CANONICAL BASES FOR FUNDAMENTAL MODULES OF QUANTIZED ENVELOPING ALGEBRAS OF TYPE A

1 Introduction

Let $\mathbf{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ of type A_n, and let U be the q-analogue of its universal enveloping algebra defined by Drinfel'd [4] and Jimbo [6]. According to [9, 3.5.6, 6.2.3 & 6.3.4], for each dominant weight λ in the weight lattice of $\mathbf{g}$ there is an irreducible, finite-dimensional highest weight U-module $V(\lambda)$ with highest weight λ. Kashiwara [7] and Lusztig [8] have independently shown the existence of a certain canonical basis $\mathbf{B}(\lambda)$ for $V(\lambda)$. Fix $r \in I = \{1, 2, \ldots, n\}$, let ω_r be the r-th fundamental weight and let V_r be the corresponding fundamental module $V(\omega_r)$ (we use the same numbering as [2, Planche I]). Let W^r be the set of distinguished left coset representatives in the Weyl group W of $\mathbf{g}$, with respect to the parabolic subgroup generated by all of the fundamental generators $s_1, s_2, \ldots, s_n$ of W except s_r.

In this paper we show that there is a natural correspondence between W^r and the canonical basis for V_r. To do this we shall examine W^r in detail and prove some properties of reduced expressions for its elements, which will enable us to define a natural map ϕ_r from W^r to U^-, the minus part of U, which takes elements of W^r to monomials. This is used to define a map ψ_r from W^r to V_r which is shown to be a bijection; its image turns out to be the canonical basis for V_r. We shall also exhibit a concrete realization for V_r, as a submodule of $V_1^{\otimes r}$. This construction provides us with a natural basis for V_r which we show to be the canonical basis, which enables us to exhibit the canonical basis in an explicit way and also to list the elements of W^r.

Note: Since completing this work, the author has learnt that in [9, 28.1], Lusztig proves a theorem which provides a description of the canonical basis for miniscule modules. It can be seen that the above bijection is a special case of this, since all of the fundamental modules in case A_n are miniscule (see [1, VIII,§7.3]). Here, we provide an alternative proof, and the canonical basis is made explicit. Along the way, various results about W^r and V_r are shown which are both interesting in their own right and give reasons why the correspondence described above should be true. We now go into more detail.

We use the treatment in [9, §§1-3]. Let $\mathbf{g}$ be a simple Lie algebra of type A_n, with root system Φ, simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, and Killing form $(\, , \,)$. Let $h_1, h_2, \ldots, h_n$ be a basis for a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, satisfying $(h_i, h) = \alpha_i^*(h)$ for all h in $\mathbf{h}$ and all $i \in I$. Let Y be the $\mathbb{Z}$-lattice spanned by $h_1, h_2, \ldots, h_n$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the fundamental weights of $\mathbf{g}$, defined by $\omega_i(h_j) = \delta_{ij}$, and let X be the $\mathbb{Z}$-lattice spanned by them (the weight lattice). For $i, j \in I$, define $i \cdot j$ to be 2 if $i = j$, to be -1 if $|i - j| = 1$, and to be 0 if $|i - j| \geq 2$. (So $i \cdot j$ is a scalar multiple of (α_i, α_j).) Then $(I, \cdot)$ is a Cartan datum as in [9, 1.1.1]. Note that we have $(i \cdot i)/2 = 1$ for all $i \in I$. For $\mu \in Y$ and $\lambda \in X$, define $\langle \mu, \lambda \rangle$ to be $\lambda(\mu)$. Define an imbedding of I into Y by $i \mapsto h_i$ and into X by $i \mapsto \alpha_i$ for all $i \in I$. We then have a root datum of type $(I, \cdot)$ as in [9, 2.2.1], with $\langle h_i, \alpha_j \rangle = \alpha_j(h_i) = A_{ij}$ the corresponding Cartan matrix.

Let $\mathbb{Q}(v)$ be the field of rational functions in an indeterminate v, and $\mathcal{A} \subseteq \mathbb{Q}(v)$ the ring $\mathbb{Z}[v, v^{-1}]$. For $N, M \in \mathbb{N}$ we define the following (which all lie in $\mathcal{A}$):

$$[N] = \frac{v^N - v^{-N}}{v - v^{-1}}, \quad [N]! = [N][N-1]\cdots[1], \quad \begin{bmatrix} M \\ N \end{bmatrix} = \frac{[M]!}{[N]![M-N]!}.$$

These are referred to as quantized integers, quantized factorials and quantized binomial coefficients, respectively. If v is specialised to 1 they specialize to the usual integers, factorials and binomial coefficients.

We define the quantized enveloping algebra U corresponding to the above data (as in [9, 3.1.1 & 33.1.5]) to be the $\mathbb{Q}(v)$-algebra U with generators $1, E_1, E_2, \ldots, E_n$, $F_1, F_2, \ldots, F_n$, and K_μ for $\mu \in Y$, subject to the relations: (for each $i, j \in I$ and $\mu, \mu' \in Y$)

$$K_0 = 1,$$

$$K_\mu K_{\mu'} = K_{\mu+\mu'},$$

$$K_\mu E_i = v^{\alpha_i(\mu)} E_i K_\mu,$$

$$K_\mu F_i = v^{-\alpha_i(\mu)} F_i K_\mu,$$

$$E_iF_i - F_iE_i = \frac{K_i - K_i^{-1}}{v - v^{-1}},$$

$$E_iF_j - F_jE_i = 0, \quad i \neq j,$$

$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \left[\begin{array}{c} 1 - A_{ij} \\ p' \end{array} \right] E_i^p E_j E_i^{p'} = 0, \quad i \neq j,$$

$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \left[\begin{array}{c} 1 - A_{ij} \\ p' \end{array} \right] F_i^p F_j F_i^{p'} = 0, \quad i \neq j,$$

(where, for $i \in I$, we put $K_i = K_{h_i}$ and $K_i^{-1} = K_{-h_i}$). In the last two summations, p and p' are restricted to the non-negative integers.

We make the following definitions (see [9, 3.1.1 & 3.1.13]). For $M \in \mathbb{N}$, and $i \in I$, we put $E_i^{(M)} = E_i^M/[M]!$, and $F_i^{(M)} = F_i^M/[M]!$, which are called *divided powers*. Let $U_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of U generated by the elements $E_i^{(N)}, F_i^{(N)}, K_\mu$ for $i \in I$, $N \in \mathbb{N}$ and $\mu \in Y$. It is called the *integral form* of U. Let U^+ be the $\mathbb{Q}(v)$-subalgebra of U generated by the E_i, $i \in I$, and $U_{\mathcal{A}}^+$ the $\mathcal{A}$-subalgebra of U generated by $E_i^{(N)}$, $i \in I, N \in \mathbb{N}$. Let U^- be the $\mathbb{Q}(v)$-subalgebra of U generated by the F_i, $i \in I$, and $U_{\mathcal{A}}^-$ the $\mathcal{A}$-subalgebra of U generated by $F_i^{(N)}$, $i \in I, N \in \mathbb{N}$. Let U^0 be the $\mathbb{Q}(v)$-subalgebra generated by the K_μ, $\mu \in Y$.

Let W be the Weyl group of $\mathbf{g}$. So W is the group:

$$W = \langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1, (s_is_j)^{m_{ij}} = 1 \ (i \neq j) \rangle$$

where $m_{ij} = 2$ if $A_{ij} = 0$ and $m_{ij} = 3$ if $A_{ij} = -1$. For $r \in I$, let W^r be the set of distinguished left coset representatives of the parabolic subgroup W_r of W obtained by 'omitting' s_r.

Let $X^+ \subseteq X$ be the set of dominant weights, i.e. those of the form $\lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n \in X$ where $\omega_1, \omega_2, \ldots, \omega_n$ are the fundamental weights of $\mathbf{g}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{N}$. If L is a U-module, $x \in L$ and $\lambda \in X$, we say that x has weight λ if $K_\mu x = v^{\lambda(\mu)}x$ for all $\mu \in Y$. We call the subspace of L consisting of all of the elements of weight λ the λ-*weight space* of L. As in [9, 3.4.1], we restrict our attention to U-modules which are direct sums of their weight spaces. We say that $x \in L$, $x \neq 0$, is a *highest* (respectively, *lowest*) weight vector if x has weight λ, for some $\lambda \in X$, $E_ix = 0$ (respectively, $F_ix = 0$) for each $i \in I$ and $U^-x = L$ (respectively, $U^+x = L$). Such a vector is uniquely determined up to a non-zero scalar multiple. We say that L is a highest weight module with highest weight λ if it contains a highest weight vector of weight λ. Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n$ be a dominant weight. We follow the construction in [9, 3.4.5 & 3.5.6]. Let J be the left ideal of U generated by the elements E_i for $i \in I$ and the elements $K_\mu - v^{(\lambda(\mu))}$ for $\mu \in Y$. Then the map from U^- to U/J taking $x \in U^-$ to $x + J$ is a $\mathbb{Q}(v)$-vector space isomorphism, which can be used to transfer the left U-module structure of U/J to U^-. The resulting U-module we denote by $M(\lambda)$; it is the called a *Verma module*. Let $T(\lambda)$ be the left ideal of $M(\lambda)$ (as a $\mathbb{Q}(v)$-algebra) generated by the elements $F_i^{\lambda_i+1}$, for $i \in I$, and let $V(\lambda)$ be the quotient module $M(\lambda)/T(\lambda)$. Then, by [9, 6.2.3 & 6.3.4], $V(\lambda)$ is an irreducible, finite-dimensional highest weight U-module with highest weight λ, unique up to isomorphism. We fix x_1 as the image of $1 \in M(\lambda)$ under the natural map from $M(\lambda)$ to $V(\lambda)$. Then x_1 is a highest weight vector for $V(\lambda)$. If λ and λ' are any two distinct dominant weights, then $V(\lambda)$ and $V(\lambda')$ are not isomorphic (see [9, 6.2.3(b)]). It is known that $V(\lambda)$ is the direct sum of its weight spaces (see [9, 3.4.1 & 3.5.6]). We also write $V(\lambda)_{\mathcal{A}} = U_{\mathcal{A}}^- x_1$, the integral form of $V(\lambda)$ (see [9, 19.3.1]). For $\mu \in Y$, we have $K_\mu V(\lambda)_{\mathcal{A}} \subseteq V(\lambda)_{\mathcal{A}}$, since K_μ always acts as an integral power of v on an element in a weight space and $V(\lambda)_{\mathcal{A}}$ is the direct sum of its weight spaces (see [9, 19.3.1]). Therefore, by [9, 19.3.2], $V(\lambda)_{\mathcal{A}}$ is a $U_{\mathcal{A}}$-module. For each $r \in I$ we denote by V_r the module $V(\omega_r)$ with highest weight ω_r. This is called the r-th *fundamental module* for U.

2 The Canonical Basis of V_r

Fix $r \in I$. By [5, 1.10], we have that every element of W can be written uniquely as a product of an element of W^r and an element of the parabolic subgroup W_r. We conclude that $|W| = |W_r||W^r|$. But $|W| = (n+1)!$. Also, W_r is a direct product of a Weyl group of type A_{r-1} and one of type A_{n-r} (the former being the subgroup generated by $s_1, s_2, \ldots, s_{r-1}$ and the latter being the subgroup generated by $s_{r+1}, s_{r+1}, \ldots, s_n$), so $|W_r| = r!(n+1-r)!$. We conclude that $|W^r| = \begin{pmatrix} n+1 \\ r \end{pmatrix}$. Note that this is also the dimension of V_r over $\mathbb{Q}(v)$ (see [1, Table 2,p214] and note that the dimension of V_r over $\mathbb{Q}(v)$ is equal to the dimension of the corresponding module for $\mathcal{U}$, by [9, 33.1.3(d)]). We note also that if $s_{i_1}s_{i_2}\cdots s_{i_t}$ is a reduced expression for an element $w \in W^r \setminus \{1\}$ then $s_{i_1}w = s_{i_2}\cdots s_{i_t} \in W^r$. This follows immediately from the fact that

$$W^r = \{w \in W : \ell(ws_i) > \ell(w)\ \forall i \in I,\ i \neq r\}.$$

We shall use the following result:

Lemma 2.1 *Suppose that* $s_{i_1}s_{i_2}\cdots s_{i_t} = s_{j_1}s_{j_2}\cdots s_{j_t}$ *are two reduced expressions for an element* $w \in W$*. Then there exists a finite sequence of braid relations which, when applied to the first expression, in order, gives the second.*

Proof: This result is due to Matsumoto; see [3, 64.20]. □

The following lemma is a key step in pinning down the elements of W^r:

Lemma 2.2 *Suppose* $s_{i_1}s_{i_2}\cdots s_{i_t}$ *is a reduced expression for an element* $w \in W^r$ *of length at least* 3*. Then, in the list* $i_1, i_2, \ldots, i_t$*, there is no subsequence* p, q, p *with* $A_{pq} = -1$*. So no long braid relation* $s_p s_q s_p = s_q s_p s_q$ *can be applied directly to this expression.*

Proof By the above remarks we can suppose that $w = s_p s_q s_p s_{i_4}\cdots s_{i_t}$ (and try to get a contradiction). By [5, p14] we see that the following positive roots are all sent to negative roots by w:

$$\begin{aligned}\beta_1 &= s_{i_t}s_{i_{t-1}}\cdots s_{i_4}s_p s_q(\alpha_p),\\ \beta_2 &= s_{i_t}s_{i_{t-1}}\cdots s_{i_4}s_p(\alpha_q),\\ \beta_3 &= s_{i_t}s_{i_{t-1}}\cdots s_{i_4}(\alpha_p).\end{aligned}$$

Since $s_p s_q(\alpha_p) = \alpha_q$ and $s_p(\alpha_q) = \alpha_p + \alpha_q$, it is easily seen that $\beta_1 + \beta_3 = \beta_2$. However, we know that for $i \neq r$, $w(\alpha_i) > 0$ while $w(\alpha_r) < 0$ (see [5, §1.6] and note that $w \in W^r$ and $w \neq 1$). Hence if α is any positive root made negative by w, the coefficient of α_r in the expression of α as an integral combination of simple roots should be at least 1. But the positive roots are precisely:

$$\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

for $1 \leq i < j \leq n$ (see [2, Planche I]), so this coefficient must be 1. It is now clear that that the above situation cannot occur, since necessarily the coefficient of α_r in β_2 is 2. We have a contradiction and the lemma is proved. □

We return now to the fundamental modules V_r, $r = 1, 2, \ldots, n$. If x_1 is our fixed weight vector in V_r, it is clear from the definition of $V(\lambda)$ (see [9, 3.5.6]), that in V_r, we have $F_i x_1 = 0$ if $i \neq r$ and $F_r^2 x_1 = 0$.

Lemma 2.3 *Suppose that* $\xi = F_{i_1} F_{i_2} \cdots F_{i_t} \in U^-$ *is a monomial (with* $t \geq 3$*) such that in the list* $i_1, i_2, \ldots, i_t$ *there is a sequence of the form* p, q, p *with* $A_{pq} = -1$. *Then* $F_{i_1} F_{i_2} \cdots F_{i_t} x_1 = 0$.

Proof: First we assume that $q = p + 1$. Without loss of generality we can assume such a sequence occurs at the start, since if $F_{i_k} F_{i_{k+1}} \cdots F_{i_t} x_1 = 0$ for some $k \geq 1$ then $F_{i_1} F_{i_2} \cdots F_{i_t} x_1 = 0$. We are considering case A_n. The following information is from [1, VIII,§13.1]. The Lie algebra $\mathbf{g}$ is isomorphic to the Lie algebra of $(n+1) \times (n+1)$ matrices of trace zero, and a Cartan subalgebra $\mathbf{h}$ is the subspace of diagonal matrices of trace zero. Let ε'_i be the map from the set of all $(n+1) \times (n+1)$ diagonal matrices to $\mathbb{C}$ taking such a matrix to its (i, i) entry, for $i \in [1, n+1]$. Then $\mathbf{h}^*$ is spanned by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}$, where ε_i is the restriction of ε'_i to $\mathbf{h}$. The weight lattice X is the subset of $\mathbf{h}^*$ consisting of integral linear combinations of the ε_i. The fundamental roots are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i \in I$. The weights of V_r are given by

$$\varepsilon_{j_1} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_r}\ , \quad (1 \leq j_1 < j_2 < \cdots < j_r \leq n+1).$$

Now, assume that $F_{i_1} F_{i_2} \cdots F_{i_t} x_1 \neq 0$. Then the weight of $F_{i_4} \cdots F_{i_t} x_1$ is $\varepsilon_{j_1} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_r}$ for some $j_1, j_2, \ldots, j_r$ satisfying $1 \leq j_1 < j_2 < \cdots < j_r \leq n+1$. Multiplying $F_{i_4} \cdots F_{i_t} x_1$ on the left by F_i subtracts $\varepsilon_i - \varepsilon_{i+1}$ from the weight. So the weight of $F_p F_{p+1} F_p F_{i_4} \cdots F_{i_t}$ is:

$$-2\varepsilon_p + \varepsilon_{p+1} + \varepsilon_{p+2} + \varepsilon_{j_1} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_r},$$

which must be of the same form. Suppose it is equal to $\varepsilon_{l_1} + \varepsilon_{l_2} + \cdots + \varepsilon_{l_r}$. Then,

$$\varepsilon_{j_1} + \cdots + \varepsilon_{j_r} - 2\varepsilon_p + \varepsilon_{p+1} + \varepsilon_{p+2} - \varepsilon_{l_1} - \cdots - \varepsilon_{l_r} = 0$$

in $\mathbf{h}^*$. Therefore,

$$\varepsilon'_{j_1} + \cdots + \varepsilon'_{j_r} - 2\varepsilon'_p + \varepsilon'_{p+1} + \varepsilon'_{p+2} - \varepsilon'_{l_1} - \cdots - \varepsilon'_{l_r} = a(\varepsilon'_1 + \cdots + \varepsilon'_{n+1})$$

for some $a \in \mathbb{C}$. Note that $\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_{n+1}$ are independent. Looking at the coefficient of ε'_p in this expression we see that a must be $-3, -2$ or -1, but looking at the coefficient of ε'_{p+1} we see that a must be $0, 1$ or 2, a contradiction. We are forced to conclude that $F_{i_1} F_{i_2} \cdots F_{i_t} x_1 = 0$. The argument when $q = p - 1$ is very similar. The lemma is proved. □

Lemma 2.4 *Suppose that* $\xi = F_i^2 F_{i_1} F_{i_2} \cdots F_{i_t} \in U^-$. *Then* $\xi x_1 = 0$.

Proof: We argue exactly as in Lemma 2.3, noting that in this case multiplying an element in a weight space of V_r on the left by F_i^2 subtracts $2\varepsilon_i - 2\varepsilon_{i+1}$ from its weight. A similar contradiction for the value of a arises. □

It is clear from Lemmas 2.3 and 2.4 that if $\xi = F_{i_1} F_{i_2} \cdots F_{i_t}$ is any monomial satisfying $\xi x_1 \neq 0$, then ξ must satisfy the following conditions:

(a) We have $F_{i_t} = F_r$. Also, after any commuting of F_i's, the expression for ξ must still end in F_r. This is clear from the action of U^- on x_1.

(b) There is no way of commuting F_i's to get a subsequence of the form p, p in the list $i_1, i_2, \ldots, i_t$.

(c) There is no way of commuting F_i's to get a subsequence of the form p, q, p with $A_{pq} = -1$ in the list $i_1, i_2, \ldots, i_t$.

We now define a map $\phi_r : W^r \to U^-$ in the following manner:
Suppose that $w = s_{i_1} s_{i_2} \cdots s_{i_t}$ is a reduced expression for an element $w \in W^r$. We put:

$$\phi_r(w) = F_{i_1} F_{i_2} \cdots F_{i_t} \in U^-.$$

In particular, $\phi_r(1) = 1 \in U^-$. This map is well-defined: if $s_{j_1} s_{j_2} \cdots s_{j_t}$ is another reduced expression for w, there exists a finite sequence of braid relations taking $s_{i_1} s_{i_2} \cdots s_{i_t}$ to $s_{j_1} s_{j_2} \cdots s_{j_t}$ (by Lemma 2.1). By Lemma 2.2, these relations must all be commutations. By applying the same commutations in U^-, we get equality of $F_{i_1} F_{i_2} \cdots F_{i_t}$ and $F_{j_1} F_{j_2} \cdots F_{j_t}$.

Lemma 2.5 *Suppose that* $\xi = F_{i_1} F_{i_2} \cdots F_{i_t}$ *is a monomial in* U^-, *satisfying* $\xi x_1 \neq 0$. *Then* $\xi \in \mathrm{Im}(\phi_r)$.

Proof: Given $\xi = F_{i_1}F_{i_2}\cdots F_{i_t}$ as above, put $w = s_{i_1}s_{i_2}\cdots s_{i_t} \in W$. It is irrelevant which expression of ξ as a monomial is given — whichever it is we define an element w in this way. We show that this expression is reduced and that $w \in W^r$. It will then follow that $\phi_r(w) = \xi$ and we will be done. We know that $\xi x_1 \neq 0$, so ξ must satisfy conditions (a), (b) and (c) given after the proof of Lemma 2.4. Conditions (b) and (c) together imply that no finite sequence of defining relations for W can be applied to $s_{i_1}s_{i_2}\cdots s_{i_t}$ to produce a shorter expression for w, so this expression must be reduced. By condition (a), no finite sequence of braid relations applied to $s_{i_1}s_{i_2}\cdots s_{i_t}$ can make it end in anything other than s_{i_t} $(= s_r)$. (These relations must be commutations, by (c)). Hence, by Lemma 2.1, any reduced expression for w must end in s_r, so $w \in W^r$, as required. □

Proposition 2.6 *The function ϕ_r defined above defines a bijection ψ_r from W^r onto a basis $\mathcal{B}_r$ for V_r, given by the formula:*
If $w = s_{i_1}s_{i_2}\cdots s_{i_t} \in W^r$ is a reduced expression, put

$$\psi_r(w) = \phi_r(w)x_1 = F_{i_1}F_{i_2}\cdots F_{i_t}x_1.$$

Proof: Define ψ_r as in the proposition. By the definition of U^- it is clear that the subset S of U^- consisting of all of the monomials in the F_i's spans U^- as a $\mathbb{Q}(v)$-vector space. So, in V_r,

$$\begin{aligned} U^-x_1 &= \left(\sum_{\xi\in S}\mathbb{Q}(v)\xi\right)x_1 \\ &= \sum_{\xi\in S}\mathbb{Q}(v)(\xi x_1). \end{aligned}$$

But, by Lemma 2.5, if $\xi \in S$ and $\xi x_1 \neq 0$, then $\xi \in \mathrm{Im}(\phi_r)$. Hence,

$$\begin{aligned} U^-x_1 &= \sum_{\xi\in \mathrm{Im}(\phi_r)}\mathbb{Q}(v)(\xi x_1) \\ &= \mathbb{Q}(v)\,\mathrm{Im}(\phi_r)x_1 \\ &= \mathbb{Q}(v)\,\mathrm{Im}(\psi_r). \end{aligned}$$

So $\mathrm{Im}(\psi_r)$ is a spanning set for V_r as a $\mathbb{Q}(v)$-vector space, since $V_r = U^-x_1$. But $|W^r| = \dim_{\mathbb{Q}(v)} V_r = \binom{n+1}{r}$, and clearly $|\mathrm{Im}(\psi_r)| \leq \binom{n+1}{r}$, since its domain W^r contains only this many elements. So, as $\mathrm{Im}(\psi_r)$ spans V_r, it must be a basis $\mathcal{B}_r$ for V_r and ψ_r must be injective, since $|\mathrm{Im}(\psi_r)| = \binom{n+1}{r}$. The result is proved. □

Theorem 2.7 *The basis $\mathcal{B}_r$ for V_r is in fact, up to sign, the canonical basis for V_r. So, up to sign, the canonical basis of V_r is:*

$$\{F_{i_1}F_{i_2}\cdots F_{i_t}x_1 \ : \ s_{i_1}s_{i_2}\cdots s_{i_t} \text{ is a reduced expression for an element in } W^r\},$$

where $F_{i_1}F_{i_2}\cdots F_{i_t}x_1 = F_{j_1}F_{j_2}\cdots F_{j_t}x_1$ if and only if $s_{i_1}s_{i_2}\cdots s_{i_t} = s_{j_1}s_{j_2}\cdots s_{j_t}$.

Proof: The statement about equality of elements follows from the existence of ψ_r and the fact that it is injective. By definition $U_{\mathcal{A}}^-$ is spanned as an $\mathcal{A}$-module by the set of elements of the form $F_{i_1}^{(r_1)}F_{i_2}^{(r_2)}\cdots F_{i_t}^{(r_t)}$. Hence, $V_{r,\mathcal{A}}$ is spanned as an $\mathcal{A}$-module by the elements of the form $F_{i_1}^{(r_1)}F_{i_2}^{(r_2)}\cdots F_{i_t}^{(r_t)}x_1$ which are non-zero. Suppose now that $\xi = F_{i_1}^{(r_1)}F_{i_2}^{(r_2)}\cdots F_{i_t}^{(r_t)} \in U_{\mathcal{A}}^-$, with $\xi x_1 \neq 0$. Then $F_{i_1}^{r_1}F_{i_2}^{r_2}\cdots F_{i_t}^{r_t}x_1 \neq 0$, so by Lemma 2.5 and its proof, each $r_k = 1$, $s_{i_1}s_{i_2}\cdots s_{i_t}$ is a reduced expression for an element in W^r, and $\phi_r(s_{i_1}s_{i_2}\cdots s_{i_t}) = F_{i_1}F_{i_2}\cdots F_{i_t} = \xi$. Hence $\mathcal{B}_r$ spans $V_{r,\mathcal{A}}$ as an $\mathcal{A}$-module and is thus an $\mathcal{A}$-basis for it.

We now use the fact that, in V_r, there are precisely $\binom{n+1}{r}$ distinct weight spaces, each of dimension 1. This is true since each of these fundamental modules is *miniscule* — see [1, VIII,§7.3], Proposition 7 and the Remark at the end of §7.3. (For λ a dominant weight, the dimensions of the weight spaces of $V(\lambda)$ are the same as the dimensions of the corresponding weight spaces of $V'(\lambda)$ — see [9, 33.1.3(d)]). Because of its monomial form, each element of $\mathcal{B}_r$ lies in a weight space, and it is known (see [8, 14.4.2]) that each element of the canonical basis lies in a weight space. So, since $\mathcal{B}_r$ and the canonical basis are $\mathcal{A}$-bases for $V_{r,\mathcal{A}}$ (see [9, 19.3.1] for the latter), each b in the canonical basis must be equal to $\eta b'$ for some $b' \in \mathcal{B}_r$ and $\eta \in \mathcal{A}$, invertible. Clearly each such η should be plus or minus a power of v. But the automorphism $^-$ of V_r fixes each element of the canonical basis (see Theorem 2.15(1)) and clearly fixes each element of $\mathcal{B}_r$, so each η must be ± 1 and we are done. □

Examples

(1) The case A_3. In this case we have:

$$\begin{aligned} W^1 &= \{1,\ s_1,\ s_2s_1,\ s_3s_2s_1\}, \\ W^2 &= \{1,\ s_2,\ s_1s_2,\ s_3s_2,\ s_1s_3s_2,\ s_2s_1s_3s_2\}, \\ W^3 &= \{1,\ s_3,\ s_2s_3,\ s_1s_2s_3\}. \end{aligned}$$

We therefore have the following: (note that $\mathcal{B}_r \subseteq V_r$ and that in each case $x_1 \in V_r$)

$$\begin{aligned} \mathcal{B}_1 &= \{x_1,\ F_1x_1,\ F_2F_1x_1,\ F_3F_2F_1x_1\}, \\ \mathcal{B}_2 &= \{x_1,\ F_2x_1,\ F_1F_2x_1,\ F_3F_2x_1,\ F_1F_3F_2x_1,\ F_2F_1F_3F_2x_1\}, \\ \mathcal{B}_3 &= \{x_1,\ F_3x_1,\ F_2F_3x_1,\ F_1F_2F_3x_1\}. \end{aligned}$$

(2) The case A_n. In this case we look at V_1 only. Its dimension is $n+1$. We have:

$$W^1 = \{1,\ s_1,\ s_2s_1,\ \ldots,\ s_ns_{n-1}\cdots s_1\},$$

whence:

$$\mathcal{B}_1 = \{x_1,\ F_1x_1,\ F_2F_1x_1,\ \ldots,\ F_nF_{n-1}\cdots F_1x_1\}.$$

Note: We shall see later that in fact $\mathcal{B}_r$ is the canonical basis for V_r. We now embark on proving this, first studying further the distinguished coset representatives.

Lemma 2.8 *Suppose that $r \in I$ and that $s_{i_1}s_{i_2}\cdots s_{i_t}$ is a reduced expression for an element $w \in W^r$, and that $s_{i_k} = s_{i_1}$ is the second appearance of s_{i_1} in the expression. Then in the set $\{s_{i_m} : 2 \le m \le k-1\}$ there is precisely one occurrence of s_{i_1-1} and precisely one occurrence of s_{i_1+1}. In particular, $i_1 \ne 1$ and $i_1 \ne n$.*

Proof: Note first that the cases $t = 1,2,3$ cannot occur. The cases $t = 1,2$ are trivial and if $t = 3$ then $w = s_{i_1}s_{i_2}s_{i_1} \notin W^r$, so the first case to consider is $t = 4$. We must have $w = s_{i_1}s_{i_2}s_{i_3}s_{i_1}$. An easy case-by-case analysis enables us to conclude that $\{s_{i_2}, s_{i_3}\} = \{s_{i_1-1}, s_{i_1+1}\}$, and we are done. We now prove the result by induction on t, having dealt already with the base case. Suppose we know the result to be true for all smaller t.

Case(I): Suppose that no s_{i_m} $(2 \le m \le k-1)$ is s_{i_1-1} or s_{i_1+1}. Then both s_{i_1} and

s_{i_k} commute with all of these s_{i_m} and by commuting them together we can apply the relation $s_{i_1}^2 = 1$ to get a shorter expression for w — a contradiction.
Case(II): Suppose that, in between s_{i_1} and s_{i_k}, at least two s_{i_1-1}'s occur. Let s_{i_a} and s_{i_b} be two such occurrences with $a < b$ and with no s_{i_1-1} occurring between them. The inductive hypothesis, applied to $s_{i_a} \cdots s_{i_b} \cdots s_{i_t}$ (also in W^r) supplies an $s_{(i_1-1)+1} = s_{i_1}$ between them — a contradiction. In a similar way we can deal with the case when there are at least two occurrences of s_{i_1+1}.
Case(III): Suppose that, in between s_{i_1} and s_{i_k}, a single s_{i_1-1} occurs but s_{i_1+1} does not. Then s_{i_1-1} is the only s_i between s_{i_1} and s_{i_k} which s_{i_1} and s_{i_k} do not commute with, so we can get a reduced expression for w of the form $\cdots s_{i_1} s_{i_1-1} s_{i_1} \cdots$, which is a contradiction to Lemma 2.2. In a similar way we can deal with the case when a single s_{i_1+1} occurs but s_{i_1-1} does not.
We are now forced to conclude that the lemma is true for this expression and so by induction, it is proved in general. □

Lemma 2.9 *Suppose that $s_{i_1} s_{i_2} \cdots s_{i_t}$ is a reduced expression for an element $w \in W^r$ of length at least 2, and that s_{i_1} occurs in the expression in the first place only. Then, in the set $\{s_{i_m} : 2 \leq m \leq t\}$, there is either:*
(a) precisely one occurrence of s_{i_1-1} or
(b) precisely one occurrence of s_{i_1+1},
but not both.

Proof: Again, we prove this by induction on t. The case $t = 2$ is easy to see, since w must be either $s_{i_1} s_{i_1-1}$ or $s_{i_1} s_{i_1+1}$. So, assume the result to be true for all smaller t.
Case(I): Assume that to the right of s_{i_1} there are at least two occurrences of s_{i_1-1}. By Lemma 2.8, if we take two of them with no s_{i_1-1} between them, we must have an $s_{(i_1-1)+1} = s_{i_1}$ between them, contradicting the hypotheses of the lemma. The case with two s_{i_1+1}'s is dealt with similarly.
Case(II): Assume that to the right of s_{i_1}, there are no s_{i_1-1}'s or s_{i_1+1}'s. Then we can commute s_{i_1} to the end of the expression, but $i_1 \neq r$, since s_{i_1} occurs only at the beginning of the original expression, and the length of w is at least 2. We have a contradiction.
Case(III): Assume that to the right of s_{i_1}, precisely one each of s_{i_1-1} and s_{i_1+1} occurs. By the inductive hypothesis, there is either an s_{i_1-2} or an s_{i_1} to the right of the s_{i_1-1} (but not both). The latter case is impossible, so we have an s_{i_1-2} to

the right of s_{i_1-1}. Take the last occurrence of s_{i_1-2}. Continue in this way until we get to the end of the expression. Each time we get an s_i with smaller subscript. The last step should give us $s_{i_t} = s_{i_1-p}$ for some $p \geq 0$. But we can apply this argument to s_{i_1+1} also and get that $s_{i_t} = s_{i_1+q}$ for some $q \geq 0$ — a contradiction unless $w = s_{i_1}$, a case we have excluded.
We are forced to conclude that the lemma holds for this expression and thus for this t. By induction, the lemma is proved. $\square$

These two lemmas give us the following useful result about $\mathcal{B}_r$:

Lemma 2.10 *Suppose that $s_{i_1}s_{i_2}\cdots s_{i_t}$ is a reduced expression for an element $w \in W^r$, and $\xi = \phi_r(w)$, so ξx_1 is the corresponding element of $\mathcal{B}_r$. Then:*

$$K_{i_1}(F_{i_2}\cdots F_{i_t}x_1) = vF_{i_2}\cdots F_{i_t}x_1.$$

Proof: We use Lemmas 2.8 and 2.9, and note that in U, for $i \in I$,

$$\begin{aligned} K_iF_i &= v^{-2}F_iK_i, \\ K_iF_{i\pm 1} &= vF_{i\pm 1}K_i, \\ K_iF_j &= F_jK_i \quad \text{if } |i-j|>1. \end{aligned} \tag{1}$$

Between each occurrence of F_{i_1}, there must occur exactly one each of F_{i_1-1} and F_{i_1+1}. Suppose that there are m occurrences of F_{i_1} in total. Let $F_{i_u}F_{i_{u+1}}\cdots F_{i_t}$ be the final string where F_{i_1} occurs only as F_{i_u}. If $u = t$ then $K_{i_1}x_1 = vx_1$ since we must have $i_1 = r$. If not, then Lemma 2.9 tells us that in this string there is precisely one occurrence of either F_{i_1-1} or F_{i_1+1} but not both. By equations (1), and the fact that, since $i_1 \neq r$, $K_{i_1}x_1 = x_1$, we see that $K_{i_1}F_{i_{u+1}}\cdots F_{i_t}x_1 = vF_{i_{u+1}}\cdots F_{i_t}x_1$. Putting all of this together we get the required result. $\square$

The following is also true:

Lemma 2.11 *Suppose that $s_{i_1}s_{i_2}\cdots s_{i_t}$ is a reduced expression for an element $w \in W^r$, and $\xi = \phi_r(w)$, so ξx_1 is the corresponding element of $\mathcal{B}_r$. Suppose that $K_i(\xi x_1) = v\xi x_1$. Then $F_iF_{i_1}F_{i_2}\cdots F_{i_t}x_1 \neq 0$. (Equivalently, $s_is_{i_1}s_{i_2}\cdots s_{i_t} \in W^r$.)*

Proof: Again we induct on t. Since the only $i \in I$ satisfying $K_ix_1 = vx_1$ is r, and $F_rx_1 \neq 0$, the $t = 0$ case holds. Assume we have the result for smaller t. We have $K_i(F_{i_1}F_{i_2}\cdots F_{i_t}x_1) = vF_{i_1}F_{i_2}\cdots F_{i_t}x_1$. Hence, by the way K_i acts on V_r (see equations (1)), we have:

$$\sum_{s=1}^{t} -A_{i,i_s} + \delta_{i,r} = 1,$$

where δ is the Kroneker delta. Suppose first that i does not occur in $i_1, i_2, \ldots, i_t$. Then:

$$\begin{aligned} E_i(F_iF_{i_1}F_{i_2}\cdots F_{i_t}x_1) &= \left[\delta_{i,r} - \sum_{s=1}^{t} A_{i,i_s}\right] F_{i_1}F_{i_2}\cdots F_{i_t}x_1 \\ &= F_{i_1}F_{i_2}\cdots F_{i_t}x_1 \neq 0. \end{aligned}$$

(using the relations of U). Therefore we must have $F_iF_{i_1}F_{i_2}\cdots F_{i_t}x_1 \neq 0$ and we are done. Now suppose that i_u is the first appearance of i in $i_1, i_2, \ldots, i_t$. Then:

$$\begin{aligned} E_i(F_iF_{i_1}F_{i_2}\cdots F_{i_t}x_1) &= \left[\delta_{i,r} - \sum_{s=1}^{t} A_{i,i_s}\right] F_{i_1}F_{i_2}\cdots F_{i_t}x_1 \\ &+ \left[\delta_{i,r} - \sum_{s=u+1}^{t} A_{i,i_s}\right] F_iF_{i_1}\cdots F_{i_{u-1}}\widehat{F_{i_u}}F_{i_{u+1}}\cdots F_{i_t}x_1 \\ &+ \text{ plus other similar terms: one for each other occurrence} \\ &\quad \text{of } i \text{ in } i_1, i_2, \ldots, i_t. \end{aligned}$$

(using the relations of U). Now, $F_{i_1}F_{i_2}\cdots F_{i_t}x_1 \neq 0$, so $F_{i_u}F_{i_{u+1}}\cdots F_{i_t}x_1 \neq 0$. Hence, by Lemma 2.10,

$$K_i(F_{i_{u+1}}\cdots F_{i_t}x_1) = K_{i_u}(F_{i_{u+1}}\cdots F_{i_t}x_1) = vF_{i_{u+1}}\cdots F_{i_t}x_1.$$

We therefore have $\sum_{s=u+1}^{t} -A_{i,i_s} + \delta_{i,r} = 1$ and similarly for the later terms in the sum. Thus:

$$\begin{aligned} E_i(F_iF_{i_1}F_{i_2}\cdots F_{i_t}x_1) &= [1]F_{i_1}F_{i_2}\cdots F_{i_t}x_1 + [1]F_iF_{i_1}\cdots \widehat{F_{i_u}}\cdots F_{i_t}x_1 + \cdots \\ &= F_{i_1}F_{i_2}\cdots F_{i_t}x_1 + F_iF_{i_1}\cdots \widehat{F_{i_u}}\cdots F_{i_t}x_1 + \cdots \end{aligned}$$

and so can be expressed as a sum of elements, each of which is a monomial acting on x_1. Each one, by Lemma 2.5, must be zero or an element of $\mathcal{B}_r$. By assumption, the first one is non-zero, so $E_i(F_iF_{i_1}F_{i_2}\cdots F_{i_t}x_1)$ is a sum of basis elements and is therefore non-zero. Hence $F_iF_{i_1}F_{i_2}\cdots F_{i_t}x_1 \neq 0$ as required. □

We can put these two results together to get:

Proposition 2.12 *If* $F_{i_1}F_{i_2}\cdots F_{i_t}x_1 \neq 0$ *in* V_r*, then*

$$K_i(F_{i_1}F_{i_2}\cdots F_{i_t}x_1) = vF_{i_1}F_{i_2}\cdots F_{i_t}x_1 \text{ if and only if } F_iF_{i_1}F_{i_2}\cdots F_{i_t}x_1 \neq 0. \ \square$$

Corollary 2.13 *Let* $s_{i_1}s_{i_2}\cdots s_{i_t}$ *be a reduced expression for an element* $w \in W^r$*, and* $\xi = F_{i_1}F_{i_2}\cdots F_{i_t}$*. Suppose that* F_{i_u} *cannot be commuted to the start of the expression for* ξ*. Then* $F_{i_1}F_{i_2}\cdots \widehat{F_{i_u}}\cdots F_{i_t}x_1 = 0$ *in* V_r*.*

Proof: We induct on t. The case $t = 1$ does not occur. If $t = 2$, we must have $\xi = F_{r\pm1}F_r$, and the lemma is simply saying that $F_{r\pm1}x_1 = 0$. So, assume the result is true for smaller t. If F_{i_u} cannot be commuted to the *second* position, we have, by induction hypothesis, $F_{i_2}\cdots \widehat{F_{i_u}}\cdots F_{i_t}x_1 = 0$, whence the result. Therefore we assume that F_{i_u} can be commuted to the second position, and in fact we can assume that is already there (that is, $u = 2$). Because it cannot be commuted to the first position, we have $F_{i_2} = F_{i_1\pm1}$. Without loss of generality, assume that $F_{i_2} = F_{i_1-1}$. We have $F_{i_1}F_{i_1-1}F_{i_3}\cdots F_{i_t}x_1 \neq 0$. Hence, by Proposition 2.12,

$$K_{i_1}F_{i_1-1}F_{i_3}\cdots F_{i_t}x_1 = vF_{i_1-1}F_{i_3}\cdots F_{i_t}x_1$$

Therefore, since $A_{i_1,i_1-1} = -1$, we have

$$K_{i_1}F_{i_3}\cdots F_{i_t}x_1 = v^0F_{i_3}\cdots F_{i_t}x_1,$$

and

$$F_{i_1}F_{i_3}\cdots F_{i_t}x_1 = 0,$$

using Proposition 2.12, and we have shown the result to be true for this expression. By induction, the lemma holds. $\square$

We are finally nearing our goal. We shall use the Kashiwara operators, $\widetilde{F}_i$, for $i \in I$, which are defined as follows.

Definition 2.14 *Suppose that* $V(\lambda)$ *is the finite-dimensional irreducible* U*-module with highest weight* λ*. Any element* $m \in V(\lambda)$ *can be written uniquely* $m = \sum_{0\leq k\leq k'} F_i^{(k)}x_{k,k'}$*, where the* $x_{k,k'}$ *satisfy* $E_ix_{k,k'} = 0$ *and* $K_ix_{k,k'} = v^{k'}x_{k,k'}$*. Then define*
$\widetilde{F}_i(m) = \sum_{0\leq k\leq k'} F_i^{(k+1)}x_{k,k'}$.

In the sequel, we shall need the following theorem, which is basically the first part of Theorem 19.3.5 in [9]. Here, $L(\lambda)$ is the R-span in $V(\lambda)$ of the set of elements of the form $\widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_p}x_1$ where $i_1, i_2, \ldots, i_p$ is a sequence in I and R is the subring of $\mathbb{Q}(v)$ of elements regular at $v^{-1} = 0$. To make this theorem correct, the extra hypothesis $b \neq 0$ is required in statement (1).

Theorem 2.15 *Let $b \in V(\lambda)$. We have $b \in \mathbf{B}(\lambda)$ if and only if*

(1) *$b \in V(\lambda)_{\mathcal{A}}$, $\overline{b} = b$, $b \neq 0$ and*

(2) *there exists a sequence $i_1, i_2, \ldots, i_p$ in I such that $b = \widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_p}x_1 \bmod v^{-1}L(\lambda)$.*

Proof: If $b \in \mathbf{B}(\lambda)$ then (1) and (2) are satisfied — the proof is as in [9, 19.3.5]. Suppose now that (1) and (2) are satisfied. The proof is almost that in [9, 19.3.5] except that the hypothesis $b \neq 0$ is required. Lusztig shows that if (1) and (2) are satisfied, then $b = ux_1$ for some element u in the canonical basis for U^-. Since $b \neq 0$, we must have $ux_1 \neq 0$, whence by [9, 14.4.11], $ux_1 \in \mathbf{B}(\lambda)$. So $b \in \mathbf{B}(\lambda)$ and the theorem is proved. □

If $b \in \mathcal{B}_r$, then by the definition of $\mathcal{B}_r$ (see Proposition 2.6), b satisfies condition (1) of Theorem 2.15. So, by Theorem 2.15, if $b \in \mathcal{B}_r$ and $b = \widetilde{F}_{j_1}\widetilde{F}_{j_2}\cdots\widetilde{F}_{j_k}x_1$ for some $j_1, j_2, \ldots, j_k$ in I, then in fact b lies in the canonical basis.

Theorem 2.16 *The basis $\mathcal{B}_r$ for V_r is the canonical basis for V_r.*

Proof: Suppose that $b \in \mathcal{B}_r$. By the above remarks it is enough to show that we have $b = \widetilde{F}_{j_1}\widetilde{F}_{j_2}\cdots\widetilde{F}_{j_k}x_1$ for some $j_1, j_2, \ldots, j_k$ in I. Suppose that $b = F_{i_1}F_{i_2}\cdots F_{i_t}x_1$, where $s_{i_1}s_{i_2}\cdots s_{i_t}$ is a reduced expression for an element $w \in W^r$. By Proposition 2.12 we know that $K_{i_1}F_{i_2}\cdots F_{i_t} = vF_{i_2}\cdots F_{i_t}$. Also, by using the relations of U, we know that $E_{i_1}(F_{i_2}\cdots F_{i_t}x_1)$ is a sum of terms, each term being $F_{i_2}\cdots F_{i_t}x_1$ with an F_{i_1} omitted. None of these F_{i_1}'s could be commuted to the start of $F_{i_2}\cdots F_{i_t}$, since then we would have $F_{i_1}^2F_{i_2}\cdots F_{i_t}x_1 \neq 0$, contradicting Lemma 2.4. Hence, by Corollary 2.13, each of these terms is zero and $E_{i_1}(F_{i_2}\cdots F_{i_t}) = 0$. Therefore, the element $F_{i_2}\cdots F_{i_t}x_1$ has the decomposition $F_{i_2}\cdots F_{i_t}x_1 = F_{i_1}^{(0)}x_{0,1}$ with respect to i_1 (see Definition 2.14), so $\widetilde{F}_{i_1}(F_{i_2}\cdots F_{i_t}x_1) = F_{i_1}^{(1)}x_{0,1} = F_{i_1}F_{i_2}\cdots F_{i_t}x_1$. Repeating this argument we see that $b = \widetilde{F}_{i_1}\widetilde{F}_{i_2}\cdots\widetilde{F}_{i_t}x_1$ as required. Therefore $\mathcal{B}_r$ is contained in the canonical basis for V_r. Since both are bases for V_r, we have equality and we are done. □

3 Quantized Exterior Powers

In the unquantized case, the fundamental modules for the Lie algebra of type A_n can be obtained by taking the exterior powers of the module corresponding to V_1 (see [1, VIII,§13.1,p188]). In this section, we present a corresponding result in the quantized case.

We note first that there is a comultiplication Δ on U, which is a $\mathbb{Q}(v)$-algebra homomorphism from U to $U \otimes U$ having the following effect on the generators of U:

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes K_i^{-1} + 1 \otimes E_i, \\
\Delta(F_i) &= F_i \otimes 1 + K_i \otimes F_i, \\
\Delta(K_\mu) &= K_\mu \otimes K_\mu, \\
\Delta(1) &= 1 \otimes 1.
\end{aligned}$$

(this is almost the same as the map Δ_- in [7, 1.4]). By checking agreement on the generators of U, it can be seen that $(\Delta\otimes 1)\Delta = (1 \otimes \Delta)\Delta : U \to U \otimes U \otimes U$. We denote this map by $\Delta^{(2)}$. We define $\Delta^{(3)} = (\Delta\otimes 1\otimes 1)(\Delta\otimes 1)\Delta$ and similarly $\Delta^{(m)}$ for any positive integer m. If V is any U-module, we can make $V^{\otimes m}$ into a U-module by defining:

$$u(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = \Delta^{(m-1)}(u)(v_1 \otimes v_2 \otimes \cdots \otimes v_m),$$

for $u \in U$. We do this in the particular case when $M = V_1$. By example (2) given after the proof of Theorem 2.7, we know that the canonical basis for V_1 is:

$$\{x_1,\ F_1x_1,\ F_2F_1x_1, \ldots,\ F_n\cdots F_1x_1\}.$$

We therefore write $b_i = F_iF_{i-1}\cdots F_1x_1$, for $i = 0, 1, \ldots, n$, so that the canonical basis is $\{b_0, b_1, \ldots, b_n\}$. Using the relations of U and also Lemma 2.5, we see that the action of U on this basis is given by:

$$\begin{array}{rcll} F_i b_j &=& \delta_{i-1,j} b_i, & \text{for } i \in I, j \in [0,n], \\ E_i b_j &=& \delta_{i,j} b_{i-1}, & \text{for } i \in I, j \in [0,n], \\ K_i b_i &=& v^{-1} b_i, & \text{for } i \in I, \\ K_i b_{i-1} &=& v b_{i-1}, & \text{for } i \in I, \\ K_i b_j &=& b_j, & \text{otherwise.} \end{array}$$

(Note that for $\mu = c_1h_1 + c_2h_2 + \cdots + c_nh_n \in Y$, we have $K_\mu = K_1^{c_1} K_2^{c_2} \cdots K_n^{c_n}$ so it is enough to describe the action of the K_i's.) We define $\bigwedge_q^r(V)$, the r-th quantized exterior power of V, to be the subspace of $V^{\otimes r}$ spanned by the elements:

$$\{b_{i_1,i_2,\ldots,i_r} \;:\; 0 \le i_1 < i_2 < \cdots < i_r \le n\},$$

where:

$$b_{i_1,i_2,\ldots,i_r} = \sum_{\sigma \in S_r} \sigma(b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_r})(-v)^{-\ell(\sigma)}.$$

(S_r, the symmetric group on r letters, acts on $V^{\otimes r}$ in the following manner:

$$\sigma(b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_r}) = b_{i_{\sigma^{-1}(1)}} \otimes b_{i_{\sigma^{-1}(2)}} \otimes \cdots \otimes b_{i_{\sigma^{-1}(r)}},$$

and $\ell(\sigma)$ denotes the length of σ as an element of S_r considered as a Coxeter group with generators $(1\ 2), (2\ 3), \ldots, (r-1\ r)$.)
Note: The idea of a quantized exterior power is well-known but the author has not seen in the literature either this definition, which arose from a discussion with R. M. Green, or the following proof that $V_r \cong \bigwedge_q^r(V)$.

Since the elements $b_{p_1} \otimes b_{p_2} \otimes \cdots \otimes b_{p_r}$, with each $p_k \in I$, form a basis for $V^{\otimes r}$, and each $b_{i_1,i_2,\ldots,i_r}$ consists of a linear combination of elements from a disjoint subset of this basis, the above spanning set for $\bigwedge_q^r(V)$ is actually a basis for $\bigwedge_q^r(V)$. Thus $\dim_{\mathbb{Q}(v)}(\bigwedge_q^r(V)) = \begin{pmatrix} n+1 \\ r \end{pmatrix}$. We now show that in fact, for each $r \in I$, $\bigwedge_q^r(V)$ is a U-submodule of $V^{\otimes r}$. To do this we need to see how the generators of U act on the above basis. We use the above description of their action on V to do this. It is clear that, if $i-1 \notin \{i_1, i_2, \ldots, i_r\}$, then $F_i b_{i_1,i_2,\ldots,i_r} = 0$. Also, if some $i_k = i-1$ but $i \notin \{i_1, i_2, \ldots, i_r\}$, then we have $F_i b_{i_1,\ldots,i_k,\ldots,i_r} = b_{i_1,\ldots,i_k+1,\ldots,i_r}$. The only remaining possibility is that, for some k, $i_k = i-1$ and $i_{k+1} = i$. Let $\xi = F_i b_{i_1,i_2,\ldots,i_r}$. We use the fact that

$$\Delta^{(r-1)}(F_i) = F_i \otimes 1 \cdots \otimes 1 + K_i^{-1} \otimes F_i \otimes 1 \cdots \otimes 1 + \cdots + K_i^{-1} \otimes \cdots \otimes K_i^{-1} \otimes F_i \in U^{\otimes r}.$$

We have:

$$\xi = F_i \sum_{\sigma \in S_r} b_{i_{\sigma^{-1}(1)}} \otimes b_{i_{\sigma^{-1}(2)}} \otimes \cdots \otimes b_{i_{\sigma^{-1}(r)}} (-v)^{-\ell(\sigma)}.$$

Note that in V_1, F_i acts as zero on every basis element but b_{i-1}. In each tensor in the sum, there is one occurrence of b_i and one of b_{i-1}. So, when F_i acts on the sum, it will produce a similar sum, but with each occurrence of b_{i-1} replaced by b_i. Since $K_i^{-1} = \tilde{K}_i^{-1}$ acts as 1 on each basis element apart from b_{i-1} and b_i, upon which it acts as v^{-1} and v respectively, the coefficient of the tensor will be the same in the new sum unless b_i occurs before b_{i-1} in the term in the original sum, in which case the coefficient is multiplied by v. This will happen if and only if $\sigma(k) > \sigma(k+1)$, that is, if and only if $\ell(\sigma\rho) < \ell(\sigma)$. Define $\gamma(\sigma)$ to be 1 in this case, and 0 otherwise. We have:

$$\xi = \sum_{\sigma \in S_r} b_{i_{\sigma^{-1}(1)}} \otimes b_{i_{\sigma^{-1}(2)}} \otimes \cdots \otimes b_{i_{\sigma^{-1}(r)}} (-v)^{-\ell(\sigma)} v^{\gamma(\sigma)}, \tag{2}$$

with each occurrence of b_{i-1} replaced with b_i. Now, as σ varies over S_r, so does $\sigma\rho$, where ρ is the transposition $(k\ k+1)$. So we can rewrite the original sum as follows:

$$\begin{aligned} \xi &= F_i \sum_{\sigma \in S_r} b_{i_{(\sigma\rho)^{-1}(1)}} \otimes b_{i_{(\sigma\rho)^{-1}(2)}} \otimes \cdots \otimes b_{i_{(\sigma\rho)^{-1}(r)}} (-v)^{-\ell(\sigma\rho)} \\ &= F_i \sum_{\sigma \in S_r} b_{i_{\rho\sigma^{-1}(1)}} \otimes b_{i_{\rho\sigma^{-1}(2)}} \otimes \cdots \otimes b_{i_{\rho\sigma^{-1}(r)}} (-v)^{-\ell(\sigma\rho)}. \end{aligned}$$

The effect of introducing ρ is to swap the order of b_{i_k} and $b_{i_{k+1}}$. After applying F_i, the sum will be a linear combination of the same tensors as those in the sum (2) but with different coefficients. If $\gamma(\sigma) = 1$, then the new coefficient will be $(-v)^{-(\ell(\sigma)-1)} = (-v)^{-\ell(\sigma)}(-v) = -(-v)^{-\ell(\sigma)} v^{\gamma(\sigma)}$. If $\gamma(\sigma) = 0$ then the new coefficient will be $(-v)^{-(\ell(\sigma)+1)}(v) = (-v)^{-\ell(\sigma)}(-v)^{-1}(v) = -(-v)^{-\ell(\sigma)} v^{\gamma(\sigma)}$. The v comes from the action of the K_i^{-1} in this case. In both cases the coefficient is the negative of the coefficient in (2). Therefore we have $\xi = -\xi$, whence $\xi = 0$.

In a similar way we can show the following:

$$E_i b_{i_1,\ldots,i_k,\ldots,i_r} = \begin{cases} 0 & \text{if } i \notin \{i_1, \ldots, i_r\}, \\ b_{i_1,\ldots,i_k-1,\ldots,i_r} & \text{if } i_k = i \text{ but } i-1 \notin \{i_1, \ldots, i_r\}, \\ 0 & \text{if } i_k = i \text{ and } i_{k-1} = i-1. \end{cases}$$

Finally, since $\Delta(K_i) = K_i \otimes K_i \otimes \cdots K_i$, we have $K_i b_{i_1,i_2,\ldots,i_r} = v^{\varepsilon_{i-1}-\varepsilon_i} b_{i_1,i_2,\ldots,i_r}$ for $i \in I$, where ε_i is 1 if $i \in \{i_1, i_2, \ldots i_r\}$ and is zero otherwise. Note that this is enough to describe the action of K_μ for any $\mu \in Y$.

We have shown that all of the generators of U map $\bigwedge_q^r(V)$ to itself. It follows that $\bigwedge_q^r(V)$ is a U-submodule of $V^{\otimes r}$.

From the above calculations, we can see that each E_i kills the vector $b_{0,1,\dots,r-1}$, and that this vector generates $\bigwedge_q^r(V)$ as a U^--module. Therefore $b_{0,1,\dots,r-1}$ is a highest weight vector for $\bigwedge_q^r(V)$. A quick check verifies that its weight is the highest weight of V_r, that is, $(0,\dots,0,1,0,\dots,0)$ (with the 1 in the r-th place). Therefore, $b_{0,1,\dots,r-1}$ generates a U-submodule of $\bigwedge_q^r(V)$ isomorphic to V_r. However, the dimensions of V_r and $\bigwedge_q^r(V)$ are the same, so we must have $V_r \cong \bigwedge_q^r(V)$. We have thus proved:

Theorem 3.1 *For each $r \in I$, the r-th quantized exterior power of V_1 is isomorphic to V_r. The isomorphism can be chosen to take $b_{0,1,\dots r-1}$ to x_1.* $\square$

We thus have a natural basis for V_r, consisting of the elements $b_{i_1,i_2,\dots,i_r}$, for $0 \le i_1 < i_2 < \cdots < i_r \le n$ (regarding them as elements of V_r by the above isomorphism). Denote this basis by $\mathcal{C}_r$.

Theorem 3.2 *We have that, for each $r \in I$, $\mathcal{C}_r$ is the canonical basis for V_r. So the elements $b_{i_1,i_2,\dots,i_r}$, for $0 \le i_1 < i_2 < \cdots < i_r \le n$, form the canonical basis for $\bigwedge_q^r(V)$, when it is viewed as V_r.*

Proof: We identify $x_1 \in V_r$ with $b_{0,1,\dots,r-1}$. For $0 \le i_1 < i_2 < \cdots < i_r \le n$, let

$$\xi = (F_{i_1}F_{i_1-1}\cdots F_1)(F_{i_2}F_{i_2-1}\cdots F_2)\cdots(F_{i_r}F_{i_r-1}\cdots F_r).$$

Then, $\xi x_1 = b_{i_1,i_2,\dots,i_r} \neq 0$. By Lemma 2.5, $\xi \in \mathrm{Im}(\phi_r)$, and so $\xi x_1 \in \mathrm{Im}(\psi_r)$. By Theorem 2.16, ξx_1 lies in the canonical basis of V_r. Therefore $\mathcal{C}_r$ is contained in the canonical basis of V_r, but both are bases for V_r, so we must have equality. $\square$

Corollary 3.3 *For each $r \in I$, the canonical basis of V_r is:*

$$\{(F_{i_1}F_{i_1-1}\cdots F_1)(F_{i_2}F_{i_2-1}\cdots F_2)\cdots(F_{i_r}F_{i_r-1}\cdots F_r)x_1 \;:\; 0 \le i_1 < \cdots < i_r \le n\}.$$

Furthermore,

$$W^r = \{(s_{i_1}s_{i_1-1}\cdots s_1)(s_{i_2}s_{i_2-1}\cdots s_2)\cdots(s_{i_r}s_{i_r-1}\cdots s_r) \;:\; 0 \le i_1 < \cdots < i_r \le n\},$$

and each of the expressions in the latter is reduced in W.

Proof: The first statement follows directly from Theorem 3.2 and its proof. Define

$$W_1^r = \{(s_{i_1} s_{i_1-1} \cdots s_1)(s_{i_2} s_{i_2-1} \cdots s_2) \cdots (s_{i_r} s_{i_r-1} \cdots s_r) \ : \ 0 \leq i_1 < \cdots < i_r \leq n\}.$$

By Lemma 2.5 each of the elements of W_1^r lies in W^r, and its expression above is reduced, since the action of the corresponding monomial in U^- on x_1 is non-zero. By the first statement, $\psi_r(W_1^r)$ is the canonical basis of V_r, but the map ψ_r is a bijection from W^r to the canonical basis of V_r, by Proposition 2.6 and Theorem 2.16, whence $W_1^r = W^r$ and we are done. □

Example

We consider the case when U is of type A_4, and $r = 2$. We have:

$$W^2 = \{(s_{j_1} \cdots s_1)(s_{j_2} \cdots s_2) \ : \ 0 \leq j_1 < j_2 \leq 4\}$$

and the canonical basis for V_2 is

$$\{(F_{j_1} \cdots F_1)(F_{j_2} \cdots F_2)x_1 \ : \ 0 \leq j_1 < j_2 \leq 4\}.$$

More explicitly:

j_1	j_2	b_{j_1,j_2}	Element of W^r
0	1	$(1)(1)x_1$	$(1)(1)$
0	2	$(1)(F_2)x_1$	$(1)(s_2)$
0	3	$(1)(F_3F_2)x_1$	$(1)(s_3s_2)$
0	4	$(1)(F_4F_3F_2)x_1$	$(1)(s_4s_3s_2)$
1	2	$(F_1)(F_2)x_1$	$(s_1)(s_2)$
1	3	$(F_1)(F_3F_2)x_1$	$(s_1)(s_3s_2)$
1	4	$(F_1)(F_4F_3F_2)x_1$	$(s_1)(s_4s_3s_2)$
2	3	$(F_2F_1)(F_3F_2)x_1$	$(s_2s_1)(s_3s_2)$
2	4	$(F_2F_1)(F_4F_3F_2)x_1$	$(s_2s_1)(s_4s_3s_2)$
3	4	$(F_3F_2F_1)(F_4F_3F_2)x_1$	$(s_3s_2s_1)(s_4s_3s_2)$

ACKNOWLEDGMENTS

This paper was written while the author was supported by a grant from the Engineering and Physical Science Research Council, and supervised by Professor R. W. Carter at the University of Warwick, England.

REFERENCES

[1] N. Bourbaki. *Groupes et algèbres de Lie, Chapitres 7 et 8.* Hermann, Paris, 1975.

[2] N. Bourbaki. *Groupes et algèbres de Lie, Chapitres 4,5 et 6.* Masson, Paris, 1981.

[3] C. W. Curtis and I. Reiner. *Methods of Representation Theory with Applications to Finite Groups and Orders*, volume II. Wiley, New York, 1987.

[4] V. G. Drinfel'd. Hopf algebras and the Yang–Baxter equation. *Soviet Math. Dokl.*, 32:254–258, 1985.

[5] J. E. Humphreys. *Reflection Groups and Coxeter Groups.* C.U.P., Cambridge, 1992.

[6] M. Jimbo. A q-difference analogue of $U(\mathbf{g})$ and the Yang–Baxter equation. *Lett. Math. Phys.*, 10:63–69, 1985.

[7] M. Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras. *Duke Math. J.*, 63(2):465–516, 1991.

[8] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3:447–498, 1990.

[9] G. Lusztig. *Introduction to Quantum Groups.* Birkhäuser, Boston, 1993.