Canonical basis linearity regions arising from quiver representations

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Abstract

In this paper we show that there is a link between the combinatorics of the canonical basis of a quantized enveloping algebra and the monomial bases of the second author [22] arising from representations of quivers. We prove that some reparametrization functions of the canonical basis, arising from the link between Lusztig's approach to the canonical basis and the string parametrization of the canonical basis, are given on a large cone by linear functions arising from these monomial bases for a quantized enveloping algebra.
Keywords: Quantum group, Lie algebra, canonical basis, parametrization functions, monomial basis, representations of quivers, degenerations, piecewise-linear functions.

1 Introduction

Let $U = U_q(\mathbf{g})$ be the quantum group associated to a semisimple Lie algebra $\mathbf{g}$ of rank n. The negative part U^- of U has a canonical basis $\mathbf{B}$ with favourable properties (see Kashiwara [8] and Lusztig [14, §14.4.6]). For example, via action on highest weight vectors it gives rise to bases for all the finite-dimensional irreducible highest weight U-modules.

We consider the reparametrization functions of the canonical basis arising from two different types of parametrization, each dependent on the choice of a reduced expression for the longest element w_0 in the Weyl group W of $\mathbf{g}$. Let $s_1, s_2, \ldots, s_n$ be the Coxeter generators of W, and let $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ be such an expression, so that $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ and N is the number of positive roots of $\mathbf{g}$. The first parametrization of $\mathbf{B}$ arises from the correspondence between $\mathbf{B}$ and a basis of PBW type associated to $\mathbf{i}$ [14, §14.4.6]. We denote it by $\varphi_{\mathbf{i}} : \mathbf{B} \to \mathbb{N}^N$. The second is the string parametrization (see [5, §2], [9] and [19, §2]) which we denote by $\psi_{\mathbf{i}} : \mathbb{B} \to X_{st}(\mathbf{i})$, where $X_{st}(\mathbf{i}) \subseteq \mathbb{N}^N$ is known as the *string cone* corresponding to $\mathbf{i}$. If $\mathbf{i}$ and $\mathbf{j}$ are two reduced expressions for w_0 then the reparametrization function $S_{\mathbf{i}}^{\mathbf{j}}$ is the composition

$$S_{\mathbf{i}}^{\mathbf{j}} = \varphi_{\mathbf{j}} \psi_{\mathbf{i}}^{-1} : X_{st}(\mathbf{i}) \to \mathbb{N}^N.$$

These reparametrization functions are useful in relating the two different types of parametrizations, and in [6] were shown to have applications to the description of the tensor product multiplicities of simple $\mathbf{g}$-modules in an approach involving totally positive varieties. They were also shown to be closely connected with the description of the canonical basis in [7] and [18].

We also consider the monomial basis of U^- as constructed in [22]. Let Γ be a quiver with underlying graph Δ, where Δ is the Dynkin diagram of $\mathbf{g}$. Let $\mathbf{i}(\Gamma) = (i_1, i_2, \ldots, i_N)$ be a reduced expression for w_0 adapted to Γ in the sense of [11]. Let $k\Gamma$ denote the path algebra of Γ (where k is an arbitrary field). Any other quiver Q of type Δ defines, in a natural way, a directed partition (see Definition 4.2) of the Auslander-Reiten quiver of $k\Gamma$. Using the representation theory of $k\Gamma$, the second author [22] has associated a function $D_Q^\Gamma : \mathbb{N}^N \to \mathbb{N}^N$ to the pair (Γ, Q), such that

$$\{F_{i_1}^{D_1(\mathbf{c})} F_{i_2}^{D_2(\mathbf{c})} \cdots F_{i_N}^{D_N(\mathbf{c})} \,:\, \mathbf{c} \in \mathbb{N}^N\}$$

is a monomial basis of U^- (where $D_Q^\Gamma(\mathbf{c}) = (D_1(\mathbf{c}), D_2(\mathbf{c}), \ldots, D_N(\mathbf{c}))$).

Let $\mathbf{i}(Q)$ be a reduced expression for w_0 adapted to Q. Our main result is that if Γ is the linearly oriented quiver in type A_n (see Figure 2), and Q is an arbitrary quiver of type A_n, then D_Q^Γ is invertible, and its inverse, E_Q^Γ, coincides with $S_{\mathbf{i}(Q)}^{\mathbf{i}(\Gamma)}$ on a large cone (which we call the *degeneration cone*) which contains the Lusztig cone $L_{st}(\mathbf{i}(Q))$ of [15] associated to $\mathbf{i}(Q)$. We conjecture that this result should also hold for any choice of quiver Γ (and indeed, for any simply-laced simple Lie algebra), and also that the degeneration cone should be an entire region of linearity for the piecewise-linear function $S_{\mathbf{i}(Q)}^{\mathbf{i}(\Gamma)}$.

A consequence of our result is that $S_{\mathbf{i}(Q)}^{\Gamma}$ is linear on the Lusztig cone $L_{st}(\mathbf{i}(Q)$. This was already known (for an alternating orientation on Γ) in types A_1, A_2, A_3 and A_4 (for type A_4 see [7, 6.1]).

We remark that our approach to these functions differs from that of Berenstein and Zelevinsky [6] in that we are particularly concerned with linearity properties of the reparametrization functions. Our result has an interesting interpretation in terms of the canonical basis, which we explain in §11. We also remark that Lusztig cones are used to describe regular functions on a reduced real double Bruhat cell of the corresponding algebraic group [26], they have links with primitive elements in the dual canonical basis (this can be seen using [5]) and therefore with the representation theory of affine Hecke algebras [10], they are known to correspond to regions of linearity of the Lusztig reparametrization functions and tight monomials (see [7]), and they can be described using the homological algebra of representations of quivers [2].

The structure of the paper is as follows. In §2 we recall the parametrizations of the canonical basis we will need, and in §3 we recall the connection between reduced expressions for the longest word in the Weyl group and quivers. In §4 we recall the monomial basis and corresponding linear functions D_Q^Γ defined in [22], as well as showing that they are invertible. In the remaining sections, we construct the following commutative diagram (where Γ is the linearly oriented quiver in type A_n and $\mathbf{k}$ is a reduced expression adapted to Γ).

$$
\begin{array}{ccc}
X_{st}(\mathbf{i}(Q)) & \xrightarrow{S^{\mathbf{k}}_{\mathbf{i}(Q)}} & \mathbb{N}^N \\
\cup & & \cup \\
C_{st}(Q) & \begin{array}{c} \xrightarrow{E_Q^\Gamma} \\ \xleftarrow{D_Q^\Gamma} \end{array} & C_{PBW}(Q) \\
\cup & & \cup \\
L_{st}(\mathbf{i}(Q)) & \begin{array}{c} \xrightarrow{E_Q^\Gamma} \\ \xleftarrow{D_Q^\Gamma} \end{array} & L_{PBW}(Q)
\end{array}
$$

The Lusztig cones $L_{st}(\mathbf{i}(Q))$ will be recalled in §5, and the cones $C_{PBW}(Q)$ will be defined in §6. The degeneration cone $C_{st}(Q)$ is defined to be $D_Q^\Gamma(C_{PBW}(Q))$. In §7 we define the cones $L_{PBW}(Q)$, and in §8 we show that $L_{PBW}(Q)$ is contained in $C_{PBW}(Q)$. In §9 we show that $E_Q^\Gamma(L_{st}(Q)) = L_{PBW}(Q)$ and finally, in §10 we show that E_Q^Γ and $S^{\mathbf{k}}_{\mathbf{i}(Q)}$ coincide on $C_{st}(Q)$.

2 Parametrizations of the canonical basis

Let $\mathbf{g}$ be the simple Lie algebra over $\mathbb{C}$ of type A_n. Let R denote the set of roots of $\mathbf{g}$ and $R^+ \subseteq R$ the positive roots. Let U be the quantized enveloping algebra of $\mathbf{g}$. Then U is a $\mathbb{Q}(v)$-algebra generated by the elements E_i, F_i, K_μ, $i \in \{1, 2, \ldots, n\}$, $\mu \in P$, the weight lattice of $\mathbf{g}$. Let U^+ be the subalgebra generated by the E_i and U^- the subalgebra generated by the F_i.

Let W be the Weyl group of $\mathbf{g}$ with Coxeter generators $s_1, s_2, \ldots s_n$. It has a unique element w_0 of maximal length. For each reduced expression $\mathbf{i}$ for w_0 there are two parametrizations of the canonical basis $\mathbf{B}$ for U^-. The first arises from Lusztig's approach to the canonical basis [14, §14.4.6], and the second arises from Kashiwara's approach [8].

Lusztig's Approach

There is an $\mathbb{Q}$-algebra automorphism of U which takes each E_i to F_i, F_i to E_i, K_μ to $K_{-\mu}$ and v to v^{-1}. We use this automorphism to transfer Lusztig's definition of the canonical basis in [11, §3] to U^-.

Let T_i, $i = 1, 2, \ldots, n$, be the automorphism of U as in [13, §1.3] given by:

$$T_i(E_j) = \begin{cases} -F_jK_j, & \text{if } i = j, \\ E_j, & \text{if } |i-j| > 1 \\ -E_iE_j + v^{-1}E_jE_i & \text{if } |i-j| = 1 \end{cases}$$

$$T_i(F_j) = \begin{cases} -K_j^{-1}E_j, & \text{if } i = j, \\ F_j, & \text{if } |i-j| > 1 \\ -F_jF_i + vF_iF_j & \text{if } |i-j| = 1 \end{cases}$$

$$T_i(K_\mu) = K_{\mu - \langle \mu, \alpha_i \rangle h_i}, \text{ for } \mu \in P,$$

where the α_i are the simple roots and the h_i are the simple coroots of $\mathbf{g}$.

For each i, let r_i be the automorphism of U which fixes E_j and F_j for $j = i$ or $|i-j| > 1$ and fixes K_μ for all μ, and which takes E_j to $-E_j$ and F_j to $-F_j$ if $|i-j| = 1$. Let $T''_{i,-1} = T_ir_i$ be the automorphism of U as in [14, §37.1.3]. Let $\mathbf{c} \in \mathbb{N}^N$, where $N = \ell(w_0)$, and $\mathbf{i}$ be a reduced expression for w_0. Let

$$F_{\mathbf{i}}^{\mathbf{c}} := F_{i_1}^{(c_1)} T''_{i_1,-1}(F_{i_2}^{(c_2)}) \cdots T''_{i_1,-1} T''_{i_2,-1} \cdots T''_{i_{N-1},-1}(F_{i_N}^{(c_N)}).$$

Define $B_{\mathbf{i}} = \{F_{\mathbf{i}}^{\mathbf{c}} : \mathbf{c} \in \mathbb{N}^N\}$. Then $B_{\mathbf{i}}$ is the basis of PBW-type corresponding to the reduced expression $\mathbf{i}$. Note that, if the reduced expression $\mathbf{i}$ is adapted to a quiver in the sense of [11], then this basis can also be constructed using the Hall algebra approach of [24]. Let $\bar{\ }$ be the $\mathbb{Q}$-algebra automorphism from U to U taking E_i to E_i, F_i to F_i, K_μ to $K_{-\mu}$, and v to v^{-1}. Lusztig proves the following result in [11, §§2.3, 3.2].

Theorem 2.1 *(Lusztig)*
The $Z[v]$-span $\mathcal{L}$ of $B_{\mathbf{i}}$ is independent of $\mathbf{i}$. Let $\pi : \mathcal{L} \to \mathcal{L}/v\mathcal{L}$ be the natural projection. The image $\pi(B_{\mathbf{i}})$ is also independent of $\mathbf{i}$; we denote it by B. The restriction of π to $\mathcal{L} \cap \overline{\mathcal{L}}$ is an isomorphism of $\mathbb{Z}$-modules $\pi_1 : \mathcal{L} \cap \overline{\mathcal{L}} \to \mathcal{L}/v\mathcal{L}$. Also $\mathbf{B} = \pi_1^{-1}(B)$ is a $\mathbb{Q}(v)$-basis of U^-, which is the canonical basis *of U^-.*

Lusztig's theorem provides us with a parametrization of $\mathbf{B}$, dependent on $\mathbf{i}$. If $b \in \mathbf{B}$, we write $\phi_{\mathbf{i}}(b) = \mathbf{c}$, where $\mathbf{c} \in \mathbb{N}^N$ satisfies $b \equiv F_{\mathbf{i}}^{\mathbf{c}} \mod v\mathcal{L}$. Note that $\phi_{\mathbf{i}}$ is a bijection.

For reduced expressions $\mathbf{j}$ and $\mathbf{j}'$ for w_0, Lusztig defines in [11, §2.6] a reparametrization function $R_{\mathbf{j}}^{\mathbf{j}'} = \phi_{\mathbf{j}'}\phi_{\mathbf{j}}^{-1} : \mathbb{N}^N \to \mathbb{N}^N$. This function was shown by Lusztig to be piecewise linear and its regions of linearity were shown to have significance for the canonical basis, in the sense that elements b of the canonical basis with $\phi_{\mathbf{j}}(b)$ in the same region of linearity of $R_{\mathbf{j}}^{\mathbf{j}'}$ often have similar form. For example, this can be seen from the explicit descriptions of the canonical basis of type A_2 and A_3, as computed by Lusztig in [11] and by Xi in [25], respectively. More evidence for the importance of these regions is their connection [5] with the multiplicativity properties of dual canonical bases.

The string parametrization

Let $\widetilde{E}_i$ and $\widetilde{F}_i$ be the Kashiwara operators on U^- as defined in [8, §3.5]. Let $\mathcal{A} \subseteq \mathbb{Q}(v)$ be the subring of elements regular at $v = 0$, and let $\mathcal{L}'$ be the $\mathcal{A}$-lattice spanned by arbitrary products $\widetilde{F}_{j_1}\widetilde{F}_{j_2}\cdots\widetilde{F}_{j_m}\cdot 1$ in U^-. We denote the set of all such elements by S. The following results were proved by Kashiwara in [8].

Theorem 2.2 *(Kashiwara)*
(i) Let $\pi' : \mathcal{L}' \to \mathcal{L}'/v\mathcal{L}'$ be the natural projection, and let $B' = \pi'(S)$. Then B' is a $\mathbb{Q}$-basis of $\mathcal{L}'/v\mathcal{L}'$ (the crystal basis).
(ii) Furthermore, $\widetilde{E}_i$ and $\widetilde{F}_i$ each preserve $\mathcal{L}'$ and thus act on $\mathcal{L}'/v\mathcal{L}'$. They satisfy $\widetilde{E}_i(B') \subseteq B' \cup \{0\}$ and $\widetilde{F}_i(B') \subseteq B'$. Also for $b, b' \in B'$ we have $\widetilde{F}_i b = b'$, if and only if $\widetilde{E}_i b' = b$.
(iii) For each $b \in B'$, there is a unique element $\widetilde{b} \in \mathcal{L}' \cap \overline{\mathcal{L}'}$ such that $\pi'(\widetilde{b}) = b$. The set of elements $\{\widetilde{b} : b \in B'\}$ forms a basis of U^-, the global crystal basis *of U^-.*

It was shown by Lusztig [12, 2.3] that the global crystal basis of Kashiwara coincides with the canonical basis of U^-.

There is a parametrization of $\mathbf{B}$ arising from Kashiwara's approach, again dependent on a reduced expression $\mathbf{i}$ for w_0. Let $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ and $b \in \mathbf{B}$. Let a_1 be maximal such that $\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'$; let a_2 be maximal such that $\widetilde{E}_{i_2}^{a_2}\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'$, and so on, so that a_N is maximal such that

$$\widetilde{E}_{i_N}^{a_N}\widetilde{E}_{i_{N-1}}^{a_{N-1}}\cdots\widetilde{E}_{i_2}^{a_2}\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'.$$

Let $\mathbf{a} = (a_1, a_2, \ldots, a_N)$. We write $\psi_{\mathbf{i}}(b) = \mathbf{a}$. This is the crystal string of b (see [5, §2], [9] and [19, §2]). It is known that $\psi_{\mathbf{i}}(b)$ uniquely determines $b \in \mathbf{B}$ (see [19, §2.5]). We have $b \equiv \widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_N}^{a_N}\cdot 1 \mod v\mathcal{L}'$. The image of $\psi_{\mathbf{i}}$ is a cone which appears in [5], known as the *string cone* corresponding to $\mathbf{i}$.

We next consider a function which links the string parametrization with the parametrization arising from Lusztig's approach. For reduced expressions $\mathbf{i}$ and $\mathbf{j}$ for w_0, consider the maps

$$X_{st}(\mathbf{i}) \xrightarrow[\psi_{\mathbf{i}}^{-1}]{} \mathbf{B} \xrightarrow[\phi_{\mathbf{j}}]{} \mathbb{N}^N$$

We define $S_{\mathbf{i}}^{\mathbf{j}} = \phi_{\mathbf{j}}\psi_{\mathbf{i}}^{-1} : X_{st}(\mathbf{i}) \to \mathbb{N}^N$, a reparametrization function. This function has appeared in, for example, [6].

Remark on Notation:

We shall use the following notation convention. A cone C which is to be regarded as a subset of X_{st}, i.e. to be thought of as a set of strings, will be given the subscript "st" to denote this, thus C_{st}. A cone C which is to be regarded as a subset of $\mathbb{N}^N$, and is to be regarded as a set of PBW parameters for the canonical basis, will be denoted with the subscript "PBW", thus C_{PBW}. In each case, it will be clear from the context which reduced expression for w_0 is being used.

3 Reduced Expressions Compatible with Quivers

In this section we shall recall from [3] an explicit description of reduced expressions for w_0 compatible with quivers in type A_n, which we shall need in studying such expressions. Given two such reduced expressions $\mathbf{i}$ and $\mathbf{i}'$, we write $\mathbf{i} \sim \mathbf{i}'$ if there is a sequence of commutations (of the form $s_i s_j = s_j s_i$ with $|i - j| > 1$) which, when applied to $\mathbf{i}$, give $\mathbf{i}'$. This is an equivalence relation on the set of reduced expressions for w_0, and the equivalence classes are called commutation classes.

Let $Q = (Q_0, Q_1)$ be a quiver, i.e. a finite oriented graph with set of vertices Q_0 and set of arrows Q_1. We assume Q to be of type A_n, i.e. that the unoriented graph underlying Q is the Dynkin diagram of type A_n. If Q is a quiver and i is a sink in Q (i.e. all the arrows incident with i have i as target), we denote by $s_i(Q)$ the quiver with all of the arrows incident with i reversed. If $\mathbf{i}$ is a reduced expression for w_0, we say (following Lusztig [11]) that $\mathbf{i}$ is compatible with Q if i_1 is a sink in Q, i_2 is a sink in $s_{i_1}(Q)$, i_3 is a sink in $s_{i_2}s_{i_1}(Q)$, ..., i_N is a sink in $s_{i_{N-1}}s_{i_{N-2}}\cdots s_{i_1}(Q)$. It is known that if Q is any quiver of type A_n, then there is always at least one reduced expression $\mathbf{i}$ compatible with Q, and that the set of reduced expressions compatible with Q is the commutation class of $\mathbf{i}$.

Berenstein, Fomin and Zelevinsky give a nice description of a reduced expression compatible with a given quiver Q in type A_n in [3, §4.4.3]. Suppose that Q is such a quiver. Number the edges of Q from 1 to $n-1$, starting from the left hand end. Berenstein, Fomin and Zelevinsky construct an arrangement as follows. Consider a square in the plane, with horizontal and vertical sides. We will draw $n+1$ 'pseudo-lines' in this square. Put $n+1$ points onto the left-hand edge of the square, equally spaced, numbered 1 to $n+1$ from top to bottom, so that 1 and $n+1$ are at the corners. Do the same for the right-hand edge, but number the points from bottom to top. Line_h will join point h on the left with point h on the right. For $h = 1, n+1$, Line_h will be a diagonal of the square. For $h \in [2, n]$, Line_h will be a union of two line segments of slopes $\pi/4$ and $-\pi/4$. There are therefore precisely two possibilities for Line_h. If edge $h-1$ in Q is oriented to the left, the left segment has positive slope, while the right one has negative slope; if edge $h-1$ is oriented to the right, it goes the other way round.

Example: Berenstein, Fomin and Zelevinsky give the following example. Consider the case A_5, with Q given by the quiver with vertices $1, 2, 3, 4, 5$ and arrows $1 \to 2$, $2 \leftarrow 3$, $3 \to 4$, and $4 \leftarrow 5$. (We shall denote such a quiver by $RLRL$, where an R (respectively, L) denotes an edge oriented to the right (respectively, left).) The Berenstein-Fomin-Zelevinsky arrangement for this quiver is shown in Figure 1.

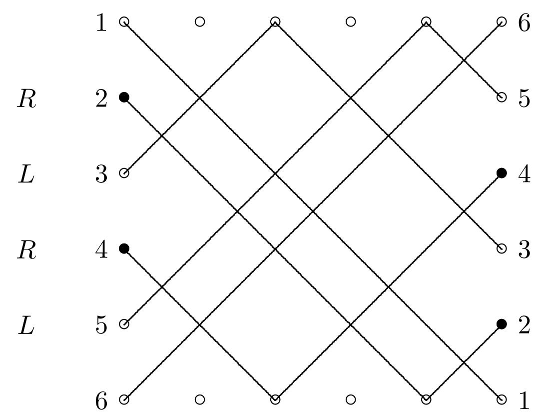

Figure 1: The Berenstein, Fomin and Zelevinsky diagram for the quiver $RLRL$

If $\mathbf{i}$ is a reduced expression for w_0 in type A_n, then the *chamber diagram* for $\mathbf{i}$ is given by a set of pseudolines numbered from 1 to n. Two sets of points numbered 1 to n are arranged on the vertical lines of a square as in the above picture. Underneath the square, the simple reflections in $\mathbf{i}$ are written from left to right. The ith pseudoline then links the point marked i on the left with the point marked i on the right, in such a way that immediately above the simple reflection i_j from $\mathbf{i}$ pseudolines i_j and i_j+1 cross. See [3, 1.4] for details. Berenstein, Fomin and Zelevinsky prove the following result.

Proposition 3.1 *(Berenstein, Fomin and Zelevinsky) A reduced expression* $\mathbf{i}$ *for* w_0 *is compatible with the quiver* Q *if and only if its chamber diagram is isotopic to the arrangement defined above corresponding to* Q*.*

Proof: see [3, §4.4.3]. □

Since each simple reflection appearing in such a reduced expression $\mathbf{i}$ corresponds to a crossing of two pseudo-lines in the diagram, each pseudo-line in the diagram (consisting of a part of positive slope and a part of negative slope) gives rise to some of the simple reflections appearing in $\mathbf{i}$; each simple reflection appears twice in this way as two lines crossing correspond to a simple reflection. We can remove this duplication by counting, for each line, only the simple reflections which arise during the part of the line which is of positive slope. In this way, each edge of the quiver, which corresponds to a line in the diagram, gives rise to some of the simple reflections in the reduced expression $\mathbf{i}$; we include also the line from the bottom left of the diagram to the top right. Each simple reflection arises for a unique edge of the quiver (or comes from the extra line).

Number the edges of Q from 1 to $n-1$, starting from the left. Suppose that the edges $l_1, l_2, \ldots, l_a$ all point to the left, and that edges $r_1, r_2, \ldots, r_b$ all point to the right, and that every edge is one of these, where $l_1 < l_2 < \cdots < l_a$ and $r_1 < r_2 < \cdots < r_b$. For $m \in \mathbb{N}$, denote by $(m \searrow 1)$ the sequence $m, m-1, \ldots, 2, 1$. Then it is easy to see that the above construction shows that the reduced expression

$$\mathbf{i}(Q) = (l_1 \searrow 1)(l_2 \searrow 1)\cdots(l_a \searrow 1)(n \searrow 1)(n \searrow n+1-r_b)(n \searrow n+1-r_{b-1})\cdots(n \searrow n+1-r_1)$$

is compatible with Q; it follows that the reduced expressions compatible with Q are precisely those commutation equivalent to $\mathbf{i}(Q)$.

4 Monomial Bases arising from Representations of Quivers

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver. We assume Γ to be of Dynkin type, which means that the unoriented graph Δ underlying Γ is a disjoint union of Dynkin diagrams of type A, D, E. Let k be an arbitrary field. Then we can form the path algebra $k\Gamma$ of Γ over k, which has the paths in Γ as a k-basis (including an 'empty' path for each vertex $i \in \Gamma_0$), and multiplication of paths given by concatenation if possible, and zero otherwise. This is a finite dimensional k-algebra since Γ, being of Dynkin type, has no oriented cycles. We form the category $\operatorname{mod} k\Gamma$ of finite-dimensional representations of $k\Gamma$. The isoclasses of simple objects S_i in $\operatorname{mod} k\Gamma$ correspond bijectively to the vertices $i \in \Gamma_0$ of Γ. Let $\mathbb{N}\Gamma$ be the free abelian semigroup spanned by elements α_i for $i \in \Gamma_0$; it can be identified with the positive root lattice. For a representation $M \in \operatorname{mod} k\Gamma$, we denote by d_i for $i \in \Gamma_0$ the Jordan-Hölder multiplicity of the simple S_i in M. This allows us to define a map $\underline{\dim}$ from the set of isoclasses in $\operatorname{mod} k\Gamma$ to $\mathbb{N}\Gamma$ by $\underline{\dim}(M) = \sum_{i \in \Gamma_0} d_i \alpha_i$. The fundamental result in the theory of Dynkin quivers is:

Theorem 4.1 *(Gabriel)*
The map $\underline{\dim}$ induces a bijection between isoclasses of indecomposable objects X_α in $\operatorname{mod} k\Gamma$ and the positive roots $\alpha \in R^+ \subset \mathbb{N}\Gamma$ of type Δ.

The Auslander-Reiten quiver of the algebra $k\Gamma$ is defined as the oriented graph having the isoclasses of indecomposable representations of $k\Gamma$ as vertices, and arrows corresponding to irreducible maps (i.e.

morphisms in mod $k\Gamma$ between indecomposable objects which can not be factored into a composition of non-split maps); see [1] for details. We can construct the Auslander-Reiten quiver of our quiver Γ in the following way (see [1]). Let Γ^{op} be the quiver Γ with the orientation of all arrows reversed. Let $\mathbb{Z}\Gamma^{op}$ be the quiver with vertices $\mathbb{Z} \times \{1, 2, \ldots, n\}$. Whenever there is an arrow $i \to j$ in Γ^{op}, we draw one arrow $(z, i) \to (z, j)$ and one arrow $(z, j) \to (z + 1, i)$ for each $z \in \mathbb{Z}$. Define $A(\Gamma)$ to be the full subquiver of $\mathbb{Z}\Gamma^{op}$ consisting of all vertices (z, i) such that $1 \le z \le (h + a_i - b_i)/2$ where, for each $i \in \{1, 2, \ldots, n\}$, a_i (respectively b_i), is the number of arrows in the unoriented path in Γ from i to $\sigma(i)$ that are directed towards i (respectively $\sigma(i)$). Here, σ is the unique permutation of the vertices of Γ such that $w_0(\alpha_i) = -\alpha_{\sigma(i)}$, and h is the Coxeter number. Then $A(\Gamma)$ is the Auslander-Reiten quiver of Γ. (We will not need the Auslander-Reiten translate here).

Example: Let Γ be the linearly oriented quiver in type A_5 (see Figure 2). Then the Auslander-Reiten quiver of Γ is given in Figure 3. The vertices in this diagram, which correspond to the isoclasses of indecomposable modules, are labelled by the corresponding positive roots (so, for example, 123 means $\alpha_1 + \alpha_2 + \alpha_3$).

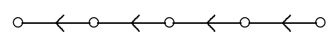

Figure 2: Linearly oriented quiver of type A_5.

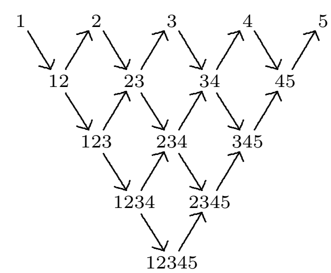

Figure 3: The Auslander-Reiten quiver of the linearly oriented quiver of type A_5

The following definition was first introduced in [22]:

Definition 4.2 An ordered partition $R^+ = I_1 \cup \ldots \cup I_s$ of R^+ into disjoint subsets I_k is called *directed* if
(i) $\mathrm{Ext}^1_{kQ}(X_\alpha, X_\beta) = 0$ for all α, β in the same part I_k,
(ii) $\mathrm{Ext}^1_{kQ}(X_\alpha, X_\beta) = 0$ and $\mathrm{Hom}_{kQ}(X_\beta, X_\alpha) = 0$ if $\alpha \in I_k$, $\beta \in I_l$, where $1 \le k < l \le s$.

The existence of (several!) directed partitions of a given quiver Γ can be seen using Auslander-Reiten theory (see [1]): if $\mathrm{Hom}_{kQ}(U, V) \neq 0$ (resp. $\mathrm{Ext}^1_{kQ}(V, U) \neq 0$) for indecomposables $U, V \in \mathrm{mod}\, k\Gamma$, then there exists a path (resp. a proper path) from $[U]$ to $[V]$ in the Auslander-Reiten quiver of $k\Gamma$.

But since this graph is directed, we can enumerate the isoclasses of indecomposables in $\mathrm{mod}\, k\Gamma$ as $[U_1], \ldots, [U_\nu]$, such that $\mathrm{Hom}_{kQ}(U_p, U_q) = 0$ for $p > q$, and $\mathrm{Ext}^1_{kQ}(U_p, U_q) = 0$ for $p \leq q$. Define roots α^p by $[U_p] = [X_{\alpha^p}]$. By definition, the partition $R^+ = \{\alpha^1\} \cup \ldots \cup \{\alpha^\nu\}$ is directed. All other directed partitions can be constructed by coarsening such a partition.

Fix a directed partition $R^+ = I_1 \cup \ldots \cup I_s$ from now on. We will associate to it a sequence $\mathbf{i} = (i_1, i_2, \ldots, i_t)$, as well as a function D from the set of isoclasses in $\mathrm{mod}\, k\Gamma$ to $\mathbb{N}^t$. Enumerate the vertices Γ_0 of Γ as $\Gamma_0 = \{1 \ldots n\}$ such that the existence of an arrow $i \to j$ in Γ implies $i < j$. Denote the multiplicity of the simple root α_s in a root α by $[\alpha : \alpha_s]$. In particular, write $s \in \alpha$ to indicate that the simple root α_s appears with non-zero coefficient in α, that is, $[\alpha : \alpha_s] \neq 0$. For each $p = 1 \ldots s$, write the subset of Γ_0 consisting of all vertices i such that $i \in \alpha$ for some root $\alpha \in I_p$ as $\{i^p_1, \ldots, i^p_{t_p}\}$, increasing with respect to the above defined ordering on Γ_0. Then the sequence $\mathbf{i}$ is defined as

$$\mathbf{i} = i^1_1 \ldots i^1_{t_1}\, i^2_1 \ldots i^2_{t_2}\, \ldots\, i^s_1 \ldots i^s_{t_s}.$$

We remark that, in general, the length of the sequence $\mathbf{i}$ is not N (but in the cases we need, we shall see that the length is N). Given an isoclass $[M]$ in $\mathrm{mod}\, k\Gamma$, we can write M as $M = \oplus_{\alpha \in R^+} X_\alpha^{c_\alpha}$, using Gabriel's Theorem and Krull-Schmidt. Let $D(M)$ be the tuple

$$\mathbf{a} = (a^1_1, \ldots a^1_{t_1}, a^2_1, \ldots a^2_{t_2}, \ldots, a^s_1, \ldots, a^s_{t_s}),$$

given by:

$$a^p_j = \sum_{\alpha \in I_p,\ i^p_j \in \alpha} [\alpha : \alpha_{i^p_j}] c_\alpha. \tag{1}$$

This defines a function D from the set of isoclasses in $\mathrm{mod}\, k\Gamma$ to $\mathbb{N}^t$, which is obviously additive, i.e. $D(M \oplus N) = D(M) + D(N)$. Identifying the set of isoclasses in $\mathrm{mod}\, k\Gamma$ with $\mathbb{N}R^+$ via $[\oplus_\alpha X_\alpha^{c_\alpha}] \mapsto \sum_\alpha c_\alpha \alpha$, we thus get a linear function $D : \mathbb{N}R^+ \to \mathbb{N}^t$. The main result of this paper is that, for suitable choices of Γ and the directed partition, the function D coincides with one of the reparametrization functions $(S^{\mathbf{j}}_{\mathbf{i}})^{-1}$ on an explicitly described region.

The original use of the function D lies in the following theorem (see [22]):

Theorem 4.3 *(Reineke) Writing $\mathbf{i} = (i_1, i_2, \ldots, i_t)$ and $D = (D_1, \ldots, D_t)$, the set*

$$\{F_{i_1}^{D_1(\mathbf{c})} \cdot \ldots F_{i_t}^{D_t(\mathbf{c})} \in U^- \ :\ \mathbf{c} \in \mathbb{N}R^+\}$$

is a basis for U^-.

Moreover, these monomial bases for U^- have good properties with respect to base change: the base change coefficients to a PBW basis (resp. to the canonical basis) form upper unitriangular matrices with respect to a certain ordering (namely, related to the degeneration ordering on quiver representations), and these coefficients have representation theoretic (resp. geometric) interpretations (see [22], [23]).

The function D is not invertible in general, but it is for special directed partitions, called regular in [22]. We now introduce a certain class of directed partitions for quivers of type A, and compute the inverse of D in these special cases.

We know from §3 that the reduced expression

$$\mathbf{k} = (1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1)$$

is adapted to the linearly oriented quiver $\Gamma_{\mathbf{k}}$, with arrows pointing to the left (see e.g. Figure 2). This is one of the most regular reduced expressions. We shall now specialise to this case.

Remark 4.4 We conjecture that our results hold, with similar proofs, if this quiver is replaced by an arbitrary Dynkin quiver of type A_n, but the Kashiwara operators are in this general case more difficult to describe combinatorially. In the special case we are considering, it is possible to handle this combinatorics. See [20] for a description of the combinatorics of Kashiwara operators on canonical basis elements using parametrizations of the canonical basis arising from PBW-bases of U^- corresponding to reduced expressions compatible with arbitrary quivers.

A reduced expression $\mathbf{i}$ for w_0 defines an ordering on the set Φ^+ of positive roots of the root system associated to W. We write $\alpha^j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1, 2, \ldots, N$. Then $\Phi^+ = \{\alpha^1, \alpha^2, \ldots \alpha^N\}$. For $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbb{Z}^N$, write $c_{\alpha^j} = c_j$. If $\alpha = \alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ with $i < j$, we also write c_{ij} for $c_{\alpha_{ij}}$.

Let $\alpha^1, \alpha^2, \ldots, \alpha^N$ be the ordering induced on Φ^+ by $\mathbf{k}$. We can write an element of $\mathbb{N}^N$ as $\mathbf{c} = (c_{ij})$, where $1 \leq i \leq j \leq n$, using the above. The canonical basis is parametrized, via the Lusztig parametrization $\phi_{\mathbf{k}} : \mathbf{B} \to \mathbb{N}^N$. We can write $\mathbf{c}$ as an array based on the Auslander-Reiten quiver for $\Gamma_{\mathbf{k}}$; see Figure 4. We write c_{ij} in place of the module corresponding to the positive root $\alpha_i + \alpha_{i+1} + \cdots \alpha_j$, with $c_{11}, c_{22}, \ldots, c_{nn}$ on the first row, $c_{12}, c_{23}, \ldots, c_{n-1,n}$ along the second, interspersing the first row elements, and so on, until $c_{1,n}$ on the last row. For example, when $n = 5$, see Figure 4. Suppose that Q is any quiver of type A_n. We can define a directed partition of $A(\Gamma_{\mathbf{k}})$ in the following

$$\begin{array}{ccccccccc}
c_{11} & & c_{22} & & c_{33} & & c_{44} & & c_{55} \\
 & c_{12} & & c_{23} & & c_{34} & & c_{45} & \\
 & & c_{13} & & c_{24} & & c_{35} & & \\
 & & & c_{14} & & c_{25} & & & \\
 & & & & c_{15} & & & &
\end{array}$$

Figure 4: Array of elements of $\mathbb{N}^{15}$

way. Fix $z \in \mathbb{Z}$, and set $z_1 = z$. Let $v_1 := (z_1, 1)$ be a vertex of $\mathbb{Z}\Gamma_{\mathbf{k}}$. For $i = 2 \ldots n$, define a vertex $v_i = (z_i, i)$ of $\mathbb{Z}\Gamma_{\mathbf{k}}$ inductively, as follows. If $i - 1 \to i$ is an arrow in Q, then let v_i be the head of the unique arrow with source v_{i-1}. If $i - 1 \leftarrow i$ is an arrow in Q, then let v_i be the source of the unique arrow with head v_{i-1}. Let $S_z = (v_1, v_2, \ldots, v_n) \subseteq \mathbb{Z}\Gamma_{\mathbf{k}}$. Then it is clear that $\mathbb{Z}\Gamma_{\mathbf{k}} = \cup_{z\in\mathbb{Z}} S_z$. It follows that $A(\Gamma_{\mathbf{k}}) = \cup_{z\in\mathbb{Z}}(S_z \cap A(\Gamma_{\mathbf{k}}))$; note that this decomposition must be finite. It is clear from the construction that this is a directed partition of $A(\Gamma_{\mathbf{k}})$. Let $T_z = S_z \cap A(\Gamma_{\mathbf{k}})$. We call each S_z a *slice* of $\mathbb{Z}\Gamma_{\mathbf{k}}$. If T_z is non-empty, we call it a slice of $A(\Gamma_{\mathbf{k}})$ (note that for this to happen, we must have $z \geq 1$).

Example: Consider the case A_5, with $Q = RLRL$. The corresponding decomposition of $A(\Gamma_{\mathbf{k}})$ into slices is given in Figure 5. Each vertex of $A(\Gamma_{\mathbf{k}})$ is denoted by a number, indicating the number of the slice it lies in.

1		2		3		4		5
	1		2		3		4	
		2		3		4		
			2		3			
				3				

Figure 5: Slice structure of $A(\Gamma_{\mathbf{k}})$ corresponding to the quiver $RLRL$

Let Q be an arbitrary quiver (of type A_n). Note that we can identify (i, j), for $1 \leq i \leq j \leq n$, with the indecomposable module with dimension vector $\alpha_i + \cdots + \alpha_j$, so each such pair lies in a corresponding slice T_z. If $v = (z, a) \in A(\Gamma_{\mathbf{k}})$, let $i(v), j(v)$ be such that the dimension vector of the module at vertex v is $\alpha_{i(v)} + \cdots + \alpha_{j(v)}$.

Recall the function $D : \mathbb{Z}R^+ \to \mathbb{Z}^t$ associated in §4 to a quiver and a directed partition. Consider now our special case where $\Gamma = \Gamma_{\mathbf{k}}$, and where the directed partition is associated to an arbitrary quiver Q of type A as above. In this case it is easy to see that the sequence $\mathbf{i}$ is given by

$$\mathbf{i}(Q) = (l_1 \searrow 1)(l_2 \searrow 1)\cdots(l_a \searrow 1)(n \searrow 1)(n \searrow n+1-r_b)(n \searrow n+1-r_{b-1})\cdots(n \searrow n+1-r_1),$$

a reduced expression for the longest word in the Weyl group compatible with Q. Here, each of the bracketed parts of $\mathbf{i}(Q)$ arises from a part of the directed partition associated to Q.

In this special case, we can easily prove the invertibility of $D_Q^{\Gamma_{\mathbf{k}}}$.

Lemma 4.5 *Let $D_Q^{\Gamma_{\mathbf{k}}}$ be the function associated to the directed partition of $A(\Gamma_{\mathbf{k}})$ corresponding to an arbitary quiver Q of type A. Then the function $D_Q^{\Gamma_{\mathbf{k}}}$ is an invertible map from $\mathbb{N}^N$ to $\mathbb{N}^N$ with inverse function $E_Q^{\Gamma_{\mathbf{k}}} = (D_Q^{\Gamma_{\mathbf{k}}})^{-1}$. Moreover, the components c_α of $\mathbf{c} = (c_\alpha)_\alpha = E_Q^{\Gamma_{\mathbf{k}}}(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{N}^N$ and $\alpha \in R^+$ are of the form $c_\alpha = a_k - a_l$ or $c_\alpha = a_k$.*

Proof: Recall from §4 that we can write

$$\mathbf{i} = (i_1^1 \ldots i_{t_1}^1 \;\ldots\; i_1^s \ldots i_{t_s}^s)$$

and that $D_Q^{\Gamma_{\mathbf{k}}}(\mathbf{c})$ is the tuple $\mathbf{a} = (a_j^p)$ given by:

$$a_j^p = \sum_{\alpha \in I_p,\ i_j^p \in \alpha} c_\alpha.$$

In our case, it is easily seen from the definition of the directed partition that we can write

$$I_p = \{\alpha_1^p, \ldots, \alpha_{t_p'}^p\},$$

such that the root α_u^p is of length u, and $\alpha_u^p = \alpha_{u-1}^p + \alpha_{j_u}$ for some simple root α_{j_u} (we formally set $\alpha_0^p = 0$). It follows that $\alpha_u^p = \alpha_{j_1} + \ldots + \alpha_{j_u}$, which in particular implies $t_p' = t_p$. We also see that

$$\{i_1^p, \ldots, i_{t_p}^p\} = \{j_1, \ldots, j_{t_p}\}.$$

It follows that $D_Q^{\Gamma_{\mathbf{k}}} : \mathbb{N}^N \to \mathbb{N}^N$ as claimed. Denote by σ the permutation defined by $j_{\sigma(u)} = i_u^p$. Then we can compute a_u^p as:

$$a_u^p = \sum_{\alpha \in I_p,\ i_j^p \in \alpha} c_\alpha = c_{\alpha_{\sigma u}^p} + c_{\alpha_{\sigma u+1}^p} \ldots + c_{\alpha_{t_p}^p}.$$

It follows that

$$c_{\alpha_u^p} = a_{\sigma^{-1}u}^p - a_{\sigma^{-1}(u+1)}^p \text{ if } u \neq t_p, \text{ and } c_{\alpha_u^p} = a_{\sigma^{-1}u}^p \text{ otherwise.}$$

This proves the claimed properties of $D_Q^{\Gamma_{\mathbf{k}}}$.

Remark 4.6 It follows from the proof of Lemma 4.5 that, in the situation of the Lemma, we have that $D_Q^{\Gamma_{\mathbf{k}}}(\mathbf{c})$ is the tuple $(a_j)_{j=1,2,\ldots,N}$ such that if the root α^j lies in I_p, then

$$a_j = \sum_{\alpha \in I_p, i_j \in \alpha} c_\alpha.$$

5 The Lusztig cones

Lusztig [15] introduced certain regions which, in low rank, give rise to canonical basis elements of a particularly simple form. The *Lusztig cone* corresponding to a reduced expression $\mathbf{i}$ for w_0 is defined to be the set of points $\mathbf{a} \in \mathbb{N}^N$ satisfying the following inequalities:

For every pair $s, s' \in [1, N]$ with $s < s'$, $i_s = i_{s'} = i$ and $i_p \neq i$ whenever $s < p < s'$, we have

$$(\sum_p a_p) - a_s - a_{s'} \geq 0, \tag{2}$$

where the sum is over all p with $s < p < s'$ such that i_p is joined to i by an edge in the Dynkin diagram. We shall denote this cone by $L_{st}(\mathbf{i})$ (as we shall regard it as a set of strings of $\mathbf{B}$ in direction $\mathbf{i}$). We shall shorten the notation $L_{st}(\mathbf{i}(Q))$ to $L_{st}(Q)$.

It was shown by Lusztig [15] that, in type A_n, if $\mathbf{a} \in L_{st}(\mathbf{i})$ then the monomial $F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_N}^{(a_N)}$ lies in the canonical basis $\mathbf{B}$, provided $n = 1, 2, 3$. The first author [17] showed that this remains true if $n = 4$, but it is false for $n \geq 5$ by [17], [21]. The Lusztig cones have been studied in the papers [7], [16] and [18] in type A for every reduced expression $\mathbf{i}$ for the longest word, and have also been studied by Bedard in [2] for arbitrary finite (simply-laced) type for reduced expressions compatible with a quiver whose underlying graph is the Dynkin diagram. Bedard describes these vectors using the Auslander-Reiten quiver of the quiver and homological algebra, showing they are closely connected to the representation theory of the quiver.

6 The Degeneration Cones

We define the cone $C_{PBW}(Q) \subseteq \mathbb{N}^N$ corresponding to a quiver Q to be the set of points $\mathbf{c} = (c_{ij})$ satisfying the inequalities (C1) and (C2) below. We define the *degeneration cone* corresponding to Q to be the cone $C_{st}(Q) = D_Q^{\Gamma_{\mathbf{k}}}(C_{PBW}(Q))$.

Define a *component* of Q to be a maximal full subgraph X of Q subject to the condition that all of the arrows of X point in the same direction. Call X a *left* (respectively, *right*) component of Q if its arrows all point to the left (respectively, right). If X is a component of Q (left or right), let $S_z(X)$ (respectively, $T_z(X)$) denote the part of the slice S_z (respectively, T_z) corresponding to X.

Example: Consider the example with $Q = RLRL$ given above. Then Q has 4 components, each containing one edge. Two are right components, and two are left components. For each component X, we indicate the subsets of slices, $T_z(X)$, in $A(\Gamma_{\mathbf{k}})$; see Figure 6. In each case, the numbers z denote elements of the sets $T_z(X)$, and the empty circles denote elements not in any subset $T_z(X)$.

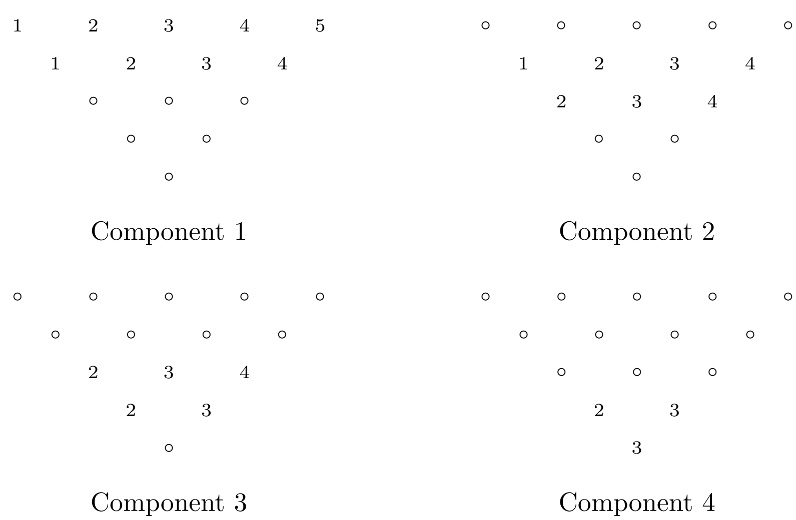

Figure 6: The subsets $T_z(X)$ of $A(\Gamma_{\mathbf{k}})$ corresponding to components of the quiver $RLRL$

Firstly, let X be a left component of Q, and suppose T_z, T_{z+1} are consecutive slices in $A(\Gamma_{\mathbf{k}})$ such that $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$ (i.e. the subsets $S_z(X)$ and $S_{z+1}(X)$ of $\mathbb{Z}\Gamma_{\mathbf{k}}$ are contained entirely inside $A(\Gamma_{\mathbf{k}})$). We can order $T_z(X)$ linearly by the second entry of the vertices appearing in it (recall from §4 that each vertex is regarded as a pair (r, i), where $r \in \mathbb{N}$ and i is a vertex of $\Gamma_{\mathbf{k}}$, i.e. an element of $\{1, 2, \ldots, n\}$). Write $T_z(X) = \{x_1, x_2, \ldots x_k\}$. Similarly, we can write $T_{z+1}(X) = \{y_1, y_2, \ldots, y_k\}$. For $t = 1 \ldots k$, let $i_t = i(x_t)$, and let $j_t = j(x_t)$. Note that $i(y_t) = i_t + 1$ and $j(y_t) = j_t + 1$. Then inequalities (C1) are:

(C1) $\sum_{r=a}^{k} c_{i_r, j_r} \geq \sum_{r=a}^{k} c_{i_r+1, j_r+1}$ for $a = 1 \ldots k$.

Secondly, let X be a right component of Q, and suppose T_z, T_{z+1} are consecutive slices in $A(\Gamma_{\mathbf{k}})$. We can order $T_z(X)$ linearly by the second component of the vertices appearing in it; write $T_z(X) = \{x_1, x_2, \ldots x_k\}$. Similarly, we can write $T_{z+1}(X) = \{y_1, y_2, \ldots, y_l\}$, where $k \geq l$, since X is a right component. For $t = 1 \ldots k$, let $i_t = i(x_t)$, and let $j_t = j(x_t)$. Note that $i(y_t) = i_t + 1$ and $j(y_t) = j_t + 1$ for $t = 1 \ldots l$. Then inequalities (C2) are:

(C2) $c_{i_r, j_r} \geq c_{i_r+1, j_r+1}$ for $r = 1 \ldots l - 1$.

The cone $C_{PBW}(Q)$ is defined to be the set of $(c_{ij}) \in \mathbb{N}^N$ satisfying all of the inequalities (C1) and (C2).

Example: In our running example (see the start of this section), the inequalities (C1) and (C2) are given as follows.

Component 1: $c_{11} \geq c_{22} \geq c_{33} \geq c_{44} \geq c_{55}$.
Component 2: $c_{13} \geq c_{24} \geq c_{35}$ and $c_{13} + c_{23} \geq c_{24} + c_{34} \geq c_{35} + c_{45}$.
Components 3 and 4 give rise to no inequalities in this example.

7 The PBW-version of the Lusztig cones

We now define a cone $L_{PBW}(Q)$. Eventually, we shall see that $S^{\mathbf{k}}_{\mathbf{i}(Q)}(L_{st}(Q)) = L_{PBW}(Q)$ (see Theorem 10.10), so this cone can be regarded as a PBW-version of the Lusztig cone $L_{st}(Q)$.

Firstly, let X be a left component of Q, and suppose T_z, T_{z+1} are consecutive slices in $A(\Gamma_{\mathbf{k}})$. We can order $T_z(X)$ linearly by the second entry of the vertices appearing in it; write $T_z(X) = \{x_1, x_2, \ldots x_k\}$. Similarly, we can write $T_{z+1}(X) = \{y_1, y_2, \ldots, y_l\}$, where $k \leq l$ (since X is a left component). For $t = 1 \ldots k$, let $i_t = i(x_t)$, and let $j_t = j(x_t)$. Note that $i(y_t) = i_t + 1$ and $j(y_t) = j_t + 1$ for $1 \leq t \leq k$. Then the defining inequalities of $L_{PBW}(Q)$ are

(L1) If $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$ then $\sum_{r=1}^{k} c_{i_r,j_r} \geq \sum_{r=1}^{k} c_{i_r+1,j_r+1}$, and
(L2) $c_{i_r,j_r} \leq c_{i_r+1,j_r+1}$ for $r = 2 \ldots k-1$.

Secondly, let X be a right component of Q, and suppose T_z, T_{z+1} are consecutive slices in $A(\Gamma_{\mathbf{k}})$. We can order $T_z(X)$ linearly by the second entry of the vertices appearing in it; write $T_z(X) = (x_1, x_2, \ldots x_k)$. Similarly, we can write $T_{z+1}(X) = (y_1, y_2, \ldots, y_l)$, where $k \geq l$ (since X is a right component). For $t = 1 \ldots l$, let $i_t = i(x_t)$, and let $j_t = j(x_t)$. Note that $i(y_t) = i_t + 1$ and $j(y_t) = j_t + 1$ for $1 \leq t \leq l$. Then the defining inequalities of $L_{PBW}(Q)$ are

(L3) If $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$ then $\sum_{r=1}^{k} c_{i_r,j_r} \leq \sum_{r=1}^{k} c_{i_r+1,j_r+1}$, and
(L4) $c_{i_r,j_r} \geq c_{i_r+1,j_r+1}$ for $r = 2 \ldots l-1$.

Finally, if X is a left component and is the leftmost component of Q, then we have:

(L5) $c_{i_1,j_1} \leq c_{i_1+1,j_1+1}$,

and if X is a right component and is the leftmost component of Q, then we have:

(L6) $c_{i_1,j_1} \geq c_{i_1+1,j_1+1}$.

8 The relationship between the Lusztig cones and the degeneration cones in the PBW-parametrization

We show in this section, that, for an arbitrary quiver Q of type A_n, we have $L_{PBW}(Q) \subseteq C_{PBW}(Q)$. Firstly, we show that certain inequalities hold in L_{PBW}.

Lemma 8.1 *Suppose* $\mathbf{c} \in L_{PBW}(Q)$. *Then the following inequalities hold:*

If X *is a left component of* Q, *then (with the notation above),*

(I) $c_{i_1,j_1} \leq c_{i_1+1,j_1+1}$, *and*
(II) $c_{i_k,j_k} \geq c_{i_k+1,j_k+1}$.

If X *is a right component, then*

(III) $c_{i_1,j_1} \geq c_{i_1+1,j_1+1}$, *and*
(IV) $c_{i_k,j_k} \leq c_{i_k+1,j_k+1}$.

The inequalities (III) and (IV) only hold if they make sense — i.e. if $k = l$ (which is the case when $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$).

Proof: Let $\mathbf{c} \in L_{PBW}(Q)$. Suppose first that the leftmost component X_1 of Q is a left component. The inequality (I) for X_1 is the defining inequality (L5) of $L_{PBW}(Q)$. Suppose that $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$. Then we have, from (L1) and (L2), that:

(a) $\sum_{r=1}^{k} c_{i_r,j_r} \geq \sum_{r=1}^{k} c_{i_r+1,j_r+1}$, and
(b) $c_{i_r,j_r} \leq c_{i_r+1,j_r+1}$ for $r = 2 \ldots k-1$.

Since (I) and (b) hold, if (II) were false, (a) would be false — a contradiction. Therefore (II) holds for X_1.

A similar argument shows that the inequalities (III) and (IV) hold for X_1 in the case when X_1 is a right component.

Suppose we are still in the case where X_1 is a left component. Let X_2 be the component immediately to the right of X_1 (if it exists); this must be a right component. Note that the inequality (II) for X_1 is inequality (III) for X_2. We can then argue as above for X_2 to deduce (IV) for X_2.

We can argue similarly in the case where X_1 is a right component. In this way we can deduce the relevant inequalities for all components of Q, by induction on the number of the component, starting from the left, and we are done. □

Proposition 8.2 *Let Q be an arbitrary quiver of type A_n, and let $L_{PBW}(Q)$ and $C_{PBW}(Q)$ be the cones as above. Then $L_{PBW}(Q) \subseteq C_{PBW}(Q)$.*

Proof: Let $\mathbf{c} \in L_{PBW}(Q)$. Suppose that X is any left component of Q, and that $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$. The following inequalities hold:

(c) $\sum_{r=1}^{k} c_{i_r,j_r} \geq \sum_{r=1}^{k} c_{i_r+1,j_r+1}$, and
(d) $c_{i_r,j_r} \leq c_{i_r+1,j_r+1}$ for $r = 1 \ldots k-1$ and $c_{i_k,j_k} \geq c_{i_k+1,j_k+1}$.

Inequality (c) comes from (L1), and inequalities (d) come from (L2) and Lemma 8.1(II). We show that the inequalities (C1) all hold. We already have the inequality $c_{i_k,j_k} \geq c_{i_k+1,j_k+1}$, from (d). If we also had $c_{i_{k-1},j_{k-1}} + c_{i_k,j_k} \leq c_{i_{k-1}+1,j_{k-1}+1} + c_{i_k+1,j_k+1}$, then this, together with the inequalities $c_{i_r,j_r} \leq c_{i_r+1,j_r+1}$ for $r = 1, \ldots, k-2$ would give $\sum_{r=1}^{k} c_{i_r,j_r} \leq \sum_{r=1}^{k} c_{i_r+1,j_r+1}$, contradicting (c). Similarly, if we had $\sum_{r=a}^{k} c_{i_r,j_r} \leq \sum_{r=a}^{k} c_{i_r+1,j_r+1}$, for some $1 \leq a \leq k-2$, the inequalities in (d) would give us $\sum_{r=1}^{k} c_{i_r,j_r} \leq \sum_{r=a}^{k} c_{i_r+1,j_r+1}$, contradicting (c). Hence $\sum_{r=a}^{k} c_{i_r,j_r} \geq \sum_{r=a}^{k} c_{i_r+1,j_r+1}$, for any $1 \leq a \leq k-1$, and we see that the defining inequalities (C1) of $C_{PBW}(Q)$ are satisfied.

Suppose next that X is any right component of Q. Then the following inequalities hold, using (L4) and Lemma 8.1(III).

(e) $c_{i_r,j_r} \geq c_{i_r+1,j_r+1}$ for $r = 1 \ldots l-1$.

In this case, (e) contains all of the defining inequalities (C2) of $C_{PBW}(Q)$. We thus see that $L_{PBW}(Q) \subseteq C_{PBW}(Q)$. □

9 The image of the Lusztig cone $L_{st}(Q)$ under $E_Q^{\Gamma_{\mathbf{k}}}$ is $L_{PBW}(Q)$

We show in this section that we have $E_Q^{\Gamma_{\mathbf{k}}}(L_{st}(Q)) = L_{PBW}(Q)$, i.e. that $E_Q^{\Gamma_{\mathbf{k}}}$ takes the Lusztig cone to the cone defined in §7. We shall see in §10 (Theorem 10.10) that $S^{\mathbf{k}}_{\mathbf{i}(Q)}(L_{st}(Q)) = L_{PBW}(Q)$, since we shall see that $S^{\mathbf{k}}_{\mathbf{i}(Q)}$ and $E_Q^{\Gamma_{\mathbf{k}}}$ are identical on $L_{st}(Q)$.

We suppose that $a = (a_1, a_2, \ldots, a_N) \in L_{st}(Q)$, and let $\mathbf{c} = (c_{ij}) = E_Q^{\Gamma_{\mathbf{k}}}(\mathbf{a})$. We will show that $\mathbf{a} \in L_{st}(Q)$ if and only if $\mathbf{c} \in L_{PBW}(Q)$. We translate the linear inequalities (2) defining $L_{st}(Q)$ using the linear function $E_Q^{\Gamma_{\mathbf{k}}}$, and show that they become the defining inequalities of $L_{PBW}(Q)$. We also show that $a_i \geq 0$ for all i if and only if $c_{ij} \geq 0$ for all $1 \leq i \leq j \leq n$. Since $E_Q^{\Gamma_{\mathbf{k}}}$ is a linear function, it will follow that $E_Q^{\Gamma_{\mathbf{k}}}(L_{st}(Q)) = L_{PBW}(Q)$.

Recall that:

$$\mathbf{i}(Q) = (l_1 \searrow 1)(l_2 \searrow 1)\cdots(l_a \searrow 1)(n \searrow 1)(n \searrow n+1-r_b)(n \searrow n+1-r_{b-1})\cdots(n \searrow n+1-r_1)$$

is compatible with Q (see the end of §3).

The inequalities defining $L_{st}(Q)$ arise from pairs of equal simple reflections occurring in $\mathbf{i}(Q)$. Note that, for the first $a+1$ factors appearing in the above, each factor is always contained in the one immediately to the right, and that, for the last $b+1$ factors, each factor is always contained in the one immediately to the left. It is clear that the defining inequalities for $L_{st}(Q)$ always arise from such pairs of factors. To make the notation clearer, let us define $l_{a+1} = r_{b+1} = n$.

Let us consider such a pair, $(l_p \searrow 1)(l_{p+1} \searrow 1)$, and suppose that $1 \leq s \leq l_p$, so s occurs both in $(l_p \searrow 1)$ and in $(l_{p+1} \searrow 1)$. Suppose s appears in position t of $\mathbf{i}$ in the first factor; it then must appear in the second factor — suppose that this is in position u in the second factor. Let us first assume that $s > 1$. The defining inequality of $L_{st}(Q)$ arising from this pair of s's is:

$$a_{t+1} + a_{u-1} \geq a_t + a_u. \tag{3}$$

We can rewrite this as:

$$a_t - a_{t+1} \leq a_{u-1} - a_u. \tag{4}$$

Since $D_Q^{\Gamma_{\mathbf{k}}}(\mathbf{c}) = \mathbf{a}$, we have (by Remark 4.6):

$$\begin{aligned}
a_t &= \sum_{\alpha \in T_{n+1-p},\ s \in \alpha} c_\alpha \\
a_{t+1} &= \sum_{\alpha \in T_{n+1-p}),\ s-1 \in \alpha} c_\alpha \\
a_{u-1} &= \sum_{\alpha \in T_{n-p},\ s+1 \in \alpha} c_\alpha \\
a_u &= \sum_{\alpha \in T_{n-p},\ s \in \alpha} c_\alpha
\end{aligned}$$

Suppose first, that $s \leq p$. Then $s-1 \in \alpha$ implies that $s \in \alpha$ for $\alpha \in T_{n+1-p}$, and $s \in \alpha$ implies that $s+1 \in \alpha$ for $\alpha \in T_{n-p}$, as $s+1 \leq p+1$ also. Thus the inequality (4) becomes:

$$\sum_{\alpha \in T_{n+1-p}, s \in \alpha, s-1 \notin \alpha} c_\alpha \leq \sum_{\alpha \in T_{n-p}, s+1 \in \alpha, s \notin \alpha} c_\alpha.$$

This is an inequality of type (L2) or (L3) for $L_{PBW}(Q)$, depending on what s is.

Next suppose that $s = 1$. Then the defining inequality (4) is the same, except that the term a_{t+1} does not appear. So we have $a_t \leq a_{u-1} - a_u$. This translates to

$$\sum_{\alpha \in T_{n+1-p}, 1 \in \alpha} c_\alpha \leq \sum_{\alpha \in T_{n-p}, 2 \in \alpha, 1 \notin \alpha} c_\alpha,$$

and we again get an inequality of type (L2) or (L3) for $L_{PBW}(Q)$.

Suppose next, that $s > p$. Then $s \in \alpha$ implies that $s-1 \in \alpha$ for $\alpha \in T_{n+1-p}$, and $s+1 \in \alpha$ implies that $s \in \alpha$ for $\alpha \in T_{n-p}$, as $s+1 \leq p+1$ also. We rewrite inequality (4) as $a_{t+1} - a_t \geq a_u - a_{u-1}$, and we have:

$$\sum_{\alpha \in T_{n+1-p}, s-1 \in \alpha, s \notin \alpha} c_\alpha \geq \sum_{\alpha \in T_{n-p}, s \in \alpha, s+1 \notin \alpha} c_\alpha.$$

It is clear that this is an inequality of type (L1) or (L4) for $L_{PBW}(Q)$, depending on what s is. We do not get a boundary case to consider in this case.

For consecutive factors appearing after $(n \searrow 1)$, a similar argument shows that they give defining inequalities for $L_{PBW}(Q)$. It is easy to see that, if all possible pairs of consecutive factors are taken, we get precisely the defining inequalities for $L_{PBW}(Q)$.

Thus, the defining inequalities for $L_{st}(Q)$ correspond, under $E_Q^{\Gamma_{\mathbf{k}}}$, to the defining inequalities for $L_{PBW}(Q)$. It remains to check that $a_i \geq 0$ for all i if and only if $c_{ij} \geq 0$ for all $1 \leq i \leq j \leq n$ (if $E_Q^{\Gamma_{\mathbf{k}}}(\mathbf{a}) = \mathbf{c}$).

By the description of the coordinates of the function $D_Q^{\Gamma_{\mathbf{k}}} = (E_Q^{\Gamma_{\mathbf{k}}})^{-1}$ (see Remark 4.6) as nonnegative combinations of the coordinate entries of $\mathbf{c}$, it is clear that if all $c_{ij} \geq 0$ then all $a_i \geq 0$. Now, suppose that all $a_i \geq 0$. Then we know by [16, 4.1] that, if $\mathbf{a}$ is labelled $(a_\alpha)_{\alpha\in\Phi^+}$ according to the ordering on the positive roots induced by $\mathbf{i}(Q)$, then, whenever $\alpha, \beta \in \Phi^+$ and $\alpha \geq \beta$ (i.e. $\alpha = \beta$ plus a nonnegative combination of simple roots), we have $a_\alpha \geq a_\beta$. By the description of the function $E_Q^{\Gamma_\mathbf{k}}$, we know that the entries on c_{ij}, regarded as functions of the coordinates of $\mathbf{a}$, are all of the form $a_\alpha - a_\beta$ where $\alpha = \beta + \gamma$, for some simple root γ. It follows that $c_{ij} \geq 0$ for all $1 \leq i \leq j \leq n$.

We have proved:

Theorem 9.1 $E_Q^{\Gamma_\mathbf{k}}(L_{st}(Q)) = L_{PBW}(Q)$.

Example: We return to our running example, with $Q = RLRL$ in type A_5. Let $\mathbf{a} \in \mathbb{N}^N$. Then $E_Q^{\Gamma_\mathbf{k}}(\mathbf{a}) = (\mathbf{c}) = (c_{ij})$ where $\mathbf{c}$ is given by the diagram in Figure 7.

$$a_2 - a_1 \qquad a_5 - a_4 \qquad a_9 - a_8 \qquad a_{13} - a_{12} \qquad a_{15}$$

$$a_1 \qquad a_4 - a_6 \qquad a_8 - a_{10} \qquad a_{12} - a_{14}$$

$$a_6 - a_3 \qquad a_{10} - a_7 \qquad a_{14}$$

$$a_3 \qquad a_7 - a_{11}$$

$$a_{11}$$

Figure 7: The function $S^{\mathbf{k}}_{\mathbf{i}(Q)}(\mathbf{a})$

We give below the correspondence between the defining inequalities of $L_{st}(Q)$ and those of $L_{PBW}(Q)$. Recall that $\mathbf{i}(Q) = (2, 1, 4, 3, 2, 1, 5, 4, 3, 2, 1, 5, 4, 3, 5)$.

Inequality of $L_{st}(Q)$	Inequality of $L_{PBW}(Q)$
$a_2 + a_4 \geq a_1 + a_5$	$c_{11} \geq c_{22}$
$a_5 \geq a_2 + a_6$	$c_{22} + c_{23} \geq c_{11} + c_{12}$
$a_4 + a_7 \geq a_3 + a_8$	$c_{23} + c_{13} \geq c_{34} + c_{24}$
$a_5 + a_8 \geq a_4 + a_9$	$c_{22} \geq c_{33}$
$a_6 + a_9 \geq a_5 + a_{10}$	$c_{33} + c_{34} \geq c_{22} + c_{23}$
$a_{10} \geq a_6 + a_{11}$	$c_{24} + c_{25} \geq c_{13} + c_{14}$
$a_8 \geq a_7 + a_{12}$	$c_{34} + c_{24} \geq c_{45} + c_{35}$
$a_9 + a_{12} \geq a_8 + a_{13}$	$c_{33} \geq c_{44}$
$a_{10} + a_{13} \geq a_9 + a_{14}$	$c_{44} + c_{45} \geq c_{33} + c_{34}$
$a_{13} \geq a_{12} + a_{15}$	$c_{44} \geq c_{55}$

10 Description of the function $S^{\mathbf{k}}_{\mathbf{i}(Q)}$

We now show that the function $S^{\mathbf{k}}_{\mathbf{i}(Q)}$ coincides with the function $E_Q^{\Gamma_{\mathbf{k}}} = (D_Q^{\Gamma_{\mathbf{k}}})^{-1}$ arising from representations of quivers, as described in §4, on the degeneration cone $C_{st}(Q)$. We first of all note that each edge of Q corresponds in a natural way to a slice of $A(\Gamma_{\mathbf{k}})$; such a slice generates the factor of $\mathbf{i}(Q)$ corresponding to this edge — see §3.

Suppose that $\mathbf{a} \in C_{st}(Q) = D_Q^{\Gamma_{\mathbf{k}}}(C_{PBW}(Q))$ (the degeneration cone). We will show that $\mathbf{a} \in X_{st}(Q)$ (i.e. the string cone $X_{st}(\mathbf{i}(Q))$) and that $S^{\mathbf{k}}_{\mathbf{i}(Q)}(\mathbf{a}) = E_Q^{\Gamma_{\mathbf{k}}}(\mathbf{a})$. It then follows that, as $E_Q^{\Gamma_{\mathbf{k}}}$ and $S^{\mathbf{k}}_{\mathbf{i}(Q)}$ are bijective, $(S^{\mathbf{k}}_{\mathbf{i}(Q)})^{-1}(\mathbf{c}) = \mathbf{a} = D_Q^{\Gamma_{\mathbf{k}}}(\mathbf{c})$ for any $\mathbf{c} \in C_{PBW}(Q)$. This is the result we would like to prove, and this section is mainly devoted to achieving this aim. In the following analysis, $\mathbf{a}$ shall denote an arbitrary element of $C_{st}(Q)$, and $\mathbf{c} = E_Q^{\Gamma_{\mathbf{k}}}(\mathbf{a})$.

In applying $S^{\mathbf{k}}_{\mathbf{i}(Q)}$ to $\mathbf{a} = (a_{ij})$, we need to compute $\mathbf{c}$ such that $\widetilde{F}_{i_1}^{a_1} \cdots \widetilde{F}_{i_N}^{a_N} \cdot 1 \equiv F_{\mathbf{k}}^{\mathbf{c}} \mod v\mathcal{L}'$, where $\mathbf{i} = \mathbf{i}(Q)$. Recall (see the end of §3) that

$$\mathbf{i}(Q) = (l_1 \searrow 1)(l_2 \searrow 1) \cdots (l_a \searrow 1)(n \searrow 1)(n \searrow n+1-r_b)(n \searrow n+1-r_{b-1}) \cdots (n \searrow n+1-r_1).$$

Let m be an edge of Q. Then in the Berenstein-Fomin-Zelevinsky arrangement corresponding to Q (see, for example, Figure 1), the line corresponding to m passes first through the lines corresponding to R's to the left of m, starting from the right, then the line from top left to bottom right, followed by the L's to the left of $E_Q^{\Gamma_{\mathbf{k}}}$ in Q, from left to right. This is how the reduced expression $\mathbf{i}(Q)$ is built up. Thus we apply the product

$$\widetilde{F}(l_1)\widetilde{F}(l_2)\cdots\widetilde{F}(l_a)\widetilde{F}(n)\widetilde{F}(r_b)\widetilde{F}(r_{b-1})\cdots\widetilde{F}(r_1) \tag{5}$$

to $1 = F_{\mathbf{k}}^{\mathbf{0}}$ where $\mathbf{0}$ denotes the zero vector, and $\widetilde{F}(m)$ is a monomial of Kashiwara operators defined as follows. Given $m \in [1,n]$, let d be maximal so that $l_d < m$ and let e be maximal so that $r_e < m$. Then we have that $d+e = m-1$, so $m-e-d = 1$, and $n-e-d = n-m+m-e-d = n+1-m$. Then, if edge m is an L, we have:

$$\widetilde{F}(m) = \widetilde{F}_m^{a_{r_e+1,m+1}}\widetilde{F}_{m-1}^{a_{r_{e-1}+1,m+1}}\cdots\widetilde{F}_{m-e+1}^{a_{r_1+1,m+1}}\widetilde{F}_{m-e}^{a_{1,m+1}}\widetilde{F}_{m-e-1}^{a_{l_1+1,m+1}}\widetilde{F}_{m-e-2}^{a_{l_2+1,m+1}}\cdots\widetilde{F}_{m-e-d}^{a_{l_d+1,m+1}},$$

and if edge m is an R, or $m = n$, we have:

$$\widetilde{F}(m) = \widetilde{F}_n^{a_{r_e+1,m+1}}\widetilde{F}_{n-1}^{a_{r_{e-1}+1,m+1}}\cdots\widetilde{F}_{n-e+1}^{a_{r_1+1,m+1}}\widetilde{F}_{n-e}^{a_{1,m+1}}\widetilde{F}_{n-e-1}^{a_{l_1+1,m+1}}\widetilde{F}_{n-e-2}^{a_{l_2+1,m+1}}\cdots\widetilde{F}_{n-e-d}^{a_{l_d+1,m+1}}.$$

We can regard the vector $\mathbf{c} \in \mathbb{N}^N$ as the vector $(c_\alpha)_{\alpha\in\Phi^+}$ (see§4.3). Each root $\alpha \in \Phi_+$ will lie in a slice T_z of Φ_+. Given $z \in \mathbb{N}$, let $\mathbf{c}(z)$ denote the vector $\mathbf{c}$ but with c_α set to zero for every α lying in a slice $T_{z'}$ with $z' < z$; thus $\mathbf{c}(n+1) = \mathbf{0}$ and $\mathbf{c}(1) = \mathbf{c}$.

For $i = 1,2,\ldots,n$, let $\widetilde{P}_i$ denote the ith product appearing in (5). Thus, for $i = 1,2,\ldots,a$, $\widetilde{P}_i = \widetilde{F}(l_i)$, $\widetilde{P}_{a+1} = \widetilde{F}(n)$, and for $i = a+2,\ldots,n$, $\widetilde{P}_i = \widetilde{F}(r_{a+2-i+b})$, and we have

$$\widetilde{P}_1\widetilde{P}_2\cdots\widetilde{P}_n = \widetilde{F}(l_1)\widetilde{F}(l_2)\cdots\widetilde{F}(l_a)\widetilde{F}(n)\widetilde{F}(r_b)\widetilde{F}(r_{b-1})\cdots\widetilde{F}(r_1).$$

We will show that, for $z = 1,2,\ldots,n$, $\widetilde{P}_z \cdot F_{\mathbf{k}}^{\mathbf{c(z+1)}} \equiv F_{\mathbf{k}}^{\mathbf{c(z)}} \mod v\mathcal{L}'$, from which it will follow that

$$\widetilde{P}_1\widetilde{P}_2\cdots\widetilde{P}_n\cdot 1 \equiv F_{\mathbf{k}}^{\mathbf{c}} \mod v\mathcal{L}',$$

as required.

We also need to show that $\mathbf{a} \in X_{st}(Q)$. We will use the following definiton: A monomial action $\widetilde{F}_{j_1}^{b_1}\widetilde{F}_{j_2}^{b_2}\cdots\widetilde{F}_{j_t}^{b_t}\cdot F_{\mathbf{k}}^{\mathbf{x}}$ is said to satisfy (STRING) provided that

$$\widetilde{E}_{j_u}\widetilde{F}_{j_{u+1}}^{b_{u+1}}\widetilde{F}_{j_{u+2}}^{b_{u+2}}\cdots\widetilde{F}_{j_t}^{b_t}\cdot F_{\mathbf{k}}^{\mathbf{x}} \equiv 0 \mod v\mathcal{L}',$$

for $u = 1,2,\ldots,t$. We will also use the description of the action of the $\widetilde{F}_i$'s on a PBW basis modulo $v\mathcal{L}$ as proved by the second author in [20]:

Proposition 10.1 *(Reineke) Suppose $\mathbf{c} = (c_{ij}) \in \mathbb{N}^N$. For each $1 \le i \le j \le n$, define*

$$f_{ij} = \sum_{k=1}^{i} c_{kj} - \sum_{k=1}^{i-1} c_{k,j-1}.$$

Let i_0 be maximal so that $f_{i_0j} = \max_i f_{ij}$. Then $\widetilde{F}_j$ increases c_{i_0j} by 1, decreases $c_{i_0,j-1}$ by 1 (unless $i_0 = j$, when this latter effect does not occur), and leaves the other c_{ij}'s unchanged.

It is easy to see, using Theorem 2.2(ii), that this implies the following description of the $\widetilde{E}_i$'s:

Proposition 10.2 *Suppose* $\mathbf{c} = (c_{ij}) \in \mathbb{N}^N$. *For each* $1 \leq i \leq j \leq n$, *define*

$$f_{ij} = \sum_{k=1}^{i} c_{kj} - \sum_{k=1}^{i-1} c_{k,j-1}.$$

Let i_0 *be minimal so that* $f_{i_0 j} = \max_i f_{ij}$. *Then* $\widetilde{E}_j$ *decreases* $c_{i_0 j}$ *by* 1, *inreases* $c_{i_0,j-1}$ *by* 1 *(unless* $i_0 = j$, *when this latter effect does not occur), and leaves the other* c_{ij}*'s unchanged, except, if* $c_{i_0 j} = 0$, *then it acts as zero modulo* $v\mathcal{L}$.

We will need the following technical Lemmas, describing the action of $\widetilde{F}_i$ and $\widetilde{E}_i$ in certain circumstances:

Lemma 10.3 *Fix* $1 \leq i \leq j \leq n$ *and* $s \in \mathbb{N}$. *Suppose the following hold for a triangle* $(c_{kl}) \in \mathbb{N}^N$:

(a) For $k = 1 \ldots i-1$, *we have* $\sum_{l=k+1}^{i} c_{lj} \geq \sum_{l=k}^{i-1} c_{l,j-1}$.
(b) For $k = i+1 \ldots j$, *we have* $s \leq \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{lj}$.

Then $\widetilde{F}_j^s$ *acts on* (c_{kl}) *by increasing* c_{ij} *by* s, *decreasing* $c_{i,j-1}$ *by* s, *and leaving the other* c_{kl}*'s unchanged.*

Proof: For $t = 0 \ldots s-1$, define a new triangle c^t_{kl} by

$$c^t_{kl} = \begin{cases} c_{ij} + t, & k = i, l = j \\ c_{i,j-1} - t, & k = i, l = j-1 \\ c_{kl}, & \text{otherwise.} \end{cases}$$

The Lemma clearly holds if for all $t = 0 \ldots i-1$, the index i_0 of Proposition 10.1 equals i. This translates into the following inequalities for the f_{kl}'s of Proposition 10.1:

$$f_{kj} \leq f_{ij} \text{ for } k = 1 \ldots i-1, \quad f_{kj} < f_{ij} \text{ for } k = i+1 \ldots j.$$

Using the definitions of (c^t_{kl}) and f_{kl}, this translates into the following conditions:

$$t \geq \sum_{l=k}^{i-1} c_{l,j-1} - \sum_{l=k+1}^{i} c_{lj} \text{ for } t = 0 \ldots s-1,\ k = 1 \ldots i-1,$$

which is equivalent to condition (a), and

$$t < \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{lj} \text{ for } t = 0 \ldots s-1,\ k = i+1 \ldots j,$$

which is equivalent to condition (b). □

Lemma 10.4 *Fix $1 \leq i \leq j \leq n$ and $s \in \mathbb{N}$. Suppose the following hold for a triangle $(c_{kl}) \in \mathbb{N}^N$:*

(a) For $k = 1 \dots i$, we have $c_{kj} = 0$.
(b) For $k = i+1 \dots j$, we have $s \leq \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{lj}$.

Then for $\widetilde{s} := s + \sum_{k=1}^{i-1} c_{k,j-1}$, we have:

$$\widetilde{F}_j^{\widetilde{s}}(c_{kl})_{kl} = \begin{cases} 0, & l = j-1, k < i \\ c_{k,j-1}, & l = j, k < i \\ c_{i,j-1} - s, & l = j-1, k = i \\ s, & l = j, k = i \\ c_{kl}, & \textit{otherwise.} \end{cases}$$

Proof: We proceed by induction on i. If $i = 1$, the claimed statement follows directly from Lemma 10.3. For arbitrary i, we set $i' = i - 1$, $s' = c_{i-1,j-1}$ and claim that we can apply the Lemma – which we assume to be already true for i' – with the same triangle (c_{kl}), but with (i, j, s) replaced by (i', j, s'). We thus have to show that the assumptions of the Lemma are satisfied:
Condition (a) is satisfied trivially. For $k = i+1 \dots j$, condition (b) gives $\sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{lj} \geq s \geq 0$, which (using $c_{ij} = 0$ by condition (a)) means

$$\sum_{l=i'}^{k-1} c_{l,j-1} - \sum_{l=i'+1}^{k} c_{lj} \geq c_{i-1,j-1} = s'.$$

For $k = i'+1 = i$, the desired condition is $s' \leq c_{i-1,j-1} - c_{ij}$, which clearly holds since $c_{ij} = 0$. Thus, we can apply the Lemma to (i', j, s'). Setting $(c'_{kl}) = \widetilde{F}_j^{\widetilde{s}'}(c_{kl})$, we find:

$$c'_{kl} = \left.\begin{cases} 0, & l = j-1, k < i' \\ c_{k,j-1}, & l = j, k < i' \\ c_{i',j-1} - s, & l = j-1, k = i' \\ s, & l = j, k = i' \\ c_{kl}, & \text{otherwise} \end{cases}\right\} = \begin{cases} 0, & l = j-1, k < i \\ c_{k,j-1}, & l = j, k < i \\ c_{kl}, & \text{otherwise.} \end{cases}$$

Since $\widetilde{s}' = \sum_{k=1}^{i-1} c_{k,j-1}$, it thus remains to show that $\widetilde{F}_j^s$ acts on (c'_{kl}) by decreasing $c_{i,j-1}$ by s, increasing $c_{ij} = 0$ by s, and leaving the rest of (c'_{kl}) unchanged. To prove this, we only have to check the assumptions of Lemma 10.3 for (c'_{kl}): for condition (a) of Lemma 10.3, this is trivial since

$c'_{k,j-1} = 0$ for $k < i$; for condition (b) of Lemma 10.3, we just use condition (b) of the present Lemma. Applying Lemma 10.3, we see that we are done. □

Lemma 10.5 *Fix $1 \leq i \leq j \leq n$ and $s \in \mathbb{N}$. Suppose the following hold for a triangle $(c_{kl}) \in \mathbb{N}^N$:*

(a) For $k = 1 \dots i$, we have $c_{kj} = 0$.
(b) For $k = i+1 \dots j$, we have $0 \leq \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{lj}$.

Then we have $\widetilde{E}_j(c_{kl}) = 0$.

Proof: For each $1 \leq p \leq j \leq n$, define

$$f_{pj} = \sum_{k=1}^{p} c_{kj} - \sum_{k=1}^{p-1} c_{k,j-1},$$

as in Proposition 10.2. By assumption (a), $f_{pj} = -\sum_{k=1}^{p-1} c_{k,j-1}$, for $p = 1, 2, \dots i$, and, using both assumptions,

$$\begin{aligned} f_{pj} &= \sum_{k=i+1}^{p} c_{kj} - \sum_{k=1}^{p-1} c_{k,j-1} \\ &= f_{ij} + \sum_{k=i+1}^{p} c_{kj} - \sum_{k=i}^{p-1} c_{k,j-1} \leq f_{ij} \end{aligned}$$

for $p = i+1, i+2, \dots, j$. Hence if p_0 is minimal so that $f_{p_0 j} = \max_p f_{pj}$, we must have $p_0 \leq i$. It follows from (a) that $c_{p_0 j} = 0$, so we conclude that $\widetilde{E}_j$ acts as zero modulo $v\mathcal{L}$. □

We start by considering the action of the monomial $\widetilde{F}(r_p)$ (where $1 \leq p \leq b+1$ — recall that $r_{b+1} = n$) on $\mathbf{c}(n+2-p)$. Recall that, for $p = 1, 2, \dots, b+1$, we have

$$\widetilde{F}(r_p) = \widetilde{F}_n^{a_{r_{p-1}+1,m+1}} \widetilde{F}_{n-1}^{a_{r_{p-2}+1,m+1}} \cdots \widetilde{F}_{n-p+2}^{a_{r_1+1,m+1}} \widetilde{F}_{n-p+1}^{a_{1,m+1}} \widetilde{F}_{n-p}^{a_{l_1+1,m+1}} \widetilde{F}_{n-p-1}^{a_{l_2+1,m+1}} \cdots \widetilde{F}_{n+1-p-d}^{a_{l_d+1,m+1}}.$$

The initial part of the computation is reasonably easy:

Lemma 10.6 *Suppose that $1 \leq p \leq b$. Let $\mathbf{d} \in \mathbb{N}^N$ be the triangle given by*

$$\begin{aligned} d_{n+1-p,n+1-p} &= a_{1,r_p+1} - a_{l_1+1,r_p+1}, \\ d_{n-p,n+1-p} &= a_{l_1+1,r_p+1} - a_{l_2+1,r_p+1}, \\ d_{n-p-1,n+1-p} &= a_{l_2+1,r_p+1} - a_{l_3+1,r_p+1}, \\ \vdots &= \vdots \\ d_{n+2-p-d,n+1-p} &= a_{l_{d-1}+1,r_p+1} - a_{l_d+1,r_p+1}, \\ d_{n+3-p-d,n+1-p} &= a_{l_d+1,r_p+1}. \end{aligned}$$

Then we have

$$\widetilde{F}_{n-p+1}^{a_{1,m+1}}\widetilde{F}_{n-p}^{a_{l_1+1,m+1}}\widetilde{F}_{n-p-1}^{a_{l_2+1,m+1}}\cdots\widetilde{F}_{n+1-p-d}^{a_{l_d+1,m+1}}\cdot F_{\mathbf{k}}^{\mathbf{c}(n+2-p)}\equiv F_{\mathbf{k}}^{\mathbf{c}(n+2-p)+\mathbf{d}}.$$

Furthermore, the action

$$\widetilde{F}_{n-p+1}^{a_{1,m+1}}\widetilde{F}_{n-p}^{a_{l_1+1,m+1}}\widetilde{F}_{n-p-1}^{a_{l_2+1,m+1}}\cdots\widetilde{F}_{n+1-p-d}^{a_{l_d+1,m+1}}\cdot F_{\mathbf{k}}^{\mathbf{c}(n+2-p)}$$

satisfies (STRING).

Proof: We first note that we have $a_{1,r+1}\geq a_{l_1+1,r+1}\geq a_{l_2+1,r+1}\geq\cdots\geq a_{l_b+1,r+1}$. These follow from the fact that $\mathbf{a}=D_Q^{\Gamma_{\mathbf{k}}}(\mathbf{c})$, since we know that all the coordinates of $\mathbf{c}$ are nonnegative: we use Remark 4.6. Let $l_0=0$. We know that, for $t=0,1,\ldots d-1$,

$$a_{l_t,r+1}=\sum_{\alpha\in T_{n+1-p},n-p+1-t\in\alpha}c_\alpha, \tag{6}$$

and

$$a_{l_{t+1},r+1}=\sum_{\alpha\in T_{n+1-p},n-p-t\in\alpha}c_\alpha. \tag{7}$$

Let $t\in\{0,1,\ldots,d-1\}$. Since $n-p+1-t\leq n-p+1$, it follows that if α_{n-p-t} appears in a root of slice p, so does $\alpha_{n-p+1-t}$. Thus, by equations (6) and (7), we have $a_{l_t,r+1}\geq a_{l_{t+1},r+1}$ as required.

Let $\mathbf{x}=\mathbf{c}(n+2-p)$. It is easy to see, using Proposition 10.1, that $\widetilde{F}_{n+1-r_p}^{a_{l_b+1,r_p+1}}$ sets $x_{n+1-r_p,n+1-r_p}$ to be a_{l_b+1,r_p+1}, and leaves the other x_{ij}'s unchanged. Similarly, it is easy to see, using Proposition 10.2, that $\widetilde{E}_{n+1-r_p}$ acts as zero on $\mathbf{x}$. In both cases this is because $x_{k,n+1-r_p}=0$ for $k=1,2,\ldots n+1-r$, and $x_{k,n-r_p}=0$ for $k=1,2,\ldots n-r_p$, so that all of the f_{ij} in Proposition 10.1 or Proposition 10.2 are zero.

Next, $\widetilde{F}_{n+2-r_p}^{a_{l_{b-1}+1,r_p+1}}$ sets $x_{n+1-r_p,n+1-r_p}=0$ and sets $x_{n+1-r_p,n+2-r_p}=a_{l_b+1,r_p+1}$. It also sets $x_{n+2-r_p,n+2-r_p}$ to be $a_{l_{b-1}+1}-a_{l_b+1,r_p+1}$. The rest of the $\widetilde{F}_i$'s act in a similar way, until we finally get the vector described in the Lemma. It is also easy to see that the given monomial action satisfies (STRING) using Proposition 10.2. □.

The next part of the computation, which involves computing $\widetilde{F}_n^{a_{r_{p-1}+1,m+1}}\widetilde{F}_{n-1}^{a_{r_{p-2}+1,m+1}}\cdots\widetilde{F}_{n-p+2}^{a_{r_1+1,m+1}}\cdot F_{\mathbf{k}}^{\mathbf{c}'(n-p+2)}$ (where $\mathbf{c}'(n-p+2)=\mathbf{c}(n-p+2)+\mathbf{d}$ where $\mathbf{d}$ is as in Lemma 10.6), is more involved. We need to consider what happens step-by-step, and to understand what is happening in detail along each slice $T_z=T_{n+1-p}$ of $A(\Gamma_{\mathbf{k}})$. Let $T_z=\{\beta_{z1},\beta_{z2},\ldots\beta_{zk_z}\}$, where the k_z roots in T_z are listed according to increasing height. Thus $\beta_{z1}=\alpha_{n+1-p}$.

Given q such that $1\leq q\leq p-1$, we define a vector $\mathbf{c}(p,q)$ as follows. (In order to simplify notation, we use the numbering arising from the edges of Q oriented to the right, rather than the usual slice numbering.) If α belongs to a slice numbered $n+2-p$ or greater (i.e. corresponding to one of the right-oriented edges numbered $r_1, r_2, \ldots, r_{p-1}$), then $c_\alpha(p,q) = c_\alpha$. We also set, for $i = 1, 2, \ldots, r_q - 1$, $c_{\beta_{zi}}(p,q) = a_{i,r_p+1} - a_{i+1,r_p+1}$, and set $c_{\beta_{r_q,i}}(p,q) = a_{r_q+1,r_p+1} - a_{l_s+1,r_p+1}$, where l_t is the number of the edge of the first L to the right of edge r_q. Suppose that $\beta_{zr_q} = \alpha_{ij}$. Then we also set $c_{i-1,j}(p,q) = a_{l_t+1,r_p+1} - a_{l_{t+1}+1,r_p+1}$, $c_{i-2,j}(p,q) = a_{l_{t+2}+1,r_p+1}, \ldots, c_{t+i+1-b,j}(p,q) = a_{l_{b-1}+1,r_p+1} - a_{l_b+1,r_p+1}$, and $c_{t+i-b,j}(p,q) = a_{l_b+1,r_p+1}$. We write $\mathbf{c}(p,0)$ for the vector $\mathbf{c}(n+2-p)+\mathbf{d}$ appearing in Lemma 10.6. The vector $\mathbf{c}(p,q-1)$ (which appears in the Lemma below) can be visualised as in Figure 8 (note that the case displayed is where edge $r_q - 1$ is oriented to the right). We have:

Lemma 10.7 *Suppose that $1 \leq p \leq b$ and that $1 \leq q \leq p-1$. We have that*

(a) $\widetilde{F}_{n+1-p+q}^{a_{r_q+1,r_p+1}} \cdot F_{\mathbf{k}}^{\mathbf{c}(p,q-1)} \equiv F_{\mathbf{k}}^{\mathbf{c}(p,q)} \mod v\mathcal{L}'$*, and that*
(b) $\widetilde{E}_{n+1-p+q}^{a_{r_q+1,r_p+1}} \cdot F_{\mathbf{k}}^{\mathbf{c}(p,q-1)} \equiv 0 \mod v\mathcal{L}'$.

Proof: Suppose that $\beta_{z,r_q-1} = \alpha_{ij}$. It is easy to check that $i = n+1-p-(t-1)$ and $j = n+1-p+q-1$, where l_t is the number of the edge of the first L to the right of edge $r_q - 1$. Let $s = a_{r_q+1,r_p+1} - a_{l_t+1,r_p+1}$, (Note that these notations were used for $\mathbf{c}(p,q)$ in the above, rather than $\mathbf{c}(p,q-1)$).

We apply Lemma 10.4 with the pair $i, j+1$ and s as above. We need to check the assumptions of the Lemma. It is clear from the definition of $\mathbf{c}(p,q-1)$ that $c_{k,j+1}(p,q-1) = 0$ for $k = 1, 2, \ldots, i$. Suppose that $i+1 \leq k \leq j+1$. We consider

$$\sum_{l=i}^{k-1} c_{l,j}(p,q-1) - \sum_{l=i+1}^{k} c_{l,j+1}(p,q-1).$$

This is a sum of terms, each of which is the sum of elements on one side on an inequality (C1) or (C2) (see §6), minus the corresponding sum of elements on the other side of the inequality. Since the vector $\mathbf{c} \in C_{PBW}(Q)$ satisfies inequalities (C1) and (C2), we have that:

$$\sum_{l=i}^{k-1} c_{l,j} - \sum_{l=i+1}^{k} c_{l,j+1} \geq 0 \tag{8}$$

(indeed, the inequalities of $C_{PBW}(Q)$ are designed to ensure this holds). By the definition of $\mathbf{c}(p,q-1)$, it is clear that for $l = i+1, \ldots j$, $c_{l,j} = c_{l,j}(p,q-1)$ and that for $l = i+1, \ldots, k$, $c_{l,j+1} = c_{l,j+1}(p,q-1)$. Furthermore, $c_{ij}(p,q-1) = a_{r_q-1+1,r_p+1} - a_{l_t+1-r_p+1} = s + (a_{r_q-1+1,r_p+1} - a_{r_q+1,r_p+1}) = s + c_{ij}$. It follows from Equation (8) that

$$\sum_{l=i}^{k-1} c_{l,j}(p,q-1) - \sum_{l=i+1}^{k} c_{l,j+1}(p,q-1) \geq s \tag{9}$$

as required in Lemma 10.4. Thus the conditions of the Lemma are all satisfied, and it applies. Let $\tilde{s} = s + \sum_{k=1}^{i-1} c_{k,j} = a_{r_q+1,r_p+1} - a_{l_t+1,r_p+1} + a_{l_t+1,r_p+1} - a_{l_{t+1}+1,r_p+1} + a_{l_{t+1}+1,r_p+1} - a_{l_{t+2}+1,r_p+1} +$

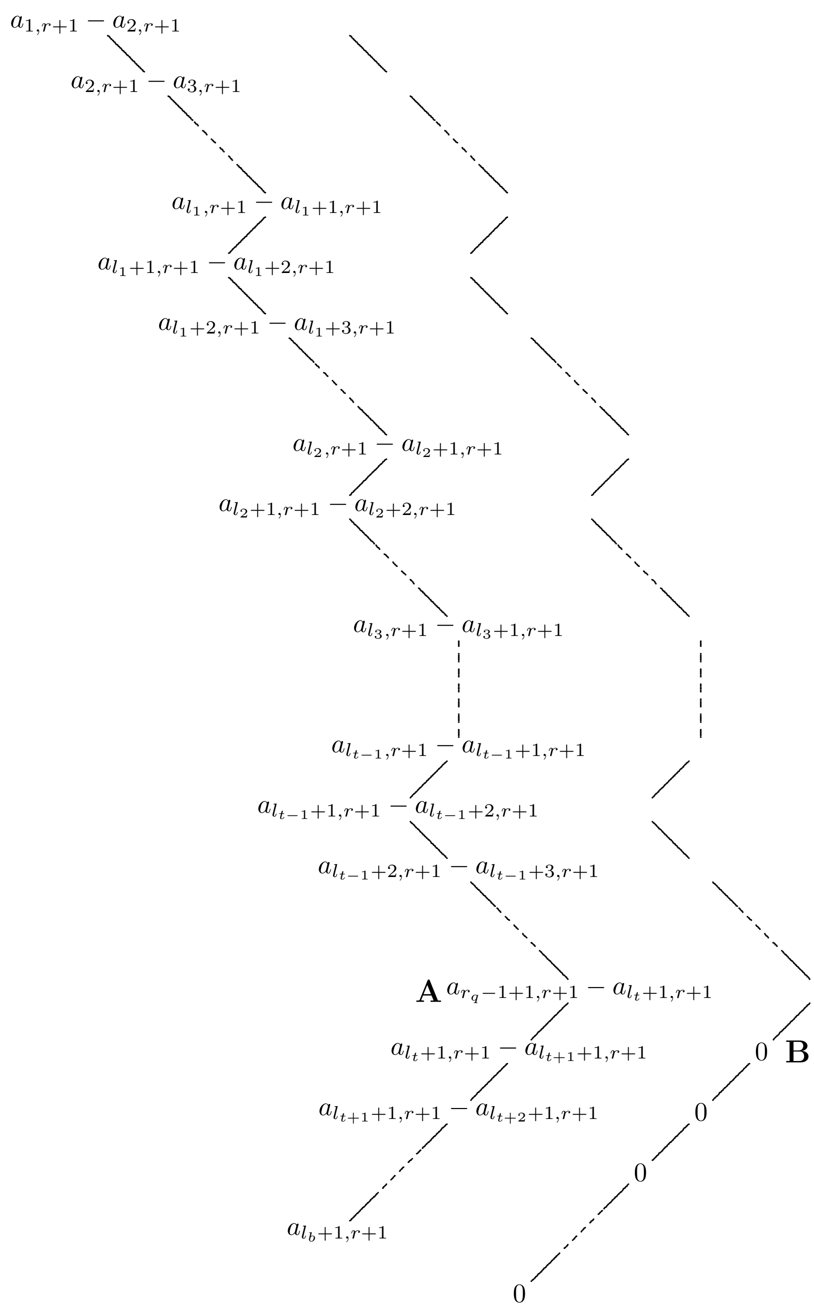

Figure 8: The vector $\mathbf{c}(p, q-1)$.

$\cdots + a_{l_{b-1}+1,r_p+1} - a_{l_b+1,r_p+1} + a_{l_b+1,r_p+1} = a_{r_q+1,r_p+1}$. Then it is easy to check that Lemma 10.4 tells us that $\widetilde{F}_{n+1-p+q}^{a_{r_q+1,r_p+1}} \cdot F_{\mathbf{k}}^{\mathbf{c}(p,q-1)} \equiv F_{\mathbf{k}}^{\mathbf{c}(p,q)} \mod v\mathcal{L}'$, giving us (a) above. This can easily be visualised in Figure 8; the value of $c_{ij}(p,q-1)$ is displayed at A in the diagram, and this is replaced by $a_{r_{q-1}+1,r_p+1} - a_{r_q+1,r_p+1}$; the value of $c_{i,j+1}(p,q-1)$ (which is zero) is displayed at B in the diagram, and this is replaced by $s = a_{r_q+1,r_p+1} - a_{l_t+1,r_p+1}$. Finally, the values on the diagonal below and to the left of A are all moved one step down and to the right (to places where the value is currently zero), forming the diagonal below and to the left of B. The new values in the diagonal below A are all zero.

In order to show (b), we apply Lemma 10.5 to $\mathbf{c}(p,q-1)$, again with the pair $i, j+1$. It is clear that the conditions of this Lemma are satisfied by $\mathbf{c}(p,q-1)$, since they are same as those in Lemma 10.4 but with s taken to be zero. The Lemma tells us that $\widetilde{E}_{n+1-p+q}^{a_{r_q+1,r_p+1}} \mathbf{c}(p,q-1) \equiv 0 \mod v\mathcal{L}'$, giving us (b) above. □

We can now prove our main result:

Theorem 10.8 *Let Q be an arbitrary quiver of type A_n, and suppose that $\mathbf{c} \in C_{PBW}(Q)$. Then*

$$(S_{\mathbf{i}(Q)}^{\mathbf{k}})^{-1}(\mathbf{c}) = D_Q^{\Gamma_{\mathbf{k}}}(\mathbf{c}).$$

Proof: Let $\mathbf{a} \in C_{st}(Q) = D_Q^{\Gamma_{\mathbf{k}}}(C_{PBW}(Q))$, and suppose $1 \leq p \leq b$. Lemma 10.6 tells us that

$$\widetilde{F}_{n-p+1}^{a_{1,m+1}} \widetilde{F}_{n-p}^{a_{l_1+1,m+1}} \widetilde{F}_{n-p-1}^{a_{l_2+1,m+1}} \cdots \widetilde{F}_{n+1-p-d}^{a_{l_d+1,m+1}} \cdot F_{\mathbf{k}}^{\mathbf{c}(n+2-p)} \equiv F_{\mathbf{k}}^{\mathbf{c}(p,0)}.$$

Lemma 10.7(a) tells us that, for $1 \leq q \leq p-1$, we have $\widetilde{F}_{n+1-p+q}^{a_{r_q+1,r_p+1}} \cdot F_{\mathbf{k}}^{\mathbf{c}(p,q-1)} \equiv F_{\mathbf{k}}^{\mathbf{c}(p,q)} \mod v\mathcal{L}'$. Repeated application of this second result (for $q = 1,2,\ldots,p-1$) tells us that $\widetilde{F}(r_p) \cdot F_{\mathbf{k}}^{\mathbf{c}(n+2-p)} \equiv F_{\mathbf{k}}^{\mathbf{c}(n+1-p)} \mod v\mathcal{L}'$, for $p = 1,2,\ldots,r_b$. It follows that $\widetilde{F}(n)\widetilde{F}(r_b)\widetilde{F}(r_{b-1}) \cdots \widetilde{F}(r_1) \cdot F_{\mathbf{k}}^{\mathbf{c}(n+1)} \equiv F_{\mathbf{k}}^{\mathbf{c}(a+1)} \mod v\mathcal{L}'$. A proof similar to the one given above can be used to show that $\widetilde{F}(l_i) \cdot F_{\mathbf{k}}^{\mathbf{c}(i+1)} \equiv F_{\mathbf{k}}^{\mathbf{c}(i)} \mod v\mathcal{L}'$ for $i = 1,2,\ldots,a$. It then follows that $\widetilde{F}(l_1)\widetilde{F}(l_2) \cdots \widetilde{F}(l_a)\widetilde{F}(n)\widetilde{F}(r_b)\widetilde{F}(r_{b-1}) \cdots \widetilde{F}(r_1) \cdot F_{\mathbf{k}}^{\mathbf{c}(n+1)} \equiv F_{\mathbf{k}}^{\mathbf{c}(1)} \equiv F_{\mathbf{k}}^{\mathbf{c}} \equiv F_{\mathbf{k}}^{E_Q^{\Gamma_{\mathbf{k}}}(\mathbf{a})} \mod v\mathcal{L}'$, as required. It also follows from Lemmas 10.6 and 10.7 (and a similar proof for the monomial $\widetilde{F}(l_1)\widetilde{F}(l_2) \cdots \widetilde{F}(l_a)$) that the monomial action $\widetilde{F}(l_1)\widetilde{F}(l_2) \cdots \widetilde{F}(l_a)\widetilde{F}(n)\widetilde{F}(r_b)\widetilde{F}(r_{b-1}) \cdots \widetilde{F}(r_1) \cdot F_{\mathbf{k}}^{\mathbf{0}} \equiv F_{\mathbf{k}}^{\mathbf{c}} \mod v\mathcal{L}'$ satisfies (STRING), from which it follows that $\mathbf{a} \in X_{st}(Q)$.

We thus have that, for all $\mathbf{a} \in C_{st}(Q) = D_Q^{\Gamma_{\mathbf{k}}}(C_{PBW}(Q))$, $S_{\mathbf{i}(Q)}^{\mathbf{k}}(\mathbf{a}) = E_Q^{\Gamma_{\mathbf{k}}}(\mathbf{a})$ and $\mathbf{a} \in X_{st}(Q)$. It follows that for all $\mathbf{c} \in C_{PBW}(Q)$, $(S_{\mathbf{i}(Q)}^{\mathbf{k}})^{-1}(\mathbf{c}) = D_Q^{\Gamma_{\mathbf{k}}}(\mathbf{c})$, as required. □

We conjecture that this theorem holds in the case when $\mathbf{k}$ is replaced by any reduced expression for w_0 compatible with a quiver.

We have the corollary:

Corollary 10.9 *Let Q be an arbitrary quiver of type A_n. Then*

$$(S^{\mathbf{k}}_{\mathbf{i}(Q)})^{-1}(C_{PBW}(Q)) = C_{st}(Q).$$

Proof: This follows from the Theorem and the definition of the degeneration cone $C_{st}(Q) = D_Q^{\Gamma_{\mathbf{k}}}(C_{PBW}(Q))$. $\square$

Finally, since (by Theorem 9.1), $E_Q^{\Gamma_{\mathbf{k}}}(L_{st}(Q)) = L_{PBW}(Q)$ and since $L_{PBW}(Q) \subseteq C_{PBW}(Q)$, we have, by Theorem 10.8, that $S^{\mathbf{k}}_{\mathbf{i}(Q)}$ and $E_Q^{\Gamma_{\mathbf{k}}}$ coincide on $L_{st}(Q)$. It follows that:

Theorem 10.10 *Let Q be an arbitrary quiver of type A_n. Then*

$$S^{\mathbf{k}}_{\mathbf{i}(Q)}(L_{st}(Q)) = L_{PBW}(Q).$$

Example: We return to our running example, with $Q = RLRL$ in type A_5. By Theorem 10.8, for $\mathbf{a}$ such that $E_Q^{\Gamma_{\mathbf{k}}}(\mathbf{a}) \in C_{PBW}(Q)$, we have $S^{\mathbf{k}}_{\mathbf{i}(Q)}(\mathbf{a}) = (\mathbf{c}) = (c_{ij})$ where c_{ij} is given in Figure 7.

Using the description of $D_Q^{\Gamma_{\mathbf{k}}}$, we have, for $\mathbf{c} \in L_{PBW}(Q)$, that $(S^{\mathbf{k}}_{\mathbf{i}(Q)})^{-1}(\mathbf{c}) = \mathbf{a}$, where $\mathbf{a} = (c_{12}, c_{11}+c_{12}, c_{14}, c_{23}+c_{13}+c_{14}, c_{22}+c_{23}+c_{13}+c_{14}, c_{13}+c_{14}, c_{25}+c_{15}, c_{34}+c_{24}+c_{25}+c_{15}, c_{33}+c_{34}+c_{24}+c_{25}+c_{15}, c_{24}+c_{25}+c_{15}, c_{15}, c_{45}+c_{35}, c_{44}+c_{45}+c_{35}, c_{35}, c_{55})$.

11 Connection with the canonical basis

Recall that $\alpha^1, \alpha^2, \ldots, \alpha^N$ is the ordering on R^+ induced by $\mathbf{k}$. Given a $k\Gamma_{\mathbf{k}}$-module $M = X_{\alpha^1}^{c_1} \oplus X_{\alpha^2}^{c_2} \oplus \cdots \oplus X_{\alpha_N}^{c_N}$, write the PBW-basis element $F_{\mathbf{k}}^{\mathbf{c}}$ as $F_{[M]}$. Let $D(M) = D_Q^{\Gamma_{\mathbf{k}}}(M) = (d_1, d_2, \ldots, d_N)$, and let $F_{\mathbf{i}(Q)}(D(M)) = F_{i_1}^{(d_1)} F_{i_2}^{(d_2)} \cdots F_{i_N}^{(d_N)}$. Let $\leq_{deg}$ denote the degeneration ordering on $k\Gamma_{\mathbf{k}}$-modules (see [4]). A special case of Lemma 4.5 of [22] states that:

Lemma 11.1 *(Reineke) Let M be a $k\Gamma_{\mathbf{k}}$-module as above. Then*

$$F_{\mathbf{i}(Q)}(D(M)) = \sum_{[N]} \gamma_N^M F_{[N]},$$

where $\gamma_N^M \in \mathbb{Z}[v, v^{-1}]$ is zero unless $M \leq_{deg} N$, and $\gamma_M^M = 1$.

which means that the monomials $F_{\mathbf{i}(Q)}(D(M))$, like the canonical basis, can naturally be indexed by the $k\Gamma_{\mathbf{k}}$-modules. For M a $k\Gamma_{\mathbf{k}}$-module, let $\widetilde{F}_{\mathbf{i}(Q)}(D(M))$ denote the monomial in Kashiwara operators obtained by replacing each divided power $F_i^{(a)}$ in $F_{\mathbf{i}(Q)}(D(M))$ with $\widetilde{F}_i^a$. Theorem 10.8 implies that:

Corollary 11.2 *Let M be a $k\Gamma_{\mathbf{k}}$-module as above. Then, if $\mathbf{c} \in C_{PBW}(Q)$, we have*

$$\widetilde{F}_{\mathbf{i}(Q)}(D(M)) \cdot 1 \equiv E_{[M]} \mod v\mathcal{L}'.$$

It thus follows that, if $B(M)$ denotes the canonical basis element corresponding to $E_{[M]}$, then

$$\widetilde{F}_{\mathbf{i}(Q)}(D(M)) \cdot 1 \equiv B(M) \mod v\mathcal{L}',$$

indicating a curious connection between the canonical basis and the monomial basis of the second author. The situation is even nicer in small cases. For example, suppose that $\mathbf{g}$ is of type A_4. Then it is known (see [7, 8.4]) that, for $\mathbf{c} \in L_{PBW}(Q)$, we have $\widetilde{F}_{\mathbf{i}(Q)}(D(M)) \cdot 1 \equiv F_{\mathbf{i}(Q)}(D(M)) \mod v\mathcal{L}'$, and that $F_{\mathbf{i}(Q)}(D(M)) \in \mathbf{B}$. Thus part of the monomial basis lies in the canonical basis in this case.

In general, it is at least known that the monomial basis is related to the canonical basis in the following way:

Proposition 11.3 *[23, Proposition 5.4] Let M be a $k\Gamma_{\mathbf{k}}$-module as above. Then*

$$F_{\mathbf{i}(Q)}(D(M)) = \sum_{[N]} \delta_N^M B(N),$$

where $\delta_N^M \in \mathbb{N}[v, v^{-1}]$ is zero unless $M \leq_{deg} N$, and $\delta_M^M = 1$.

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References

[1] M. Auslander, I. Reiten and S.O. Smalø. *Representation theory of Artin algebras.* Cambridge studies in advanced mathematics 36, Cambridge University Press, Cambridge, 1997.

[2] R. Bedard. On the spanning vectors of Lusztig cones. *Represent. Theory*, 4:306–329, 2000.

[3] A. Berenstein, S. Fomin, and A. Zelevinsky. Parametrizations of canonical bases and totally positive matrices. *Adv. Math.*, 122(1):49–149, 1996.

[4] K. Bongartz. On degenerations and extensions of finite dimensional modules. *Adv. Math.* 121(2):245–287, 1996.

[5] A. Berenstein and A. Zelevinsky. String bases for quantum groups of type A_r. *Adv. Soviet Math.*, 16, Part 1:51–89, 1993.

[6] A. Berenstein and A. Zelevinsky. Tensor product multiplicities, canonical bases and totally positive varieties. *Invent. Math.*, 143(1):77–128, 2001.

[7] R. W. Carter and R. J. Marsh. Regions of linearity, Lusztig cones and canonical basis elements for the quantized enveloping algebra of type A_4. *J. Algebra* 234(2):545–603, 2000.

[8] M. Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras. *Duke Math. J.*, 63(2):465–516, 1991.

[9] M. Kashiwara. The crystal base and Littelmann's refined Demazure character formula. *Duke Math. J.*, 71(3):839–858, 1993.

[10] B. Leclerc, M. Nazarov and J.-Y. Thibon. Induced representations of affine Hecke algebras and canonical bases of quantum groups, Preprint arXiv:math.QA/0011074, 2000.

[11] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3:447–498, 1990.

[12] G. Lusztig. Canonical bases arising from quantized enveloping algebras, II. *Prog. Theor. Phys. Sul.*, 102:175–201, 1990.

[13] G. Lusztig. Finite dimensional Hopf algebras arising from quantized universal enveloping algebras. *J. Amer. Math. Soc.*, 3:257–296, 1990.

[14] G. Lusztig. *Introduction to Quantum Groups*. Birkhäuser, Boston, 1993.

[15] G. Lusztig. Tight monomials in quantized enveloping algebras. In *Quantum deformations of algebras and their representations*, volume 7 of *Israel Math. Conf. Proc.*, pages 117–132, 1993.

[16] R. J. Marsh. The Lusztig cones of quantized enveloping algebras of type A. *J. Algebra* 244(1):59–75, 2001.

[17] R. J. Marsh. More tight monomials in quantized enveloping algebras. *J. Algebra*, 204(2):711–732, 1998.

[18] R. J. Marsh. Rectangle diagrams for the Lusztig cones of quantized enveloping algebras of type A. J. Algebra 249(2):377–401, 2002.

[19] T. Nakashima and A. Zelevinsky. Polyhedral realizations of crystal bases for quantized Kac-Moody algebras. *Adv. in Math.*, 131(1):253–278, 1997.

[20] M. Reineke. On the coloured graph structure of Lusztig's canonical basis. *Math. Ann.*, 307(4):705–723, 1997.

[21] M. Reineke. Monomials in canonical bases of quantum groups and quadratic forms. *J. Pure Appl. Algebra*, 157:301–309, 2001.

[22] M. Reineke. Feigin's map and monomial bases for quantized enveloping algebras. *Math. Z.*, 237:639-667, 2001.

[23] M. Reineke. Quivers, desingularizations, and canonical bases, *in* Studies in memory of Issai Schur, eds. A. Joseph, A. Melnikov and R. Rentschler, Progress in Mathematics 210, Birkhäuser, Boston, 2002.

[24] C.M. Ringel. Hall algebras and quantum groups. *Invent. Math.*, 101:583–592, 1990.

[25] N. Xi. Canonical basis for type A_3. *Comm. Algebra*, 27(11):5703–5710, 1999.

[26] A. Zelevinsky. Connected components of real double Bruhat cells. *Internat. Math. Res. Notices*, 21: 1131-1154, 2000.