MATH1022, INTRODUCTORY GROUP THEORY
Question Sheet 6: Normal Subgroups and Quotient Groups

Not for assessment

Q1 Let $G = \mathbb{Z}_{14}^*$ and consider the subset $N = \{1, 13\}$ of G. Prove that N is a normal subgroup of G. List the elements of G/N (i.e. the right cosets of N) in the form Nx for some $x \in G$, and compute the Cayley table of G/N.

Q2 Find the centre Z of the dihedral group D_4 of order 8, and compute the Cayley table of D_4/Z.

Q3 Prove that the centre of a group G is a normal subgroup of G.

Q4 Prove that each of the following maps is a group homomorphism, and find the kernel and the image in each case.
(i) The map $\theta : \mathbb{Z} \to \mathbb{Z}$ given by $\theta(x) = 3x$, where the operation in $\mathbb{Z}$ is addition.
(ii) The map $\theta : \mathbb{R}^* \to \mathbb{R}^*$ given by $\theta(x) = x^2$, where $\mathbb{R}^*$ denotes the group of non-zero real numbers under multiplication.
(iii) The map $\theta : \mathbb{Z} \to \mathbb{Z}_n$ given by $\theta(a) = r$, where r is the remainder on dividing a by n. Here the operation in $\mathbb{Z}$ is addition and the operation in $\mathbb{Z}_n$ is addition modulo n, as usual.

Q5 For each homomorphism in Question Q4, write down the conclusion of the First Isomorphism Theorem.

Q6 (a) Use Fermat's Little Theorem to show that 323 is not prime. Confirm this by writing 323 as a nontrivial product.
(b) Show that $2^{560} \equiv 1 \mod 561$. Can you conclude anything from Fermat's Little Theorem in this case?