MATH1022, INTRODUCTORY GROUP THEORY
Question Sheet 5: Symmetric Groups

To be handed in by Friday 25th April

Q1. Write each of the following permutations as a product of *disjoint* cycles:

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 1 & 3 & 7 & 9 & 8 & 2 & 6 & 5 \end{pmatrix}$ (in S_9),

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & 3 & 6 & 8 & 9 & 5 & 1 & 10 & 7 & 11 & 2 \end{pmatrix}$ (in S_{11})

Q2. Write each of the following products of permutations as a product of disjoint cycles (see the example in T2 overleaf):

(a) $(1\ 8\ 2\ 4)(3\ 5\ 9\ 7)(2\ 7)(1\ 9\ 3)$,

(b) $(2\ 5\ 3\ 7)(1\ 3\ 7)(2\ 6\ 4)(2\ 7\ 3\ 5)$.

Q3. Show that the identity together with the three elements $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, and $(1\ 4)(2\ 3)$ form a subgroup V_4 of S_4 of order 4, and that V_4 is isomorphic to the group of symmetries of a (non-square) rectangle.

Q4. Find the orders of the following elements of S_{15}, and determine which pairs of them are conjugate:

(a) $(1\ 2\ 4\ 8)(3\ 6\ 12)(5\ 10\ 15)$,

(b) $(1\ 2)(4\ 8)(3\ 12\ 11)(5\ 13\ 7)$,

(c) $(1\ 7\ 9)(11\ 12\ 13)(5\ 10\ 8\ 6)$.

Q5. Given that a cycle of odd length is even, and a cycle of even length is odd, determine which of the following members of S_{10} are even, or odd:

(a) $(1\ 9\ 3)(2\ 6)(4\ 5\ 10)(7\ 8)$,

(b) $(1\ 9\ 3\ 2\ 6)(1\ 3\ 9\ 6\ 2\ 4)$,

(c) $(1\ 9\ 3\ 2\ 6\ 4\ 5\ 10)$.

Q6. Prove that the subgroup V_4, considered in question Q3 above, is in fact a normal subgroup of S_4.

Q7. Let H be the subset of S_n consisting of those elements which can be written as a product of k transpositions, where k is some multiple of 3 (including zero). Prove that H is a subgroup of S_n. Find H in the case when $n = 3$. Can you say anything about the general case?

MATH1022, INTRODUCTORY GROUP THEORY
Tutorial Exercises 5: Symmetric Groups

To be discussed in the tutorial in the week beginning Monday 21 April

T1. Write each of the following permutations as a product of *disjoint* cycles:

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 9 & 8 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$ (in S_9),

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 \end{pmatrix}$ (in S_{11}).

T2. Write each of the following products of permutations as a product of cycles: (remember to start with the rightmost cycle; see example below).

(a) (1 2 4 8 16)(2 3 5 9)(3 4 6 10)(4 5 7 11),

(b) (1 5)(2 9)(1 2 3 4 5 6 7 8 9)(1 5)(2 9).

Example: $\pi =$ (1 4 2 3)(1 2). Since (1 2) sends 1 to 2 and (1 4 2 3) sends 2 on to 3, overall we have $\pi(1) = 3$. We then see what $\pi(3)$ is. Since (1 2) fixes 3 and 3 is mapped to 1 by (1 4 2 3), we have $\pi(3) = 1$. So we have a cycle (1 3) in the disjoint cycle decomposition of π.

Since (1 2) sends 2 to 1 and (1 4 2 3) sends 1 on to 4, we have $\pi(2) = 4$. We then see what $\pi(4)$ is. Since (1 2) fixes 4 and (1 4 2 3) send 4 on to 2, we see that $\pi(4) = 2$. Hence we have a cycle (2 4) in the disjoint cycle decomposition of π.

Putting this together we see that $\pi =$ (1 3)(2 4) (the order of these two cycles is not important, since they are disjoint).

T3. Show that the set of all elements of S_4 which fix 3 (i.e. map 3 to 3) form a subgroup H of S_4 of order 6, which is isomorphic to S_3.

T4. Find the orders of the following elements of S_{16}, and determine which pairs of them are conjugate:

(a) (1 5 7 16)(2 11 13)(3 6)(8 10 12),

(b) (3 4)(5 9)(6 14)(7 10 11),

(c) (1 14)(2 4 8)(5 10)(6 7).

T5. Show that a cycle of odd length is an even permutation, and a cycle of even length is an odd permutation. Hence determine which of the following members of S_{11} are even, or odd:

(a) (1 2)(3 6 8)(4 11 10 5 9 7),

(b) (1 3 5 7 9 11 2 4 6 8),

(c) (1 2 3 4)(1 2 4 3)(1 4 2 3).

T6. Prove that the subgroup $\langle(1\ 2\ 3)\rangle$ of S_4 (i.e. the subgroup generated by (1 2 3)) is *not* a normal subgroup of S_4.

T7. Write the 12 elements of R_{tet}, the group of rotations of a regular tetrahedron, as permutations of the vertices A, B, C, D, and hence show that R_{tet} is isomorphic to the alternating group A_4.