MATH1022, INTRODUCTORY GROUP THEORY
Question Sheet 2: Subgroups; examples

To be handed in by Friday 15th February

Q1. Find all subgroups of the group $\{I, R, S, T, U, V\}$ of rotations of a regular hexagon (see Question Sheet 1, Question 2).

Q2. In the group $\{1, 2, 3, \ldots, 10\}$ under multiplication mod 11, find which of the following sets are subgroups:
(i) $H_1 := \{1, 3, 4, 5, 9\}$,
(ii) $H_2 := \{1, 3, 5, 7, 8\}$,
(iii) $H_3 := \{1, 8\}$,
(iv) $H_4 := \{1, 10\}$,
(v) $H_5 := \{1, 3, 10\}$.
(Reminder: to check that H is a subgroup see (a) if it is closed under the group operation, (b) whether the identity lies in H, (c) whether the inverses of all elements of H lie in H.)

Q3. Let G be the additive group of integers $(\mathbb{Z}, +)$. Find two subgroups of G whose union is not a subgroup.

Q4. A *regular tetrahedron* is a solid with 4 vertices and 4 faces, all of them equilateral triangles.
Draw a diagram of a regular tetrahedron, and describe all rotations of three-dimensional space which preserve a tetrahedron.
(Hint; there are three different kinds of rotation: the identity, and rotations through $2\pi/3$ and π. The key is locating the possible axes for rotations.)
Calculate the order of the rotation group R_{tet} of a regular tetrahedron and find the orders of all elements of R_{tet}. Which elements of R_{tet} are equal to their own inverse?

Q5. Show that the set of all complex numbers of modulus 1 forms a subgroup of the group $\mathbb{C}^*$ of non-zero complex numbers under multiplication.

Q6. Prove that a group G is abelian if and only if for every $a, b \in G$ and positive integer n, $(ab)^n = a^n b^n$.

MATH1022, INTRODUCTORY GROUP THEORY
Tutorial Exercises 2: Subgroups; examples

To be discussed in the tutorial in the week beginning Monday 11 February

T1. Find all subgroups of the group $G := \{1, 5, 7, 11\}$ under $\times$ mod 12 (see Question Sheet 1, Question 3(iii)).
(*Note that your answer should include a proof that all of the subgroups of G appear in your list*).

T2. In the group $G := \{1, 2, 3, 4, 5, 6\}$ under multiplication mod 7, find which of the following subsets of G are subgroups of G:
(i) $H_1 := \{1, 3, 4, 5\}$,
(ii) $H_2 := \{2, 4, 6\}$,
(iii) $H_3 := \{1, 6\}$,
(iv) $H_4 := \{1, 2, 4\}$,
(v) $H_5 := \varphi$ (the empty subset).

(Reminder: to check that a subset H of G is a subgroup of G see (a) if it is closed under the group operation, (b) whether the identity lies in H, (c) whether the inverses of all elements of H lie in H.)

T3. Prove that the intersection of two subgroups of a group G is also a subgroup of G.
Let n be a positive integer, and let $n\mathbb{Z}$ denote the set of all integers which are divisible by n. Prove that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, and show that the intersection of $m\mathbb{Z}$ and $n\mathbb{Z}$ is equal to $l\mathbb{Z}$ where l is the least common multiple of m and n.

T4. Describe all rotations of three-dimensional space that preserve a cube.
(Hint: there are three different kinds of axis).
Calculate the order of the group R_{cube} of rotations of a cube, and find the orders of all of its elements.
(Recall that the order of a group is the number of elements in it, while the order of an element g is the smallest positive integer n such that $g^n = e$, or infinity if no such n exists.)

T5. Show that the set of all non-singular upper triangular matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with real a, b, and c, forms a subgroup of $GL_2(\mathbb{R})$. Generalize to $GL_n(\mathbb{R})$.

T6. Let G be a group in which the square of every element is equal to the identity. Prove that G is abelian. Is the converse true? Justify your answer.