

# MATH1022, INTRODUCTORY GROUP THEORY

## Question Sheet 5: Symmetric Groups

To be handed in by Friday 27th April

Q1. Write each of the following permutations as a product of disjoint cycles:

(a)  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 9 & 8 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$  (in  $S_9$ ),

(b)  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 \end{pmatrix}$  (in  $S_{11}$ ).

Q2. Write each of the following products of permutations as a product of disjoint cycles: (remember to start from the right):

(a)  $(1\ 2\ 4\ 8\ 16)(2\ 3\ 5\ 9)(3\ 4\ 6\ 10)(4\ 5\ 7\ 11)$ ,

(b)  $(1\ 5)(2\ 9)(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)(1\ 5)(2\ 9)$ .

Q3. Show that the set of all elements of  $S_4$  which fix 3 (i.e. map 3 to 3) form a subgroup  $H$  of  $S_4$  of order 6, which is isomorphic to  $S_3$ .

Q4. Find the orders of the following members of  $S_{16}$ , and determine which pairs of them are conjugate:

(a)  $(1\ 5\ 7\ 16)(2\ 11\ 13)(3\ 6)(8\ 10\ 12)$ ,

(b)  $(3\ 4)(5\ 9)(6\ 14)(7\ 10\ 11)$ ,

(c)  $(1\ 14)(2\ 4\ 8)(5\ 10)(6\ 7)$ .

Q5. Show that a cycle of odd length is an even permutation, and a cycle of even length is an odd permutation. Hence determine which of the following members of  $S_{11}$  are even, or odd:

(a)  $(1\ 2)(3\ 6\ 8)(4\ 11\ 10\ 5\ 9\ 7)$ ,

(b)  $(1\ 3\ 5\ 7\ 9\ 11\ 2\ 4\ 6\ 8)$ ,

(c)  $(1\ 2\ 3\ 4)(1\ 2\ 4\ 3)(1\ 4\ 2\ 3)$ .

Q6. Prove that the subgroup  $V_4$ , considered in question T3 overleaf, is in fact a normal subgroup of  $S_4$ .

Q7. Let  $H$  be the subset of  $S_n$  consisting of those elements which can be written as a product of  $k$  transpositions, where  $k$  is some multiple of 3 (including zero). Prove that  $H$  is a subgroup of  $S_n$ .

Find  $H$  in the case when  $n = 3$ . Can you say anything about the general case?

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## Tutorial Exercises 5: Symmetric Groups

To be discussed in the tutorial in the week 23 April – 27 April

T1. Write each of the following permutations as a product of disjoint cycles:

(a)  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 1 & 3 & 7 & 9 & 8 & 2 & 6 & 5 \end{pmatrix}$  (in  $S_9$ ),

(b)  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & 3 & 6 & 8 & 9 & 5 & 1 & 10 & 7 & 11 & 2 \end{pmatrix}$  (in  $S_{11}$ )

T2. Write each of the following products of permutations as a product of disjoint cycles: (remember to start from the right):

(a)  $(1\ 8\ 2\ 4)(3\ 5\ 9\ 7)(2\ 7)(1\ 9\ 3)$ ,

(b)  $(2\ 5\ 3\ 7)(1\ 3\ 7)(2\ 6\ 4)(2\ 7\ 3\ 5)$ .

T3. Show that the identity together with the three elements  $(1\ 2)(3\ 4)$ ,  $(1\ 3)(2\ 4)$ , and  $(1\ 4)(2\ 3)$  form a subgroup  $V_4$  of  $S_4$  of order 4, and that  $V_4$  is isomorphic to the group of symmetries of a (non-square) rectangle.

T4. Find the orders of the following members of  $S_{15}$ , and determine which pairs of them are conjugate:

(a)  $(1\ 2\ 4\ 8)(3\ 6\ 12)(5\ 10\ 15)$ ,

(b)  $(1\ 2)(4\ 8)(3\ 12\ 11)(5\ 13\ 7)$ ,

(c)  $(1\ 7\ 9)(11\ 12\ 13)(5\ 10\ 8\ 6)$ .

T5. Given that a cycle of odd length is even, and a cycle of even length is odd, determine which of the following members of  $S_{10}$  are even, or odd:

(a)  $(1\ 9\ 3)(2\ 6)(4\ 5\ 10)(7\ 8)$ ,

(b)  $(1\ 9\ 3\ 2\ 6)(1\ 3\ 9\ 6\ 2\ 4)$ ,

(c)  $(1\ 9\ 3\ 2\ 6\ 4\ 5\ 10)$ .

T6. Prove that the subgroup  $\langle(1\ 2\ 3)\rangle$  of  $S_4$  (i.e. the subgroup generated by  $(1\ 2\ 3)$ ) is *not* a normal subgroup of  $S_4$ .

T7. Write the 12 elements of  $R_{\text{tet}}$ , the group of rotations of a regular tetrahedron, as permutations of the vertices  $A, B, C, D$ , and hence show that  $R_{\text{tet}}$  is isomorphic to the alternating group  $A_4$ .