PARAMETRIZATIONS OF FLAG VARIETIES

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Abstract. For the flag variety G/B of a reductive algebraic group G we define and describe explicitly a certain (set-theoretical) cross-section $\phi : G/B \to G$. The definition of ϕ depends only on a choice of reduced expression for the longest element w_0 in the Weyl group W. It assigns to any gB a representative $g \in G$ together with a factorization into simple root subgroups and simple reflections. The cross-section ϕ is continuous along the components of Deodhar's decomposition of G/B [6]. We introduce a generalization of the Chamber Ansatz of [2] and give formulas for the factors of $g = \phi(gB)$. These results are then applied to parametrize explicitly the components of the totally nonnegative part of the flag variety $(G/B)_{\geq 0}$ defined by Lusztig [10], giving a new proof of Lusztig's conjectured cell decomposition of $(G/B)_{\geq 0}$. We also give minimal sets of inequalities describing these cells.

1. Introduction

Consider a simply connected reductive algebraic group G over $\mathbb{C}$ (or Chevalley group over $\mathbb{K}$) with opposite Borel subgroups B^+ and B^-. So for example $G = SL_d(\mathbb{C})$ with the subgroups of upper- and lower-triangular matrices. The flag variety G/B^+ may be embedded in the projective space of a sufficiently general representation of G, say $V = V(\rho)$, by

$$G/B^+ \hookrightarrow \mathbb{P}(V) \ : \ gB^+ \to \langle g \cdot \xi \rangle_{\mathbb{C}} \,,$$

where ξ is a highest weight vector. Then to any element gB^+ we may associate the highest and lowest extremal weights, $v\rho$ and $w\rho$, such that $g \cdot \xi$ has nonzero component in the corresponding weight space. Equivalently, the Weyl group elements v and w determine the intersection of opposed Bruhat cells

$$B^- vB^+/B^+ \cap B^+wB^+/B^+$$

in which gB^+ lies. Now fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ for the longest element of the Weyl group. Following V. Deodhar [6], there is a finer datum that can be associated to gB^+. The element gB^+ can be successively reduced, compatibly with this reduced expression, to give a sequence $\big(B^+, g_{(1)}B^+, \ldots, g_{(n-1)}B^+, gB^+\big)$ in the flag variety, or a sequence of intermediate lines

$$L_0 = \langle \xi \rangle \ , \, L_1 = \left\langle g_{(1)} \cdot \xi \right\rangle \ , \, \ldots \, , \, L_{n-1} = \left\langle g_{(n-1)} \cdot \xi \right\rangle \ , \, L_n = \langle g \cdot \xi \rangle$$

in $V(\rho)$. For example, if we write gB^+ as bwB^+ for $b \in B^+$, then L_{n-1} is the line $\langle bws_i \cdot \xi \rangle$, where s_i is the right-most simple reflection in the reduced expression for w_0 such that $ws_i < w$ (see Section 4.4). Given all the intermediate lines L_k, the further data associated to gB^+ is now the collection $(v_{(1)}, \ldots, v_{(n)})$ of Weyl group elements such that $v_{(k)}\rho$ is the *highest* extremal weight for which $g_{(k)} \cdot \xi$ has non-zero weight space component. The set of gB^+ in B^+wB^+/B^+ with fixed $(v_{(1)}, \ldots, v_{(n)})$ is called a Deodhar component of the flag variety.

Consider the special case where the element gB^+ from above has $v = 1$. Then $gB^+ = uB^+$ for some unipotent $u \in B^-$. If also $v_{(i)} = 1$ for all i, then u may be factorized into negative simple root subgroups as $u = y_{j_1}(t_1) \cdots y_{j_n}(t_n)$ for some nonzero parameters $t_i \in \mathbb{C}$ (where $s_{j_1} \cdots s_{j_n}$ is a reduced expression for w governing the construction of the intermediate lines L_i). If we write $uB^+ = zwB^+$ for some unipotent

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$z \in B^+$, then A. Berenstein and A. Zelevinsky's Chamber Ansatz [2] gives formulas for the t_i in terms of minors of z.

In this paper we generalize the above result by describing factorizations, and hence parametrizations, for a general Deodhar component and giving formulas for the parameters (Proposition 5.2 and Theorem 7.1). Our formulas for the nonzero parameters, analogous to the t_k above, are obtained by a direct generalization of the Chamber Ansatz. However a general Deodhar component also has another type of parameter which runs through $\mathbb{K}$. The formulas for these involve the generalized Chamber Ansatz along with a correction term.

The Chamber Ansatz used in the formulas for the parameters depends on the Deodhar component in which an element zwB^+ lies. Therefore we also give a simple algorithm to determine this component (Section 6). The algorithm in a sense 'generates' the chambers in the Chamber Ansatz for zwB^+ recursively. We illustrate how this works with a very explicit type A example in Section 10.

In Section 11 we set $\mathbb{K} = \mathbb{R}$ and use these results to examine the totally nonnegative part $(G/B^+)_{\geq 0}$ of the flag variety. This is the closure in G/B^+ of the set $\{y_{i_1}(t_1)\cdots y_{i_N}(t_N)B^+ \,|\, t_i \in \mathbb{R}_{>0}\}$. We explicitly describe the intersection of $(G/B^+)_{\geq 0}$ with each of the sets $\mathcal{R}_{v,w} = B^-vB^+/B^+ \cap B^+wB^+/B^+$. Namely in $\mathcal{R}_{v,w}$ there is a unique open dense Deodhar component which is isomorphic to $(\mathbb{R}^*)^{\ell(w)-\ell(v)}$. And the totally nonnegative part $\mathcal{R}_{v,w}^{>0}$ of $\mathcal{R}_{v,w}$ is shown to be the subset of the above Deodhar component where all of the parameters are positive.

This in particular reproves a result of the second author conjectured by G. Lusztig, that $\mathcal{R}_{v,w}^{>0}$ is a semi-algebraic cell. However the new proof presented in this paper gives for the first time explicit parametrizations of these totally nonnegative parts (depending on a choice of reduced expression of w). And it has the advantage of being independent of the theory of canonical bases, which was required in the previous proof. Moreover the parameters of $\mathcal{R}_{v,w}^{>0}$ can all be computed by the generalized Chamber Ansatz (without correction term).

Finally in Section 11 we give an efficient description for $\mathcal{R}_{v,w}^{>0}$ in terms of minor inequalities, generalizing a result of Berenstein and Zelevinsky from the $v = 1$ case. For any choice of reduced expression for w we obtain a set of $\ell(w) - \ell(v)$ inequalities. This set of inequalities and $\ell(v)$ minor equalities, that can also be given explicitly, describe $\mathcal{R}_{v,w}^{>0}$ as a semi-algebraic subset of the (real) Bruhat cell B^+wB^+/B^+.

Remark 1.1. The case of intersections of opposite Bruhat cells $\mathcal{R}_{v,w}$ in the flag variety which we treat in this paper is not to be confused with intersections of opposite Bruhat double cosets

$$G^{v,w} = B^+wB^+ \cap B^-vB^-$$

in the group. These other intersections were studied by Fomin and Zelevinsky [7], who obtained a different generalization of the Chamber Ansatz in that setting (using it also to give parametrizations and minimal sets of inequalities for their corresponding totally positive parts in the group).

Our study of parametrizations in flag varieties compatible with the $\mathcal{R}_{v,w}$ is substantially different from the problems in the group considered in [7], for example already where total positivity is concerned. An immediate and obvious difference between total positivity questions in the two cases lies in the fact that $(G/B^+)_{\geq 0}$ is the *closure* of the image in G/B^+ of the totally nonnegative part $G_{\geq 0}$ of the group, and is actually generally larger than this image. So it is clear that the totally positive cells $\mathcal{R}_{v,w}^{>0}$ in $(G/B^+)_{\geq 0}$ cannot all come from totally positive cells in $G_{\geq 0}$. In fact, the cell decomposition of $G_{\geq 0}$, which is studied in detail in [7], was first obtained by Lusztig in [10] where the analogous problem for flag varieties was formulated only as a conjecture. One has to depart significantly from the study of total positivity in the group in order to study total positivity in the flag variety.

The overlap between the two parametrization problems, ours and the one from [7], is precisely the joint special case covered in [2]. In that case one has $G^{1,w} \cong \mathcal{R}_{1,w} \times T$, where the maximal torus factor T is irrelevant for the parametrization problem. Otherwise, unless $v = 1$ or symmetrically $w = w_0$, the varieties $G^{v,w}$ have no sensible counterpart in the flag variety. Moreover both [2] and [7] parametrize and give formulas only for an open dense subset of the varieties they study. So Theorem 7.1 already adds to these results in the joint special case, since it determines parameters for *any* element in $\mathcal{R}_{1,w}$.

It is an interesting open problem to extend our results from $\mathcal{R}_{1,w}$, and hence $G^{1,w}$, also to the remaining varieties $G^{v,w}$ in the group. That is, similarly to find a way to parametrize every element of the group G. This should involve finding appropriate stratifications of the $G^{v,w}$ for arbitrary v, w (the open strata being the ones already understood by [7]), and then extending the Chamber Ansatz from [7] to all the remaining strata.

2. Notation and basic definitions

Let $\mathbb{K}$ be a field. Let $G_{\mathbb{K}}$ be a split, connected, simply connected, semisimple algebraic $\mathbb{K}$-group (or Chevalley group over $\mathbb{K}$). See [8] Section II.1 or any of [5], [13], [14]. Fix a $\mathbb{K}$-split maximal torus $T_{\mathbb{K}}$. We write $\mathbb{K}^*$ for the multiplicative group $G_m(\mathbb{K})$ and $\mathbb{K}$ for the additive group $G_a(\mathbb{K})$. As we will always be concerned with the $\mathbb{K}$-valued points we will write G for $G(\mathbb{K})$ and T for $T(\mathbb{K})$, and so forth. In later sections we will take $\mathbb{K}$ to be $\mathbb{R}$.

Let $X(T) = \mathrm{Hom}(T, \mathbb{K}^*)$ and $R \subset X(T)$ the set of roots. Choose a system of positive roots R^+. We denote by B^+ the Borel subgroup corresponding to R^+, and by U^+ its unipotent radical. We also have the opposite Borel B^- such that $B^+ \cap B^- = T$, and its unipotent radical U^-.

Denote the set of simple roots by

$$\Pi = \{\alpha_i \mid i \in I\} \subset R^+ \subset R \subset X(T).$$

For every $\alpha_i \in \Pi$ there is an associated homomorphism

$$\varphi_i : SL_2 \to G.$$

Consider the 1-parameter subgroups in G (landing in U^+, U^- and T respectively) defined by

$$x_i(m) = \varphi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad y_i(m) = \varphi_i \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \quad \alpha_i^\vee(t) = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

where $m \in \mathbb{K}$, $t \in \mathbb{K}^*$, and $i \in I$. The datum $(T, B^+, B^-, x_i, y_i; i \in I)$ for G is called a *pinning* in [10]. The standard pinning for SL_d consists of the diagonal, upper-triangular and lower-triangular matrices, along with the simple root subgroups $x_i(m) = I_d + mE_{i,i+1}$ and $y_i(m) = I_d + mE_{i+1,i}$, where I_d is the identity matrix, and $E_{i,j}$ has a 1 in position (i, j) and zeroes elsewhere.

Next consider the cocharacter lattice $Y(T) = \mathrm{Hom}(\mathbb{K}^*, T)$. It is dually paired with $X(T)$ in the standard way by $<\,,\,>$: $X(T) \times Y(T) \to \mathrm{Hom}(\mathbb{K}^*, \mathbb{K}^*) \cong \mathbb{Z}$. The $\alpha_i^\vee$ viewed as elements of $Y(T)$ are the simple coroots, and the Cartan matrix $A = (a_{ij}) \in \mathbb{Z}^{I \times I}$ is given by $a_{ij} = <\alpha_j, \alpha_i^\vee>$.

That G is simply connected means the $\alpha_i^\vee$ freely generate $Y(T) \cong \mathbb{Z}^I$. The dual basis in $X(T)$ is the set of fundamental weights $\{\omega_i \mid i \in I\}$. Let $X(T)_+$ be the set of dominant weights and $\rho = \sum_{i \in I} \omega_i \in X(T)_+$. For a dominant weight λ let $V(\lambda)$ denote the Weyl module with highest weight λ, see [8] II 2.13. In characteristic 0 this is just the irreducible representation with highest weight λ.

The Weyl group $W = N_G(T)/T$ acts on $X(T)$ permuting the roots R. We denote the action of $w \in W$ on $\alpha \in X(T)$ by $w\alpha$. The simple reflections $s_i \in W$ are given explicitly by $s_i := \dot{s}_i T$, where

$$\dot{s}_i := \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and any $w \in W$ can be expressed as a product $w = s_{i_1} \cdots s_{i_m}$ with a minimal number of factors $m = \ell(w)$. We set

$$\dot{w} = \dot{s}_{i_1} \dot{s}_{i_2} \cdot \ldots \cdot \dot{s}_{i_m}$$

to get a representative of w in $N_G(T)$. It is well known that this product is independent of the choice of reduced expression $s_{i_1} \cdots s_{i_m}$ for w. Let $<$ denote the Bruhat order on W. The unique maximal element of W is denoted w_0.

We note for future reference the following identity ([8] II 1.3)

$$\alpha_i^\vee(t^{-1})\dot{s}_i = x_i(-t^{-1}) y_i(t) x_i(-t^{-1}), \tag{2.1}$$

which can be checked in $SL_2(\mathbb{K})$.

Finally, for every root we introduce the corresponding root subgroup. Let $U^+_{\alpha_i}$ be the simple root subgroup in G given explicitly by $\{x_i(t) \mid t \in \mathbb{K}\}$. For an arbitrary root α there is a $w \in W$ and simple root α_i such that $\alpha = w\alpha_i$. Then the one-dimensional subgroup corresponding to α may be defined as $\dot{w}U^+_{\alpha_i}\dot{w}^{-1}$. If $\alpha \in R^+$ this subgroup lies in U^+ and we write $U^+_\alpha = \dot{w}U^+_{\alpha_i}\dot{w}^{-1}$. Otherwise the subgroup is called $U^-_\alpha = \dot{w}U^+_{\alpha_i}\dot{w}^{-1}$ and lies in U^-.

3. Subexpressions of reduced expressions

Consider a reduced expression in W, say $s_3s_2s_1s_3s_2s_3$ in type A_3. Informally, a subexpression is what is obtained by choosing some of the factors. So for example choosing the underlined factors in

$$\underline{s_3}\,\underline{s_2}\,s_1\,\underline{s_3}\,\underline{s_2}\,s_3 \tag{3.1}$$

gives a subexpression for s_2s_3 in the word $s_3s_2s_1s_3s_2s_3$.

It will be useful to represent expressions, like $s_3s_2s_1s_3s_2s_3$ or its subexpression $s_3s_2\,1\,s_3s_2\,1$, by their sequences of partial products

$$\begin{array}{lllllllll} (& 1, & s_3, & s_3s_2, & s_3s_2s_1, & s_3s_2s_1s_3, & s_3s_2s_1s_3s_2, & s_3s_2s_1s_3s_2s_3 &), \\ (& 1, & s_3, & s_3s_2, & s_3s_2, & s_3s_2s_3, & s_2s_3, & s_2s_3 &). \end{array}$$

We formalize this below.

Definition 3.1. Let us define an *expression* for $w \in W$ to be a sequence

$$\mathbf{w} = \big(w_{(0)}, w_{(1)}, w_{(2)}, \dots, w_{(n)}\big)$$

in W, such that $w_{(0)} = 1$, $w_{(n)} = w$ and

$$w_{(j)} = \begin{cases} w_{(j-1)}, & \text{or} \\ w_{(j-1)}s_i, & \text{for some simple reflection } s_i \end{cases}$$

for $j = 1, \dots, n$. The expression $\mathbf{w}$ may equivalently be specified by its *sequence of factors*,

$$(w_{(1)}, w_{(1)}^{-1}w_{(2)}, \dots, w_{(n-1)}^{-1}w_{(n)}),$$

which has entries in $\{s_i \mid i \in I\} \cup \{1\}$.

Definition 3.2. For an expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ define

$$\begin{aligned} J^+_{\mathbf{w}} &= \{k \in \{1, \dots, n\} \mid w_{(k-1)} < w_{(k)}\}, \\ J^\circ_{\mathbf{w}} &= \{k \in \{1, \dots, n\} \mid w_{(k-1)} = w_{(k)}\}, \\ J^-_{\mathbf{w}} &= \{k \in \{1, \dots, n\} \mid w_{(k)} < w_{(k-1)}\}. \end{aligned}$$

An expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ is called *non-decreasing* if $w_{(j-1)} \le w_{(j)}$ for all $j = 1, \dots, n$, so $J^-_{\mathbf{w}} = \emptyset$. It is called *reduced* if $w_{(j-1)} < w_{(j)}$ for all $j = 1, \dots, n$. Clearly, any non-decreasing expression $\mathbf{w}$ for w gives rise to a reduced expression $\widehat{\mathbf{w}}$ of w by discarding all $w_{(j)}$ with $j \in J^\circ_{\mathbf{w}}$.

The following definition is taken from [6, Definition 2.3].

Definition 3.3 (Distinguished subexpressions)**.** Let $\mathbf{w}$ be a reduced expression for $w \in W$ with factors $(s_{i_1}, \dots, s_{i_n})$. Let $v \le w$. Then by a *subexpression* for v in $\mathbf{w}$, we mean an expression $\mathbf{v} = (v_{(0)}, v_{(1)}, v_{(2)}, \dots, v_{(n)})$ such that

$$v_{(j)} \in \big\{v_{(j-1)},\, v_{(j-1)}s_{i_j}\big\} \qquad \text{for all } j = 1, \dots n,$$

and $v_{(n)} = v$. In particular there is always the "empty" subexpression $(1, \dots, 1)$ for 1.

A subexpression $\mathbf{v}$ of $\mathbf{w}$ as above is called *distinguished* if we have

$$v_{(j)} \le v_{(j-1)}\ s_{i_j} \quad \text{for all } j \in \{1, \dots, n\}. \tag{3.2}$$

In other words, if right multiplication by s_{i_j} decreases the length of $v_{(j-1)}$, then in a distinguished subexpression the component $v_{(j)}$ must be given by $v_{(j)} = v_{(j-1)}s_{i_j}$.

We write $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v}$ is a distinguished subexpression of $\mathbf{w}$.

Examples. For $w = w_0$ in A_3 and the reduced expression $\mathbf{w}$ with factors $(s_3, s_2, s_1, s_3, s_2, s_3)$, the only distinguished subexpression for $s_2 s_3$ is

$$\mathbf{v} = (1, 1, 1, 1, 1, s_2, s_2 s_3). \tag{3.3}$$

In particular, the subexpression indicated in (3.1) is not distinguished. If $v = s_2$, then we have four distinguished subexpressions for v in $\mathbf{w}$,

$$\mathbf{v} = (1, 1, 1, 1, 1, s_2, s_2), \qquad (s_3\, s_2\, s_1\, s_3\, \underline{s_2}\, s_3), \tag{3.4}$$

$$\mathbf{v} = (1, s_3, s_3, s_3, 1, s_2, s_2), \qquad (\underline{s_3}\, s_2\, s_1\, \underline{s_3}\, \underline{s_2}\, s_3), \tag{3.5}$$

$$\mathbf{v} = (1, s_2, s_2, s_2 s_3, s_2 s_3, s_2), \qquad (s_3\, \underline{s_2}\, s_1\, \underline{s_3}\, s_2\, \underline{s_3}), \tag{3.6}$$

$$\mathbf{v} = (1, s_3, s_3 s_2, s_3 s_2, s_3 s_2 s_3, s_2 s_3, s_2), \qquad (\underline{s_3}\, \underline{s_2}\, s_1\, \underline{s_3}\, \underline{s_2}\, \underline{s_3}). \tag{3.7}$$

Definition 3.4 (Positive subexpressions)**.** Let $\mathbf{w}$ be a reduced expression with factors $(s_{i_1}, \ldots, s_{i_n})$. We call a subexpression $\mathbf{v}$ of $\mathbf{w}$ *positive* if

$$v_{(j-1)} < v_{(j-1)} s_{i_j} \tag{3.8}$$

for all $j = 1, \ldots, n$.

Note that (3.8) is equivalent to $v_{(j-1)} \le v_{(j)} \le v_{(j-1)} s_{i_j}$. So in other words a positive subexpression is one that is distinguished and non-decreasing. In the examples above only (3.3) and (3.4) are positive.

Lemma 3.5. *Given $v \le w$ in W and a reduced expression $\mathbf{w}$ for w, then there is a unique positive subexpression $\mathbf{v}_+$ for v in $\mathbf{w}$.*

Proof. We construct $\mathbf{v}_+ = (v_{(0)}, \ldots, v_{(n)})$ starting from the right with $v_{(n)} = v$. The inequality $v_{(j-1)} < v_{(j-1)} s_{i_j}$ says that $v_{(j-1)}$ cannot have a reduced expression ending in s_{i_j}. If $v_{(j)}$ has such a reduced expression then we must set $v_{(j-1)} = v_{(j)} s_{i_j}$. If $v_{(j)}$ does not, then $v_{(j-1)} = v_{(j)}$. To summarize, $v_{(j-1)}$ is given by

$$v_{(j-1)} = \begin{cases} v_{(j)} s_{i_j} & \text{if } v_{(j)} s_{i_j} < v_{(j)}, \\ v_{(j)} & \text{otherwise.} \end{cases}$$

This along with $v_{(n)} = v$ clearly defines (uniquely) the desired positive subexpression of $\mathbf{w}$. $\square$

The positive subexpression $\mathbf{v}_+$ is in a sense the right-most subexpression for v in $\mathbf{w}$ that is non-decreasing.

4. Deodhar's decomposition

4.1. **Bruhat decomposition.** Let us identify the flag variety with the variety $\mathcal{B}$ of Borel subgroups, via

$$gB^+ \longleftrightarrow g \cdot B^+ := gB^+ g^{-1}.$$

We have the Bruhat decompositions,

$$\mathcal{B} = \bigsqcup_{w \in W} B^+ \dot{w} \cdot B^+ = \bigsqcup_{w \in W} B^- \dot{w} \cdot B^-,$$

of $\mathcal{B}$ into B^+-orbits called *Bruhat cells*, and B^--orbits called *opposite Bruhat cells*. Let $\alpha^1, \ldots, \alpha^n$ be the positive roots made negative by w^{-1}. Recall that the Bruhat cell $B^+ \dot{w} \cdot B^+$ can be identified with the product of root subgroups

$$U^+ \cap \dot{w} U^- \dot{w}^{-1} = U^+_{\alpha^1} U^+_{\alpha^2} \cdots U^+_{\alpha^n} \cong \mathbb{K}^n \tag{4.1}$$

via $u \mapsto u\dot{w} \cdot B^+$. Moreover,

$$U^+ = (U^+ \cap \dot{w} U^- \dot{w}^{-1})(U^+ \cap \dot{w} U^+ \dot{w}^{-1}),$$

where the second factor is a product of the remaining positive root subgroups. Given a reduced expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ with factors $(s_{i_1}, \ldots, s_{i_n})$, the positive roots sent to negative roots by w^{-1} can be listed as

$$\alpha^1_{\mathbf{w}} = \alpha_{i_1}\ , \quad \alpha^2_{\mathbf{w}} = w_{(1)} \cdot \alpha_{i_2}\ , \quad \alpha^3_{\mathbf{w}} = w_{(2)} \cdot \alpha_{i_3}\ , \ \ldots\ldots\ , \quad \alpha^n_{\mathbf{w}} = w_{(n-1)} \cdot \alpha_{i_n}. \tag{4.2}$$

Therefore another way to write down the parametrization of the Bruhat cell $B^+\dot w\cdot B^+$ is by

$$\mathbb{K}^n \xrightarrow{\sim} B^+\dot w\cdot B^+ \ :\quad (m_1,\dots,m_n)\mapsto x_{i_1}(m_1)\dot s_{i_1}\cdots x_{i_n}(m_n)\dot s_{i_n}\cdot B^+. \tag{4.3}$$

If one moves all the simple reflections to the right (conjugating the intermediate simple root subgroups), then what remains on the left is a product of root subgroups corresponding to precisely the roots listed in (4.2).

4.2. **Relative position.** Consider the product $\mathcal{B}\times\mathcal{B}$ with G acting diagonally. Let $B_1 = g_1\cdot B^+$ and $B_2 = g_2\cdot B^+$. Then there is a unique $w\in W$ such that ${g_1}^{-1}g_2\cdot B^+\in B^+\dot w\cdot B^+$. Equivalently, w is the unique Weyl group element such that

$$(B_1,B_2)\ \in\ G\text{-orbit of }(B^+,\dot w\cdot B^+).$$

We call w the *relative position* of (B_1,B_2) and write

$$B_1\xrightarrow{w}B_2.$$

For example $B_1\xrightarrow{1}B_2$ implies $B_1=B_2$. And $B^+\xrightarrow{w}B$ says that B lies in the Bruhat cell $B^+\dot w\cdot B^+$. While $B^-\xrightarrow{w}B$ means that B lies in the opposite Bruhat cell $B^-\dot w_0\dot w\dot w_0\cdot B^-$. We will also use the notation

$$(B_1,B_2)\sim(B_1',B_2')$$

to indicate that (B_1,B_2) and (B_1',B_2') in $\mathcal{B}\times\mathcal{B}$ are conjugate under G.

The following assertions follow from the definitions and standard properties of the Bruhat decomposition.

(1) If $B_1\xrightarrow{w}B_2$ and $g\in G$, then also $g\cdot B_1\xrightarrow{w}g\cdot B_2$.
(2) If $B_1\xrightarrow{s}B_2\xrightarrow{s}B_3$ for a simple reflection s, then $B_1\xrightarrow{s}B_3$ or $B_1=B_3$.
(3) If $B_1\xrightarrow{v}B_2\xrightarrow{w}B_3$ and $\ell(vw)=\ell(v)+\ell(w)$, then $B_1\xrightarrow{vw}B_3$.
(4) If $B_1\xrightarrow{w}B_2$, then $B_2\xrightarrow{w^{-1}}B_1$.

We will make use of these properties freely.

4.3. **Reduction maps.** Suppose $w=vv'$ with $\ell(w)=\ell(v)+\ell(v')$. Then the set of positive roots sent to negative roots by v^{-1} is a subset of the positive roots made negative by w^{-1}, by (4.2). Under these circumstances one can define a morphism

$$\pi^w_v : B^+\dot w\cdot B^+\to B^+\dot v\cdot B^+$$
$$b\dot w\cdot B^+\mapsto b\dot v\cdot B^+,$$

where $b\in B^+$. The map π^w_v is well-defined since $B^+\cap\dot wB^+\dot w^{-1}\ \subseteq\ B^+\cap\dot vB^+\dot v^{-1}$. For a given $B\in B^+\dot w\cdot B^+$, the element $\pi^w_v(B)$ is characterized by the property

$$B^+\ \xrightarrow{v}\ \pi^w_v(B)\ \xrightarrow{v^{-1}w}\ B. \tag{4.4}$$

Let us call π^w_v a reduction map.

4.4. **Deodhar's theorem.**

Definition 4.1. For $v,w\in W$ define

$$\mathcal{R}_{v,w} := B^+\dot w\cdot B^+\cap B^-\dot v\cdot B^+ = \{B\in\mathcal{B}\mid B^+\xrightarrow{w}B\xleftarrow{w_0v}B^-\}.$$

The intersection $\mathcal{R}_{v,w}$ is non-empty precisely if $v\le w$. And in that case Kazhdan and Lusztig proved that over an algebraically closed field it is irreducible of dimension $\ell(w)-\ell(v)$, see [9] §1. If $v=w$ then $\mathcal{R}_{w,w}=\{\dot w\cdot B^+\}$.

Suppose now that $\mathbf{w}$ is a reduced expression for $w\in W$ with factors $(s_{i_1},\dots,s_{i_n})$, and $B\in\mathcal{R}_{v,w}$. Using the reduction maps we can associate to B uniquely a sequence of 'intermediate' Borel subgroups

$$B^+=B_0\xrightarrow{s_{i_1}}B_1\xrightarrow{s_{i_2}}B_2\xrightarrow{s_{i_3}}\cdots\xrightarrow{s_{i_n}}B_n=B,$$

where $B_k = \pi^w_{w_{(k)}}(B)$. By construction $B^+ \xrightarrow{w_{(k)}} B_k$. However, the position of B_k with respect to B^-, or the opposite Bruhat cell containing B_k, defines a new element $v_{(k)} \in W$ by

$$B_k \in B^- v_{(k)} \cdot B^+.$$

For $\mathbf{w}$ as above and a sequence $\mathbf{v} := (v_{(0)}, \dots, v_{(n)})$ we define the *Deodhar component* $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ in $\mathcal{B}$ by

$$\begin{aligned} \mathcal{R}_{\mathbf{v},\mathbf{w}} &:= \{ B \in \mathcal{R}_{v,w} \mid \pi^w_{w_{(k)}}(B) \in B^- v_{(k)} \cdot B^+ \} \\ &= \{ B \in \mathcal{R}_{v,w} \mid \pi^w_{w_{(k)}}(B) \in \mathcal{R}_{v_{(k)},w_{(k)}} \}. \end{aligned} \tag{4.5}$$

Theorem 4.2 ([6] Theorem 1.1). *Suppose $w \in W$ and $B \in B^+\dot{w} \cdot B^+$, and fix a reduced expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ for w.*

(1) *The Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ is nonempty if and only if $\mathbf{v}$ is a distinguished subexpression of $\mathbf{w}$.*
(2) *If $\mathbf{v} \prec \mathbf{w}$, then $\mathcal{R}_{\mathbf{v},\mathbf{w}} \cong (\mathbb{K}^*)^{|J^\circ_{\mathbf{v}}|} \times \mathbb{K}^{|J^-_{\mathbf{v}}|}$, where $J^\circ_{\mathbf{v}}$ and $J^-_{\mathbf{v}}$ are as in Definition 3.2.*

Another proof of this theorem will be contained in the next section. If the reduced expression $\mathbf{w}$ is fixed, then as a corollary of the theorem one has a decomposition

$$\mathcal{R}_{v,w} = \bigsqcup_{\mathbf{v}} \mathcal{R}_{\mathbf{v},\mathbf{w}}, \tag{4.6}$$

where the union is over all distinguished subexpressions for v in $\mathbf{w}$. Note that the Deodhar component $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$ corresponding to the unique positive subexpression for v in $\mathbf{w}$ has dimension $|J^\circ_{\mathbf{v}_+}| = \ell(w) - \ell(v)$. So if $\mathbb{K}$ is algebraically closed then it is dense in $\mathcal{R}_{v,w}$. This also holds for $\mathbb{K} = \mathbb{R}$ since $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}(\mathbb{R})$ is Zariski dense in $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}(\mathbb{C})$. Finally, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, it holds that $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$ is open dense in $\mathcal{R}_{v,w}$ with respect to the usual Hausdorff topology.

Suppose we fix a reduced expression $\mathbf{w}_0$ for the longest element w_0. Then for any $w \in W$ the positive subexpression for w in $\mathbf{w}_0$ determines a reduced expression $\widehat{\mathbf{w}}_+$ for w. Therefore we have a decomposition of the whole flag variety,

$$\mathcal{B} = \bigsqcup_{w \in W} \left(\bigsqcup_{\mathbf{v} \prec \widehat{\mathbf{w}}_+} \mathcal{R}_{\mathbf{v},\widehat{\mathbf{w}}_+} \right), \tag{4.7}$$

which we may call the Deodhar decomposition of $\mathcal{B}$ corresponding to $\mathbf{w}_0$.

Remark 4.3. The varieties $\mathcal{R}_{v,w}$ may be defined over a finite field $\mathbb{K} = \mathbb{F}_q$. In this setting the number of points determine the R-polynomials $R_{v,w}(q) = \#(\mathcal{R}_{v,w}(\mathbb{F}_q))$ introduced by Kazhdan and Lusztig [9] to give a recursive formula for the Kazhdan-Lusztig polynomials. This is the origin of the notation $\mathcal{R}_{v,w}$ as well as Deodhar's original application of the theorem. The decompositions (4.6) together with the isomorphisms $\mathcal{R}_{\mathbf{v},\mathbf{w}}(\mathbb{F}_q) \cong (\mathbb{F}_q^*)^{|J^\circ_{\mathbf{v}}|} \times {\mathbb{F}_q}^{|J^-_{\mathbf{v}}|}$ give formulas for the R-polynomials.

5. Explicit parametrizations of Deodhar components

Let $\mathbf{w}$ be a reduced expression with factors $(s_{i_1}, \dots, s_{i_n})$, and $\mathbf{v} \prec \mathbf{w}$.

Definition 5.1. Define a subset $G_{\mathbf{v},\mathbf{w}}$ in G by

$$G_{\mathbf{v},\mathbf{w}} = \left\{ g = g_1 g_2 \cdots g_n \;\middle|\; \begin{array}{lll} g_k = x_{i_k}(m_k)\dot{s}_{i_k}^{-1} & \text{if } k \in J^-_{\mathbf{v}}, & \\ g_k = y_{i_k}(t_k) & \text{if } k \in J^\circ_{\mathbf{v}}, & \text{for } t_k \in \mathbb{K}^*,\ m_k \in \mathbb{K}. \\ g_k = \dot{s}_{i_k} & \text{if } k \in J^+_{\mathbf{v}}, & \end{array} \right\}. \tag{5.1}$$

There is an obvious map $(\mathbb{K}^*)^{J^\circ_{\mathbf{v}}} \times \mathbb{K}^{J^-_{\mathbf{v}}} \to G_{\mathbf{v},\mathbf{w}}$ defined by the parameters t_k and m_k in (5.1). For $\mathbf{v} = \mathbf{w} = (1)$ we define $G_{\mathbf{v},\mathbf{w}} = \{1\}$.

The following proposition gives an explicit parametrization for the Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$.

Proposition 5.2. *The map* $(\mathbb{K}^*)^{J^\circ_{\mathbf{v}}} \times \mathbb{K}^{J^-_{\mathbf{v}}} \to G_{\mathbf{v},\mathbf{w}}$ *from Definition 5.1 is an isomorphism. The set* $G_{\mathbf{v},\mathbf{w}}$ *lies in* $U^-\dot{v} \cap B^+\dot{w}B^+$*, and the assignment* $g \mapsto g \cdot B^+$ *defines an isomorphism*

$$G_{\mathbf{v},\mathbf{w}} \xrightarrow{\sim} \mathcal{R}_{\mathbf{v},\mathbf{w}}. \tag{5.2}$$

The special case $v = 1$ and $w = w_0$ of this proposition already appears in [11] Proposition 2.5. The proof below is analogous to the one we gave for that special case, and of course also similar to Deodhar's proof of Theorem 4.2.(2), although his is ultimately a different isomorphism.

Proof. Let $\mathbf{w} = (w_{(0)}, \dots, w_{(n)})$ be a reduced expression with factors $(s_{i_1}, \dots, s_{i_n})$, and $\mathbf{v} = (v_{(0)}, \dots, v_{(n)})$. The proof is by induction on n. If $n = 0$ then $\mathbf{v} = \mathbf{w} = (1)$ and the isomorphism (5.2) is the trivial one $1 \mapsto B^+$. There is nothing more to check. For $n > 0$ let $\mathbf{w}' := (w_{(0)}, \dots, w_{(n-1)})$ and similarly $\mathbf{v}' = (v_{(0)}, \dots, v_{(n-1)})$, the truncations of $\mathbf{v}$ and $\mathbf{w}$. Also set $w' = w_{(n-1)}$ and $v' = v_{(n-1)}$. We may assume the proposition is true for $\mathbf{v}', \mathbf{w}'$.

It is easy to check that $G_{\mathbf{v}',\mathbf{w}'} \times \mathbb{K} \xrightarrow{\sim} (\pi^w_{w'})^{-1}(\mathcal{R}_{\mathbf{v}',\mathbf{w}'})$ via the map $(g', m) \mapsto g' x_{i_n}(m)\dot{s}_{i_n} \cdot B^+$, using for example (4.4) and properties of relative position. And we have a commutative diagram

$$\begin{array}{ccc} G_{\mathbf{v}',\mathbf{w}'} \times \mathbb{K} & \xrightarrow{\sim} & (\pi^w_{w'})^{-1}(\mathcal{R}_{\mathbf{v}',\mathbf{w}'}) \\ pr_1 \downarrow & & \downarrow \pi^w_{w'} \\ G_{\mathbf{v}',\mathbf{w}'} & \xrightarrow{\sim} & \mathcal{R}_{\mathbf{v}',\mathbf{w}'}. \end{array} \tag{5.3}$$

Now let $B \in (\pi^w_{w'})^{-1}(\mathcal{R}_{\mathbf{v}',\mathbf{w}'})$, so $B = g' x_{i_n}(m)\dot{s}_{i_n} \cdot B^+$ for some $g' \in G_{\mathbf{v}',\mathbf{w}'}$ and $m \in \mathbb{K}$. We consider two cases.

(i) Suppose $m = 0$. Then $B = g'\dot{s}_{i_n} \cdot B^+$. Since $g' \in U^-\dot{v}'$ we have $B \in B^-\dot{v}'\dot{s}_{i_n} \cdot B^+$.

(ii) Suppose $m \neq 0$. Then the identity (2.1) implies $x_{i_n}(m)\dot{s}_{i_n} \cdot B^+ = y_{i_n}(m^{-1}) \cdot B^+$. So we may write B in two different ways,

$$B = g' x_{i_n}(m)\dot{s}_i \cdot B^+ = g' y_{i_n}(m^{-1}) \cdot B^+.$$

- If $v's_{i_n} > v'$, then $\dot{v}' y_{i_n}(m^{-1})\dot{v}'^{-1} \in U^-$. In this case we have

  $$g := g' y_{i_n}(m^{-1}) \in U^-\dot{v}'$$

  and $B = g \cdot B^+ \in B^-\dot{v}' \cdot B^+$.
- If $v's_{i_n} < v'$, then $\dot{v}' x_{i_n}(m)\dot{v}'^{-1} \in U^-$. Therefore we have

  $$g := g' x_{i_n}(m)\dot{s}_{i_n}^{-1} \in U^-\dot{v}'\dot{s}_{i_n}^{-1}$$

  and $B = g \cdot B^+ \in B^-\dot{v}'\dot{s}_{i_n}^{-1} \cdot B^+$.

Note that in both cases, (i) and (ii), if $v's_{i_n} < v'$ we have $B \in B^-\dot{v}'\dot{s}_{i_n} \cdot B^+$. This explains Theorem 4.2.(1). We now use the above to analyze the possibilities for an element $B \in \mathcal{R}_{\mathbf{v},\mathbf{w}} \subseteq (\pi^{w'}_w)^{-1}(\mathcal{R}_{\mathbf{v}',\mathbf{w}'})$ and complete the proof of the proposition.

(1) Suppose $n \in J^-_{\mathbf{v}}$. Then both (i) and (ii) are possible. Therefore $\mathcal{R}_{\mathbf{v},\mathbf{w}} = (\pi^{w'}_w)^{-1}(\mathcal{R}_{\mathbf{v}',\mathbf{w}'})$, and we have

$$G_{\mathbf{v},\mathbf{w}} \cong G_{\mathbf{v}',\mathbf{w}'} \times \mathbb{K} \xrightarrow{\sim} \mathcal{R}_{\mathbf{v},\mathbf{w}},$$

via $g_1 \dots g_{n-1} x_{i_n}(m_n)\dot{s}_{i_n}^{-1} \mapsto (g_1 \dots g_{n-1}, m_n)$ and (5.3).

(2) If $n \in J^\circ_{\mathbf{v}}$ then $v_{(n)} = v_{(n-1)}$ so only case (ii) is possible. Then (5.3) restricts to give

$$G_{\mathbf{v},\mathbf{w}} \cong G_{\mathbf{v}',\mathbf{w}'} \times \mathbb{K}^* \xrightarrow{\sim} \mathcal{R}_{\mathbf{v},\mathbf{w}},$$

where the identification $G_{\mathbf{v},\mathbf{w}} \cong G_{\mathbf{v}',\mathbf{w}'} \times \mathbb{K}^*$ is via $g_1 \dots g_{n-1} y_{i_n}(t_n) \mapsto (g_1 \dots g_{n-1}, t_n^{-1})$.

(3) Finally, if $n \in J^+_{\mathbf{v}}$ then only case (i) is possible and $B = g'\dot{s}_{i_n} \cdot B^+$. Therefore (5.3) induces

$$G_{\mathbf{v},\mathbf{w}} \cong G_{\mathbf{v}',\mathbf{w}'} \times \{ 1 \} \xrightarrow{\sim} R_{\mathbf{v},\mathbf{w}}.$$

In each case $G_{\mathbf{v},\mathbf{w}} \subset U^-\dot{v}_{(n)}$, where we note that in (1) above, $v_{(n)} = v's_{i_n} < v'$ implies $\dot{v}_{(n)} = \dot{v}'\dot{s}_{i_n}^{-1}$. The inclusion $G_{\mathbf{v},\mathbf{w}} \subset B^+\dot{w}B^+$ is clear. □

Remark 5.3. Let $\mathbf{w}_0$ be a fixed reduced expression for w_0. Then Deodhar's decomposition (4.7) of $\mathcal{B}$ together with Proposition 5.2 gives rise to a set theoretic crossection

$$\phi : \mathcal{B} \longrightarrow G,$$

defined on each Deodhar component $\mathcal{R}_{\mathbf{v},\widehat{\mathbf{w}}_+} \subset \mathcal{B}$ as the inverse of $G_{\mathbf{v},\widehat{\mathbf{w}}_+} \xrightarrow{\sim} \mathcal{R}_{\mathbf{v},\widehat{\mathbf{w}}_+}$. To describe the map ϕ more explicitly we must, firstly, explain how to determine the Deodhar component of an element of $\mathcal{B}$ and, secondly, give formulas for the individual maps $\mathcal{R}_{\mathbf{v},\mathbf{w}} \to G_{\mathbf{v},\mathbf{w}}$.

6. Deodhar components in terms of minors

Suppose B lies in a particular Bruhat cell, $B = z\dot{w} \cdot B^+$ for $z \in U^+$. In this section we determine the conditions on z for B to lie in a Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$. The conditions will be expressed in terms of (generalized) minors of z.

Let $V(\lambda)$ be the Weyl module of G with highest weight λ. In the following λ will often be a fundamental weight ω_i. Consider the weight space decomposition $V(\lambda) = \oplus_\mu V(\lambda)_\mu$, and denote by $pr_\mu : V(\lambda) \to V(\lambda)_\mu$ the corresponding projections. Let us fix a highest weight vector ξ_λ. Then the element $\dot{w} \cdot \xi_\lambda \in V(\lambda)$ for $w \in W$ spans the extremal weight space $V(\lambda)_{w\lambda}$. In this way, the choice of highest weight vector gives rise to a canonical choice of basis vectors for all the extremal weight spaces.

Lemma 6.1. *If $w\lambda = w'\lambda$, then $\dot{w} \cdot \xi_\lambda = \dot{w}' \cdot \xi_\lambda$.*

Proof. It is necessary only to check that $\dot{v} \cdot \xi_\lambda = \xi_\lambda$ whenever $v\lambda = \lambda$. Since the stabilizer of λ is a parabolic subgroup of W we may assume v is a simple reflection s_i. Then $\dot{s}_i = x_i(-1)y_i(1)x_i(-1)$ and the statement is clear. □

Definition 6.2 (Generalized minors)**.** For $\eta \in V(\lambda)$ define $\langle \eta, \dot{w} \cdot \xi_\lambda \rangle$ to be the coefficient in η of the extremal weight vector $\dot{w} \cdot \xi_\lambda$. That is, with notation as above,

$$pr_{w\lambda}(\eta) = \langle \eta, \dot{w} \cdot \xi_\lambda \rangle \;\; \dot{w} \cdot \xi_\lambda.$$

For two extremal weights $w\lambda$ and $w'\lambda$ we then have a regular function $\Delta^{w\lambda}_{w'\lambda}$ on G defined by

$$\Delta^{w\lambda}_{w'\lambda}(g) := \langle g\dot{w}' \cdot \xi_\lambda \, , \, \dot{w} \cdot \xi_\lambda \rangle \, .$$

Since any weight lies in the Weyl group orbit of a unique dominant weight, this notation is unambiguous.

It is not hard to see that $\Delta^{w\lambda}_{w'\lambda}$ coincides with the regular function $\Delta_{w\lambda,w'\lambda}$ defined in [7, Definition 1.4].

The functions $\Delta^{w\omega_i}_{w'\omega_i}$, where ω_i ranges through the set of fundamental weights, are called *minors* or *generalized minors*. If $G = SL_d$ with the standard pinning then $\Delta^{w\omega_i}_{w'\omega_i}$ is precisely the usual $i \times i$ minor, where $w\omega_i$ encodes the row set and $w'\omega_i$ the column set.

Definition 6.3 (Chamber minors)**.** Suppose $\mathbf{w}$ is a reduced expression and $\mathbf{v} \prec \mathbf{w}$ a distinguished subexpression.

(1) The minors $\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}$ for $k = 0, 1, \ldots, n$ are called the *standard chamber minors* for $\mathbf{v}$ and $\mathbf{w}$.
(2) The minors $\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}$ for $k \in J^-_{\mathbf{v}} \cup J^+_{\mathbf{v}}$ are called the *special chamber minors* for $\mathbf{v}$ and $\mathbf{w}$.

Note that $\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}} = \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}$ if $k \in J^\circ_{\mathbf{v}}$.

Proposition 6.4. *Let $B = z\dot{w}\cdot B^+$ for $z \in U^+$, and $\mathbf{w}$ be a reduced expression with factors $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$. Then B lies in the the Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ where $\mathbf{v} = (v_{(0)}, v_{(1)}, \ldots, v_{(n)})$ is constructed as follows. Let $v_{(0)} = 1$. Suppose that $k \geq 1$ and $v_{(k-1)}$ has already been defined.*

(a) *If $v_{(k-1)}s_{i_k} > v_{(k-1)}$ and $\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) \neq 0$, then $v_{(k)} = v_{(k-1)}$.*
(b) *If $v_{(k-1)}s_{i_k} > v_{(k-1)}$ and $\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) = 0$, then $v_{(k)} = v_{(k-1)}s_{i_k}$.*
(c) *If $v_{(k-1)}s_{i_k} < v_{(k-1)}$, then $v_{(k)} = v_{(k-1)}s_{i_k}$.*

Remark 6.5. Note that in the situation of the proposition $\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\neq 0$ for all $k=1,\dots,n$, as follows from the definition of the $v_{(k)}$. The chamber minors give rise to well-defined maps (which we denote in the same way),

$$\begin{array}{llll} \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}: & \mathcal{R}_{\mathbf{v},\mathbf{w}}\to\mathbb{K}^*: & z\dot w\cdot B^+\mapsto \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) & k=1,\dots,n,\\ \Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}: & \mathcal{R}_{\mathbf{v},\mathbf{w}}\to\mathbb{K}: & z\dot w\cdot B^+\mapsto \Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) & k\in J^-_{\mathbf{v}}.\end{array}$$

Proof. By Theorem 4.2.(1) we have that $\mathbf{v}$ is a distinguished subexpression of $\mathbf{w}$. Therefore (c) holds. Now suppose $v_{(k-1)}s_{i_k}>v_{(k-1)}$. We have either

(1) $v_{(k)}=v_{(k-1)}$ and $z\dot w_{(k)}\cdot B^+\in B^-\dot v_{(k-1)}\cdot B^+$, or
(2) $v_{(k)}=v_{(k-1)}s_{i_k}$ and $z\dot w_{(k)}\cdot B^+\in B^-\dot v_{(k-1)}\dot s_{i_k}\cdot B^+$.

We can distinguish between these two cases by looking just at the representation $V_{\omega_{i_k}}$. In the first case, the highest weight occurring in $z\dot w_{(k)}\cdot\xi_{\omega_{i_k}}$ is $v_{(k-1)}\omega_{i_k}$, and hence $\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\neq 0$. In the second case, the highest weight occurring in $z\dot w_{(k)}\cdot\xi_{\omega_{i_k}}$ is $v_{(k-1)}s_{i_k}\omega_{i_k}$, which is lower than $v_{(k-1)}\omega_{i_k}$ since $v_{(k-1)}s_{i_k}>v_{(k-1)}$. Therefore we have $\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)=0$. □

As a reformulation of Proposition 6.4 we have the following description of $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ inside the Bruhat cell $B^+\dot w\cdot B^+$.

Corollary 6.6. *Suppose* $\mathbf{w}$ *is a reduced expression of* w *and* $\mathbf{v}\prec\mathbf{w}$ *a distinguished subexpression, with* $J^+_{\mathbf{v}}$ *and* $J^\circ_{\mathbf{v}}$ *as in Definition 3.2. Then* $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ *may be described by*

$$\mathcal{R}_{\mathbf{v},\mathbf{w}}=\left\{z\dot w\cdot B^+\ \middle|\ z\in U^+;\ \begin{array}{ll}\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)=0 & \text{for all } k\in J^+_{\mathbf{v}},\\ \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\neq 0 & \text{for all } k\in J^\circ_{\mathbf{v}}\end{array}\right\}.$$

7. The generalized Chamber Ansatz

By Proposition 5.2 a Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ comes with isomorphisms

$$(\mathbb{K}^*)^{J^\circ_{\mathbf{v}}}\times\mathbb{K}^{J^-_{\mathbf{v}}}\xrightarrow{\sim}G_{\mathbf{v},\mathbf{w}}\xrightarrow{\sim}\mathcal{R}_{\mathbf{v},\mathbf{w}}. \tag{7.1}$$

The aim of this section is to describe an inverse to (7.1). The following theorem generalizes the Chamber Ansatz of Berenstein and Zelevinsky [2].

Theorem 7.1. *(Generalized Chamber Ansatz)*
Let $B=z\dot w\cdot B^+\in\mathcal{R}_{v,w}$*, where* $z\in U^+$*,* $v,w\in W$ *and* $v\le w$*. Let* $\mathbf{w}=(w_{(0)},w_{(1)},\dots,w_{(n)})$ *be a reduced expression for* w *with factors* $(s_{i_1},s_{i_2},\dots,s_{i_n})$*. Then* B *lies in a Deodhar component* $\mathcal{R}_{\mathbf{v},\mathbf{w}}$*, where* $\mathbf{v}=(v_{(0)},v_{(1)},\dots,v_{(n)})$ *is a distinguished subexpression for* v *in* $\mathbf{w}$*. By Proposition 5.2, there is* $g\in G_{\mathbf{v},\mathbf{w}}$ *such that* $B=g\cdot B^+$*. By Definition 5.1 we can write* $g=g_1g_2\cdots g_n\in U^-\dot v\cap B^-\dot wB^+$*, where*

$$g_k=\begin{cases} y_{i_k}(t_k) & k\in J^\circ_{\mathbf{v}},\\ \dot s_{i_k} & k\in J^+_{\mathbf{v}},\\ x_{i_k}(m_k)\dot s_{i_k}^{-1} & k\in J^-_{\mathbf{v}}.\end{cases}$$

For each k*, let* $g_{(k)}=g_1g_2\cdots g_k$ *denote the partial product. Then the following hold.*

(1) *For* $k\in J^\circ_{\mathbf{v}}$*, we have:*

$$t_k=\frac{\prod_{j\neq i_k}\Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}}{\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\,\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}$$

(2) *For* $k\in J^-_{\mathbf{v}}$*, we have:*

$$m_k=\frac{\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\,\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}{\prod_{j\neq i_k}\Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}}-\Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)}).$$

Remark 7.2. It is easy to check that the minors appearing in Theorem 7.1(1) are the standard chamber minors of Definition 6.3(1). The formula for the m_k also involves the special chamber minors, as well as a correction term, $\Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)})$, which can be computed recursively. It is an open problem to find a closed formula in terms of minors of z for this correction term. See Section 8 for another interpretation of the formulas in Theorem 7.1.

In order to prove Theorem 7.1, we will rewrite the chamber minors as minors of $g_{(k)} = g_1 g_2 \cdots g_k$, for $k = 0, 1, \ldots, n$ (Lemma 7.4). We will then compute these minors (Lemma 7.5) and substitute these formulas back into the expressions in Theorem 7.1(1) and (2), finally showing that they reduce to the coefficients t_k and m_k as claimed.

Lemma 7.3. *For $k \in \{0, 1, \ldots, n\}$ we have $g_{(k)} \cdot B^+ = z\dot{w}_{(k)} \cdot B^+$.*

Proof. Consider the sequence $B^+, g_{(1)} \cdot B^+, g_{(2)} \cdot B^+, \ldots, g_{(n)} \cdot B^+$. Then clearly

$$B^+ \xrightarrow{s_{i_1}} g_{(1)} \cdot B^+ \xrightarrow{s_{i_2}} g_{(2)} \cdot B^+ \xrightarrow{s_{i_3}} \cdots \xrightarrow{s_{i_n}} g_{(n)} \cdot B^+.$$

It follows from Section 4.3 that $g_{(k)} \cdot B^+ = \pi^w_{w_{(k)}}(B) = z\dot{w}_{(k)} \cdot B^+$ for $k = 0, 1, \ldots, n$ as required. □

We note that Lemma 7.3 gives two expressions for the intermediate Borel subgroups $B_k = \pi^w_{w_{(k)}}(B)$. Suppose that λ is a dominant weight. Let L_k denote the line in the module $V(\lambda)$ stabilised by B_k. Then, by the above, we have $L_k =< z\dot{w}_{(k)} \cdot \xi_\lambda >=< g_{(k)} \cdot \xi_\lambda >$, where $< \eta >$ denotes the line spanned by $\eta \in V(\lambda)$. We use this fact in the following lemma to compute minors of z in terms of minors of $g_{(k)}$.

Lemma 7.4. *Let λ be a dominant weight. Then*

(1) *For $k \in \{0, 1, \ldots, n\}$, we have*

$$\Delta^{v_{(k)}\lambda}_{w_{(k)}\lambda}(z) = \frac{1}{\Delta^{w_{(k)}\lambda}_{\lambda}(g_{(k)})}, \quad \textit{and}$$

(2) *For $k \in \{1, \ldots, n\}$, we have*

$$\Delta^{v_{(k-1)}\lambda}_{w_{(k)}\lambda}(z) = \frac{\Delta^{v_{(k-1)}\lambda}_{\lambda}(g_{(k)})}{\Delta^{w_{(k)}\lambda}_{\lambda}(g_{(k)})}.$$

Proof. Let $L_k =< z\dot{w}_{(k)} \cdot \xi_\lambda >=< g_{(k)} \cdot \xi_\lambda >$ be the line in $V(\lambda)$ defined above. Since $z \in U^+$, we have $\langle z\dot{w}_{(k)} \cdot \xi_\lambda, \dot{w}_{(k)} \cdot \xi_\lambda \rangle = 1$. Therefore

$$z\dot{w}_{(k)} \cdot \xi_\lambda = \left(\frac{1}{\langle g_{(k)} \cdot \xi_\lambda, \dot{w}_{(k)} \cdot \xi_\lambda \rangle} \right) g_{(k)} \cdot \xi_\lambda. \tag{7.2}$$

Comparing coefficients of $\dot{v}_{(k-1)} \cdot \xi_\lambda$ on both sides, (2) immediately follows, and comparing coefficients of $\dot{v}_{(k)} \cdot \xi_\lambda$ on both sides, we obtain

$$\Delta^{v_{(k)}\lambda}_{w_{(k)}\lambda}(z) = \frac{\Delta^{v_{(k)}\lambda}_{\lambda}(g_{(k)})}{\Delta^{w_{(k)}\lambda}_{\lambda}(g_{(k)})}.$$

However, by Proposition 5.2, $g_{(k)} \in U^- \dot{v}_{(k)}$, so $\langle g_{(k)} \cdot \xi_\lambda, \dot{v}_{(k)} \cdot \xi_\lambda \rangle = \Delta^{v_{(k)}\lambda}_{\lambda}(g_{(k)}) = 1$, and (1) follows. □

We now compute the minors of $g_{(k)}$ from Lemma 7.4.

Lemma 7.5. *Let $k \in \{0, 1, \ldots, n\}$ and let λ be a dominant weight. Then we have:*

(1)

$$\Delta^{w_{(k)}\lambda}_{\lambda}(g_{(k)}) = \prod_{l=1,\ l \in J^\circ_{\mathbf{v}}}^{k} t_l^{\langle s_{i_{l+1}} s_{i_{l+2}} \cdots s_{i_k} \lambda, \alpha^\vee_{i_l} \rangle} \prod_{l=1,\ l \in J^-_{\mathbf{v}}}^{k} (-1)^{\langle s_{i_{l+1}} s_{i_{l+2}} \cdots s_{i_k} \lambda, \alpha^\vee_{i_l} \rangle}.$$

(2) *If $k \in J^-_{\mathbf{v}}$, then*

$$m_k = -\Delta^{v_{(k-1)}\omega_{i_k}}_{\omega_{i_k}}(g_{(k)}) - \Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)}).$$

Proof. (1) We prove the result for $g = g_{(n)} = g_1 g_2 \cdots g_n$. The result for arbitrary k follows since $g_{(k)}$ is defined in terms of the distinguished subexpression $\mathbf{v}_{(k)} = (v_{(0)}, v_{(1)}, v_{(2)}, \ldots, v_{(k)})$ for $v_{(k)}$ in $w_{(k)}$ in the same way that g is defined in terms of the reduced subexpression $\mathbf{v}$ for v in $\mathbf{w}$. For $l = 1, 2, \ldots, n+1$, let $g^{(l)} = g_l g_{l+1} \cdots g_n$. We note that $g^{(l)} \in B^+ \dot{w}^{(l)} B^+$ for $l = 1, 2, \ldots, n+1$, where $w^{(l)} = \dot{s}_{i_l} \dot{s}_{i_{l+1}} \cdots \dot{s}_{i_n}$. We prove, by reverse induction on l, that

$$\Delta_\lambda^{w^{(l)}\lambda}(g^{(l)}) = \prod_{j=l,\ j\in J_{\mathbf{v}}^\circ}^{n} t_j^{\left\langle s_{i_{j+1}} s_{i_{j+2}} \cdots s_{i_n}\lambda, \alpha_{i_j}^\vee \right\rangle} \prod_{j=l,\ j\in J_{\mathbf{v}}^-}^{n} (-1)^{\left\langle s_{i_{j+1}} s_{i_{j+2}} \cdots s_{i_n}\lambda, \alpha_{i_j}^\vee \right\rangle}.$$

The start of the induction is clear. Suppose that the result holds for $l+1$, i.e. for $g^{(l+1)} = g_{l+1} \cdots g_n$, and consider $g^{(l)} = g_l g_{l+1} \cdots g_n$. Since $g^{(l+1)} \in B^+ \dot{w}^{(l+1)} B^+$, $g^{(l+1)} \cdot \xi_\lambda$ is a linear combination of elements of $V(\lambda)$ of weight $\mu \geq w^{(l+1)}\lambda$.

Case (I). Suppose that $l \in J_{\mathbf{v}}^\circ$, so that $g_l = y_{i_l}(t_l)$. Then, using that $w^{(l+1)}\lambda$ and $w^{(l)}\lambda$ are extremal weights, and $w^{(l)}\lambda = s_{i_l} w^{(l+1)}\lambda \leq w^{(l+1)}\lambda$ we have

$$\begin{aligned}
\Delta_\lambda^{w^{(l)}\lambda}(g^{(l)}) &= \Delta_\lambda^{w^{(l)}\lambda}(y_{i_l}(t_l) g^{(l+1)}) \\
&= \left\langle y_{i_l}(t_l) g^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l)} \cdot \xi_\lambda \right\rangle \\
&= \left\langle y_{i_l}(t_l) \dot{w}^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l)} \cdot \xi_\lambda \right\rangle \left\langle g^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l+1)} \cdot \xi_\lambda \right\rangle.
\end{aligned}$$

By equation (2.1),

$$\begin{aligned}
\Delta_\lambda^{w^{(l)}\lambda}(g^{(l)}) &= \left\langle x_{i_l}(t_l^{-1}) \alpha_{i_l}^\vee(t_l^{-1}) \dot{s}_{i_l} x_{i_l}(t_l^{-1}) \dot{w}^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l)} \cdot \xi_\lambda \right\rangle \left\langle g^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l+1)} \xi_\lambda \right\rangle \\
&= t_l^{\left\langle s_{i_{l+1}} \cdots s_{i_n}\lambda, \alpha_{i_l}^\vee \right\rangle} \Delta_\lambda^{w^{(l+1)}\lambda}(g^{(l+1)}).
\end{aligned}$$

Case (II). Suppose that $l \in J_{\mathbf{v}}^+$, so that $g_l = \dot{s}_{i_l}$. Then

$$\begin{aligned}
\Delta_\lambda^{w^{(l)}\lambda}(g^{(l)}) &= \Delta_\lambda^{w^{(l)}\lambda}(\dot{s}_{i_l} g^{(l+1)}) \\
&= \left\langle \dot{s}_{i_l} g^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l)} \cdot \xi_\lambda \right\rangle.
\end{aligned}$$

It is thus clear that

$$\Delta_\lambda^{w^{(l)}\lambda}(g^{(l)}) = \Delta_\lambda^{w^{(l+1)}\lambda}(g^{(l+1)}).$$

Case (III). Suppose that $l \in J_{\mathbf{v}}^-$, so that $g_l = x_{i_l}(m_l) \dot{s}_{i_l}^{-1}$. Then, using the fact that that $w^{(l+1)}\lambda \geq w^{(l)}\lambda = s_{i_l} w^{(l+1)}\lambda$ are extremal weights, we have:

$$\begin{aligned}
\Delta_\lambda^{w^{(l)}\lambda}(g^{(l)}) &= \Delta_\lambda^{w^{(l)}\lambda}(x_{i_l}(m_l) \dot{s}_{i_l}^{-1} g^{(l+1)}) \\
&= \left\langle x_{i_l}(m_l) \dot{s}_{i_l}^{-1} g^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l)} \cdot \xi_\lambda \right\rangle \\
&= \left\langle x_{i_l}(m_l) \dot{s}_{i_l}^{-1} \dot{w}^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l)} \cdot \xi_\lambda \right\rangle \left\langle g^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l+1)} \cdot \xi_\lambda \right\rangle \\
&= \left\langle \dot{s}_{i_l}^{-1} \dot{w}^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l)} \cdot \xi_\lambda \right\rangle \left\langle g^{(l+1)} \cdot \xi_\lambda, \dot{w}^{(l+1)} \cdot \xi_\lambda \right\rangle \\
&= (-1)^{\left\langle s_{i_{l+1}} \cdots s_{i_n}\lambda, \alpha_{i_l}^\vee \right\rangle} \Delta_\lambda^{w^{(l+1)}\lambda}(g_{(l+1)}).
\end{aligned}$$

The last equality follows from the fact that $\dot{s}_{i_l}^{-1} = \alpha_{i_l}^\vee(-1) \dot{s}_{i_l}$.

The result for l now follows (in each case) from the inductive hypothesis and we are done.

(2) We have:

$$\begin{aligned}
\Delta_{\omega_{i_k}}^{v_{(k-1)}\omega_{i_k}}(g_{(k)}) &= \left\langle g_{(k)}\cdot\xi_{\omega_{i_k}}, \dot{v}_{(k-1)}\cdot\xi_{\omega_{i_k}}\right\rangle \\
&= \left\langle g_{(k-1)}x_{i_k}(m_k)\dot{s}_{i_k}^{-1}\cdot\xi_{\omega_{i_k}}, \dot{v}_{(k-1)}\cdot\xi_{\omega_{i_k}}\right\rangle \\
&= -\left\langle g_{(k-1)}\dot{s}_{i_k}\cdot\xi_{\omega_{i_k}}, \dot{v}_{(k-1)}\cdot\xi_{\omega_{i_k}}\right\rangle - m_k\left\langle g_{(k-1)}\cdot\xi_{\omega_{i_k}}, \dot{v}_{(k-1)}\cdot\xi_{\omega_{i_k}}\right\rangle, \\
&= -\Delta_{s_{i_k}\omega_{i_k}}^{v_{(k-1)}\omega_{i_k}}(g_{(k-1)}) - m_k,
\end{aligned}$$

noting that, since $g_{(k-1)} \in U^-\dot{v}_{(k-1)}$, we have that $\langle g_{(k-1)}\cdot\xi_{\omega_{i_k}}, \dot{v}_{(k-1)}\cdot\xi_{\omega_{i_k}}\rangle = 1$. The result follows. □

Remarks 7.6. (1) Let $t_k = -1$ for $k \in J^-_{\mathbf{v}}$ and let $t_k = 1$ for $k \in J^+_{\mathbf{v}}$ (so that now t_k is defined for $k = 1, 2, \ldots, n$). Then the formula in Lemma 7.5(1) can be rewritten as:

$$\Delta_{\lambda}^{w_{(k)}\lambda}(g_{(k)}) = \prod_{l=1}^{k} t_l^{\left\langle s_{i_{l+1}}s_{i_{l+2}}\cdots s_{i_k}\lambda, \alpha_{i_l}^\vee\right\rangle}.$$

(2) The following lemma gives an expression for m_k which is simpler than the Chamber Ansatz version, Theorem 7.1(2). However, the Chamber Ansatz formula for the m_k will be more useful in Section 8.

Lemma 7.7. *For $k \in J^-_{\mathbf{v}}$, we have:*

$$m_k = -\frac{\Delta_{w_{(k)}\omega_{i_k}}^{v_{(k-1)}\omega_{i_k}}(z)}{\Delta_{w_{(k)}\omega_{i_k}}^{v_{(k)}\omega_{i_k}}(z)} - \Delta_{s_{i_k}\omega_{i_k}}^{v_{(k-1)}\omega_{i_k}}(g_{(k-1)}).$$

Proof. This follows immediately from Lemma 7.4 and Lemma 7.5(2). □

Proof of Theorem 7.1. We can now prove Theorem 7.1 by using Lemmas 7.4 and 7.5 and Remark 7.6(1) to substitute for the minors appearing in the expressions on the right hand sides of Theorem 7.1(1) and (2). We first claim that, for $k = 1, 2, \ldots, n$, we have:

$$t_k = \frac{\prod_{j\neq i_k}\Delta_{w_{(k)}\omega_j}^{v_{(k)}\omega_j}(z)^{-a_{j,i_k}}}{\Delta_{w_{(k)}\omega_{i_k}}^{v_{(k)}\omega_{i_k}}(z)\Delta_{w_{(k-1)}\omega_{i_k}}^{v_{(k-1)}\omega_{i_k}}(z)}. \tag{7.3}$$

(See [2, 4.3] for a similar proof of this statement in the special case where $J^+_{\mathbf{v}} = \emptyset$). We have, using Lemma 7.4(1) and Remark 7.6(1):

$$\frac{\prod_{j\neq i_k}\Delta_{w_{(k)}\omega_j}^{v_{(k)}\omega_j}(z)^{-a_{j,i_k}}}{\Delta_{w_{(k)}\omega_{i_k}}^{v_{(k)}\omega_{i_k}}(z)\Delta_{w_{(k-1)}\omega_{i_k}}^{v_{(k-1)}\omega_{i_k}}(z)} = \frac{\prod_{j\neq i_k}\prod_{l=1}^{k} t_l^{a_{j,i_k}\left\langle s_{i_{l+1}}\cdots s_{i_k}\omega_j, \alpha_{i_l}^\vee\right\rangle}}{\prod_{l=1}^{k} t_l^{-\left\langle s_{i_{l+1}}\cdots s_{i_k}\omega_{i_k}, \alpha_{i_l}^\vee\right\rangle}\prod_{l=1}^{k-1} t_l^{-\left\langle s_{i_{l+1}}\cdots s_{i_{k-1}}\omega_{i_k}, \alpha_{i_l}^\vee\right\rangle}}.$$

The exponent of t_k is given by

$$\begin{aligned}
\sum_{j\neq i_k} a_{j,i_k}\left\langle\omega_j, \alpha_{i_k}^\vee\right\rangle + \left\langle\omega_{i_k}, \alpha_{i_k}^\vee\right\rangle &= \left\langle\sum_{j\neq i_k} a_{j,i_k}\omega_j + \omega_{i_k}, \alpha_{i_k}^\vee\right\rangle \\
&= \left\langle\alpha_{i_k} - \omega_{i_k}, \alpha_{i_k}^\vee\right\rangle = 1.
\end{aligned}$$

If $k < k'$, the exponent of $t_{k'}$ is clearly zero. If $k' < k$, then the exponent of $t_{k'}$ is given by

$$\begin{aligned}
&\left(\sum_{j\neq i_k} a_{j,i_k}\left\langle s_{i_{k'+1}}\cdots s_{i_k}\omega_j, \alpha_{i_{k'}}^\vee\right\rangle\right) + \left\langle s_{i_{k'+1}}\cdots s_{i_k}\omega_{i_k}, \alpha_{i_{k'}}^\vee\right\rangle + \left\langle s_{i_{k'+1}}\cdots s_{i_{k-1}}\omega_{i_k}, \alpha_{i_{k'}}^\vee\right\rangle \\
&= \left\langle s_{i_{k'+1}}\cdots s_{i_k}\left(\left(\sum_{j\neq i_k} a_{j,i_k}\omega_j\right) + 2\omega_{i_k} - \alpha_{i_k}\right), \alpha_{i_{k'}}^\vee\right\rangle \\
&= \left\langle s_{i_{k'+1}}\cdots s_{i_k}(\alpha_{i_k} - \alpha_{i_k}), \alpha_{i_{k'}}^\vee\right\rangle = 0,
\end{aligned}$$

and the claim (7.3) is proved; Theorem 7.1(1) is a special case.

We now prove Theorem 7.1(2). Suppose that $k \in J_{\mathbf{v}}^-$. Using Lemma 7.4 and (7.3) (noting that $t_k = -1$), we see that

$$\begin{aligned}
\frac{\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}{\prod_{j\neq i_k}\Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}} &= \frac{\Delta^{v_{(k-1)}\omega_{i_k}}_{\omega_{i_k}}(g_{(k)})\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}{\Delta^{w_{(k)}\omega_{i_k}}_{\omega_{i_k}}(g_{(k)})\prod_{j\neq i_k}\Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}}\\
&= \frac{\Delta^{v_{(k-1)}\omega_{i_k}}_{\omega_{i_k}}(g_{(k)})\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}{\prod_{j\neq i_k}\Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}}\\
&= -\Delta^{v_{(k-1)}\omega_{i_k}}_{\omega_{i_k}}(g_{(k)}).
\end{aligned}$$

Theorem 7.1(2) now follows from Lemma 7.5(2), and the proof of Theorem 7.1 is complete. □

8. A change of coordinates

We can gain some more insight into the structure of the formulas from Theorem 7.1 if we consider the standard and special chamber minors as providing an alternative system of coordinates on $\mathcal{R}_{\mathbf{v},\mathbf{w}}$.

Proposition 8.1. *Let $\mathbf{v} \prec \mathbf{w}$. With notation as above the map*

$$\begin{aligned}
\mathcal{R}_{\mathbf{v},\mathbf{w}} &\longrightarrow (\mathbb{K}^*)^{J^\circ_{\mathbf{v}}} \times (\mathbb{K})^{J^-_{\mathbf{v}}}\\
z\dot{w}\cdot B^+ &\mapsto \left((\Delta^{v_{(j)}\omega_{i_j}}_{w_{(j)}\omega_{i_j}}(z))_{j\in J^\circ_{\mathbf{v}}}, (\Delta^{v_{(j-1)}\omega_{i_j}}_{w_{(j)}\omega_{i_j}}(z))_{j\in J^-_{\mathbf{v}}}\right)
\end{aligned}$$

is an isomorphism.

For the special case $\mathbf{v} = (1,\dots,1)$ see also Theorem 4.3 and Corollary 4.4 in [2].

Proof. Let $\mathbf{w}_{(k)} = (w_{(0)},\dots,w_{(k)})$ be the reduced expression for $w_{(k)}$ obtained from $\mathbf{w}$ by truncation, and $\mathbf{v}_{(k)}$ the corresponding truncation of $\mathbf{v}$. The proof of the proposition is by induction on k and using Theorem 7.1. The start of the induction is trivial, so let us assume the proposition is true for $\mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}$. We have three cases for k.

(1) If $k \in J^\circ_{\mathbf{v}}$, then as in the proof of Proposition 5.2 we have

$$\begin{aligned}
\mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}\times\mathbb{K}^* &\xrightarrow{\sim} \mathcal{R}_{\mathbf{v}_{(k)},\mathbf{w}_{(k)}},\\
(g_{(k-1)}\cdot B^+, t_k) &\mapsto g_{(k-1)}y_{i_k}(t_k)\cdot B^+, \qquad g_{(k-1)}\in G_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}.
\end{aligned}$$

Compose this map with

$$\begin{aligned}
\mathcal{R}_{\mathbf{v}_{(k)},\mathbf{w}_{(k)}} &\longrightarrow \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}\times\mathbb{K}^*,\\
z\dot{w}_{(k)}\cdot B^+ &\mapsto \left(z\dot{w}_{(k-1)}\cdot B^+, \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\right)
\end{aligned}\tag{8.1}$$

to get a map $\psi_k : \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}\times\mathbb{K}^* \to \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}\times\mathbb{K}^*$. The Chamber Ansatz says that t_k can be recovered from $z\dot{w}_{(k)}\cdot B^+$ by

$$t_k = a_k\left(z\dot{w}_{(k-1)}\cdot B^+\right)\ \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)^{-1},\tag{8.2}$$

where

$$a_k(z\dot{w}_{(k-1)}\cdot B^+) := \frac{\prod_{j\neq i_k}\Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}}{\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}, \qquad z\dot{w}_{(k-1)}\cdot B^+ \in \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}.$$

Note that this gives a well-defined map $a_k : \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}} \to \mathbb{K}^*$, since a_k is made up of standard chamber minors for $(\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)})$. Now the formula (8.2) gives rise to an inverse to ψ_k. Hence also (8.1) is an isomorphism, and the proposition holds for $\mathcal{R}_{\mathbf{v}_{(k)},\mathbf{w}_{(k)}}$ by the induction hypothesis.

(2) Suppose $k \in J^-_{\mathbf{v}}$. Then we have an isomorphism

$$\begin{array}{rcll} \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}} \times \mathbb{K} & \xrightarrow{\sim} & \mathcal{R}_{\mathbf{v}_{(k)},\mathbf{w}_{(k)}}, & \\ (g_{(k-1)} \cdot B^+, m_k) & \mapsto & g_{(k-1)} x_{i_k}(m_k) \dot{s}_{i_k}^{-1} \cdot B^+, & g_{(k-1)} \in G_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}, \end{array}$$

which we can compose with

$$\begin{array}{rcl} \mathcal{R}_{\mathbf{v}_{(k)},\mathbf{w}_{(k)}} & \longrightarrow & \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}} \times \mathbb{K}, \\ z\dot{w}_{(k)} \cdot B^+ & \mapsto & \left(z\dot{w}_{(k-1)} \cdot B^+, \Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) \right) \end{array} \tag{8.3}$$

to get a map $\psi_k : \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}} \times \mathbb{K} \to \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}} \times \mathbb{K}$. By Theorem 7.1 one can recover the m_k coordinate from $z\dot{w}_{(k)} \cdot B^+$ by

$$m_k = a_k \left(z\dot{w}_{(k-1)} \cdot B^+ \right) \Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) - b_k \left(z\dot{w}_{(k-1)} \cdot B^+ \right), \tag{8.4}$$

where $a_k : \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}} \to \mathbb{K}^*$ and $b_k : \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}} \to \mathbb{K}$ are given by

$$\begin{array}{rcll} a_k(z\dot{w}_{(k-1)} \cdot B^+) & = & \dfrac{\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}{\prod_{j \neq i_k} \Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}}, & z\dot{w}_{(k-1)} \cdot B^+ \in \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}, \\ b_k(g_{(k-1)} \cdot B^+) & = & \Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)}), & g_{(k-1)} \in G_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}. \end{array}$$

Now the identity (8.4) gives an inverse to the map ψ_k. So (8.3) is an isomorphism and the proposition holds for $\mathcal{R}_{\mathbf{v}_{(k)},\mathbf{w}_{(k)}}$.

(3) If $k \in J^+_{\mathbf{v}}$ then $\mathcal{R}_{\mathbf{v}_{(k)},\mathbf{w}_{(k)}} \cong \mathcal{R}_{\mathbf{v}_{(k-1)},\mathbf{w}_{(k-1)}}$ and we are done.

□

Remark 8.2. Assuming Proposition 8.1, much of the structure of Theorem 7.1 is already determined. The proposition says that we may take the $(\Delta^{v_{(j)}\omega_{i_j}}_{w_{(j)}\omega_{i_j}}(z))_{j \in J^\circ_{\mathbf{v}}}$ in $\mathbb{K}^*$, and the $(\Delta^{v_{(j-1)}\omega_{i_j}}_{w_{(j)}\omega_{i_j}}(z))_{j \in J^-_{\mathbf{v}}}$ in $\mathbb{K}$ together as coordinates for $\mathcal{R}_{\mathbf{v},\mathbf{w}}$. And then Theorem 7.1 can be interpreted roughly as the transition from these coordinates to the coordinates $(t_j)_{j \in J^\circ_{\mathbf{v}}}$ and $(m_j)_{j \in J^-_{\mathbf{v}}}$ from the factorization. From the outset these two sets of coordinates have to be quite closely related, since they are both compatible with reduction. In either setting the map $\pi^w_{w_{(k)}} : \mathcal{R}_{\mathbf{v},\mathbf{w}} \to \mathcal{R}_{\mathbf{v}_{(k)},\mathbf{w}_{(k)}}$ corresponds to the projection onto the first k coordinates.

Let us consider explicitly an element $g \cdot B^+ = z\dot{w} \cdot B^+$ with fixed reduction $g_{(k)} \cdot B^+ = z\dot{w}_{(k)} \cdot B^+$. Then this is equivalent to fixing coordinates t_j and m_j, or to fixing the minors $\Delta^{v_{(j)}\omega_{i_j}}_{w_{(j)}\omega_{i_j}}(z)$ and $\Delta^{v_{(j-1)}\omega_{i_j}}_{w_{(j)}\omega_{i_j}}(z)$, where $j \leq k$ in $J^\circ_{\mathbf{v}}$ or $J^-_{\mathbf{v}}$. If $k+1 \in J^\circ_{\mathbf{v}}$ then the change of coordinate from $\Delta^{v_{(k+1)}\omega_{i_{k+1}}}_{w_{(k+1)}\omega_{i_{k+1}}}(z)$ to t_{k+1} amounts to an invertible map $\mathbb{K}^* \to \mathbb{K}^*$, which depends only on the earlier coordinates. So it has to be of the form $z \mapsto az^{\pm 1}$ for some $a \in \mathbb{K}^*$. The Chamber Ansatz simply says the map is of the form $z \mapsto az^{-1}$ and describes the coefficient a explicitly in terms of earlier chamber minors of z.

If $k \in J^-_{\mathbf{v}}$ then the change of the coordinate $\Delta^{v_{(k)}\omega_{i_{k+1}}}_{w_{(k+1)}\omega_{i_{k+1}}}(z)$ to m_{k+1} amounts to an invertible map $\mathbb{K} \to \mathbb{K}$, which depends only on the earlier coordinates. Therefore this map must be of the form $z \mapsto az + b$ for $a \in \mathbb{K}^*$ and $b \in \mathbb{K}$. Here again a is computed by the Chamber Ansatz, and b is the correction term in Theorem 7.1.

9. The generalized Chamber Ansatz for SL_d

Suppose we are given $B = z\dot{w} \cdot B^+$, with $z \in U^+$, with fixed reduced expression $\mathbf{w}$ for w. We can use Proposition 6.4 to determine which Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ contains B, where $\mathbf{v}$ is a distinguished subexpression for v in $\mathbf{w}$. We shall give an explicit example in Section 10 of how to do this. Then $B = g \cdot B^+$, where $g = g_1 g_2 \cdots g_n$, and

$$g_k = \begin{cases} y_{i_k}(t_k) & k \in J^\circ_{\mathbf{v}}, \\ \dot{s}_{i_k} & k \in J^+_{\mathbf{v}}, \\ x_{i_k}(m_k)\dot{s}_{i_k}^{-1} & k \in J^-_{\mathbf{v}}. \end{cases}$$

The generalized Chamber Ansatz (Theorem 7.1) gives formulas for the t_k and m_k in terms of minors of z (and the minor $\Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)})$ of $g_{(k-1)}$). We write Theorem 7.1(2) in the form

$$m_k = r_k - \Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)}),$$

where

$$r_k = \frac{\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k-1)}\omega_{i_k}}(z)}{\prod_{j\neq i_k}\Delta^{v_{(k)}\omega_j}_{w_{(k)}\omega_j}(z)^{-a_{j,i_k}}}.$$

This is the term $a_k\left(z\dot{w}_{(k-1)}\cdot B^+\right)\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)$ in equation (8.4). In this section, we give a graphical algorithm (generalizing that of [3]) for determining the coefficients t_k and r_k, in the case where $G = SL_d$. The coefficients m_k can then be computed by computing the minors $\Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)})$ inductively, noting that $g_{(k-1)}$ depends only on the coefficients t_j and m_j for $j \leq k-1$ (an example of this will be given in section 10.2).

We employ a generalised version of the pseudoline arrangements used in [3], in which two pseudolines can either intersect, as in [3], or pass over or under each other (see below for examples). These can also be regarded as diagrams of singular braids [1, 4].

The main idea is to associate such an arrangement (which we call the *ansatz arrangement*) to the pair $\mathbf{v}, \mathbf{w}$. For example, if $\mathbf{w} = (1, s_3, s_3s_2, s_3s_2s_1, s_3s_2s_1s_3, s_3s_2s_1s_3s_2)$, and $\mathbf{v}$ is the distinguished subexpression $(1, s_3, s_3, s_3, 1, s_2)$ for s_2 in $\mathbf{w}$, then the arrangement is as in Figure 1.

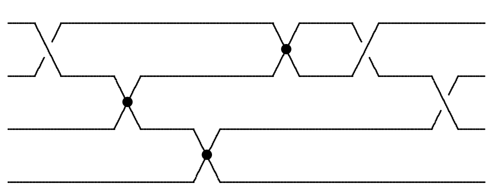


Figure 1. Ansatz arrangement (unlabeled) for $\underline{s_3}s_2s_1\underline{s_3}\underline{s_2}$. Note that $g = \dot{s}_3 y_2(t_2) y_1(t_3) x_3(m_4) \dot{s}_3^{-1}\dot{s}_2$.

The pair $\mathbf{v}$, $\mathbf{w}$ determines the factors g_k of g, which in turn determine the ansatz arrangement in the following way. It consists of d pseudolines, numbered $1, 2, \ldots, d$, from bottom to top on the left hand side of the arrangement. Each factor $x_{i_k}(m_k)$, $y_{i_k}(t_k)$, $\dot{s}_{i_k}$ or $\dot{s}_{i_k}^{-1}$ of g gives rise to a constituent of the arrangement, in which pseudolines at level i_k from the bottom of the arrangement are either braided or cross at singular point. Note that $x_{i_k}(m_k)$ and $\dot{s}_{i_k}^{-1}$ are treated as separate factors. The rules for how this is done are given in Figure 2.

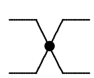
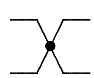
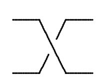
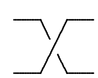
Factor of g	$x_{i_k}(m_k)$	$y_{i_k}(t_k)$	$\dot{s}_{i_k}$	$\dot{s}_{i_k}^{-1}$
Constituent of ansatz arrangement				

Figure 2. Constituents of the ansatz arrangement.

As usual, we define a *chamber* of a generalised pseudoline arrangement to be a component of the complement of the union of the pseudolines in the arrangement (for this definition we interpret the under- and over-crossings as singular points). In order to label the chambers, we need two auxilliary pseudoline arrangements associated to the pair $\mathbf{v}, \mathbf{w}$, which we call the *upper* and *lower arrangements* (since, as will be seen, they will determine upper and lower subscripts of chamber minors). These arrangements are defined in the same way as the ansatz arrangement, except that different rules for the factors of g are employed. These are described in Figure 3, and the upper and lower arrangements for the example above are given in Figures 4 and 5. The chambers for these arrangements are labeled with the labels of the strings passing below them.

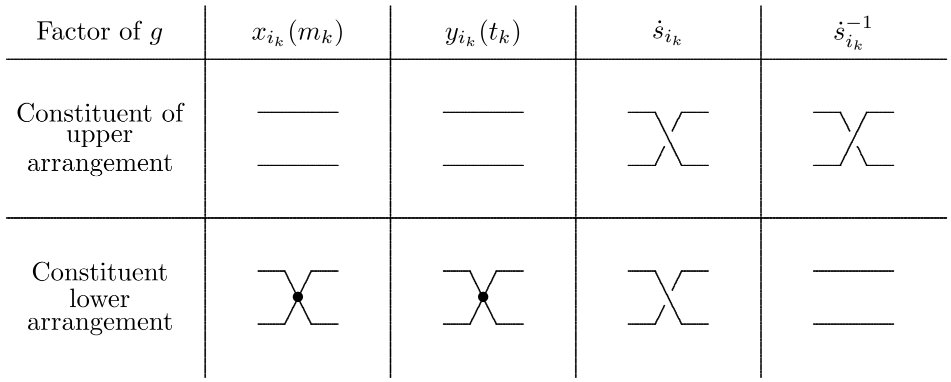


FIGURE 3. Constituents of the auxilliary arrangements.

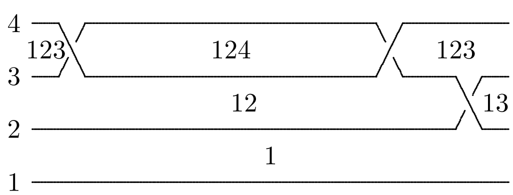


FIGURE 4. Upper arrangement for $\underline{s_3}s_2s_1\underline{s_3s_2}$. Note that $g = \dot{s}_3y_2(t_2)y_1(t_3)x_3(m_4)\dot{s}_3^{-1}\dot{s}_2$.

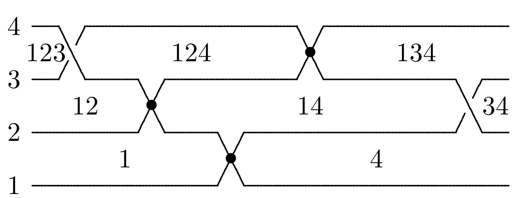


FIGURE 5. Lower arrangement for $\underline{s_3}s_2s_1\underline{s_3s_2}$. Note that $g = \dot{s}_3y_2(t_2)y_1(t_3)x_3(m_4)\dot{s}_3^{-1}\dot{s}_2$.

We note that, since $G = SL_d$, the generalized minors of Definition 6.2 coincide with the usual minors of matrices. We denote by Δ_S^R the minor with row set R and column set S (interpreted as 1 if $R = S = \emptyset$). Suppose that X is a chamber of the ansatz arrangement. Let $R(X)$ be the label of the chamber containing the corresponding part of the upper arrangement, and let $S(X)$ be the label of the chamber containing the corresponding part of the lower arrangement (these corresponding parts can be obtained by overlaying the ansatz arrangement with the upper and lower arrangements respectively). We label X with the minor $\Delta_{S(X)}^{R(X)}$. The resulting labeled ansatz arrangement for our example is given in Figure 6.

Next, we note that the singular points in the ansatz arrangement correspond precisely to the factors of g of the form $x_{i_k}(m_k)$ and $y_{i_k}(t_k)$. We label these (beneath the arrangment) with t_k and m_k, respectively, for

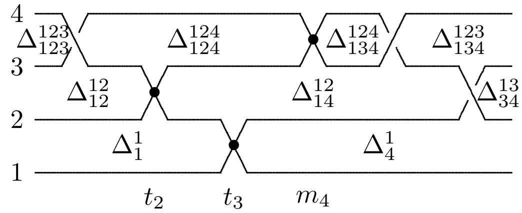


FIGURE 6. Ansatz arrangement for $\underline{s_3}s_2s_1\underline{s_3s_2}$. Note that $g = \dot{s}_3 y_2(t_2)y_1(t_3)x_3(m_4)\dot{s}_3^{-1}\dot{s}_2$.

convenience. The ansatz arrangement can then be used to compute the coefficients t_k and m_k as follows. Suppose $k \in J^-_{\mathbf{v}} \cup J^\circ_{\mathbf{v}}$. Let A_k, B_k, C_k and D_k be the minors labelling the chambers surrounding the singular point in the ansatz arrangement corresponding to k, with A_k and D_k above and below it, and B_k and C_k on the same horizontal level (see Figure 7). It is easy to check that Theorem 7.1 implies that, for $k \in J^\circ_{\mathbf{v}}$,

$$t_k = \frac{A_k(z)D_k(z)}{B_k(z)C_k(z)},$$

and, for $k \in J^-_{\mathbf{v}}$,

$$r_k = \frac{B_k(z)C_k(z)}{A_k(z)D_k(z)}.$$

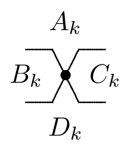


FIGURE 7. Chambers surrounding the singular point corresponding to $k \in J^-_{\mathbf{v}} \cup J^\circ_{\mathbf{v}}$.

10. Determining Deodhar components for SL_d

10.1. **Graphical algorithm for Deodhar components.** In this section we show that Proposition 6.4 also gives rise to a graphical algorithm for SL_d to determine the Deodhar component of an element $B \in \mathcal{B}$. Suppose that $B = z\dot{w} \cdot B^+$, where $w \in W$ and $z \in U^+$, and that a reduced expression $\mathbf{w}$ for w is chosen. Then the graphical algorithm for computing the distinguished subexpression $\mathbf{v}$ of $\mathbf{w}$ such that $B \in \mathcal{R}_{\mathbf{v},\mathbf{w}}$ is as follows.

Firstly we draw the usual pseudoline diagram for the reduced expression $\mathbf{w}$ of w as in [3] (call this the *classical pseudoline arrangement* for $\mathbf{w}$). Each factor s_{i_k} of $\mathbf{w}$ corresponds to a singular crossing between the i_kth and i_{k+1}st pseudolines from the bottom (see below for an example). We define $v_{(0)}$ to be 1. Suppose that $v_{(0)}, v_{(1)}, \ldots, v_{(k-1)}$ have already been computed, and that if $k > 1$ we have drawn the upper arrangement for the pair $\mathbf{v}_{(k-1)} = (v_{(1)}, v_{(2)}, \ldots, v_{(k-1)}), \mathbf{w}_{(k-1)} = (w_{(0)}, w_{(1)}, \ldots, w_{(k-1)})$.

We compute $v_{(k)}$ in the following way. Consider the upper arrangement for the pair $\mathbf{v}_{(k-1)}, \mathbf{w}_{(k-1)}$. If $k > 1$, let R_k be the label of the unbounded chamber at the right hand end of this arrangement between the i_kth and i_{k+1}st pseudolines (counting from bottom to top). If $k = 1$, we take R_1 to be $\{1, 2, \ldots, i_k\}$.

Let S_k be the label of the chamber in the classical arrangement for $\mathbf{w}$ immediately to the right of the kth crossing. It is clear that $|R_k| = |S_k| = i_k$. Now the minor $\Delta^{R_k}_{S_k}(z)$ determines the value of $v_{(k)}$. We have, by Proposition 6.4:

(*a*) If $v_{(k-1)}s_{i_k} > v_{(k-1)}$ and $\Delta^{R_k}_{S_k}(z) \neq 0$, then $v_{(k)} = v_{(k-1)}$.
(*b*) If $v_{(k-1)}s_{i_k} > v_{(k-1)}$ and $\Delta^{R_k}_{S_k}(z) = 0$, then $v_{(k)} = v_{(k-1)}s_{i_k}$.

(*c*) If $v_{(k-1)}s_{i_k} < v_{(k-1)}$, then $v_{(k)} = v_{(k-1)}s_{i_k}$.

Thus $v_{(k)}$ is computed, and we draw the upper arrangement for the pair $\mathbf{v}_{(k)}, \mathbf{w}_{(k)}$, by building on the upper arrangement for $\mathbf{v}_{(k-1)}, \mathbf{w}_{(k-1)}$ if $k > 1$. We are thus ready for the next step.

In this way, all of the $v_{(k)}$ are determined inductively. We also note that at the end we have drawn the upper arrangement for the pair $\mathbf{v}, \mathbf{w}$. So, after drawing the lower arrangement for $\mathbf{v}, \mathbf{w}$, we are ready to apply the method in Section 9 to compute the factors of g explicitly.

10.2. **An Example.** In this section we give an explicit example of the graphical algorithm described above. We consider the element

$$z = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in SL_4(\mathbb{C}),$$

and set $w = s_3s_2s_1s_3s_2$, so we have the reduced expression $\mathbf{w} = (1, s_3, s_3s_2, s_3s_2s_1, s_3s_2s_1s_3, s_3s_2s_1s_3s_2)$ for w. The classical arrangement for $\mathbf{w}$ is given in Figure 8. We start with $v_{(0)} = 1$.

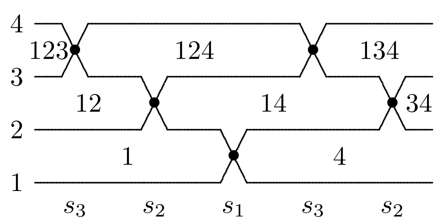


FIGURE 8. Classical arrangement for $\mathbf{w} = s_3s_2s_1s_3s_2$.

Step 1.: We have $v_{(0)}s_3 = s_3 > v_{(0)}$. Here, $R_1 = \{1, 2, 3\}$ and $S_1 = \{1, 2, 4\}$. Since $i_1 = 3$ and $\Delta^{123}_{124}(z) = 0$, we are in case (b) and $v_{(1)} = v_{(0)}s_3 = s_3$.

Step 2.: We have $v_{(1)}s_2 = s_3s_2 > v_{(1)}$. The upper arrangement for the pair $\mathbf{v}_{(1)}, \mathbf{w}_{(1)}$ is shown in Figure 9. Here, $R_2 = \{1, 2\}$ and $S_2 = \{1, 4\}$. Since $i_2 = 2$ and $\Delta^{12}_{14}(z) = 2 \neq 0$, we are in case (a) and $v_{(2)} = v_{(1)} = s_3$.

Step 3.: We have $v_{(2)}s_1 = s_3s_1 > v_{(2)}$. The upper arrangement for the pair $\mathbf{v}_{(2)}, \mathbf{w}_{(2)}$ is shown in Figure 10. Here, $R_3 = \{1\}$ and $S_3 = \{4\}$. Since $i_3 = 1$ and $\Delta^{1}_{4}(z) = 1$, we are in case (a) and $v_{(3)} = v_{(2)} = s_3$.

Step 4.: We have $v_{(3)}s_3 = s_3s_3 = 1 < v_{(2)}$. The upper arrangement for the pair $\mathbf{v}_{(3)}, \mathbf{w}_{(3)}$ is shown in Figure 11. We are in case (c) and $v_{(4)} = v_{(3)}s_3 = 1$.

Step 5.: We have $v_{(4)}s_2 = s_2 > v_{(4)}$. The upper arrangement for the pair $\mathbf{v}_{(4)}, \mathbf{w}_{(4)}$ is shown in Figure 12. Here, $R_5 = \{1, 2\}$ and $S_5 = \{3, 4\}$. Since $i_5 = 2$ and $\Delta^{12}_{34}(z) = 0$, we are in case (b) and $v_{(5)} = v_{(4)}s_2 = s_2$. The upper arrangement for the pair $\mathbf{v}_{(5)}, \mathbf{w}_{(5)}$ is shown in Figure 4.

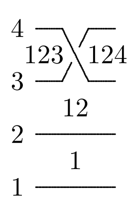


FIGURE 9. Upper arrangement for $\underline{s_3}$. Note that $g_{(1)} = \dot{s}_3$.

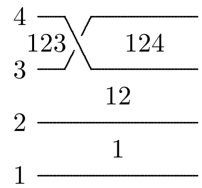


FIGURE 10. Upper arrangement for $\underline{s_3}s_2$. Note that $g_{(2)} = \dot{s}_3 y_2(t_2)$.

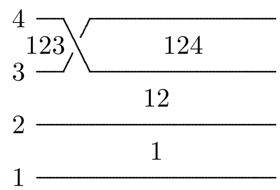


FIGURE 11. Upper arrangement for $\underline{s_3}s_2s_1$. Note that $g_{(3)} = \dot{s}_3 y_2(t_2) y_1(t_3)$.

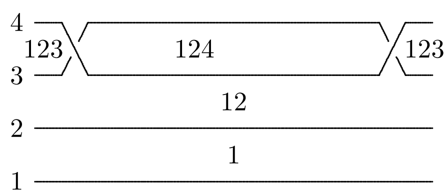


FIGURE 12. Upper arrangement for $\underline{s_3}s_2s_1\underline{s_3}$. Note that $g_{(4)} = \dot{s}_3 y_2(t_2) y_1(t_3) x_3(m_4) \dot{s}_3^{-1}$.

We conclude that B lies in the Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$, where $\mathbf{v} = (1, s_3, s_3, s_3, 1, s_2)$ and $\mathbf{w} = (1, s_3, s_3s_2, s_3s_2s_1, s_3s_2s_1s_3, s_3s_2s_1s_3s_2)$.

We remark that we can use the above computation to determine criteria for $z\dot{w} \cdot B^+$ to lie in $\mathcal{R}_{\mathbf{v},\mathbf{w}}$, where z is an arbitrary matrix in U^+, in terms of minors of z (see Corollary 6.6). Suppose that

$$z = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain that

$$\mathcal{R}_{\mathbf{v},\mathbf{w}} = \{z\dot{w} \cdot B^+ \,:\, \Delta^{123}_{124}(z) = a_{34} = 0, \Delta^{12}_{14}(z) = a_{24} \neq 0, \Delta^{1}_{4}(z) = a_{14} \neq 0, \Delta^{12}_{34}(z) = a_{13}a_{24} - a_{14}a_{23} = 0\}.$$

We note that no condition is obtained on the entry a_{12} of z. No such condition could arise, since $z\dot{w} \cdot B^+$ doesn't depend on a_{12}, as $w^{-1}\alpha_1$ is positive.

Finally, we note that in our example, B lies in the Deodhar component $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ considered in Section 9, so we can apply the graphical algorithm given there in order to compute the coefficients t_k and m_k in the factorisation of $g = \dot{s}_3 y_2(t_2) y_1(t_3) x_3(m_4) \dot{s}_3^{-1} \dot{s}_2 \in G_{\mathbf{v},\mathbf{w}}$ (where $B = g \cdot B^+$). We obtain:

$$t_2 = \frac{\Delta^{124}_{124}(z)\Delta^{1}_{1}(z)}{\Delta^{12}_{12}(z)\Delta^{12}_{14}(z)} = 1/2,$$

$$t_3 = \frac{\Delta^{12}_{14}(z)\Delta^{\emptyset}_{\emptyset}(z)}{\Delta^{1}_{1}(z)\Delta^{1}_{4}(z)} = 2,$$

and

$$r_4 = \frac{\Delta^{124}_{124}(z)\Delta^{124}_{134}(z)}{\Delta^{1234}_{1234}(z)\Delta^{12}_{14}(z)} = 2.$$

Finally, we show that the minor $\Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)})$, that would appear as correction term vanishes on this Deodhar component, so that $m_k = r_k$. We have $\Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)}) = \Delta^{v_{(3)}\omega_{i_4}}_{s_{i_4}\omega_{i_4}}(g_{(3)}) = \Delta^{s_3\omega_3}_{s_3\omega_3}(\dot{s}_3 y_2(t_2) y_1(t_3))$. We note that $g_{(3)}\dot{s}_3\omega_3$ is a linear combination of elements of weight ω_3 and $s_3(s_3\omega_3 - \alpha_2) = \omega_3 - \alpha_2 - \alpha_3$, and therefore has zero component in the $s_3\omega_3$-weight space, from which it follows that $\Delta^{s_3\omega_3}_{s_3\omega_3}(\dot{s}_3 y_2(t_2) y_1(t_3)) = 0$, so $m_4 = r_4 - 0 = r_4$. Thus, in this case, $m_4 = 2$. We finally see that

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \dot{s}_3\dot{s}_2\dot{s}_1\dot{s}_3\dot{s}_2 \cdot B^+ = \dot{s}_3 y_2(1/2) y_1(2) x_3(2) \dot{s}_3^{-1}\dot{s}_2 \cdot B^+.$$

Remark 10.1. It is possible that the minor $\Delta^{v_{(k-1)}\omega_{i_k}}_{s_{i_k}\omega_{i_k}}(g_{(k-1)})$ is non-zero. This happens, for example, for $G = SL_4$. Let $v = 1$ and $w = w_0$, and take $\mathbf{v} = (1, 1, 1, 1, s_1, s_1, 1)$, a subexpression for v in the reduced expression for w with factors $(s_1, s_2, s_3, s_1, s_2, s_1)$. Then $g_{(5)} = y_1(t_1) y_2(t_2) y_3(t_3) \dot{s}_1 y_2(t_5)$, and it is easy to check that

$$\Delta^{v_{(5)}\omega_{i_6}}_{s_{i_6}\omega_{i_6}}(g_{(5)}) = -t_1.$$

11. Total positivity

From now on let $\mathbb{K} = \mathbb{R}$. We view the group G and the flag variety $\mathcal{B}$ as real manifolds, with the corresponding Hausdorff topology.

Definition 11.1 ([10])**.** The totally nonnegative part $U^-_{\geq 0}$ of U^- is defined to be the semigroup in U^- generated by the $y_i(t)$ for $t \in \mathbb{R}_{\geq 0}$. The totally nonnegative part of $\mathcal{B}$ is defined by

$$\mathcal{B}_{\geq 0} := \overline{\{u \cdot B^+ \mid u \in U^-_{\geq 0}\}}$$

where the closure is taken inside $\mathcal{B}$ in its real topology.

Let us collect below some useful facts, see [10] and also [2] for (1).

(1) For any braid relation such as $s_i s_j s_i = s_j s_i s_j$ in W there is a subtraction-free rational transformation relating the parameters of the corresponding parametrizations. For example in type A_2,

$$y_i(a) y_j(b) y_i(c) = y_j\left(\frac{bc}{a+c}\right) y_i(a+c)\, y_j\left(\frac{ab}{a+c}\right).$$

(2) For $w \in W$ and a reduced expression $w = s_{i_1} \dots s_{i_n}$ define

$$U^-_{>0}(w) := \{y_{i_1}(t_1) \cdot \dots \cdot y_{i_n}(t_n) \mid t_1, \dots, t_n \in \mathbb{R}_{>0}\}.$$

By (1) this set is independent of the reduced expression chosen. Moreover any product of $y_i(t)$'s for positive parameters t can be transformed until it is seen to lie in some $U^-_{>0}(w)$. Therefore

$$U^-_{\geq 0} = \bigsqcup_{w \in W} U^-_{>0}(w).$$

This is of course precisely the decomposition of $U^-_{\geq 0}$ induced by Bruhat decomposition, that is, $U^-_{>0}(w) = U^-_{\geq 0} \cap B^+ \dot{w} B^+$.

(3) The totally positive part of U^- may be defined as $U^-_{>0} = U^-_{\geq 0}(w_0)$. For the flag variety the totally positive part is taken to be

$$\mathcal{B}_{>0} := \{u \cdot B^+ \mid u \in U^-_{>0}\}.$$

Clearly $\mathcal{B}_{>0}$ is open dense in $\mathcal{B}_{\geq 0}$.

(4) Let $u \in U^-_{\geq 0}$. The semigroup property $uU^-_{\geq 0} \subset U^-_{\geq 0}$ implies, by continuity, that

$$u \cdot \mathcal{B}_{\geq 0} \subset \mathcal{B}_{\geq 0}.$$

Definition 11.2. For $v, w \in W$ with $v \leq w$, let

$$\mathcal{R}^{>0}_{v,w} := \mathcal{R}_{v,w} \cap \mathcal{B}_{\geq 0}.$$

In the special case where $v = 1$ we have $\mathcal{R}_{1,w} \cong U^- \cap B^+\dot{w}B^+$ and $\mathcal{R}^{>0}_{1,w} = U^-_{>0}(w) \cdot B^+$ (see property (2) above). From this observation Lusztig [10] conjectured that also $\mathcal{R}^{>0}_{v,w}$ is a semi-algebraic cell. The first proof of this is in [12]. However this proof does not provide an explicit parametrization and uses deep properties of canonical bases. We will now give a different proof which gives parametrizations and is completely elementary.

Let us choose a reduced expression $\mathbf{w}$ for w with factors $(s_{i_1}, \dots, s_{i_n})$. To $v \leq w$ we may associate the positive subexpression $\mathbf{v}_+$ for v in $\mathbf{w}$ as in Lemma 3.5. Note that $\mathbf{v}_+$ is non-decreasing, so $J^-_{\mathbf{v}_+} = \emptyset$. We define

$$G^{>0}_{\mathbf{v}_+,\mathbf{w}} := \left\{ g = g_1 g_2 \cdots g_n \;\middle|\; \begin{array}{ll} g_k = y_{i_k}(t_k) \text{ for } t_k \in \mathbb{R}_{>0}, & \text{if } k \in J^\circ_{\mathbf{v}_+} \\ g_k = \dot{s}_{i_k}, & \text{if } k \in J^+_{\mathbf{v}_+} \end{array} \right\}$$

Then $G^{>0}_{\mathbf{v}_+,\mathbf{w}} \cong \mathbb{R}^{\ell(w)-\ell(v)}_{>0}$ is a semi-algebraic cell in G. The aim of this section is to prove the following theorem.

Theorem 11.3. *The isomorphism $G_{\mathbf{v}_+,\mathbf{w}} \xrightarrow{\sim} \mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$ restricts to an isomorphism of real semi-algebraic varieties*

$$G^{>0}_{\mathbf{v}_+,\mathbf{w}} \xrightarrow{\sim} \mathcal{R}^{>0}_{v,w}.$$

Note that if $v = 1$ then $G^{>0}_{\mathbf{v}_+,\mathbf{w}} = U^-_{>0}(w)$ and, as a subset of G, does not depend on the reduced expression $\mathbf{w}$. This is no longer true if $v \neq 1$ as can be seen already in type A_2. We begin the proof of Theorem 11.3 with a simple observation about minors.

Lemma 11.4. *Suppose $B = z\dot{w} \cdot B^+$ lies in $\mathcal{B}_{\geq 0}$ with $z \in U^+$ and $w \in W$. For any dominant weight λ and $v \in W$,*

$$\Delta^{v\lambda}_{w\lambda}(z) \geq 0.$$

Proof. Since $B \in \mathcal{B}_{\geq 0}$ we can find a sequence $u_n \cdot B^+$ with $u_n \in U^-_{>0}$ that converges to $B = z\dot{w} \cdot B^+$. Note that for any $u = y_{i_1}(t_1) \dots y_{i_N}(t_N) \in U^-_{>0}$ and $x \in W$, the element $u \cdot \xi_\lambda$ in $V(\lambda)$ has a positive projection to the $x\lambda$ weight space, using that $s_{i_1} \dots s_{i_N} = w_0$ has a subexpression for x. Now we have

$$\frac{u_n \cdot \xi_\lambda}{\langle u_n \cdot \xi_\lambda, \dot{w} \cdot \xi_\lambda \rangle} \to z\dot{w} \cdot \xi_\lambda \qquad (n \to \infty),$$

where the denominator $\langle u_n \cdot \xi_\lambda, \dot{w} \cdot \xi_\lambda \rangle$ is just the required normalization factor. It follows that

$$0 \leq \lim_{n\to\infty} \frac{\langle u_n \cdot \xi_\lambda, \dot{v} \cdot \xi_\lambda \rangle}{\langle u_n \cdot \xi_\lambda, \dot{w} \cdot \xi_\lambda \rangle} = \langle z\dot{w} \cdot \xi_\lambda, \dot{v} \cdot \xi_\lambda \rangle = \Delta^{v\lambda}_{w\lambda}(z).$$

□

We need to recall one lemma.

Lemma 11.5 ([12] Lemma 2.3)**.** *Suppose $w = w_1 w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$. Consider the reduction map $\pi^w_{w_1} : B^+\dot{w} \cdot B^+ \to B^+\dot{w}_1 \cdot B^+$. If $B \in B^+\dot{w} \cdot B^+$ lies in $\mathcal{B}_{\geq 0}$ then so does $\pi^w_{w_1}(B)$.*

This lemma is easy to see if $B \in \mathcal{R}_{1,w}^{>0}$: In that case using Lusztig's parametrization we may write $B = y_{i_1}(t_1)\dots y_{i_n}(t_n)\cdot B^+$ for some positive t_i, being careful to choose a reduced expression $s_{i_1}\cdots s_{i_n}$ of w such that $w_1 = s_{i_1}\cdots s_{i_m}$, where $m = \ell(w_1)$. The element $\pi_{w_1}^w(B) = y_{i_1}(t_1)\dots y_{i_m}(t_m)\cdot B^+$ is then clearly totally nonnegative again. The property extends from the dense open part $\mathcal{R}_{1,w}$ to the whole Bruhat cell essentially by continuity (see [12] for a more careful argument).

Now we are ready to show one part of the theorem.

Lemma 11.6. *If $\mathcal{R}_{\mathbf{v},\mathbf{w}} \cap \mathcal{B}_{\geq 0} \neq \emptyset$, then $\mathbf{v}$ is a positive subexpression of $\mathbf{w}$.*

Proof. Let $B \in \mathcal{R}_{\mathbf{v},\mathbf{w}} \cap \mathcal{B}_{\geq 0}$ and write $B = z\dot{w}\cdot B^+$ for $z \in U^+$. Suppose $\mathbf{v} \prec \mathbf{w}$ is not a positive subexpression. Then $J_{\mathbf{v}}^- \neq \emptyset$. Let $k \in J_{\mathbf{v}}^-$. The equation (7.3) together with Remark 7.6.(1) gives

$$-1 = \frac{\prod_{j\neq i_k} \Delta_{w_{(k)}\omega_j}^{v_{(k)}\omega_j}(z)^{-a_{j,i_k}}}{\Delta_{w_{(k)}\omega_{i_k}}^{v_{(k)}\omega_{i_k}}(z)\Delta_{w_{(k-1)}\omega_{i_k}}^{v_{(k-1)}\omega_{i_k}}(z)}.$$

Therefore at least one of the minors in this formula must be negative. By Lemma 11.4 this implies that one of the two elements $z\dot{w}_{(k)}\cdot B^+$ and $z\dot{w}_{(k-1)}\cdot B^+$ does not lie in $\mathcal{B}_{\geq 0}$. Since these are both reductions of B we have a contradiction to Lemma 11.5. □

Remark 11.7. Suppose $z\dot{w}\cdot B^+ \in \mathcal{R}_{v,w}^{>0}$ and $\mathbf{w}$ is a reduced expression for w with positive subexpression $\mathbf{v}_+$ for v. Then by a combination of the above lemmas we have,

$$\Delta_{w_{(k)}\omega_{i_k}}^{v_{(k)}\omega_{i_k}}(z) > 0, \qquad k = 0, 1, \dots, \ell(w).$$

Recall that these minors, as standard chamber minors, are automatically nonzero, hence the strict positivity. Since $\mathbf{v}_+$ is non-decreasing there are no special chamber minors. So the observation is that if $z\dot{w}\cdot B^+$ lies in $\mathcal{B}_{\geq 0}$, then all of the associated chamber minors are positive.

The following lemma is a technical tool we will need to finish the proof of the theorem.

Lemma 11.8. *Let $v \leq w$ in W and suppose α_{i_0} is a simple root such that $u^{-1}\alpha_{i_0} > 0$ for all $v \leq u \leq w$. Then for all $g\cdot B^+ \in \mathcal{R}_{v,w}$ and any $m \in \mathbb{R}$,*

$$x_{i_0}(m)g\cdot B^+ = g\cdot B^+. \tag{11.1}$$

In other words, if $u^{-1}\alpha_{i_0} > 0$ for all $v \leq u \leq w$, then $\mathcal{R}_{v,w}$ is contained in the Springer fiber of $x_{i_0}(m)$.

Note that it is easy to see that the condition on α_{i_0} is also necessary. Suppose $m \neq 0$. If $x_{i_0}(m)$ fixes the elements of $\mathcal{R}_{v,w}$ then it also fixes the elements of the closure. So in particular, $x_{i_0}(m)\dot{u}\cdot B^+ = \dot{u}\cdot B^+$ for $v \leq u \leq w$. This implies $u^{-1}\alpha_{i_0} > 0$.

Proof. Let $\mathbf{v}_+$ be the positive subexpression for v of a reduced expression $\mathbf{w}$ for w. Since the corresponding Deodhar component $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$ is dense in $\mathcal{R}_{v,w}$, it suffices to show that $x_{i_0}(m)g\cdot B^+ = g\cdot B^+$ for $g\cdot B^+ \in \mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$. In other words we may assume $g \in G_{\mathbf{v}_+,\mathbf{w}}$.

By the defining property (3.8) for positive subexpressions we have that $v_{(j-1)}\alpha_{i_j} > 0$ for all j. Also $J_{\mathbf{v}_+}^- = \emptyset$. So we may write $g \in G_{\mathbf{v}_+,\mathbf{w}}$ as

$$g = \left(\prod_{j\in J_{\mathbf{v}_+}^{\circ}} y_{v_{(j-1)}\alpha_{i_j}}(t_j)\right)\dot{v},$$

where $y_{v_{(j-1)}\alpha_{i_j}}(t) := \dot{v}_{(j-1)}y_{i_j}(t)\dot{v}_{(j-1)}^{-1} \in U^-$. Let us set $y = \prod_{j\in J_{\mathbf{v}_+}^{\circ}} y_{v_{(j-1)}\alpha_{i_j}}(t_j)$. Then we have

$$g\cdot B^+ = y\dot{v}\cdot B^+ = z\dot{w}\cdot B^+,$$

where $z \in U^+$. Let us also write the reductions $\pi_{w_{(k)}}^w(g\cdot B^+)$ as

$$g_{(k)}\cdot B^+ = y_{(k)}\dot{v}_{(k)}\cdot B^+ = z\dot{w}_{(k)}\cdot B^+,$$

where $y_{(k)} = \prod_{j\in J_{\mathbf{v}_+}^{\circ}, j\leq k} y_{v_{(j-1)}\alpha_{i_j}}(t_j)$ and otherwise the notation is as usual.

We will now show that the conditions on α_{i_0} imply the following assertions.

(i) If $x \in W$ satisfies $v_{(j)} \le x \le w_{(j)}$ for some $j = 1, \ldots, n$, then $x^{-1}\alpha_{i_0} > 0$.
(ii) $v_{(j-1)}\alpha_{i_j} \neq \alpha_{i_0}$ for all $j = 1, \ldots, n$.

For $x \in W$ let $R^+(x) := \{\alpha \in R^+ \,|\, x^{-1}\alpha < 0\}$. Suppose we have an $x \in W$ with $v_{(j)} \le x \le w_{(j)}$. We want to show that there exists x' with $v \le x' \le w$ such that $R^+(x) \subseteq R^+(x')$. This will imply (i). Set $x_{(j)} := x$. It suffices if we can construct $x_{(j+1)}$ with $v_{(j+1)} \le x_{(j+1)} \le w_{(j+1)}$ and $R^+(x_{(j)}) \subseteq R^+(x_{(j+1)})$. For this there are two cases. If already $v_{(j+1)} \le x_{(j)}$, then we may set $x_{(j+1)} := x_{(j)}$. Otherwise we must be in the situation $v_{(j+1)} = v_{(j)}s_{i_{j+1}}$ and we need to take (at least) $x_{(j+1)} := x_{(j)}s_{i_{j+1}}$ to obtain $v_{(j+1)} \le x_{(j+1)} \le w_{(j+1)}$. Because $v_{(j+1)} \le x_{(j+1)}$ and $v_{(j+1)} \not\le x_{(j)}$, we find that $x_{(j)} < x_{(j+1)}$ and also $R^+(x_{(j)}) \subset R^+(x_{(j+1)})$. So $x_{(j+1)}$ has been constructed successfully.

Now consider $x = v_{(j-1)}s_{i_j}$. Clearly $v_{(j)} \le x \le w_{(j)}$ is satisfied, and so by (i) we have $s_{i_j}^{-1}v_{(j-1)}^{-1}\alpha_{i_0} > 0$. Therefore

$$\ell(s_{i_0}v_{(j-1)}s_{i_j}) = \ell(v_{(j-1)}s_{i_j}) + 1 = \ell(v_{(j-1)}) + 2$$

using also that $\mathbf{v}_+$ is non-decreasing. It follows that $s_{i_0}v_{(j-1)}s_{i_j} > s_{i_0}v_{(j-1)}$ and $s_{i_0}v_{(j-1)}\alpha_{i_j} > 0$. This implies (ii).

Finally we can put everything together to show that $x_{i_0}(m)g \cdot B^+$ lies in $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$ and equals $g \cdot B^+$. By Corollary 6.6 and Theorem 7.1 we know exactly which minors to check. Namely we only have to prove

(1)
$$\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(x_{i_0}(m)z) = 0, \qquad k \in J^+_{\mathbf{v}_+},$$

(2)
$$\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(x_{i_0}(m)z) = \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z), \qquad k = 1, \ldots, n.$$

Suppose $l \in \mathbb{Z}_{\ge 0}$ and $\mu = v_{(k-1)}\omega_{i_k} - l\alpha_{i_0}$. Let $\zeta = pr_\mu(y_{(k)}\dot{v}_{(k)} \cdot \xi_{\omega_{i_k}})$ in $V(\omega_{i_k})$. If $\zeta \neq 0$ then the weight μ must be of the form

$$v_{(k-1)}\omega_{i_k} - l\alpha_{i_0} = v_{(k)}\omega_{i_k} - \sum_{j \le k, j \in J^\circ_{\mathbf{v}}} c_j v_{(j-1)}\alpha_{i_j}$$

with $c_j \ge 0$, since the factors of $y_{(k)}$ are $y_{v_{(j-1)}\alpha_{i_j}}(t_j)$ for $j \in J^\circ_{\mathbf{v}_+}$ with $j \le k$. Simplifying this equation we get

$$l\alpha_{i_0} = v_{(k-1)}\alpha_{i_k} + \sum_{j \le k, j \in J^\circ_{\mathbf{v}_+}} c_j v_{(j-1)}\alpha_{i_j}.$$

But the right hand side is a non-zero sum of positive roots α not equal to α_{i_0} by (ii) above. Therefore we have a contradiction and so $\zeta = 0$. Since $z\dot{w}_{(k)} \cdot \xi_{\omega_{i_k}}$ and $y_{(k)}\dot{v}_{(k)} \cdot \xi_{\omega_{i_k}}$ are collinear in $V(\omega_{i_k})$ this implies that also

$$pr_{v_{(k-1)}\omega_{i_k} - l\alpha_{i_0}}(z\dot{w}_{(k)} \cdot \xi_{\omega_{i_k}}) = 0.$$

for all $l \ge 0$. Therefore $\left\langle x_{i_0}(m)z\dot{w}_{(k)} \cdot \xi_{\omega_{i_k}} \,,\ \dot{v}_{(k-1)} \cdot \xi_{\omega_{i_k}} \right\rangle = 0$ and (1) holds.

We can now prove (2) in a completely analogous way. Let $l \in \mathbb{Z}_{>0}$ and consider $\zeta = pr_{v_{(k)}\omega_{i_k} - l\alpha_{i_0}}(y_{(k)}\dot{v}_{(k)} \cdot \xi_{\omega_{i_k}})$. Then $\zeta \neq 0$ only if

$$l\alpha_{i_0} = \sum_{j \le k, j \in J^\circ_{\mathbf{v}_+}} c_j v_{(j-1)}\alpha_{i_j}$$

for $c_j \ge 0$. And again this is impossible by (ii). So $\zeta = 0$ and with it

$$pr_{v_{(k)}\omega_{i_k} - l\alpha_{i_0}}(z\dot{w}_{(k)} \cdot \xi_{\omega_{i_k}}) = 0$$

for all $l > 0$. This implies (2). □

Lemma 11.9. *If $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ then $g \cdot B^+ \in \mathcal{R}^{>0}_{v,w}$.*

Proof. By Proposition 5.2 we have $g \cdot B^+ \in \mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$. We need to show that $g \cdot B^+ \in \mathcal{B}_{\geq 0}$. We have $g = g_1 \dots g_n$ with

$$g_j = \begin{cases} y_{i_j}(t_j) \text{ for } t_j \in \mathbb{R}_{>0}, & \text{if } j \in J^\circ_{\mathbf{v}_+} \\ \dot{s}_{i_j}, & \text{if } j \in J^+_{\mathbf{v}_+} \end{cases}$$

Clearly $g_n \cdot B^+ \in \mathcal{B}_{\geq 0}$. We will prove that $g_k \dots g_n \cdot B^+$ lies in $\mathcal{B}_{\geq 0}$ for all k by descending induction. Suppose we know $g_{k+1} \dots g_n \cdot B^+ \in \mathcal{B}_{\geq 0}$. There are two possibilities for g_k. The first case, $k \in J^\circ_{\mathbf{v}_+}$, is clear. In that case $g_k = y_{i_k}(t_k) \in U^-_{\geq 0}$ and so $g_k g_{k+1} \dots g_n \cdot B^+$ again lies in $\mathcal{B}_{\geq 0}$.

Let us now consider the other case. So $g_k = \dot{s}_{i_k}$ and $k \in J^+_{\mathbf{v}_+}$. From (2.1) we get the formula

$$x_i(t)\dot{s}_i = \alpha_i^\vee(t) y_i(t) x_i(-t^{-1}).$$

We apply this element for $i = i_k$ to $g_{k+1} \dots g_n \cdot B^+$ to get

$$x_{i_k}(t)\dot{s}_{i_k} g_{k+1} \dots g_n \cdot B^+ = \alpha_{i_k}^\vee(t) y_{i_k}(t) x_{i_k}(-t^{-1}) g_{k+1} \dots g_n \cdot B^+.$$

Let $v' = v_{(k)}^{-1} v$ and $w' = w_{(k)}^{-1} w$. Since $\mathbf{v}$ is the positive subexpression of $\mathbf{w}$, so the right-most reduced subexpression and $k \in J^+_{\mathbf{v}}$, it follows that $s_{i_k} u > u$ for all $v' \leq u \leq w'$. Applying Lemma 11.8 we get

$$x_{i_k}(-t^{-1}) g_{k+1} \dots g_n \cdot B^+ = g_{k+1} \dots g_n \cdot B^+.$$

Therefore in total

$$x_{i_k}(t)\dot{s}_{i_k} g_{k+1} \dots g_n \cdot B^+ = \alpha_{i_k}^\vee(t) y_{i_k}(t) g_{k+1} \dots g_n \cdot B^+. \tag{11.2}$$

Now one can see that the right hand side of (11.2) lies in $\mathcal{B}_{\geq 0}$ for all $t > 0$. However as $t \to 0$ the left hand side converges to $\dot{s}_{i_k} g_{k+1} \dots g_n \cdot B^+$. Therefore $\dot{s}_{i_k} g_{k+1} \dots g_n \cdot B^+ \in \mathcal{B}_{\geq 0}$ and the lemma follows. □

Proof of Theorem 11.3. By Lemma 11.9 the map

$$G^{>0}_{\mathbf{v}_+,\mathbf{w}} \to \mathcal{R}^{>0}_{v,w} \tag{11.3}$$

is well defined. Lemma 11.6 implies that $\mathcal{R}^{>0}_{v,w} \subset \mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$. Now if $B = z\dot{w} \cdot B^+ \in \mathcal{R}^{>0}_{v,w}$, then Lemma 11.5 and Lemma 11.4 together with Corollary 6.6 imply that the chamber minors of z (for the subexpression $\mathbf{v}_+$) are all positive. Note also that $J^-_{\mathbf{v}_+} = \emptyset$. It follows that the map $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}} \to G_{\mathbf{v}_+,\mathbf{w}}$ described in Theorem 7.1 (see Proposition 5.2) restricts to

$$\mathcal{R}^{>0}_{v,w} \to G^{>0}_{\mathbf{v}_+,\mathbf{w}},$$

giving the inverse to (11.3). □

12. Total positivity criteria

We can use the Chamber Ansatz together with Theorem 11.3 to characterize $\mathcal{R}^{>0}_{v,w}$ by inequalities. In the case where $v = 1$ the criteria below reduce to the total positivity criteria for $U^+ \cap B^- \dot{w} B^-$ of Berenstein and Zelevinsky, [2] Theorem 6.9.

Proposition 12.1. *Consider $w \in W$ with a fixed reduced expression $\mathbf{w} = (w_{(0)}, \dots, w_{(n)})$. Let $v \leq w$ in W and $\mathbf{v}_+ = (v_{(0)}, \dots, v_{(n)})$ be the positive subexpression for v in $\mathbf{w}$. Then*

$$\begin{aligned} \mathcal{R}^{>0}_{v,w} &= \left\{ z\dot{w} \cdot B^+ \;\middle|\; z \in U^+;\ \begin{array}{ll} \Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) = 0, & k \in J^+_{\mathbf{v}_+} \\ \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) > 0, & k \in J^\circ_{\mathbf{v}_+} \end{array} \right\} \\ &= \left\{ z\dot{w} \cdot B^+ \in \mathcal{R}_{v,w} \;\middle|\; z \in U^+;\ \Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) > 0,\ k \in J^\circ_{\mathbf{v}_+} \right\}. \end{aligned}$$

Proof. Let us call the two sets in question S_1 and S_2, respectively. We want to show $\mathcal{R}^{>0}_{v,w} = S_1 = S_2$. The inclusions in one direction, $R^{>0}_{v,w} \subseteq S_1 \subseteq S_2$, are clear from Remark 11.7 and Corollary 6.6.

Moreover by Corollary 6.6,

$$\Delta^{v_{(k-1)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) = 0, \quad \text{for } k \in J^+_{\mathbf{v}_+},$$

is true for all $z\dot{w} \cdot B^+$ in $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$. And since $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$ is dense in $\mathcal{R}_{v,w}$, this equality holds for all $z\dot{w} \cdot B^+ \in \mathcal{R}_{v,w}$. Therefore we also have the inclusions $S_2 \subseteq S_1 \subseteq \mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$.

It remains to show the final inclusion, of S_1, say, into $\mathcal{R}_{v,w}^{>0}$. Consider $z\dot{w}\cdot B^+ \in \mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$ with $\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) > 0$ for all $k \in J^{\circ}_{\mathbf{v}_+}$. By Theorem 11.3 and Theorem 7.1.(1) we need only show that the remaining chamber minors, $\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)$ for $k \in J^{+}_{\mathbf{v}_+}$, are also positive. Suppose indirectly that $\Delta^{v_{(k_0)}\omega_{i_{k_0}}}_{w_{(k_0)}\omega_{i_{k_0}}}(z) < 0$ for some $k_0 \in J^{+}_{\mathbf{v}_+}$. We may choose k_0 to be minimal with this property. From Remark 7.6.(1) and equation (7.3) one obtains

$$\Delta^{v_{(k_0)}\omega_{i_{k_0}}}_{w_{(k_0)}\omega_{i_{k_0}}}(z) = \frac{\prod_{j\neq i_{k_0}} \Delta^{v_{(k_0)}\omega_j}_{w_{(k_0)}\omega_j}(z)^{-a_{j,i_{k_0}}}}{\Delta^{v_{(k_0-1)}\omega_{i_{k_0}}}_{w_{(k_0-1)}\omega_{i_{k_0}}}(z)}. \tag{12.1}$$

Since the right hand side is made up of chamber minors for smaller k, it must be positive. So we have a contradiction. □

Remark 12.2. Note that by this proposition, $\mathcal{R}_{v,w}^{>0}$ is given inside $\mathcal{R}_{v,w}$ by $\dim(\mathcal{R}_{v,w}) = \ell(w) - \ell(v)$ inequalities. This is the ideal situation. Indeed, it is easy to see that our set of inequalities,

$$\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z) > 0 \qquad (k \in J^{\circ}_{\mathbf{v}_+}), \tag{12.2}$$

is minimal. Using Proposition 8.1 we can describe the situation by the following commutative diagram

$$\begin{array}{cccc} \Delta: & \mathcal{R}_{\mathbf{v}_+,\mathbf{w}} & \xrightarrow{\sim} & (\mathbb{R}^*)^{\ell(w)-\ell(v)} \\ & \uparrow & & \uparrow \\ & \mathcal{R}_{v,w}^{>0} & \xrightarrow{\sim} & (\mathbb{R}_{>0})^{\ell(w)-\ell(v)}, \end{array}$$

where the horizontal maps are given by the chamber minors,

$$\Delta\left(z\dot{w}\cdot B^+\right) = \left(\Delta^{v_{(k)}\omega_{i_k}}_{w_{(k)}\omega_{i_k}}(z)\right)_{k\in J^{\circ}_{\mathbf{v}_+}},$$

and the vertical maps are inclusions. From this picture it is clear that (12.2), or Proposition 12.1, has no redundant inequalities.

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