MATH1022, INTRODUCTORY GROUP THEORY
Question Sheet 3: Cyclic groups, isomorphism

To be handed in by Friday 2nd March

You may assume on this question sheet that any two cyclic groups of the same order are isomorphic.

Q1. Which of the following groups are cyclic?

(i) $\mathbb{R}$ under $+$.

(ii) the set $\{A^n : n \in \mathbb{Z}\}$ where A is an invertible $n \times n$ matrix with real entries, under matrix multiplication.

(iii) The group of units of $\mathbb{Z}_{12}$ (denoted $\mathbb{Z}_{12}^*$), under multiplication mod 12.

(iv) $\mathbb{Z}_{11}^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ under multiplication mod 11.

Q2. Show that the group D_4 of symmetries of a square (see Question Sheet 1, Question 2) and the group $\mathbb{Z}_{16}^*$ are non-isomorphic groups.

Q3. Show that $\mathbb{Z}_{14}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_7$, but that $\mathbb{Z}_{24}$ is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_6$.

Q4. Find which pairs of the following groups are isomorphic:

(i) the dihedral group D_3 (symmetries of an equilateral triangle), under composition of functions,

(ii) the (complex) matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix},$ where $\omega^3 = 1$, $\omega \neq 1$, under matrix multiplication,

(iii) $\{1, 2, 3, 4, 5, 6\}$ under $\times$ mod 7.

Q5. Let G be a cyclic group of order 8. Calculate the number of elements of G which are generators of G. Repeat the question for a cyclic group of order 9.

Q6. Prove that the set of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for a a non-zero real number, forms a subgroup of $GL_2(\mathbb{R})$ isomorphic to the multiplicative group of non-zero reals.

Q7. Prove that the group of real numbers under addition is isomorphic to the group of positive real numbers under multiplication. (Hint; how can you change $+$ into $\times$?)

MATH1022, INTRODUCTORY GROUP THEORY
Tutorial Exercises 3: Cyclic groups, isomorphism

To be discussed in the tutorial in the week 26 February — 2 March

You may assume on this tutorial sheet that any two cyclic groups of the same order are isomorphic.

T1. Which of the following groups are cyclic?
(i) $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ under $\times$.
(ii) The set of functions from $\mathbb{R}$ to $\mathbb{R}$ of the form $f(x) = x + a$ for some $a \in \mathbb{Z}$, under composition of functions.
(iii) $\{1, 2, 4, 7, 8, 11, 13, 14\}$ under multiplication mod 15.
(iv) $\{1, 3, 5, 9, 11, 13\}$ under multiplication mod 14.

T2. Let G_1 be the group of all integers in $\{1, 3, 5, 7, 9, 11, 13, 15\}$ under $\times$ mod 16, and G_2 be the group of all integers in $\{1, 2, 4, 7, 8, 11, 13, 14\}$ under $\times$ mod 15. Find the orders of G_1 and G_2, and of all their elements. Are G_1, G_2 abelian? Are they isomorphic? (If you think they are isomorphic, you should try to give a bijection from G_1 onto G_2 which is an isomorphism.)

T3. Show that $\mathbb{Z}_{10}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_5$, but that $\mathbb{Z}_8$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

T4. Find which pairs of the following groups are isomorphic:
(i) the dihedral group D_6 (symmetries of a regular hexagon),
(ii) $\mathbb{Z}_6$ under + mod 6,
(iii) $\mathbb{Z}_{12}$ under + mod 12,
(iv) the group of rotations of a regular hexagon,
(v) the group of rotations of a regular tetrahedron.

T5. Let G be a cyclic group of order 7. Calculate the number of elements of G which are generators of G. Repeat the question for a cyclic group of order 10. (Note that by the assumption at the start of this question sheet, it is enough to do this question for the groups $\mathbb{Z}_7$ under addition mod 7 and $\mathbb{Z}_{10}$ under addition mod 10 respectively.)

T6. Prove that the set of all 2×2 matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for a a real number, forms a subgroup of $GL_2(\mathbb{R})$ isomorphic to the additive group of reals.

T7. Let G be the group of real numbers under addition, and let H be the group of all non-zero real numbers under $\times$. Show that H has an element of order 2, but that G does not, and deduce that G and H are not isomorphic.