of a cluster-tilted algebra of finite representation type if and only if either $(a, b) = (2, m)$, for any value of $m \geq 2$ or $(a, b) = (3, m')$ for $m' \in \{3, 4, 5\}$. To see this one uses that $G(2, m)$ is mutation equivalent to a quiver of type D_{m+2}, and that $G(3, m')$ is mutation equivalent to a quiver of type $E_{3+m'}$. Finally, we remark that partial results in this direction were also obtained in [**CCS2**].

5. Cluster mutation via cluster categories

As mentioned in the introduction, one of the main motivations for the theory of cluster categories is to further develop the links between cluster algebras and the representation theory of quivers. One of the main ingredients of the definition of cluster algebras is matrix mutation, which is an involutive operation on skew-symmetrizable matrices. We consider only skew-symmetric matrices here. Skew-symmetric integer matrices correspond to quivers with no loops and no cycles of length two. Recall that the correspondence is given by assigning to an $n \times n$-matrix $X = (x_{ij})$, a quiver with n vertices $1, \ldots, n$ and with x_{ij} arrows from i to j, for positive values of x_{ij}.

5.1. Mutation via quiver representations. Here we give the main result of [**BMR2**]. It gives a module theoretic interpretation of matrix mutation.

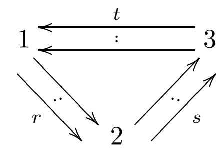
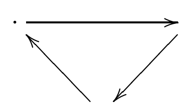


FIGURE 6. The quiver $C_{r,s,t}$.

THEOREM 5.1. *Let $\bar{T}$ be an almost complete tilting object with two complements M and M^*. Let $\Gamma = \mathrm{End}_{\mathcal{C}}(\bar{T} \amalg M)^{\mathrm{op}}$ and let $\Gamma' = \mathrm{End}_{\mathcal{C}}(\bar{T} \amalg M^*)^{\mathrm{op}}$. Let k be the vertex of Γ corresponding to M. Then the quivers Q_Γ and $Q_{\Gamma'}$, or equivalently the matrices $X_\Gamma = (x_{ij})$ and $X_{\Gamma'} = (x'_{ij})$, are related by the formula*

$$x'_{ij} = \begin{cases} -x_{ij} \\ x_{ij} + \frac{|x_{ik}|x_{kj} + x_{ik}|x_{kj}|}{2} \end{cases}$$

Let us mention an interesting aspect of the proof of this result. The idea is to reduce to the case where where n, the number of vertices of Q is three, where $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}} = kQ/I$. More precisely, it is sufficient to prove the claim of the theorem for all cluster categories $\mathcal{C}_H$ corresonding to hereditary algebras H with three simple modules. For general algebras the proof of the claim is quite involved. For algebras of finite type, however, it is easy, since we can give precise classification of all cluster-tilted algebras of finite type with three simple modules. For any cluster-tilted algebra there are no cycles in the quiver of Q of length 1 or 2. For cluster-tilted algebras of finite type there are in addition no double arrows. This means that we either have that $\Gamma = kQ/I$ is hereditary, and thus that the underlying graph of Q is A_3, or that Q must be the quiver C_3:



In this case it is easy to see that the only possible relation-space I such that kC_3/I is cluster-tilted is J^2, where J is the radical, i.e. the ideal generated by the arrows.

In the general case, it is not clear how to characterize all cluster-tilted algebras of rank 3 in terms of quivers with relations. One can show that the algebras that occur are either hereditary or have a quiver $C_{r,s,t}$ as in Figure 6.

In this case, there are no known necessary and sufficient conditions that I should satisfy in order for $kC_{r,s,t}/I$ to be a cluster-tilted algebra. However, one can still obtain enough information to prove Theorem 5.1 in this case. It is proved that the relations are homogeneous, and that $J^6 \subset I$. Let $\Gamma = kQ/I$ and $\Gamma' = kQ'/I'$ be cluster-tilted algebras related by one exchange (i.e. related as in the assumptions of the theorem), with radicals J and J'. One compares the intersections $I \cap J^2$ and $I' \cap {J'}^2$. It is sufficient to consider $I \cap J^2$ instead of I because when matrix mutation is considered as quiver mutation, only paths of length at most 2 appear.

The fact that makes it possible to reduce the proof to the case $n = 3$ is the following, which is also proved in [**BMR2**].

THEOREM 5.2. *Let Γ be a cluster-tilted algebra, and e an idempotent. Then the factor algebra $\Gamma/\Gamma e \Gamma$ is also cluster-tilted.*

A direct consequence of Theorem 5.1 is the following.

Corollary 5.3. *Let Q be an acyclic finite quiver Q, and $H = kQ$ the path algebra. Then there is a cluster-tilted algebra $\Gamma = \mathrm{End}_{\mathcal{C}_H}(T)^{\mathrm{op}}$ with quiver Q' if and only if Q' is mutation equivalent to Q.*

5.2. Cluster algebras via quiver representations. Combining results from Section 3.6 with Theorem 5.1 one obtains a strong link between tilting in cluster categories and the combinatorics of cluster algebras. Recall that for an exchange pair T_i and T_i^* in a cluster category there are the connecting triangles $T_i^* \to B \to T_i \to$ and $T_i \to B' \to T_i^* \to$, with $B, B' \in \operatorname{add} \bar{T}$, where $\bar{T}$ is an almost complete tilting object having T_i and T_i^* as complements. Let T be a tilting object in $\mathcal{C}$, and let Q_T be the quiver of the cluster-tilted algebra $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. Then we call the pair (T, Q_T) a *tilting seed*.

We also have exchange pairs for cluster algebras. Two cluster variables x_i and x_i^* are said to form an exchange pair if there are $n-1$ cluster variables $\{x_1, \dots, \hat{x_i}, \dots, x_n\}$ (where $\hat{x_i}$ denotes omission) such that $\{x_1, \dots, x_i, \dots, x_n\}$ and $\{x_1, \dots, x_i^*, \dots, x_n\}$ are clusters.

Recall that for Dynkin quivers we have a 1–1 correspondence φ between cluster variables and indecomposable objects, inducing a correspondence between clusters and tilting objects. Thus it also induces a correspondence between tilting seeds and seeds in the cluster algebra. If φ identifies x_i and x_i^* with T_i and T_i^*, respectively, we have the following.

Theorem 5.4. *For a cluster algebra of finite type, let φ be the above correspondence between seeds and tilting seeds, and between cluster variables and indecomposable objects in the cluster category.*

(a) *For any $i \in \{1, \dots, n\}$ we have a commutative diagram:*

$$\begin{array}{ccc} (\underline{x}', B') & \xrightarrow{\varphi} & (T', Q_{T'}) \\ {\scriptstyle \mu_i}\downarrow & & \downarrow{\scriptstyle \delta_i} \\ (\underline{x}'', B'') & \xrightarrow{\varphi} & (T'', Q_{T''}) \end{array}$$

(b) *Identify the cluster variables with the indecomposable objects in $\mathcal{C}$ via φ. We have*

$$T_i T_i^* = \prod (T_j)^{a_j} + \prod (T_k^*)^{c_k}$$

for an exchange pair T_i and T_i^* where the a_j and c_k appear in the unique non-split triangles

$$T_i^* \to \amalg T_j^{a_j} \to T_i \to,$$

and

$$T_i \to \amalg T_k^{c_k} \to T_i^* \to$$

in $\mathcal{C}$.

Here μ_i denotes cluster mutation, while δ_i is *tilting mutation*, that is passing from one tilting seed to another by exchanging the indecomposable T_i with T_i^*.

In the general case, where there is no known correspondence between the tilting seeds and the cluster seeds, one obtains a "local" variant of the above. The precise formulation is in [**BMR2**]. This was conjectured in [**BMRRT**]. In [**BRS**] a slightly more general version of Theorem 5.4 is given. This is used to give further links between cluster algebras and tilting theory.

6. Cluster algebras and the Canonical Basis

6.1. The Canonical Basis. Let $\mathbf{g}$ be a finite dimensional simple Lie algebra with corresponding universal enveloping algebra $U(\mathbf{g})$. In 1985, Drinfel'd [**D**] and Jimbo [**J**] defined a quantized version of $U(\mathbf{g})$, known as the quantized enveloping algebra $U_q(\mathbf{g})$ of $\mathbf{g}$, which is an algebra over the field of rational polynomials $\mathbb{Q}(q)$ in an indeterminate q. This algebra has had an enormous impact on the study of Lie algebras and in Lie theory and related areas in general. One reason for this was the definition of the canonical basis (Lusztig [**Lu1**], [**Lu3**]) or global crystal basis (Kashiwara [**Ka**]) of $U_q(g)$ around 1990. The canonical basis provides a natural setting for describing classical combinatorial results in the representation theory of semisimple Lie algebras. In particular, in Kashiwara's approach, the combinatorics (in the form of beautiful crystal graphs) are obtained from the defining algebraic structure by passing to the *crystal limit*, in which q is specialised to zero in an appropriate manner. There are many other interpretations, links and applications of the canonical basis which space precludes us from discussing here.

However, there is still much which is not known about canonical bases. In particular, a complete description of the basis has been given in the literature only in a few cases, i.e. types A_1 and A_2 [**Lu1**], type A_3 [**Lu4, X1**] and B_2 [**X2**]. In type A_4, partial information is known [**CM, HY, HYY, M**]. It is likely that the method in [**HY**] will eventually yield the entire canonical basis in type A_4.

The canonical basis $\mathbb{B}$ is actually a basis for the negative part $U_q^- = U_q^-(\mathbf{g})$ of the quantized enveloping algebra. Let us suppose for simplicity that $\mathbf{g}$ has a simply-laced Dynkin diagram. For $m \in \mathbb{N}$ let $[m] = (q^m - q^{-m})/(q - q^{-1})$ and $[m]! = [m][m-1]\cdots[2][1]$ be elements of $\mathbb{Q}(q)$. The algebra U_q^- has Chevalley generators $F_1, F_2, \ldots, F_n$ (where n is the rank of $\mathbf{g}$), subject to relations:

$$\begin{aligned} F_iF_j - F_jF_i &= 0, \quad A_{ij} = 0, \\ F_i^{(2)}F_j - F_iF_jF_i + F_jF_i^{(2)} &= 0, \quad A_{ij} = -1, \end{aligned}$$

where $A = (A_{ij})$ is the Cartan matrix of $\mathbf{g}$. Here $F_i^{(m)}$ denotes the divided power $F_i^m/[m]!$. The canonical basis in type A_1 is simply given by

$$\mathbb{B}(A_1) = \{F_1^{(m)} \,:\, m \in \mathbb{N}\}.$$

In type A_2, Lusztig shows that

$$\mathbb{B}(A_2) = \{F_1^{(a)}F_2^{(b)}F_1^{(c)} \,:\, b \geq a+c\} \cup \{F_2^{(a)}F_1^{(b)}F_2^{(c)} \,:\, b \geq a+c\},$$

where the two subsets do not overlap except that $F_1^{(a)}F_2^{(a+c)}F_1^{(c)} = F_2^{(c)}F_1^{(a+c)}F_2^{(a)}$. Thus in this case, the canonical basis falls "naturally" into two distinct regions — but it is not clear what a region is. In type A_3 the situation is similar; there are 14 regions, 8 of which are monomial, corresponding to the 8 reduced expressions for the longest element w_0 in the Weyl group of type A_3 up to commutation; these are known as the tight monomials. The other 6 regions consist of elements which can be described as 'one-parameter' sums of monomials. In type A_4 there are 62 monomial regions [**CM, M, HYY**]; evidence suggests there will be 672 regions in total.

6.2. The dual canonical basis. Study of the dual canonical basis gives some insight into the regions described above and recent efforts to understand the canonical basis have passed to the dual because it seems to be more tractable. The algebra U_q^- is isomorphic to its dual $\mathbb{C}_q[N]$, the quantized function algebra on the unipotent radical N of a Borel subgroup of the corresponding semisimple algebraic group. There is a corresponding nondegenerate bilinear form on each weight space of U_q^-; see [**Ros, T**] for details of this form. The dual canonical basis can thus be regarded as the basis of U_q^- dual to the canonical basis. For $b \in \mathbb{B}$ we denote the corresponding element of the dual canonical basis by b^*. See e.g. [**BZ1**] and [**LNT**] for discussion of the dual canonical basis.

Key elements in the dual canonical basis are the *quantum flag minors*, which are q-versions of the usual flag minors in $\mathbb{C}[N]$. They can be defined in the following way. Let λ be a dominant weight, and $V(\lambda)$ the corresponding highest weight $U_q(\mathbf{g})$-module, with highest weight vector v_λ. Lusztig has shown that $\mathbb{B}v_\lambda \setminus \{0\}$ is a basis for $V(\lambda)$ (known as the canonical basis for $V(\lambda)$). Let

$$\mathbb{B}(\lambda) = \{b \in \mathbb{B} \,:\, bv_\lambda \neq 0\}$$

be the corresponding subset of $\mathbb{B}$. Let $\omega_1, \omega_2, \ldots, \omega_n$ denote the fundamental weights of $\mathbf{g}$. Then a quantum flag minor is an element of the form b^* such that $b \in \mathbb{B}(\omega_i)$ corresponds to an element in an extremal weight space of $V(\omega_i)$.

Berenstein and Zelevinsky [**BZ1**] showed that the dual canonical basis could be described as monomials in quantum minors in types A_1, A_2 and A_3, relating this to Kashiwara's parametrization of the canonical basis via root operators and crystal graphs [**Ka**]. They also showed that the dual canonical basis has the structure of a *string basis* in U_q^-.

6.3. Cluster algebras and adapted algebras. Caldero has used the quantum flag minors to define *adapted subalgebras* [**Ca1**] of $\mathbb{B}$. These subalgebras fit very nicely into a cluster algebra approach to the dual canonical basis, so we shall use them to introduce the links between cluster algebras and the dual canonical basis, although the two theories were developed independently. Adapted subalgebras are obtained in the following way. Let W be the Weyl group of $\mathbf{g}$, with Coxeter generators $s_1, s_2, \ldots, s_n$. Let $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ be a reduced expression for the longest element w_0 in W, i.e. so that the expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ is reduced. For $k = 1, 2, \ldots, N$, let $\Delta_{\mathbf{i}}^k$ denote the dual of the element $b \in \mathbb{B}$ corresponding to the one-dimensional extremal weight space of weight $s_{i_1} s_{i_2} \cdots s_{i_k} \omega_{i_k}$ in $V(\omega_{i_k})$. Two elements x, y of U_q^- are said to *q-commute* if $xy = q^a yx$ for some integer a.

Caldero shows that:

THEOREM 6.1. *Let $\mathbf{i}$ be a reduced expression for w_0, and let $b_1 := \Delta_{\mathbf{i}}^1, b_2 := \Delta_{\mathbf{i}}^2, \ldots, b_N := \Delta_{\mathbf{i}}^N$ be the corresponding set of quantum flag minors. Then b_i and b_j q-commute for all i, j. Thus the monomials in $b_1, b_2, \ldots, b_N$ span a $\mathbb{Q}(q)$-subalgebra $A(\mathbf{i})$ of U_q^-. Up to powers of q, all such monomials lie in the dual canonical basis of U_q^-. Furthermore, such a set of monomials forms a maximal q-commuting subset of the dual canonical basis.*

The algebras arising in the theorem are known as *adapted subalgebras*, since they are adapted to the dual canonical basis. For $1 \leq j \leq n$, let i_{r_j} be the last occurrence of j in $\mathbf{i}$. Let $z_j = b_{r_j}^*$. Then the span of the monomials in the z_j forms the *q-centre* of U_q^-, i.e. the subspace of U_q^- spanned by the elements which

q-commute with all homogeneous elements of U_q^-. It is easy to see that for any two reduced expressions for w_0 related by a braid relation of the form $s_i s_j s_i = s_j s_i s_j$ (where $A_{ij} = -1$), the corresponding generating subsets of the adapted algebras as defined above coincide on all but one element (not lying in the q-centre).

The connection between adapted algebras and cluster algebras is explained in [**Ca2**], the result of discussions between Caldero and Zelevinsky. On specialisation of q to 1, the algebra $\mathbb{C}_q[N]$ (which we regard as containing the dual canonical basis) specialises to the usual coordinate ring $\mathbb{C}[N]$, which is a cluster algebra [**BFZ2**]. Each adapted subalgebra $A(\mathbf{i})$ should specialise to a subalgebra generated by a cluster. The generators of $A(\mathbf{i})$, which are quantum minors, should specialise to the cluster variables in the cluster, except that the elements of of the q-centre defined above should behave as coefficients in the cluster algebra set-up. For clusters corresponding to reduced expressions for w_0, the exchange relation should correspond to a classical three-term relation between minors; see [**BZ2, BFZ1**].

The process of changing a reduced expression $\mathbf{i}$ for w_0 by a braid relation as described above is an example of cluster mutation, and can be described very nicely in terms of pseudoline arrangements; see [**BFZ1, BZ1**]. However, in general, there should be more clusters than those obtained from the adapted algebras, i.e. monomials in clusters containing dual canonical basis elements which are not quantum flag minors. The mutation process will produce many more dual canonical basis elements in general. A conjecture implicit in the work of Berenstein, Fomin and Zelevinsky is that any monomial in a fixed cluster of $\mathbb{C}[N]$ should lie in the (specialisation of) the dual canonical basis of $\mathbb{C}[N]$; indeed this was a key motivation in the definition of cluster algebras. Fomin and Zelevinsky have shown that in type A_r, $r \leq 4$, the cluster monomials actually form the dual canonical basis of $\mathbb{C}[N]$. For type A_r, $r \geq 5$, and almost all other cases, Leclerc has shown that there are elements in the dual canonical basis which are not cluster monomials [**Le**]. The cluster type of $\mathbb{C}[N]$ is trivial, A_1, A_3 and D_6 for types A_1, A_2, A_3 and A_4, respectively, and the number of clusters is 1,2,14 and 672 respectively. In types A_1, A_2 and A_3 each set of monomials in a fixed cluster corresponds to a region of the canonical basis as desribed above, and the 672 clusters in type A_4 should correspond to the 672 'regions' of the canonical basis in this case.

We also note that Berenstein and Zelevinsky [**BZ3, Ze3**] have introduced quantum deformations of cluster algebras, known as *quantum cluster algebras*. It is expected that $\mathbb{C}_q[N]$ is a quantum cluster algebra, that each adapted algebra is a subalgebra generated by a cluster, and that all quantum cluster monomials lie in the dual canonical basis (possibly up to a power of q). We remark that bases in cluster algebras of rank two have been investigated in [**SZ**].

6.4. A Hall algebra approach to cluster categories. Let Q be an orientation of a simply-laced Dynkin quiver. Caldero and Chapoton [**CC**], motivated by cluster categories, have given formulas for cluster variables in terms of the Euler form on the category of representations of Q and a corresponding initial cluster. This gives a Hall algebra-type definition of the cluster algebra of the same Dynkin type, a combinatorial description of all of the cluster variables in type A_n and a connection with Coxeter-Conway friezes. This kind of approach, together with further development in this direction by Caldero-Keller, is likely to give further information concerning the multiplicative properties of the dual canonical basis.

6.5. Preprojective algebras and the dual canonical basis. We restrict now to the simply-laced case. We note that by [**KS, Lu2**], the irreducible components of the module varieties over the preprojective algebra are in one-to-one correspondence with the elements of the canonical basis of the same type. Geiss, Leclerc and Schröer [**GLS2**] have shown that the module category over a preprojective algebra of Dynkin type is connected to cluster algebras in a way similar to the connection between cluster categories and cluster algebras. The preprojective algebra approach is particularly suited for cluster algebras associated to algebraic groups, as it is not necessary to start with a finite-dimensional hereditary algebra. A (finite dimensional) module M over a preprojective algebra Λ is called *rigid* provided that $\mathrm{Ext}^1_\Lambda(M,M)=0$; it is *complete rigid* if it has a maximal number, k, of nonisomorphic indecomposable direct summands. A rigid module with $k-1$ nonisomorphic indecomposable direct summands is called *almost complete rigid*. Suppose $\overline{T}$ is an almost complete rigid Λ-module. GLS show that there are precisely two indecomposable modules (complements) M, M^* up to isomorphism such that $\overline{T}\oplus M$ is complete rigid. We say that M is *exchanged* with M^*. Let Γ be the quiver of $\mathrm{End}_\Lambda(\overline{T}\oplus M)$ and let Γ' be the quiver of $\mathrm{End}(\overline{T}\oplus M^*)$. GLS show that Γ and Γ' are related by cluster mutation. We remark that every indecomposable projective Λ-module must be a summand of a complete rigid Λ-module; thus such modules are not exchangeable in the above sense. They can be regarded as playing a role similar to that of coefficients in the cluster algebra setting.

Lusztig's result [**Lu6**] has been dualised by Geiss, Leclerc and Schröer, and states that for all modules M over Λ there is a geometrically defined function $f_M\in\mathbb{C}[N]$ corresponding to M such that the set of functions arising in this way spans $\mathbb{C}[N]$. If the module M is restricted to be generic, the corresponding set of elements of $\mathbb{C}[N]$ forms a basis of $\mathbb{C}[N]$, known as the *dual semicanonical basis*. In [**GLS1**], it is shown that the dual canonical basis and the dual semicanonical basis coincide if and only if Q is of type A_n for $1\leq n\leq 4$. These are precisely the cases when the preprojective algebra is of finite representation type. In these cases, all of the indecomposable modules are generic (as they have no self-extensions), and every dual semicanonical basis element arises as a monomial in the f_M, M indecomposable.

The algebra $\mathbb{C}[N]$ is a cluster algebra [**BFZ2**], so the mutation result [**GLS2**] suggests strong links between the dual semicanonical basis, preprojective algebras and the cluster algebra structure on $\mathbb{C}[N]$. In fact, GLS are able to show that for each reduced expression **i** adapted to a quiver, there is an explicit basic complete rigid module $M=M_1\amalg M_2\amalg\cdots\amalg M_N$ whose quiver, together with the functions $\{f_{M_1},f_{M_2},\ldots,f_{M_r}\}$ constitutes the seed corresponding to **i** in [**BFZ2**]. The authors conjecture that the cluster variables of $\mathbb{C}[N]$ correspond to the indecomposable rigid Λ-modules, the clusters correspond to the basic complete rigid Λ-modules, and the cluster monomials correspond to the rigid Λ-modules. We note that the number of basic complete rigid Λ modules in types A_1 to A_4 coincides with the number of clusters given above.

The approach in [**GLS2**] employs some intriguing recent work of Iyama [**I1, I2**] involving the concept of maximal orthogonal modules and generalisations of the Auslander-Reiten translate to longer sequences.

7. Other Developments

7.1. Calabi-Yau categories. Suppose $\mathcal{T}$ is a k-linear triangulated category, with shift [1]. If there is an auto-equivalence ν of $\mathcal{T}$ such that for all objects X and Y of $\mathcal{T}$, $D\mathrm{Hom}_{\mathcal{T}}(X,Y) \simeq \mathrm{Hom}_{\mathcal{T}}(Y,\nu X)$, ν is said to be a *Serre functor*, and $\mathcal{T}$ is said to have *Serre duality*; see [**BvdB, RvdB**]. If in addition, $\nu \simeq [d]$ as triangle functors, $\mathcal{T}$ is said to be *Calabi-Yau* of dimension d [**Ko3**]. The original example for this is the derived category of coherent sheaves on a smooth projective Calabi-Yau variety of dimension d, which has Calabi-Yau dimension d in the above sense. There are many other interesting examples given in [**Ke**], and Calabi-Yau categories associated with 3-dimensional rings are investigated in [**IR**]. There is a strong connection to mathematical physics; see [**Co, Ko1, Ko2**].

We have seen that the cluster category is defined as the orbit category obtained from action on the derived category of modules over a hereditary algebra of finite representation type by the autoequivalence $\tau^{-1}[1]$. It follows that the cluster category is Calabi-Yau of dimension two; this was pointed out by Colin Ingalls. Keller [**Ke**] shows that the quotient category $D^b(kQ)/\tau^{-1}[d]$ is triangulated and thus Calabi-Yau of dimension $d+1$. Calabi-Yau categories also arise from preprojective algebras. The stable module category of the preprojective algebra has Calabi-Yau dimension 2 [**AR**] and Geiss-Keller [**GK**] have shown that if Λ is a preprojective algebra of type A_n, the stable module category over the endomorphism algebra of a certain complete rigid module is Calabi-Yau of dimension 3.

There is another connection between cluster categories and mathematical physics. Idun Reiten has pointed out that there is a close connection between cluster categories and superpotentials arising in string theory; see [**BD**]. A superpotential determines an algebra defined by a quiver with relations, and often cluster-tilted algebras coincide with these algebras. There are also connections with the *perverse Morita equivalences* appearing in Chuang-Rouquier's work [**Rou**].

7.2. Representation theory and cluster algebras. In [**Zh1**], Zhu shows that the correspondence [**BMRRT**, §4] between almost positive roots and indecomposable objects in the cluster category inducing a correspondence between clusters and cluster-tilting objects also holds in the non-simply laced case. He also shows that the BGP-tilting functors on the derived category induce functors on the cluster category which realise the piecewise-linear versions of the simple transpositions arising in cluster combinatorics [**FZ2**].

In [**Zh2**], Zhu gives a simplified proof of Theorem 4.1. He also shows that two hereditary categories are derived equivalent if and only if their cluster categories are equivalent, provided that one of them is derived equivalent to a hereditary algebra. He gives an alternative definition of a cluster-tilted algebra more directly, as the trivial extension of the corresponding quasi-tilted algebra by a certain bimodule. In [**Zh3**], he exploits the connections between representation theory and cluster algebras to give descriptions of cluster variables.

7.3. Toric varieties. In [**FZ2**], a simiplicial fan Δ is associated to any cluster algebra of finite type. The cones are generated by the compatible subsets of almost positive roots inside $\mathbb{R} \otimes \mathbb{Z}\Phi$. In [**CFZ**] it is shown that the corresponding smooth toric variety X is projective. This fan can be seen to be the special case of a more general construction starting from any orientation of the corresponding Dynkin diagram using [**MRZ, BMRRT**]; the original fan corresponds to the alternating quiver. Chapoton [**Ch**] conjectures that these fans should be smooth (i.e. each cone is spanned by an integral base), and that the cone of ample divisors in $H^2(X,\mathbb{Z})$ should be smooth in the same sense, giving a natural basis of the cohomology group and natural generators of the cohomology ring giving a nice quadratic presentation. He obtains such a basis and presentation for the linear orientation in type A.

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Department of Mathematics, Faculty of Information Technology, Mathematics and Electrical Engineering, NTNU, NO-7491, Trondheim, Norway

E-mail address: aslakb@math.ntnu.no

Department of Mathematics, University of Leicester, University Road, Leicester, LE1 7RH, UK

E-mail address: rjm25@mcs.le.ac.uk