

MATH1022, INTRODUCTORY GROUP THEORY

Question Sheet 2: Subgroups; rotations of a cube

To be handed in by Friday 16th February

Q1. Find all subgroups of the group $G := \{1, 5, 7, 11\}$ under $\times \pmod{12}$ (see Question Sheet 1, Question 3(iii)).

(Note that your answer should include a proof that all of the subgroups of G appear in your list).

Q2. In the group $G := \{1, 2, 3, 4, 5, 6\}$ under multiplication mod 7, find which of the following subsets of G are subgroups of G :

(i) $H_1 := \{1, 3, 4, 5\}$,

(ii) $H_2 := \{2, 4, 6\}$,

(iii) $H_3 := \{1, 6\}$,

(iv) $H_4 := \{1, 2, 4\}$,

(v) $H_5 := \varphi$ (the empty subset).

(Reminder: to check that a subset H of G is a subgroup of G see (a) if it is closed under the group operation, (b) whether the identity lies in H , (c) whether the inverses of all elements of H lie in H .)

Q3. Prove that the intersection of two subgroups of a group G is also a subgroup of G .

Let n be a positive integer, and let $n\mathbb{Z}$ denote the set of all integers which are divisible by n . Prove that $n\mathbb{Z}$ is a subgroup of \mathbb{Z} , and show that the intersection of $m\mathbb{Z}$ and $n\mathbb{Z}$ is equal to $l\mathbb{Z}$ where l is the least common multiple of m and n .

Q4. Describe all rotations of three-dimensional space that preserve a cube.

(Hint: there are three different kinds of axis).

Calculate the order of the group R_{cube} of rotations of a cube, and find the orders of all of its elements. (Recall that the order of a group is the number of elements in it, while the order of an element g is the smallest positive integer n such that $g^n = e$, or infinity if no such n exists.)

Q5. Show that the set of all non-singular upper triangular matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with real a , b , and c , forms a subgroup of $GL_2(\mathbb{R})$. Generalize to $GL_n(\mathbb{R})$.

Q6. Let G be a group in which the square of every element is equal to the identity. Prove that G is abelian. Is the converse true? Justify your answer.

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Tutorial Exercises 2

To be discussed in the tutorial in the week 12-16 February

T1. Find all subgroups of D_4 , the group of symmetries of a square (see Question Sheet 1, Question 2).

T2. In the group $\{1, 2, 3, \dots, 10\}$ under multiplication mod 11, find which of the following sets are subgroups:

- (i) $H_1 := \{1, 3, 4, 5, 9\}$,
- (ii) $H_2 := \{1, 3, 5, 7, 8\}$,
- (iii) $H_3 := \{1, 8\}$,
- (iv) $H_4 := \{1, 10\}$,
- (v) $H_5 := \{1, 3, 10\}$.

(Reminder: to check that H is a subgroup see (a) if it is closed under the group operation, (b) whether the identity lies in H , (c) whether the inverses of all elements of H lie in H .)

T3. Let G be the additive group of integers $(\mathbb{Z}, +)$. Find two subgroups of G whose union is not a subgroup.

T4. A *regular tetrahedron* is a solid with 4 vertices and 4 faces, all of them equilateral triangles.

Draw a diagram of a regular tetrahedron, and describe all rotations of three-dimensional space which preserve a tetrahedron.

(Hint; there are three different kinds of rotation: the identity, and rotations through $2\pi/3$ and π .)

Calculate the order of the rotation group R_{tet} of a regular tetrahedron and find the orders of all elements of R_{tet} . Which elements of R_{tet} are equal to their own inverse?

T5. Show that the set of all $n \times n$ real matrices of determinant 1 forms a subgroup of $GL_n(\mathbb{R})$.

(Hint: use a formula for $\det(AB)$ in terms of $\det(A)$ and $\det(B)$.) The notation for this group is $SL_n(\mathbb{R})$.

T6. Prove that a group G is abelian if and only if for every $a, b \in G$ and positive integer n , $(ab)^n = a^n b^n$.