MATH1022, INTRODUCTORY GROUP THEORY
Question Sheet 1: Introduction

To be handed in by Friday 1st February

Q1. Draw up the Cayley table for the group $G = \{1, 2, 3, 4, 5, 6\}$ under $\times$ mod 7. List all the inverses of elements of G, and find an element x of G such that every member of G is a power of x.

Q2. Draw up the Cayley table for the group of rotations of a regular hexagon, $G = \{I, R, S, T, U, V\}$ where I is the identity, and R, S, T, U, V are clockwise rotations through $\pi/3, 2\pi/3, \pi, 4\pi/3$, and $5\pi/3$ respectively. State what the inverses of each of the six elements are.

Q3. Which of the following are groups? For those which are not groups, explain why not (it is enough to find one axiom that fails); for those which are, show that all the axioms hold, and also state whether or not they are abelian.

(i) $\mathbb{Q}$ under $\times$;

(ii) $\{q \in \mathbb{Q} : q > 0\}$ under $\times$;

(iii) $\{q \in \mathbb{Q} : q > 0\}$ under $\div$;

(iv) $\{1, 2, 3, 4, 5, 6, 7\}$ under $\times$ mod 8;

(v) $\{1, 3, 5, 7\}$ under $\times$ mod 8;

(vi) $\{2, 4, 6, 8, 10, 12\}$ under $\times$ mod 14;

(vii) The set $\{0, 1, 2, 3, 4, 5\}$ under the operation $x \circ y = |x + y - 5|$.

Q4. Show that the set of all real numbers not equal to -1, together with the operation $*$ given by $x * y = xy + x + y$, forms a group.

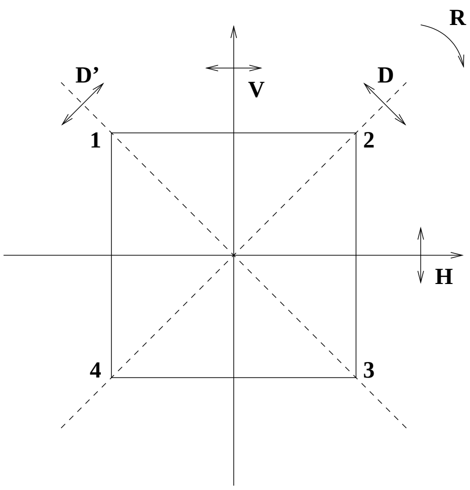
Q5. Let G be a group of order 3 and identity e. If the other elements are a and b, use the 'Latin square' property to draw up the Cayley table for G, and deduce that $a^3 = e$.

MATH1022, INTRODUCTORY GROUP THEORY
Tutorial Exercises 1: Introduction

To be discussed in the tutorial in the week beginning Monday 28th January

T1. Draw up the Cayley table for the group $G = \{1, 3, 5, 9, 11, 13\}$ under $\times$ mod 14. List all the inverses of elements of G, and find an element x of G such that every element of G is a power of x.

T2. Draw up the Cayley table for the group D_4 of symmetries of a square centred at the origin, $D_4 = \{I, R, R^2, R^3, H, V, D, D'\}$ where I is the identity, R is a clockwise rotation through $\pi/2$, H and V are reflections in the horizontal and vertical, and D, D' are reflections in the diagonals $y = x$ and $y = -x$ respectively.
Hint: Don't compute all the entries individually. First fill in the first row and column. The top left-hand 4×4 corner, and the main diagonal, are also quite easy. Then choose two or three other entries to do, and fill in the rest using the 'Latin square' property. Don't forget that, for example, RV means "do V and then R" to the rectangle, as we write maps on the left of their arguments.



T3. Which of the following are groups? For those which are not groups, explain why not (it is enough to find one axiom that fails); for those which are, show that all the axioms hold. For each one which is a group, state also whether or not it is abelian.
(i) the set of all real numbers under multiplication;
(ii) the set of all complex numbers of modulus 1 under multiplication;
(iii) $\{1, 5, 7, 11\}$ under $\times$ mod 12;
(iv) $\{2, 4, 6, 8\}$ under $\times$ mod 10;
(v) $\{2, 4, 6, 8, 10, 12, 14\}$ under $\times$ mod 16;
(vi) the set of all subsets of a set X, where the operation is $\cap$ (intersection).

T4. Prove that the set of all functions f from $\mathbb{R}$ to itself of the form $f(x) = ax + b$ where a and b are real numbers with $a \neq 0$, forms a group under function composition.

T5. Show that for elements g and h of a group G, the following are equivalent:
(i) $gh = hg$,
(ii) $(gh)^2 = g^2h^2$,
(iii) $(gh)^{-1} = g^{-1}h^{-1}$.