

# Wavelets in Applied and Pure Mathematics

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ABSTRACT. The course gives an overview of *wavelets* (or *coherent states*) construction and its realisations in applied and pure mathematics. After a short introduction to wavelets based on the representation theory of groups we will consider:

- Spaces of analytic functions with reproducing kernels: the Hardy and the Bergman spaces, etc.;
- The Fock-Segal-Bargmann space and Berezin-Toeplitz quantisation;
- Functional calculus of self-adjoint operators;
- Elements of signal processing.

The variety of applications is essentially grouped just around three groups: the Heisenberg group,  $SL_2(\mathbb{R})$ , and  $\mathfrak{a}x + \mathfrak{b}$  group.

## Course Outline

List of Figures	5
Preface	7
Preface	7
Lecture 1. What Are Wavelets and What Are They Good for?	9
1. Fourier Transform and Bases in Hilbert Spaces	9
2. Complex Analysis and Reproducing Kernels	10
3. Quantum Mechanics and Quantisation	12
4. Signal Processing	13
5. Functional Calculus	13
6. Discussion	13
Lecture 2. Groups and Homogeneous Spaces	17
1. Basics of Group Theory	17
2. Homogeneous Spaces and Invariant Measures	18
Lecture 3. Elements of the Representation Theory	23
1. Representations of Groups	23
2. Decomposition of Representations	25
3. Invariant Operators and Schur's Lemma	26
Lecture 4. Wavelets on Groups and Square Integrable Representations	29
1. Wavelet Transform on Groups	29
2. Square Integrable Representations	31
Lecture 5. Wavelets on Homogeneous Spaces: the Segal-Bargmann Space	35
1. Quantum Mechanical Setting	35
2. Fundamentals of Wavelets on Homogeneous Spaces	36
3. Advanced Properties	41
Lecture 6. Wavelets in Banach Spaces and Functional Calculus	43
1. Coherent States for Banach Spaces	43
2. Wavelets in Operator Algebras	49
3. Examples	55
Acknowledgements	61
Bibliography	63
Index	67



## List of Figures

1	The Gaussian function	12
2	An example of Windowed Fourier Transform	14



## Preface

The purpose of this course is to sketch in ten lectures a huge area related to *wavelets*. There are no precise boundaries of this area: it overlaps with many other subjects in pure mathematics and many applications in physics and engineering. Moreover there are many different approaches to wavelets based on rather different techniques. However this course is not intended to be complete and encyclopedic. Our main goal is to generate an interest in wavelets and to show that much of the theory and applications are related to *groups* and *symmetries*.

The word “wavelets” came to fashion about 15 years ago and is very popular now. On the other hand the notion of wavelets resemble *coherent states* used in quantum mechanics for 75 years already. Future analysis shows that many classic objects (e.g. from complex analysis) known at least from XIX century are essentially wavelets-coherent states too. This indicates that a significance of wavelets is above just a current fashion.

We apply the name “wavelets” to the whole range of related objects to stress their common origin and nature. Meanwhile the common usage of this term is much narrower. Objects called “wavelets” by us usually appear as *coherent states* (CS) in the literature. While commonly “wavelets” are coherent states related to  $ax + b$  group.



## LECTURE 1

# What Are Wavelets and What Are They Good for?

In this introductory lecture one would only sketch an answer to the above question: even the whole course could not be enough for that. Now we just list several instances of *wavelets* appeared in different areas. All mentioned topics will be considered in greater details in following lectures.

Most of the listed facts should be well known to reader, we are just presenting them in a way highlighting the common structure. Similarities and differences of these instances of wavelets will be discussed in the [final section 6](#).

### 1. Fourier Transform and Bases in Hilbert Spaces

We start from two basic examples which were at the beginning of harmonic and functional analysis.

**1.1. Fourier Series and Basis in Hilbert Space.** Consider the space  $L_2[-\pi, \pi]$  of square integrable functions on  $[-\pi, \pi]$  with the Lebesgue measure. It is a *Hilbert space* with a scalar product

$$(1.1) \quad \langle f_1, f_2 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(x) \bar{f}_2(x) dx.$$

Let us introduce the set of functions

$$(1.2) \quad e_0(x) = \frac{1}{2}, \quad e_{2n}(x) = \cos nx, \quad e_{2n-1} = \sin nx, \quad \text{where } n \in \mathbb{N}.$$

It is a straightforward calculation that

$$(1.3) \quad \langle e_i, e_j \rangle = \delta_{i-j},$$

where  $\delta_{i-j}$  is the Kronecker delta. Moreover the set (1.2) is a maximal family of functions in  $L_2[-\pi, \pi]$  with the above property. In fact (1.3) could be taken as a definition of *orthonormal base* in a Hilbert space  $H$ .

It worths to state main properties of the family (1.2) in the generality of an arbitrary orthonormal base. For such a base  $e_j$  the following is true [32, § III.5.1]:

- A base  $e_i$  define a *linear continuous mapping*  $\mathcal{W} : H \rightarrow \ell_2(Z)$  of a vector  $f \in H$  to a sequence of coefficients in  $\ell_2(Z)$  by the formula:

$$(1.4) \quad \hat{f}_n = \langle f, e_n \rangle.$$

- The above mapping is an *isometry* of the Hilbert spaces (the *Parseval(-Pythagoras) identity*):

$$(1.5) \quad \langle f, f' \rangle = \sum_{j=-\infty}^{\infty} \hat{f}_j \overline{\hat{f}'_j}.$$

- The mapping  $\mathcal{W}$  could be inverted by an operator  $\mathcal{M} : \ell_2(\mathbb{Z}) \rightarrow H$ , namely we could reconstruct a vector  $f$  from its sequence of coefficients as a linear combination of  $e_j$ :

$$(1.6) \quad f = \sum_{j=-\infty}^{\infty} f_j e_j.$$

- From the above we could define a reproducing operator  $P = \mathcal{M}\mathcal{W}$  on  $H$ , symbolically written by the Dirac *bra and ket* notations:

$$P = \sum_{j=-\infty}^{\infty} |e_j\rangle \langle e_j|.$$

**1.2. Fourier Transform.** It is useful to compare the above properties of the Fourier series with the Fourier integral transform. The later is defined in  $L_2(\mathbb{R})$  with the scalar product

$$(1.7) \quad \langle f_1, f_2 \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t) \bar{f}_2(t) dt.$$

by means of functions:

$$(1.8) \quad e_a(t) = e^{iat}, \quad a \in \mathbb{R}.$$

A replacement of the orthonormal property (1.3) is the following identity (cf. (1.10)):

$$(1.9) \quad \langle e_a, e_b \rangle = \delta(a - b),$$

where  $\delta(a - b)$  is the Dirac delta function and the identity is true in the sense of distributions [32, § III.4.4].

The following is true [32, § IV.2.3]:

- Functions  $e_a$  define a *linear continuous mapping*  $\mathcal{W} : L_2(\mathbb{Z}) \rightarrow L_2(\mathbb{Z})$  by the formula:

$$\hat{f}(a) = \langle f, e_a \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-iat} dt.$$

- The above mapping is an *isometry* of the Hilbert spaces (the *Plancherel identity*):

$$\langle f, f' \rangle = \langle \hat{f}, \hat{f}' \rangle.$$

- The mapping  $\mathcal{W}$  could be inverted by an operator  $\mathcal{M} : L_2(\mathbb{Z}) \rightarrow L_2(\mathbb{Z})$ , i.e. we could reconstruct a function  $f$  from its Fourier transform as a continuous linear combination of  $e_a(t)$ :

$$f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(a) e^{iat} da.$$

- The composition of the above two operators  $P = \mathcal{M}\mathcal{W}$  gives an integral resolution (in the distributional sense) of the Dirac delta function  $\delta(u-t)$ :

$$(1.10) \quad f(u) = \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} e^{i(u-t)a} da dt.$$

## 2. Complex Analysis and Reproducing Kernels

We move to the classic Hilbert spaces in complex analysis which are examples of wavelets in pure mathematics. Particularly the first example named after G.H. Hardy, probably the purest mathematician of all times and nations.

**2.1. The Hardy Space.** Let  $H_2(\mathbb{T})$  be the *Hardy space* of  $L_2$  functions on the *unit circle*  $\mathbb{T}$  with an analytic continuation inside the *unit disk*  $\mathbb{D}$ . The scalar product is defined as follows:

$$(2.1) \quad \langle f, f' \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \bar{f}'(t) dt.$$

We could consider a set of functions in  $H_2(\mathbb{T})$  parametrised by a point  $a$  of  $\mathbb{D}$ :

$$(2.2) \quad e_a(t) = \frac{1}{\bar{a}e^{it} - 1}.$$

Then we could find similarly to cases of the Fourier series and integral that:

- Functions  $e_a(t)$  define a *linear continuous mapping*  $\mathcal{W} : H_2(\mathbb{T}) \rightarrow H_2(\mathbb{D})$  of a function on  $\mathbb{T}$  to an analytic function in  $\mathbb{D}$ :

$$(2.3) \quad \begin{aligned} \hat{f}(a) = [\mathcal{W}f](a) &= \langle f, e_a \rangle \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \overline{\left( \frac{1}{\bar{a}e^{it} - 1} \right)} dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t)}{a - e^{it}} i e^{it} dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t)}{a - z} dz, \end{aligned}$$

This is the *Cauchy integral formula*, of course.

- The above mapping is an *isometry* of the Hilbert spaces  $H_2(\mathbb{T})$  and  $H_2(\mathbb{D})$ , where the scalar product on  $H_2(\mathbb{D})$  defined as usual:

$$(2.4) \quad \langle f, f' \rangle = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) \bar{f}'(re^{it}) dt.$$

- The mapping  $\mathcal{W}$  could be inverted by an operator  $\mathcal{M} : H_2(\mathbb{D}) \rightarrow H_2(\mathbb{T})$ , with a very simple definition:

$$(2.5) \quad f(t) = \lim_{r \rightarrow 1} \hat{f}(re^{it}).$$

- From the above we could define a reproducing operator  $P = \mathcal{M}\mathcal{W}$  on  $H_2(\mathbb{T})$ , which is essentially the *Szegö singular integral operator*. Considered on  $L_2(\mathbb{T})$  the operator  $P$  is an orthogonal projection on its closed subspace  $H_2(\mathbb{T})$ .

**2.2. The Bergman Space.** We consider the *Bergman space* in a way very similar to the Hardy space above. Let  $L_2(\mathbb{D})$  be the space of square integrable function on  $\mathbb{D}$ . There is a closed linear subspace—the *Bergman space*  $B_2(\mathbb{D})$ —of analytic functions in  $L_2(\mathbb{D})$ . We define a family of functions

$$e_a(z) = \frac{1}{(\bar{a}z - 1)^2}, \quad a \in \mathbb{D}.$$

- Functions  $e_a(z)$  define a *linear continuous mapping*  $\mathcal{W} : L_2(\mathbb{D}) \rightarrow B_2(\mathbb{D})$  of a square integrable function on  $\mathbb{D}$  to an analytic function in  $\mathbb{D}$ :

$$(2.6) \quad \begin{aligned} \hat{f}(a) = [\mathcal{W}f](a) &= \langle f, e_a \rangle \\ &= \int_{\mathbb{T}} \frac{f(t)}{(\bar{a}z - 1)^2} dz, \end{aligned}$$

- The above mapping is an *isometry* of the Hilbert spaces if restricted to  $B_2(\mathbb{D}) \subset L_2(\mathbb{D})$ , in fact it is the identity operator on  $B_2(\mathbb{D})$ .

- Consequently  $\mathcal{W}$  could be trivially “inverted” by the identity operator  $\mathcal{M} : B_2(\mathbb{D}) \rightarrow B_2(\mathbb{D})$ .
- It follows from the above that the operator  $P = \mathcal{M}\mathcal{W} = \mathcal{M} = \mathcal{W}$  is reproducing on  $B_2(\mathbb{D})$  and is orthogonal projection  $L_2(\mathbb{D})$  onto  $B_2(\mathbb{D})$ . This is the *Bergman projection*.

REMARK 2.1. In both cases of the Hardy and the Bergman spaces we meet orthogonal projection  $P$  from the spaces of square integrable functions onto their subspace of analytic functions. Let  $M_b$  be a (bounded) operator on  $L_2$  of multiplication by a bounded function  $b$ . It is easy to see that for any such  $b$  the *Töplitz operator*  $T_b = PM_b$  is a bounded operator on the subspace of analytical functions. We will link later such operators with the wavelet theory.

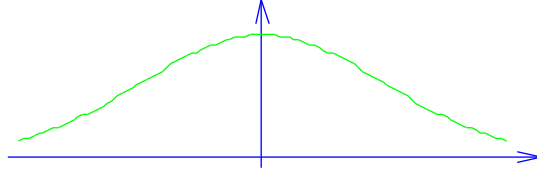
### 3. Quantum Mechanics and Quantisation

Now we turn to the object which combines the beauty of the mentioned above classic spaces of complex analysis and importance in applied area of quantum mechanics. As in case of the Fourier integral we start from  $L_2(\mathbb{R})$  with the scalar product (1.7). Let us consider the family of functions:

$$e_z(t) = e^{-(z^2+t^2)/2+\sqrt{2}zt}, \quad t \in \mathbb{R}, \quad z \in \mathbb{C}.$$

Note that  $e_0(t) = e^{-t^2/2}$  is the celebrated Gaussian shown on Figure 1. All other

FIGURE 1. The Gaussian function  $e^{-x^2/2}$ .



functions obtained from it by horizontal shifts and multiplication by a function  $e^{ipt}$  which takes value on the unit circle in  $\mathbb{C}$ . In quantum mechanical language the function  $e_z(t)$  with  $z = q + ip$  describes a state of a particle with an expectation of its coordinate equal to  $q$ , an expectation of its momentum— $p$ , and the minimal value of product of coordinate and momentum dispersions [25, § 1.3]. We will discuss a physical meaning in details letter on.

We again find a similar structure:

- Functions  $e_z(t)$  define a *linear continuous mapping*  $\mathcal{W} : L_2(\mathbb{R}) \rightarrow SB_2(\mathbb{C})$  of square integrable function  $f(t)$  on  $\mathbb{R}$  to an analytic function in  $\mathbb{C}$ :

$$(3.1) \quad \begin{aligned} \hat{f}(z) = [\mathcal{W}f](z) &= \langle f, e_z \rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-(z^2+t^2)/2+\sqrt{2}zt} dt. \end{aligned}$$

Such analytic functions are square integrable on  $\mathbb{C}$  with respect to the Gaussian measure  $d\beta(z) = e^{-|z|^2} dz$  and form *Segal-Bargmann space*  $SB_2(\mathbb{C})$ .

- The above mapping is an *isometry* of the Hilbert spaces  $L_2(\mathbb{R})$  and  $SB_2(\mathbb{C})$ , where the scalar product on  $SB_2(\mathbb{C})$  defined as follows:

$$(3.2) \quad \langle f, f' \rangle = \int_{\mathbb{C}} f(z) \bar{f}'(z) d\beta(z).$$

- The mapping  $\mathcal{W}$  could be inverted by an operator  $\mathcal{M} : SB_2(\mathbb{C}) \rightarrow L_2(\mathbb{R})$  such that the original functions is a linear combination of  $e_z(t)$ :

$$(3.3) \quad f(t) = \int_{\mathbb{C}} \hat{f}(z) e^{-(z^2+t^2)/2+\sqrt{2}zt} d\beta(z).$$

- From the above we could define a reproducing operator  $P = \mathcal{M}\mathcal{W}$  on  $L_2(\mathbb{R})$  and  $P' = \mathcal{W}\mathcal{M}$  on  $SB_2(\mathbb{C})$ . The former gives yet another integral resolution of the delta function, cf. the Fourier integral case. The later is *Segal-Bargmann projection*. Considered on  $L_2(\mathbb{C}, d\beta(z))$  the operator  $P'$  is an orthogonal projection on its closed subspace  $SB_2(\mathbb{C})$ . We again could consider Töplitz operator of the form  $T_b = P'M_b$  for a bounded function  $b$ , cf. Remark 2.1.

#### 4. Signal Processing

The Fourier series and integral appeared as a tool for decomposition of an arbitrary oscillation (or signals) into a superposition of harmonic oscillations with a fixed frequencies. This technique is quite successful in the cases then spectrum of frequencies is independent from time or changes very slowly. But in many common situation like music, speech, etc. this is not true and the Fourier transformation is out of help.

To improve performance it is useful to introduce *Windowed Fourier Transform* (WFT). It analyses the spectrum of frequency of not entire signal but only a part "seen" through a small windows. The position and size of the windows are among parameters of WFT. An example of such a transformation is shown on Figure 2 which is taken from the book [41], it is also instructional to view other [pictures from this book on-line](#).

The word "wavelets" is commonly attributed to the area of signal processing. Decompositions of that type are of huge importance in *signal processing* and are under active investigation. We will discuss this topic in details due to course.

#### 5. Functional Calculus

In the above consideration we oftenly meet a decomposition of an arbitrary function in to linear superposition of elementary ones, cf. (1.6), (2.6), (3.3). Because functions are used as models for operators such formulas could be employed for constructions of functional calculi. Particularly the Cauchy integral formula (2.2) inspires the *Riesz-Dunford functional calculus* defined by the integral formula:

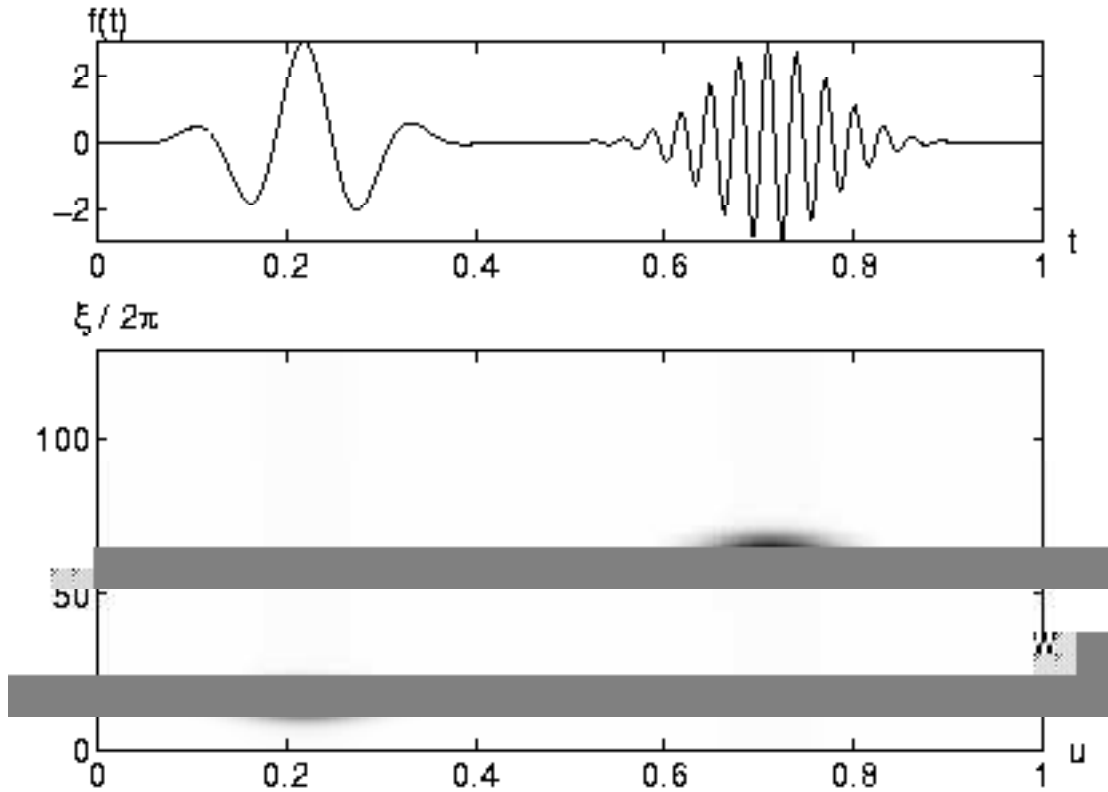
$$(5.1) \quad f(A) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{A - z} dz,$$

for an operator  $A$ .

#### 6. Discussion

The above consideration could rise many questions. We list now our answers to some of them:

**A Wavelet Tour of Signal Processing**  
**Stéphane Mallat, Academic Press 1999 (2nd edition)**



**Figure 4.20:** Choi-William distribution  $P_a f(u, \xi)$  of the two Gabor atoms shown at the top. The interference term that appears in the Wigner-Ville distribution of Figure 4.18 has nearly disappeared.

FIGURE 2. An example of Windowed Fourier Transform

- *Why is there a common pattern in the above different examples?*  
 Opinions vary. Our feeling that the common structure related to the symmetries. In each of the above case there is a group (or even several groups) which is represented by transformations in the function spaces. The groups are:
 

the Fourier series	the group of integers $\mathbb{Z}$ ;
the Fourier integral	the group of reals $\mathbb{R}$ ;
the Hardy space	the $SL_2(\mathbb{R})$ group;
the Bergman space	the $SL_2(\mathbb{R})$ group;
the Segal-Bargmann space	the Heisenberg group $\mathbb{H}^1$ ;
the signal processing	the $ax + b$ group.

For example all functions  $a_x$ , which are essentially *wavelets* or *coherent states*, could be obtained from the function  $e_0$  (*mother wavelet* or *vacuum vector*) by means of the corresponding group.

- *Why are there significant differences?* (e.g. in the Hardy space the inverse operator  $\mathcal{M}$  (2.5) is not defined as an integral)

The above groups are different with different properties, therefore pictures generated by them even within a common scheme could show significant differences. Even the same group could have representations with very different properties. For example we will see later that the same group  $SL_2(\mathbb{R})$  could generate analytic function theories of “elliptic” and “hyperbolic” types. By the way the mentioned operator  $\mathcal{M}$  (2.5) could be expressed as an integral similar to the scalar product (2.4).

- *Do groups provide the ultimate explanations in the above examples?*

Probably not. One could expect the “ultimate explanation” only in a very simple situation and we hope that the above examples are more complicated and consequently interesting. But groups do explain many fundamental properties of the mentioned objects and allow to put many different cases within a common framework (cf. with the Erlangen program of F. Klein ;-).

- *In section 2 we meet the Cauchy integral formula. Are wavelets related to other objects of complex analysis (Cauchy-Riemann equation, Laplacian, Taylor and Laurent expansion, etc.)?*

Yes. We will see it later. For the moment we will mention that the Taylor expansion is a close relative of the Fourier series from the range of our examples.

- *Are groups useful in classification of known types of wavelets or they could help to discover new ones?*

We already mentioned above a few new objects derived from the group approach: hyperbolic complex analysis and new types of functional calculi of operators.

The following lecture should give answers to more questions. But before we could proceed we will need a short overview of the representation theory.



## LECTURE 2

# Groups and Homogeneous Spaces

The group theory and the representation theory are two enormous and interesting subjects themselves. However they are auxiliary in our consideration and we are forced to restrict ourselves only to brief and very dry overview.

Besides introduction to that areas presented in [42, 56] we recommend additionally the books [31, 55]. The representation theory intensively uses tools of functional analysis and on the other hand inspires its future development. We use the book [32] for references on functional analysis here and recommend it as a nice reading too.

### 1. Basics of Group Theory

We start from the definition of central object which formalizes the universal notion of symmetries.

DEFINITION 1.1. A *transformation group*  $G$  is a nonvoid set of mappings of a certain set  $X$  into itself with the following properties:

- (i) if  $g_1 \in G$  and  $g_2 \in G$  then  $g_1 g_2 \in G$ ;
- (ii) if  $g \in G$  then  $g^{-1}$  exists and belongs to  $G$ .

EXERCISE 1.2. List all transformation groups on a set of three elements.

EXERCISE 1.3. Verify that the following are groups in fact:

- (i) Group of permutations of  $n$  elements;
- (ii) Group of  $n \times n$  matrixes with non zero determinant over a field  $\mathbb{F}$  under matrix multiplications;
- (iii) Group of **rotations** of the unit circle  $\mathbb{T}$ ;
- (iv) Groups of **shifts** of the real line  $\mathbb{R}$  and plane  $\mathbb{R}^2$ ;
- (v) Group of linear fractional transformations of the extended complex plane.

DEFINITION 1.4. An *abstract group* (or simply *group*) is a nonvoid set  $G$  on which there is a law of *group multiplication* (i.e. mapping  $G \times G \rightarrow G$ ) with the properties

- (i) *associativity*:  $g_1(g_2 g_3) = (g_1 g_2)g_3$ ;
- (ii) the existence of *identity*:  $e \in G$  such that  $eg = ge = g$  for all  $g \in G$ ;
- (iii) the existence of *inverse*: for every  $g \in G$  there exists  $g^{-1} \in G$  such that  $gg^{-1} = g^{-1}g = e$ .

EXERCISE 1.5. Check that any transformation group is an abstract group.

EXERCISE 1.6. Check that the following transformation groups (cf. Example 1.3) have the same law of multiplication, i.e. are equivalent as abstract groups:

- (i) The group of isometric mapping of an equilateral triangle onto itself;
- (ii) The group of all permutations of a set of free elements;
- (iii) The group of invertible matrix of order 2 with coefficients in the field of integers modulo 2;
- (iv) The group of linear fractional transformations of the extended complex plane generated by the mappings  $z \mapsto z^{-1}$  and  $z \mapsto 1 - z$ .

EXERCISE\* 1.7. Expand the list in the above exercise.

It is simpler to study groups with the following additional property.

DEFINITION 1.8. A group  $G$  is *commutative* if for all  $g_1, g_2 \in G$ , we have  $g_1g_2 = g_2g_1$ .

However, most of interesting and important groups are *noncommutative*.

EXERCISE 1.9. (i) Which groups among found in Exercise 1.2 are commutative?

(ii) Which groups among listed in Exercise 1.3 are noncommutative?

Groups could have some additional *analytical* structures, e.g. they could be a topological sets with a corresponding notion of *limit*. We always assume that our groups are *locally compact* [31, § 2.4].

DEFINITION 1.10. If for a group  $G$  the group multiplication and the taking of inverse are *continuous* mappings then  $G$  is *continuous group*.

Even a better structure could be found among *Lie groups* [31, § 6], e.g. groups with a *differentiable* law of multiplication. Investigating such groups we could employ the whole arsenal of analytical tools, thereafter most of groups studied in this notes will be Lie groups.

EXERCISE 1.11. Check that the following are noncommutative Lie (and thus continuous) groups:

(i) [55, Chap. 7] The *ax + b group*: set of elements  $(a, b)$ ,  $a \in \mathbb{R}_+$ ,  $b \in \mathbb{R}$  with the group law:

$$(a, b) * (a', b') = (aa', ab' + b).$$

The identity is  $(1, 0)$ , and  $(a, b)^{-1} = (a^{-1}, -b/a)$ .

(ii) The *Heisenberg group* [26], [55, Chap. 1]: a set of triples of real numbers  $(s, x, y)$  with the group multiplication:

$$(1.1) \quad (s, x, y) * (s', x', y') = (s + s' + \frac{1}{2}(x'y - xy'), x + x', y + y').$$

The identity is  $(0, 0, 0)$ , and  $(s, x, y)^{-1} = (-s, -x, -y)$ .

(iii) The  $SL_2(\mathbb{R})$  group [28, 39]: a set of  $2 \times 2$  matrixes  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with real entries  $a, b, c, d \in \mathbb{R}$ , the determinant  $\det = ad - bc$  equal to 1 and the group law coinciding with matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

The identity is the unit matrix and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The above three groups are behind many important results of real and complex analysis [26, 28, 39] and we meet them many times in these notes.

## 2. Homogeneous Spaces and Invariant Measures

While *abstract group* are a suitable language for investigation of their general properties we meet groups in applications as *transformation groups* acting on a set  $X$ .

Let  $X$  be a set and let be defined an operation  $G : X \rightarrow X$  of  $G$  on  $X$ . There is an *equivalence relation* on  $X$ , say,  $x_1 \sim x_2 \Leftrightarrow \exists g \in G : gx_1 = x_2$ , with respect to which  $X$  is a disjoint union of distinct *orbits* [38, § I.5].

EXERCISE 2.1. Let action of  $SL_2(\mathbb{R})$  group on  $\mathbb{C}$  by means of *linear-fractional transformations*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

Show that there three orbits: the real axis  $\mathbb{R}$ , *upper (lower) half plane*  $\mathbb{R}_{\pm}^{\uparrow}$ :

$$\mathbb{R}_{\pm}^{\uparrow} = \{x \pm iy \mid x, y \in \mathbb{R}, y > 0\}.$$

Thus from now on, without lost of a generality, we assume that the operation of  $G$  on  $X$  is *transitive*, i. e. for every  $x \in X$  we have

$$Gx := \bigcup_{g \in G} g(x) = X.$$

In this case  $X$  is  $G$ -homogeneous space.

EXERCISE 2.2. Show that for any group  $G$  we could define its action on  $X = G$  as follows:

- (i) The *conjugation*  $g : x \mapsto gxg^{-1}$  (which is even a group homomorphism, but is trivial for all commutative groups).
- (ii) The *left shift*  $\lambda(g) : x \mapsto gx$  and the *right shift*  $\rho(g) : x \mapsto xg^{-1}$ .

If we fix a point  $x \in X$  then the set of elements  $G_x = \{g \in G \mid g(x) = x\}$  obviously forms the *isotropy (sub)group* of  $x$  in  $G$  [38, § I.5]. The set  $X$  is in the bijection with the factor set  $G/G_x$  for any  $x \in X$ .

EXERCISE 2.3. Find a subgroup which correspond to the given action of  $G$  on  $X$ :

- (i) Action of  $ax + b$  group on  $\mathbb{R}$  by the formula:  $(a, b) : x \mapsto ax + b$ .
- (ii) Action of  $SL_2(\mathbb{R})$  group on one of three orbit from Exercise 2.1.

To do some analysis on groups we need suitably defined basic operation: *differentiation* and *integration*. The first operation is naturally defined for Lie group. If  $G$  is a Lie group then the homogeneous space  $G/G_x$  is a smooth manifold (and a *loop* as an algebraic object) for every  $x \in X$ . Therefore the one-to-one mapping  $G/G_x \rightarrow X : g \mapsto g(x)$  induces a structure of  $C^\infty$ -manifold on  $S$ . Thus the class  $C_0^\infty(X)$  of smooth functions with compact supports on  $x$  has the evident definition.

In order to perform an integration we need a suitable *measure*. A smooth measure  $d\mu$  on  $X$  is called (left) *invariant measure* with respect to an operation of  $G$  on  $X$  if

$$(2.1) \quad \int_X f(x) d\mu(x) = \int_X f(g(x)) d\mu(x), \quad \text{for all } g \in G, f(x) \in C_0^\infty(X).$$

EXERCISE 2.4. Show that measure  $y^{-2} dy dx$  on the upper half plane  $\mathbb{R}_+^2$  is invariant under action from Exercise 2.1.

Left invariant measures on  $X = G$  is called the *Haar measure*. It always exists and is uniquely defined up to a scalar multiplier [55, § 0.2]. An equivalent formulation of (2.1) is:  $G$  operates on  $L_2(X, d\mu)$  by *unitary operators*. We will transfer the Haar measure  $d\mu$  from  $G$  to  $\mathfrak{g}$  via the exponential map  $\exp : \mathfrak{g} \rightarrow G$  and will call it as the *invariant measure on a Lie algebra*  $\mathfrak{g}$ .

EXERCISE 2.5. Check that the following are Haar measures for corresponding groups:

- (i) The *Lebesgue measure*  $dx$  on the real line  $\mathbb{R}$ .
- (ii) The *Lebesgue measure*  $d\phi$  on the unit circle  $\mathbb{T}$ .
- (iii)  $dx/x$  is a Haar measure on the multiplicative group  $\mathbb{R}_+;$

- (iv)  $dx dy/(x^2 + y^2)$  is a Haar measure on the multiplicative group  $\mathbb{C} \setminus \{0\}$ , with coordinates  $z = x + iy$ .
- (v)  $a^{-2} da db$  and  $a^{-1} da db$  are the left and right invariant measure on **ax + b group**.
- (vi) The Lebesgue measure  $ds dx dy$  of  $\mathbb{R}^3$  for the **Heisenberg group**  $\mathbb{H}^1$ .

In this notes we assume *all integrations on groups performed over the Haar measures*.

EXERCISE 2.6. Show that invariant measure on a compact group  $G$  is finite and thus may be normalized to total measure 1.

The above simple result has surprisingly important consequences for representation theory of compact groups, which we state here without a proof.

THEOREM 2.7. [31, § 9.2]

- (i) Every topologically irreducible representation of a compact group  $G$  is finite-dimensional and unitarizable.
- (ii) If  $T_1$  and  $T_2$  are two inequivalent irreducible representations, then every **matrix element** of  $T_1$  is orthogonal in  $L_2(G)$  to every matrix element of  $T_2$ .
- (iii) For a compact group  $G$  its dual space  $\hat{G}$  is discrete.

DEFINITION 2.8. The left convolution  $f_1 * f_2$  of two functions  $f_1(g)$  and  $f_2(g)$  defined on a group  $G$  is

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g) dh$$

EXERCISE 2.9. Let  $k(g) \in L_1(G, d\mu)$  and operator  $K$  on  $L_1(G, d\mu)$  is the left convolution operator with  $k$ , i.e.  $K : f \mapsto k * f$ . Show that  $K$  commutes with all **right shifts** on  $G$ .

The following Lemma characterizes *linear subspaces* of  $L_2(G, d\mu)$  invariant under shifts in the term of *ideals of convolution algebra*  $L_2(G, d\mu)$  and is of the separate interest.

LEMMA 2.10. A closed linear subspace  $H$  of  $L_2(G, d\mu)$  is invariant under left (right) shifts if and only if  $H$  is a left (right) ideal of the right group convolution algebra  $L_2(G, d\mu)$ .

A closed linear subspace  $H$  of  $L_2(G, d\mu)$  is invariant under left (right) shifts if and only if  $H$  is a right (left) ideal of the left group convolution algebra  $L_2(G, d\mu)$ .

PROOF. Of course we consider only the “right-invariance and right-convolution” case. Then the other three cases are analogous. Let  $H$  be a closed linear subspace of  $L_2(G, d\mu)$  invariant under right shifts and  $k(g) \in H$ . We will show the inclusion

$$(2.2) \quad [f * k]_r(h) = \int_G f(g)k(hg) d\mu(g) \in H,$$

for any  $f \in L_2(G, d\mu)$ . Indeed, we can treat integral (2.2) as a limit of sums

$$(2.3) \quad \sum_{j=1}^N f(g_j)k(hg_j)\Delta_j.$$

But the last sum is simply a linear combination of vectors  $k(hg_j) \in H$  (by the invariance of  $H$ ) with coefficients  $f(g_j)$ . Therefore sum (2.3) belongs to  $H$  and this is true for integral (2.2) by the closeness of  $H$ .

Otherwise, let  $H$  be a right ideal in the group convolution algebra  $L_2(G, d\mu)$  and let  $\phi_j(g) \in L_2(G, d\mu)$  be an approximate unit of the algebra [21, § 13.2], i. e. for any  $f \in L_2(G, d\mu)$  we have

$$[\phi_j * f]_r(h) = \int_G \phi_j(g)f(hg) d\mu(g) \rightarrow f(h), \text{ when } j \rightarrow \infty.$$

Then for  $k(g) \in H$  and for any  $h' \in G$  the right convolution

$$[\phi_j * k]_r(hh') = \int_G \phi_j(g)k(hh'g) d\mu(g) = \int_G \phi_j(h'^{-1}g')k(hg') d\mu(g'), \quad g' = h'g,$$

from the first expression is tensing to  $k(hh')$  and from the second one belongs to  $H$  (as a right ideal). Again the closeness of  $H$  implies  $k(hh') \in H$  that proves the assertion.  $\square$



## Elements of the Representation Theory

### 1. Representations of Groups

Objects unveil their nature in actions. Groups act on other sets by means of *representations*. A representation of a group  $G$  is a group homomorphism of  $G$  in a transformation group of a set. It is a fundamental observation that *linear* objects are easier to study. Therefore we begin from linear representations of groups.

DEFINITION 1.1. A linear continuous *representation of a group*  $G$  is a continuous function  $T(g)$  on  $G$  with values in the group of non-degenerate linear continuous transformation in a linear space  $H$  (either finite or infinite dimensional) such that  $T(g)$  satisfies to the functional identity:

$$(1.1) \quad T(g_1 g_2) = T(g_1) T(g_2).$$

REMARK 1.2. If we have a representation of a group  $G$  by its action on a set  $X$  we can use the following *linearization procedure*. Let us consider a linear space  $L(X)$  of functions  $X \rightarrow \mathbb{C}$  which may be restricted by some additional requirements (e.g. integrability, boundedness, continuity, etc.). There is a natural representation of  $G$  on  $L(X)$  which produced by its action on  $X$ :

$$(1.2) \quad g : f(x) \mapsto \rho_g f(x) = f(g \cdot x), \quad \text{where } g \in G, x \in X.$$

Clearly this representation is already linear. However in many practical cases the formula for linearization (1.2) has some additional terms which are required to make it, for example, unitary.

EXERCISE 1.3. Show that  $T(g^{-1}) = T^{-1}(g)$  and  $T(e) = I$ , where  $I$  is the identity operator on  $B$ .

EXERCISE 1.4. Show that these are linear continuous representations of corresponding groups:

(i) Operators  $T(x)$  such that  $[T(x) f](t) = f(t+x)$  form a representation of  $\mathbb{R}$  in  $L_2(\mathbb{R})$ .

(ii) Operators  $T(n)$  such that  $T(n)a_k = a_{k+n}$  form a representation of  $\mathbb{Z}$  in  $\ell_2$ .

(iii) Operators  $T(a, b)$  defined by

$$(1.3) \quad [T(a, b) f](x) = \sqrt{a} f(ax + b), \quad a \in \mathbb{R}_+, b \in \mathbb{R}$$

form a representation of **ax + b group** in  $L_2(\mathbb{R})$ .

(iv) Operators  $T(s, x, y)$  defined by

$$(1.4) \quad [T(s, x, y) f](t) = e^{i(2s - \sqrt{2}yt + xy)} f(t - \sqrt{2}x)$$

form *Schrödinger representation* of the **Heisenberg group**  $\mathbb{H}^1$  in  $L_2(\mathbb{R})$ .

(v) Operators  $T(g)$  defined by

$$(1.5) \quad [T(g)f](t) = \frac{1}{ct+d} f\left(\frac{at+b}{ct+d}\right), \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

form a representation of **SL<sub>2</sub>(ℝ)** in  $L_2(\mathbb{R})$ .

In the sequel a representation *always means* linear continuous representation.  $T(g)$  is an *exact representation* (or *faithful representation* if  $T(g) = I$  only for  $g = e$ ). The opposite case when  $T(g) = I$  for all  $g \in G$  is a *trivial representation*. The space  $H$  is *representation space* and in most cases will be a *Hilber space* [32, § III.5]. If dimensionality of  $H$  is finite then  $T$  is a *finite dimensional representation*, in the opposite case it is *infinite dimensional representation*.

We denote the *scalar product* on  $H$  by  $\langle \cdot, \cdot \rangle$ . Let  $\{e_j\}$  be an (finite or infinite) *orthonormal basis* in  $H$ , i.e.

$$\langle e_j, e_j \rangle = \delta_{jk},$$

where  $\delta_{jk}$  is the *Kroneker delta*, and linear span of  $\{e_j\}$  is dense in  $H$ .

DEFINITION 1.5. The *matrix elements*  $t_{jk}(g)$  of a representation  $T$  of a group  $G$  (with respect to a basis  $\{e_j\}$  in  $H$ ) are complex valued functions on  $G$  defined by

$$(1.6) \quad t_{jk}(g) = \langle T(g)e_j, e_k \rangle.$$

EXERCISE 1.6. Show that [56, § 1.1.3]

- (i)  $T(g)e_k = \sum_j t_{jk}(g)e_j$ .
- (ii)  $t_{jk}(g_1g_2) = \sum_n t_{jn}(g_1)t_{nk}(g_2)$ .

It is typical mathematical questions to determine identical objects which may have a different appearance. For representations it is solved in the following definition.

DEFINITION 1.7. Two representations  $T_1$  and  $T_2$  of the same group  $G$  in spaces  $H_1$  and  $H_2$  correspondingly are *equivalent representations* if there exist a linear operator  $A : H_1 \rightarrow H_2$  with the continuous inverse operator  $A^{-1}$  such that:

$$T_2(g) = A T_1(g) A^{-1}, \quad \forall g \in G.$$

EXERCISE 1.8. Show that representation  $T(a, b)$  of **ax + b group** in  $L_2(\mathbb{R})$  from Exercise 1.4iii is equivalent to the representation

$$(1.7) \quad [T_1(a, b)f](x) = \frac{e^{i\frac{b}{a}}}{\sqrt{a}} f\left(\frac{x}{a}\right).$$

HINT. Use the Fourier transform. □

The *relation of equivalence* is reflexive, symmetric, and transitive. Thus it splits the set of all representations of a group  $G$  into *classes of equivalent representations*. In the sequel we study group representations up to their equivalence classes only.

EXERCISE 1.9. Show that equivalent representations have the same **matrix elements** in appropriate basis.

DEFINITION 1.10. Let  $T$  be a representation of a group  $G$  in  $H$  The *adjoint representation*  $T'(g)$  of  $G$  in  $H$  is defined by

$$T'(g) = (T(g^{-1}))^*,$$

where  $*$  denotes the adjoint operator in  $H$ .

EXERCISE 1.11. Show that

- (i)  $T'$  is indeed a representation.
- (ii)  $t'_{jk}(g) = \bar{t}_{kj}(g^{-1})$ .

Recall [32, § III.5.2] that a bijection  $U : H \rightarrow H$  is a *unitary operator* if

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \quad \forall x, y \in H.$$

EXERCISE 1.12. Show that  $UU^* = I$ .

DEFINITION 1.13.  $T$  is a *unitary representation* of a group  $G$  in a space  $H$  if  $T(g)$  is a unitary operator for all  $g \in G$ .  $T_1$  and  $T_2$  are *unitary equivalent representations* if  $T_1 = UT_2U^{-1}$  for a unitary operator  $U$ .

- EXERCISE 1.14. (i) Show that all representations from Exercises 1.4 are unitary.  
 (ii) Show that representations from Exercises 1.4iii and 1.8 are unitary equivalent.

HINT. Take that the Fourier transform is unitary for granted.  $\square$

EXERCISE 1.15. Show that if a Lie group  $G$  is represented by unitary operators in  $H$  then its Lie algebra  $\mathfrak{g}$  is represented by self-adjoint (possibly unbounded) operators in  $H$ .

The following definition have a sense for *finite* dimensional representations.

DEFINITION 1.16. A *character of representation*  $T$  is equal  $\chi(g) = \text{tr}(T(g))$ , where  $\text{tr}$  is the *trace* [32, § III.5.2 (Probl.)] of operator.

EXERCISE 1.17. Show that

- (i) Characters of a representation  $T$  are constant on the **adjoint elements**  $g^{-1}hg$ , for all  $g \in G$ .  
 (ii) Character is an algebra homomorphism from an algebra of representations with Kronecker's (tensor) multiplication [56, § 1.9] to complex numbers.

HINT. Use that  $\text{tr}(AB) = \text{tr}(BA)$ ,  $\text{tr}(A + B) = \text{tr} A + \text{tr} B$ , and  $\text{tr}(A \otimes B) = \text{tr} A \text{tr} B$ .  $\square$

For *infinite* dimensional representation characters could be defined either as distributions [31, § 11.2] or in infinitesimal terms of Lie algebras [31, § 11.3].

The characters of a representation should not be confused with the following notion.

DEFINITION 1.18. A *character of a group*  $G$  is a one-dimensional representation of  $G$ .

- EXERCISE 1.19. (i) Let  $\chi$  be a **character of a group**  $G$ . Show that a **character of representation**  $\chi$  coincides with it and thus is a character of  $G$ .  
 (ii) A **matrix element** of a group character  $\chi$  coincides with  $\chi$ .  
 (iii) Let  $\chi_1$  and  $\chi_2$  be **characters of a group**  $G$ . Show that  $\chi_1 \otimes \chi_2 = \chi_1\chi_2$  and  $\chi'(g) = \chi_1(g^{-1})$  are again characters of  $G$ . In other words *characters of a group form a group themselves*.

## 2. Decomposition of Representations

The important part of any mathematical theory is classification theorems on structural properties of objects. Very well known examples are:

- (i) The main theorem of arithmetics on unique representation an integer as a product of powers of prime numbers.  
 (ii) Jordan's normal form of a matrix.

The similar structural results in the representation theory are very difficult. The easiest (but still rather difficult) questions are on classification of **unitary representations** up to **unitary equivalence**.

DEFINITION 2.1. Let  $T$  be a representation of  $G$  in  $H$ . A linear subspace  $L \subset H$  is *invariant subspace* for  $T$  if for any  $x \in L$  and any  $g \in G$  the vector  $T(g)x$  again belong to  $L$ .

There are always two trivial invariant subspaces: the null space and entire  $H$ . All other are *nontrivial invariant subspaces*.

DEFINITION 2.2. If there are only two trivial invariant subspaces then  $T$  is *irreducible representation*. Otherwise we have *reducible representation*.

For any nontrivial invariant subspace we could define the *restriction of representation* of  $T$  on it. In this way we obtain a *subrepresentation* of  $T$ .

EXAMPLE 2.3. Let  $T(a)$ ,  $a \in \mathbb{R}_+$  be defined as follows:  $[T(a)]f(x) = f(ax)$ . Then spaces of **even and odd functions** are invariant.

DEFINITION 2.4. If the closure of linear span of all vectors  $T(g)v$  is dense in  $H$  then  $v$  is called *cyclic vector* for  $T$ .

EXERCISE 2.5. Show that for an irreducible representation any non zero vector is cyclic.

The important property of unitary representation is complete reducibility.

EXERCISE 2.6. Let a unitary representation  $T$  has an invariant subspace  $L \subset H$ , then its orthogonal completion  $L^\perp$  is also invariant.

THEOREM 2.7. [31, § 8.4] Any unitary representation  $T$  of a locally compact group  $G$  could be decomposed in a (continuous) direct sum irreducible representations:  $T = \int_X T_x d\mu(x)$ .

The necessity of continuous sums appeared in very simple examples:

EXERCISE 2.8. Let  $T$  be a representation of  $\mathbb{R}$  in  $L_2(\mathbb{R})$  as follows:  $[T(a)f](x) = e^{iax}f(x)$ . Show that

- (i) Any measurable set  $E \subset \mathbb{R}$  define an invariant subspace of functions vanishing outside  $E$ .
- (ii)  $T$  does not have invariant irreducible subrepresentations.

DEFINITION 2.9. The set of equivalence classes of unitary irreducible representations of a group  $G$  is denoted by  $\hat{G}$  and called *dual object* (or *dual space*) of the group  $G$ .

DEFINITION 2.10. A left *regular representation*  $\Lambda(g)$  of a group  $G$  is the representation by **left shifts** in the space  $L_2(G)$  of square-integrable function on  $G$  with the left **Haar measure**

$$(2.1) \quad \Lambda g : f(h) \mapsto f(g^{-1}h).$$

The *main problem of representation theory* is to decompose a left regular representation  $\Lambda(g)$  into irreducible components.

### 3. Invariant Operators and Schur's Lemma

It is a pleasant feature of an abstract theory that we obtain important general statements from simple observations. **Finiteness of invariant measure** on a compact group is one such example. Another example is **Schur's Lemma** presented here.

To find different classes of representations we need to compare them each other. This is done by *intertwining operators*.

DEFINITION 3.1. Let  $T_1$  and  $T_2$  are representations of a group  $G$  in a spaces  $H_1$  and  $H_2$  correspondingly. An operator  $A : H_1 \rightarrow H_2$  is called an *intertwining operator* if

$$A T_1(g) = T_2(g) A, \quad \forall g \in G.$$

If  $T_1 = T_2 = T$  then  $A$  is *interntwinig operator* or *commuting operator* for  $T$ .

EXERCISE 3.2. Let  $G, H, T(g)$ , and  $A$  be as above. Show the following: [56, § 1.3.1]

- (i) Let  $x \in H$  be an eigenvector for  $A$  with eigenvalue  $\lambda$ . Then  $T(g)x$  for all  $g \in G$  are eigenvectors of  $A$  with the same eigenvalue  $\lambda$ .
- (ii) All eigenvectors of  $A$  with a fixed eigenvalue  $\lambda$  form a linear subspace invariant under all  $T(g)$ ,  $g \in G$ .
- (iii) If an operator  $A$  is commuting with irreducible representation  $T$  then  $A = \lambda I$ .

HINT. Use the spectral decomposition of selfadjoint operators [32, § V.2.2].  $\square$

The next result has very important applications.

LEMMA 3.3 (Schur). [31, § 8.2] *If two representations  $T_1$  and  $T_2$  of a group  $G$  are irreducible, then every intertwining operator between them is either zero or invertible.*

HINT. Consider subspaces  $\ker A \subset H_1$  and  $\text{im } A \subset H_2$ .  $\square$

EXERCISE 3.4. Show that

- (i) Two irreducible representations are either equivalent or disjoint.
- (ii) All operators commuting with an irreducible representation form a field.
- (iii) Irreducible representations of commutative groups are one-dimensional.
- (iv) If  $T$  is a unitary irreducible representation in  $H$  and  $B(\cdot, \cdot)$  is a bounded semi-linear form in  $H$  invariant under  $T$ :  $B(T(g)x, T(g)y) = B(x, y)$  then  $B(\cdot, \cdot) = \lambda \langle \cdot, \cdot \rangle$ .

HINT. Use that  $B(\cdot, \cdot) = \langle A \cdot, \cdot \rangle$  for some  $A$  [32, § III.5.1].  $\square$



## LECTURE 4

# Wavelets on Groups and Square Integrable Representations

A matured mathematical theory looks like a tree. There is a solid trunk which supports all branches and leaves but could not be alive without them. In the case of group approach to wavelets the trunk of the theory is a construction of wavelets from a *square integrable representation* [11], [2, Chap. 8]. We begin from this trunk which is a model for many different generalisations and will continue with some smaller “generalising” branches later.

### 1. Wavelet Transform on Groups

Let  $G$  be a group with a left Haar measure  $d\mu$  and let  $\rho$  be a unitary irreducible representation of a group  $G$  by operators  $\rho_g, g \in G$  in a Hilbert space  $H$ .

DEFINITION 1.1. Let us fix a vector  $w_0 \in H$ . We call  $w_0 \in H$  a *vacuum vector* or a *mother wavelet* (other less-used names are *ground state, fiducial vector*, etc.). We will say that set of vectors  $w_g = \rho(g)w_0, g \in G$  form a family of *coherent states* (*wavelets*).

EXERCISE 1.2. If  $\rho$  is irreducible then  $w_g, g \in G$  is a *total set* in  $H$ , i.e. the linear span of these vectors is dense in  $H$ .

The *wavelet transform* could be defined as a mapping from  $H$  to a space of functions over  $G$  via its representational coefficients

$$(1.1) \quad \mathcal{W} : v \mapsto \hat{v}(g) = \langle \rho(g^{-1})v, w_0 \rangle = \langle v, \rho(g)w_0 \rangle = \langle v, w_g \rangle.$$

EXERCISE 1.3. Show that the wavelet transform  $\mathcal{W}$  is a continuous linear mapping and the image of a vector is a bounded continuous function on  $G$ . The linear space of all such images is denoted by  $W(G)$ .

EXERCISE 1.4. Let a Hilbert space  $H$  has a basis  $e_j, j \in \mathbb{Z}$  and a unitary representation  $\rho$  of  $G = \mathbb{Z}$  defined by  $\rho(k)e_j = e_{j+k}$ . Write a formula for wavelet transform with  $w_0 = e_0$  and characterise  $W(\mathbb{Z})$ .

ANSWER.  $\hat{v}(n) = \langle v, e_n \rangle$ . □

EXERCISE 1.5. Let  $G$  be  $ax + b$  group and  $\rho$  is given by (cf. (1.3)):

$$(1.2) \quad [\Gamma(a, b) f](x) = \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right),$$

in  $L_2(\mathbb{R})$ . Show that

- (i) The representation is reducible and describe its irreducible components.
- (ii) for  $w_0(x) = \frac{1}{2\pi i(x+i)}$  coherent states are  $v_{(a,b)}(x) = \frac{\sqrt{a}}{2\pi i(x-(b+ia))}$ .
- (iii) Wavelet transform is given by

$$\hat{v}(a, b) = \frac{\sqrt{a}}{2\pi i} \int_{\mathbb{R}} \frac{v(x)}{x - (b + ia)} dx,$$

which resembles the *Cauchy integral formula*.

(iv) Give a characteristic of  $W(G)$ .

PROPOSITION 1.6. *The wavelet transform  $\mathcal{W}$  intertwines  $\rho$  and the left regular representation  $\Lambda$  (2.1) of  $G$ :*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}.$$

PROOF. We have:

$$\begin{aligned} [\mathcal{W}(\rho(g)\mathbf{w})](\mathbf{h}) &= \langle \rho(\mathbf{h}^{-1})\rho(g)\mathbf{v}, \mathbf{w}_0 \rangle \\ &= \langle \rho((g^{-1}\mathbf{h})^{-1})\mathbf{v}, \mathbf{w}_0 \rangle \\ &= [\mathcal{W}\mathbf{v}](g^{-1}\mathbf{h}) \\ &= [\Lambda(g)\mathcal{W}\mathbf{v}](\mathbf{h}). \end{aligned}$$

□

COROLLARY 1.7. *The function space  $W(G)$  is invariant under the representation  $\Lambda$  of  $G$ .*

Wavelet transform maps vectors of  $H$  to functions on  $G$ . We can consider a map in the opposite direction sends a function on  $G$  to a vector in  $H$ .

DEFINITION 1.8. *The inverse wavelet transform  $\mathcal{M}$  associated with a vector  $w'_0 \in H$  maps  $L_1(G)$  to  $H$  and is given by the formula:*

$$\begin{aligned} \mathcal{M} : L_1(G) \rightarrow H : \hat{v}(g) \mapsto \mathcal{M}[\hat{v}(g)] &= \int_G \hat{v}(g)w'_g \, d\mu(g) \\ (1.3) \qquad \qquad \qquad &= \int_G \hat{v}(g)\rho(g) \, d\mu(g) w'_0, \end{aligned}$$

where in the last formula the integral express an operator acting on vector  $w'_0$ .

EXERCISE 1.9. Write inverse wavelet transforms for Exercises 1.4 and 1.5.

ANSWER. (i) For Exercises 1.4:  $v = \sum_{-\infty}^{\infty} \hat{v}(n)e_n$ .

(ii) For Exercises 1.5:

$$v(x) = \frac{1}{2\pi i} \int_{\mathbb{R}_+^2} \frac{\hat{v}(a, b)}{x - (b - ia)} \frac{da \, db}{a^{\frac{3}{2}}}.$$

□

LEMMA 1.10. *If the wavelet transform  $\mathcal{W}$  and inverse wavelet transform  $\mathcal{M}$  are defined by the same vector  $w_0$  then they are adjoint operators:  $\mathcal{W}^* = \mathcal{M}$ .*

PROOF. We have:

$$\begin{aligned} \langle \mathcal{M}\hat{v}, w_g \rangle &= \left\langle \int_G \hat{v}(g')w_{g'} \, d\mu(g'), w_g \right\rangle \\ &= \int_G \hat{v}(g') \langle w_{g'}, w_g \rangle \, d\mu(g') \\ &= \int_G \hat{v}(g') \overline{\langle w_g, w_{g'} \rangle} \, d\mu(g') \\ &= \langle \hat{v}, \mathcal{W}w_g \rangle, \end{aligned}$$

where the scalar product in the first line is on  $H$  and in the last line is on  $L_2(G)$ . Now the result follows from the totality of coherent states  $w_g$  in  $H$ . □

PROPOSITION 1.11. *The inverse wavelet transform  $\mathcal{M}$  intertwines the representation  $\Lambda$  (2.1) on  $L_2(G)$  and  $\rho$  on  $H$ :*

$$\mathcal{M}\Lambda(g) = \rho(g)\mathcal{M}.$$

PROOF. We have:

$$\begin{aligned}
\mathcal{M}[\Lambda(g)\hat{v}(h)] &= \mathcal{M}[\hat{v}(g^{-1}h)] \\
&= \int_G \hat{v}(g^{-1}h)w_g \, d\mu(h) \\
&= \int_G \hat{v}(h)w'_{gh'} \, d\mu(h') \\
&= \rho(g) \int_G \hat{v}(h)w'_h \, d\mu(h') \\
&= \rho(g)\mathcal{M}[\hat{v}(h')],
\end{aligned}$$

where  $h' = g^{-1}h$ . □

COROLLARY 1.12. *The image  $\mathcal{M}(L_1(G)) \subset H$  of subspace under the inverse wavelet transform  $\mathcal{M}$  is invariant under the representation  $\rho$ .*

The following proposition explain the usage of the name “inverse” (not “adjoint” as it could be expected from Lemma 1.10) for  $\mathcal{M}$ .

THEOREM 1.13. *The operator*

$$(1.4) \quad \mathcal{P} = \mathcal{M}\mathcal{W} : H \rightarrow H$$

*maps  $H$  into its linear subspace for which  $w'_0$  is cyclic. Particularly if  $\rho$  is an irreducible representation then  $\mathcal{P}$  is  $cI$  for some constant  $c$  depending from  $w_0$  and  $w'_0$ .*

PROOF. It follows from Propositions 1.6 and 1.11 that operator  $\mathcal{M}\mathcal{W} : H \rightarrow H$  intertwines  $\rho$  with itself. Then Corollaries 1.7 and 1.12 imply that the image  $\mathcal{M}\mathcal{W}$  is a  $\rho$ -invariant subspace of  $H$  containing  $w_0$ . From irreducibility of  $\rho$  by Schur’s Lemma [31, § 8.2] one concludes that  $\mathcal{M}\mathcal{W} = cI$  on  $C$  for a constant  $c \in \mathbb{C}$ . □

REMARK 1.14. From Exercises 1.4 and 1.9 it follows that irreducibility of  $\rho$  is not necessary for  $\mathcal{M}\mathcal{W} = cI$ , it is sufficient that  $w_0$  and  $w'_0$  are **cyclic** only.

We have similarly

THEOREM 1.15. *Operator  $\mathcal{W}\mathcal{M}$  is up to a complex multiplier a projection of  $L_1(G)$  to  $W(G)$ .*

## 2. Square Integrable Representations

So far our consideration of wavelets was mainly algebraic. Usually in analysis we wish that the wavelet transform could preserve an analytic structure, e.g. values of scalar product in Hilbert spaces. This accomplished if a representation  $\rho$  possesses the following property.

DEFINITION 2.1. [31, § 9.3] Let a group  $G$  with a left Haar measure  $d\mu$  have a unitary representation  $\rho : G \rightarrow \mathcal{L}(H)$ . A vector  $v \in H$  is called *admissible vector* if the function  $\hat{v}(g) = \langle \rho(g)v, v \rangle$  is non void and square integrable on  $G$  with respect to  $d\mu$ :

$$(2.1) \quad 0 < c^2 = \int_G \langle \rho(g)v, v \rangle \langle v, \rho(g)v \rangle \, d\mu(g) < \infty.$$

If an admissible vector exists then  $\rho$  is a *square integrable representation*.

Square integrable representations of groups have many interesting properties (see [22, § 14] for unimodular groups and [23], [2, Chap. 8] for not unimodular generalisation) which are crucial in the construction of wavelets. For example, for a square integrable representation all functions  $\langle \rho(g)v_1, v_2 \rangle$  with an admissible vector  $v_1$  and any  $v_2 \in H$  are square integrable on  $G$ ; such representation belong to *discrete series*; etc.

EXERCISE 2.2. Show that

- (i) Admissible vectors form a linear space.
- (ii) For an irreducible  $\rho$  the set of admissible vectors is dense in  $H$  or empty.

HINT. The set of all admissible vectors is an  $\rho$ -invariant subspace of  $H$ .  $\square$

- EXERCISE 2.3. (i) Find a condition for a vector to be admissible for the representation (1.2) (and therefore the representation is square integrable).  
(ii) Show that  $w_0(x) = \frac{1}{2\pi i(x+i)}$  is not admissible.

For an admissible vector  $w$  we take its normalisation  $w_0 = \frac{\|w\|}{c}w$  to obtain:

$$(2.2) \quad \int_G |\langle \rho(g)w_0, w_0 \rangle|^2 d\mu(g) = \|w_0\|.$$

Such a  $w_0$  as a vacuum state produces many useful properties.

PROPOSITION 2.4. *If both wavelet transform  $\mathcal{W}$  and inverse wavelet transform  $\mathcal{M}$  for an irreducible square integrable representation  $\rho$  are defined by the same admissible vector  $w_0$  then the following three statements are equivalent:*

- (i)  $w_0$  satisfy (2.2);
- (ii)  $\mathcal{M}\mathcal{W} = I$ ;
- (iii) for any vectors  $v_1, v_2 \in H$ :

$$(2.3) \quad \langle v_1, v_2 \rangle = \int_G \hat{v}_1(g) \overline{\hat{v}_2(g)} d\mu(g).$$

PROOF. We **already knew** that  $\mathcal{M}\mathcal{W} = cI$  for a constant  $c \in \mathbb{C}$ . Then (2.2) exactly says that  $c = 1$ . Because  $\mathcal{W}$  and  $\mathcal{M}$  **are adjoint operators** it follows from  $\mathcal{M}\mathcal{W} = I$  on  $H$  that:

$$\langle v_1, v_2 \rangle = \langle \mathcal{M}\mathcal{W}v_1, v_2 \rangle = \langle \mathcal{W}v_1, \mathcal{M}^*v_2 \rangle = \langle \mathcal{W}v_1, \mathcal{W}v_2 \rangle,$$

which is exactly the isometry of  $\mathcal{W}$  (2.3). Finally condition (2.2) is a particular case of general isometry of  $\mathcal{W}$  for vector  $w_0$ .  $\square$

EXERCISE 2.5. Write the isometry conditions (2.3) for wavelet transforms for  $\mathbb{Z}$  and  $ax + b$  groups (Exercises 1.4 and 1.5).

Wavelets from square integrable representation closely related to the following notion:

DEFINITION 2.6. A *reproducing kernel* on a set  $X$  with a measure is a function  $K(x, y)$  such that:

$$(2.4) \quad K(x, x) > 0, \quad \forall x \in X,$$

$$(2.5) \quad K(x, y) = \overline{K(y, x)},$$

$$(2.6) \quad K(x, z) = \int_X K(x, y)K(y, z) dy.$$

PROPOSITION 2.7. *The image  $W(G)$  of the wavelet transform  $\mathcal{W}$  has a reproducing kernel  $K(g, g') = \langle w_g, w_{g'} \rangle$ . The reproducing formula is in fact a convolution:*

$$(2.7) \quad \begin{aligned} \hat{v}(g') &= \int_G K(g', g) \hat{v}(g) d\mu(g) \\ &= \int_G \hat{w}_0(g^{-1}g') \hat{v}(g) d\mu(g) \end{aligned}$$

with a wavelet transform of the vacuum vector  $\hat{w}_0(g) = \langle w_0, \rho(g)w_0 \rangle$ .

PROOF. Again we have a simple application of the previous formulas:

$$\begin{aligned}
 \hat{v}(g') &= \langle \rho(g'^{-1})v, w_0 \rangle \\
 (2.8) \quad &= \int_G \langle \rho(h^{-1})\rho(g'^{-1})v, w_0 \rangle \overline{\langle \rho(h^{-1})w_0, w_0 \rangle} d\mu(h) \\
 &= \int_G \langle \rho((g'h)^{-1})v, w_0 \rangle \langle \rho(h)w_0, w_0 \rangle d\mu(h) \\
 &= \int_G \hat{v}(g'h) \hat{w}_0(h^{-1}) d\mu(h) \\
 &= \int_G \hat{v}(g) \hat{w}_0(g^{-1}g') d\mu(g),
 \end{aligned}$$

where transformation (2.8) is due to (2.3).  $\square$

EXERCISE 2.8. Write reproducing kernels for wavelet transforms for  $\mathbb{Z}$  and  $ax + b$  groups (Exercises 1.4 and 1.5).

EXERCISE\* 2.9. Operator (2.7) of convolution with  $\hat{w}_0$  is an orthogonal projection of  $L_2(G)$  onto  $W(G)$ .

HINT. Use that an left invariant subspace of  $L_2(G)$  is in fact an right ideal in convolution algebra, see Lemma 2.10.  $\square$

REMARK 2.10. To possess a reproducing kernel—is a well-known property of spaces of analytic functions. The space  $W(G)$  shares also another important property of analytic functions: it belongs to a kernel of a certain first order differential operator with Clifford coefficients (the Dirac operator) and a second order operator with scalar coefficients (the Laplace operator) [5, 35, 34, 36], which we will consider that later too.

We consider only fundamentals of the wavelet construction here. There are much results which could be stated in an abstract level. To avoid repetition we will formulate it later on together with an interesting examples of applications.

The construction of wavelets from square integrable representations is general and straightforward. However we could not use it everywhere we may wish:

- (i) Some important representations are not square integrable.
- (ii) Some groups, e.g.  $\mathbb{H}^n$ , do not have square representations at all.
- (iii) Even if representation is square integrable, some important vacuum vectors are not admissible, e.g.  $w_0(x) = \frac{1}{2\pi i(x+i)}$  in 2.3ii.
- (iv) Sometimes we are interested in Banach spaces, while unitary square integrable representations are acting only on Hilbert spaces.

To be vivid the trunk of the wavelets theory should split into several branches adopted to particular cases and we describe some of them in the next lectures.



## Wavelets on Homogeneous Spaces: the Segal-Bargmann Space

We investigate a situation when a representation  $\rho$  of  $G$  is not square integrable in the sense of the previous Lecture but is *square integrable modulo subgroup*. An example which we use for illustration is the classic construction from quantum mechanics and is origin of coherent states.

### 1. Quantum Mechanical Setting

We begin from a statement of quantum problem [46, 6] which could be naturally solved in the terms of wavelet transform. Mathematical formulation of quantum mechanics could be founded for example in [40], [32, § V.3].

The *states* of a quantum mechanical system of  $n$  degrees of freedom (e.g. particle) are usually described by a function from the space  $H = L_2(\mathbb{R}^n)$ . Depending on a physical interpretation it could be considered either as *configuration space* with real variables  $(q_1, q_2, \dots, q_n)$  describing coordinates of the particle or *momentum space* with real variables  $(p_1, p_2, \dots, p_n)$  describing its momenta. One could move from one description to another by the Fourier transform. Elements of  $H$  are also called *wave functions*.

The *observables* of the system are self-adjoint (possibly unbounded) operators on  $H$ . The result of *measurement* of an observable  $A$  on a state (wave function)  $\phi$  with  $\|\phi\| = 1$  is a random distribution with an expectation  $\langle A \rangle_\phi$ :

$$\langle A \rangle_\phi = \langle A\phi, \phi \rangle$$

EXERCISE 1.1. Let we could find all expectations  $\langle A \rangle_\phi$  for a fixed  $\phi$  and any self-adjoint  $A$ . Show that we may calculate the random distributions of measurement.

HINT. Use expectation  $\langle \chi_{[a,b]}(A)\phi, \phi \rangle$ , where  $\chi_{[a,b]}(A)$  is a spectral projection of  $A$  on the interval  $[a, b]$  [32, § V.1.3] to find a probability that the result of measurement of  $A$  on  $\phi$  will be within interval  $[a, b]$ .  $\square$

Among observables there is a special set of  $2n$  primary ones: these are  $n$  observables of coordinates  $q_1, \dots, q_n$  and  $n$  observables of momentum  $p_1, \dots, p_n$ . All other observables usually could be expressed by means of primary ones: either as functions [40] or as wavelets [35]. The relation between primary observables are given by the *Heisenberg commutation relations*, i.e. the only non-trivial commutators among them are:

$$(1.1) \quad [q_j, p_k] = i\hbar\delta_{jk}I.$$

EXERCISE 1.2. Check that the Heisenberg commutation relations (1.1) define a representation of the Lie Algebra of the **Heisenberg group**  $\mathbb{H}^n$ .

Therefore a realisation of primary observables as self-adjoint operators  $H$  is **connected** with a unitary representation of the Heisenberg group in  $L_2(\mathbb{R}^n)$ . We

already met it: this is the *Schrödinger representation* (1.4):

$$(1.2) \quad [\rho_{\hbar}(s, x, y) f](q) = e^{i(2\hbar s - \sqrt{2\hbar}xq + \hbar xy)} f(q - \sqrt{2\hbar}y).$$

(the representation (1.4) correspond to the case  $\hbar = 1$ ).

The important result is the following theorem which asserts that we know all possible realisation of the Heisenberg commutation relations (1.1).

**THEOREM 1.3** (Stone-von Neumann). [31, § 18.4], [55, § 1.2] *All unitary irreducible representations of the Heisenberg group  $\mathbb{H}^n$  up to unitary equivalence are as follows*

- (i) *For any  $\hbar \in (0, \infty)$  the Schrödinger irreducible noncommutative unitary representations in  $L_2(\mathbb{R}^n)$*

$$(1.3) \quad \rho_{\pm\hbar}(s, x, y) = e^{i(\pm s \cdot \hbar I \pm x \cdot \hbar^{1/2} M + y \cdot \hbar^{1/2} D)},$$

*where  $xM$  and  $yD$  are such unbounded self-adjoint operators on  $L_2(\mathbb{R}^n)$ :*

$$(1.4) \quad (x \cdot \hbar^{1/2} M)u(q) = \hbar^{1/2} \sum x_j q_j u(q),$$

$$(1.5) \quad (y \cdot \hbar^{1/2} D)u(q) = \frac{\hbar^{1/2}}{i} \sum y_j \frac{\partial u(q)}{\partial q_j}.$$

- (ii) *For  $(q, p) \in \mathbb{R}^{2n}$  commutative one-dimensional representations on  $\mathbb{C}$ :*

$$(1.6) \quad \rho_{(q,p)}(s, x, y)u = e^{i(qx + py)}u, \quad u \in \mathbb{C}.$$

Therefore there is essentially unique model for a quantum mechanical particle. Nevertheless it is worthwhile to look for some models which can act as alternatives for the Schrödinger representation. In particular, the Segal-Bargmann representation [6, 46] serves to

- give a realisation of states by “true” functions, not an equivalent classes from  $L_2(\mathbb{R}^n)$ .
- give a geometric representation of the dynamics of the [harmonic oscillators](#);
- present a nice model for the *creation* and *annihilation operator*:

$$(1.7) \quad a_j^+ = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad a_j^- = \frac{1}{\sqrt{2}}(q_j - ip_j),$$

which are important for quantum field theory;

- allow applying [tools of analytic function theory](#).

The huge abilities of the Segal-Bargmann (or Fock [24]) model are not yet completely employed, see for example new ideas in a recent paper [43].

Since the Segal-Bargmann model should give a representation  $\mathbb{H}^n$  which is unitary equivalent to the Schrödinger one then it is naturally to construct an intertwining operator between them as a wavelet transform.

## 2. Fundamentals of Wavelets on Homogeneous Spaces

Let  $G$  be a group and  $G_0$  be its closed normal subgroup. Let  $X = G/G_0$  be the corresponding [homogeneous space](#) with a left invariant measure  $d\mu$ . Let  $s : X \rightarrow G$  be a Borel section in the principal bundle  $G \rightarrow G/G_0$ . Let  $\rho$  be a continuous representation of a group  $G$  by invertible unitary operators  $\rho(g)$ ,  $g \in G$  in a Hilbert space  $H$ .

For any  $g \in G$  there is a unique decomposition of the form  $g = s(x)h$ ,  $h \in G_0$ ,  $x \in X$ . We will define  $r : G \rightarrow G_0 : r(g) = h = (s^{-1}(g))^{-1}g$  from the previous

equality and write a formal notation  $x = s^{-1}(g)$ . Then there is a geometric action of  $G$  on  $X \rightarrow X$  defined as follows

$$g : x \mapsto g^{-1} \cdot x = s^{-1}(g^{-1}s(x)).$$

EXAMPLE 2.1. As a subgroup  $G_0$  we select now the center of  $\mathbb{H}^n$  consisting of elements  $(t, 0)$ . Of course  $X = G/G_0$  isomorphic to  $\mathbb{C}^n$  and mapping  $s : \mathbb{C}^n \rightarrow G$  simply is defined as  $s(z) = (0, z)$ . The Haar measure on  $\mathbb{H}^n$  coincides with the standard Lebesgue measure on  $\mathbb{R}^{2n+1}$  [55, § 1.1] thus the invariant measure on  $X$  also coincides with the Lebesgue measure on  $\mathbb{C}^n$ . Note also that composition law  $s^{-1}(g \cdot s(z))$  reduces to Euclidean shifts on  $\mathbb{C}^n$ . We also find  $s^{-1}((s(z_1))^{-1} \cdot s(z_2)) = z_2 - z_1$  and  $r((s(z_1))^{-1} \cdot s(z_2)) = \frac{1}{2}\mathfrak{I}\bar{z}_1 z_2$ .

DEFINITION 2.2. Let  $G, G_0, X = G/G_0, s : X \rightarrow G, \rho : G \rightarrow \mathcal{L}(H)$  be as above. We say that  $w_0 \in H$  is a *vacuum vector* if it satisfies to the following two conditions:

$$(2.1) \quad \rho(h)w_0 = \chi(h)w_0, \quad \chi(h) \in \mathbb{C}, \quad \text{for all } h \in G_0;$$

$$(2.2) \quad \int_X |\langle w_0, \rho(s(x))w_0 \rangle|^2 dx = \|w_0\|^2.$$

We will say that set of vectors  $w_x = \rho(x)w_0, x \in X$  form a family of *coherent states*.

Note that mapping  $h \rightarrow \chi(h)$  from (2.1) defines a character of the subgroup  $G_0$ . The condition (2.2) could be easily achieved by a renormalisation  $w_0$  as soon as we sure that the integral in the left hand side is finite.

CONVENTION 2.3. In that follow we will usually write  $x \in X$  and  $x^{-1} \in X$  instead of  $s(x) \in G$  and  $s(x)^{-1} \in G$  correspondingly. The right meaning of “ $x$ ” could be easily found from the context (whether an element of  $X$  or  $G$  is expected there).

EXAMPLE 2.4. As a “vacuum vector” we will select the original *vacuum vector of quantum mechanics*—the Gauss function  $w_0(q) = e^{-q^2/2}$  (see Figure 1), which belongs to all  $L_2(\mathbb{R}^n)$ . Its transformations are defined as follow:

$$\begin{aligned} w_g(q) = [\rho_{(s,z)}w_0](q) &= e^{i(2s-\sqrt{2}xq+xy)} e^{-(q-\sqrt{2}y)^2/2} \\ &= e^{2is-(x^2+y^2)/2} e^{((x+iy)^2-q^2)/2-\sqrt{2}i(x+iy)q} \\ &= e^{2is-z\bar{z}/2} e^{(z^2-q^2)/2-\sqrt{2}izq}. \end{aligned}$$

Particularly  $[\rho_{(t,0)}w_0](q) = e^{-2it}w_0(q)$ , i.e., it really is a vacuum vector in the sense of our definition with respect to  $G_0$ .

EXERCISE 2.5. Check the square integrability condition (2.2) for  $w_0(q) = e^{-q^2/2}$ .

The wavelet transform (similarly to **the group case**) could be defined as a mapping from  $G_0$  to a space of bounded continuous functions over  $G$  via representational coefficients

$$v \mapsto \hat{v}(g) = \langle \rho(g^{-1})v, w_0 \rangle = \langle v, \rho(g)^*w_0 \rangle.$$

Due to (2.1) such functions have simple transformation properties along orbits  $gG_0$ , i.e.  $\hat{v}(gh) = \bar{\chi}(h)\hat{v}(g), g \in G, h \in G_0$ . Thus they are completely defined by their values indexed by points of  $X = G/G_0$ . Therefore we prefer to consider so called reduced wavelet transform.

DEFINITION 2.6. The *reduced wavelet transform*  $\mathcal{W}$  from a Hilbert space  $G_0$  to a space of function  $W(X)$  on a homogeneous space  $X = G/G_0$  defined by a representation  $\rho$  of  $G$  on  $G_0$ , a vacuum vector  $w_0$  is given by the formula

$$(2.3) \quad \mathcal{W} : H \rightarrow W(X) : v \mapsto \hat{v}(x) = [\mathcal{W}v](x) = \langle \rho(x^{-1})v, w_0 \rangle = \langle v, \rho^*(x)w_0 \rangle.$$

EXAMPLE 2.7. The transformation (2.3) with the kernel  $[\rho_{(0,z)}w_0](q)$  is an embedding  $L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{C}^n)$  and is given by the formula

$$\begin{aligned}
 \hat{f}(z) &= \langle f, \rho_{s(z)}f_0 \rangle \\
 &= \pi^{-n/4} \int_{\mathbb{R}^n} f(q) e^{-z\bar{z}/2} e^{-(z^2+q^2)/2+\sqrt{2}zq} dq \\
 (2.4) \quad &= e^{-z\bar{z}/2} \pi^{-n/4} \int_{\mathbb{R}^n} f(q) e^{-(z^2+q^2)/2+\sqrt{2}zq} dq.
 \end{aligned}$$

Then  $\hat{f}(g)$  belongs to  $L_2(\mathbb{C}^n, dg)$  or its preferably to say that function  $\check{f}(z) = e^{z\bar{z}/2}\hat{f}(t_0, z)$  belongs to space  $L_2(\mathbb{C}^n, e^{-|z|^2} dg)$  because  $\check{f}(z)$  is analytic in  $z$ . Such functions form the Segal-Bargmann space  $F_2(\mathbb{C}^n, e^{-|z|^2} dg)$  of functions [6, 46], which are analytic by  $z$  and square-integrable with respect to the Gaussian measure  $e^{-|z|^2} dz$ . We use notation  $\check{W}$  for the mapping  $v \mapsto \check{v}(z) = e^{z\bar{z}/2}\mathcal{W}v$ . Analyticity of  $\check{f}(z)$  is equivalent to the condition  $(\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j I)\hat{f}(z) = 0$ . The integral in (2.4) is the well-known Segal-Bargmann transform [6, 46].

EXERCISE 2.8. Check that  $\check{w}_0(z) = 1$  for the vacuum vector  $w_0(q) = e^{-q^2/2}$ .

There is a natural representation of  $G$  in  $W(X)$ . It could be obtained if we first lift functions from  $X$  to  $G$ , apply the left regular representation  $\Lambda$  and then pull them back to  $X$ . The result defines a representation  $\lambda(g) : W(X) \rightarrow W(X)$  as follow

$$(2.5) \quad [\lambda(g)f](x) = \chi(r(g^{-1} \cdot x))f(g^{-1} \cdot x).$$

We recall that  $\chi(h)$  is a character of  $G_0$  defined in (2.1) by the vacuum vector  $w_0$ . Of course, for the case of trivial  $G_0 = \{e\}$  (2.5) becomes the left regular representation  $\Lambda(g)$  of  $G$ .

PROPOSITION 2.9. The reduced wavelet transform  $\mathcal{W}$  intertwines  $\rho$  and the representation  $\lambda$  (2.5) on  $W(X)$ :

$$\mathcal{W}\rho(g) = \lambda(g)\mathcal{W}.$$

PROOF. We have with obvious adjustments in comparison with Proposition 1.6:

$$\begin{aligned}
 [\mathcal{W}(\rho(g)v)](x) &= \langle \rho(x^{-1})\rho(g)v, w_0 \rangle \\
 &= \langle \rho((g^{-1}s(x))^{-1})v, w_0 \rangle \\
 &= \langle \rho(r(g^{-1} \cdot x)^{-1})\rho(s(g^{-1} \cdot x)^{-1})v, w_0 \rangle \\
 &= \langle \rho(s(g^{-1} \cdot x)^{-1})v, \rho^*(r(g^{-1} \cdot x)^{-1})w_0 \rangle \\
 &= \chi(r(g^{-1} \cdot x)^{-1})[\mathcal{W}v](g^{-1}x) \\
 &= \lambda(g)[\mathcal{W}v](x).
 \end{aligned}$$

□

COROLLARY 2.10. The function space  $W(X)$  is invariant under the representation  $\lambda$  of  $G$ .

EXAMPLE 2.11. Integral transformation (2.4) intertwines the Schrödinger representation (1.2) with the following realization of representation (2.5):

$$\begin{aligned}
 \lambda(s, z)\hat{f}(u) &= \hat{f}_0(z^{-1} \cdot u)\bar{\chi}(s + r(z^{-1} \cdot u)) \\
 (2.6) \quad &= \hat{f}_0(u - z)e^{is+i\mathcal{J}(\bar{z}u)}
 \end{aligned}$$

EXERCISE 2.12. (i) Using relation  $\check{W} = e^{-|z|^2/2}\mathcal{W}$  derive from above that  $\check{W}$  intertwines the Schrödinger representation with the following:

$$\check{\lambda}(s, z)\check{f}(u) = \check{f}_0(u - z)e^{2is - \bar{z}u - |z|^2/2}.$$

(ii) Show that infinitesimal generators of representation  $\check{\lambda}$  are:

$$\partial\check{\lambda}(s, 0, 0) = iI, \quad \partial\check{\lambda}(0, x, 0) = -\partial_u - uI, \quad \partial\check{\lambda}(0, 0, y) = i(-\partial_z + zI)$$

We again introduce a transform adjoint to  $\mathcal{W}$ .

DEFINITION 2.13. The *inverse wavelet transform*  $\mathcal{M}$  from  $W(X)$  to  $H$  is given by the formula:

$$(2.7) \quad \begin{aligned} \mathcal{M} : W(X) \rightarrow H : \hat{v}(x) \mapsto \mathcal{M}[\hat{v}(x)] &= \int_X \hat{v}(x) w_x \, d\mu(x) \\ &= \int_X \hat{v}(x) \rho(x) \, d\mu(x) w_0. \end{aligned}$$

PROPOSITION 2.14. The *inverse wavelet transform*  $\mathcal{M}$  intertwines the representation  $\lambda$  on  $W(X)$  and  $\rho$  on  $H$ :

$$\mathcal{M}\lambda(g) = \rho(g)\mathcal{M}.$$

PROOF. We have:

$$\begin{aligned} \mathcal{M}[\lambda(g)\hat{v}(x)] &= \mathcal{M}[\chi(r(g^{-1} \cdot x))\hat{v}(g^{-1} \cdot x)] \\ &= \int_X \chi(r(g^{-1} \cdot x))\hat{v}(g^{-1} \cdot x) w_x \, d\mu(x) \\ &= \chi(r(g^{-1} \cdot x)) \int_X \hat{v}(x') w_{g \cdot x'} \, d\mu(x') \\ &= \rho_g \int_X \hat{v}(x') w_{x'} \, d\mu(x') \\ &= \rho_g \mathcal{M}[\hat{v}(x')], \end{aligned}$$

where  $x' = g^{-1} \cdot x$ . □

COROLLARY 2.15. The *image*  $\mathcal{M}(W(X)) \subset H$  of subspace  $W(X)$  under the *inverse wavelet transform*  $\mathcal{M}$  is *invariant* under the representation  $\rho$ .

EXAMPLE 2.16. Inverse transformation to (2.4) is given by a realization of (2.7):

$$(2.8) \quad \begin{aligned} f(q) &= \int_{\mathbb{C}^n} \hat{f}(z) f_{s(z)}(q) \, dz \\ &= \int_{\mathbb{C}^n} \hat{f}(x, y) e^{iy(x - \sqrt{2}y)} e^{-(q - \sqrt{2}y)^2/2} \, dx \, dy \\ &= \int_{\mathbb{C}^n} \check{f}(z) e^{-(\bar{z}^2 + q^2)/2 + \sqrt{2}\bar{z}q} e^{-|z|^2} \, dz. \end{aligned}$$

The transformation (2.8) intertwines the representations (2.6) and the Schrödinger representation (1.2) of the Heisenberg group.

The following proposition explain the usage of the name for  $\mathcal{M}$ .

THEOREM 2.17. The operator

$$(2.9) \quad \mathcal{P} = \mathcal{M}\mathcal{W} : H \rightarrow H$$

is a projection of  $H$  to its linear subspace for which  $w_0$  is cyclic. Particularly if  $\rho$  is an irreducible representation then the *inverse wavelet transform*  $\mathcal{M}$  is a left inverse operator on  $H$  for the wavelet transform  $\mathcal{W}$ :

$$\mathcal{M}\mathcal{W} = I.$$

PROOF. It follows from Propositions 2.9 and 2.14 that operator  $\mathcal{M}\mathcal{W} : H \rightarrow H$  intertwines  $\rho$  with itself. Then Corollaries 2.10 and 2.15 imply that the image  $\mathcal{M}\mathcal{W}$  is a  $\rho$ -invariant subspace of  $H$  containing  $w_0$ . Because of  $\mathcal{M}\mathcal{W}w_0 = w_0$  we conclude that  $\mathcal{M}\mathcal{W}$  is a projection.

From irreducibility of  $\rho$  by Schur's Lemma [31, § 8.2] one concludes that  $\mathcal{M}\mathcal{W} = cI$  on  $H$  for a constant  $c \in \mathbb{C}$ . Particularly

$$\mathcal{M}\mathcal{W}w_0 = \int_X \langle \rho(x^{-1})w_0, w_0 \rangle \rho(x)w_0 \, d\mu(x) = cw_0.$$

From the condition (2.2) it follows that  $\langle cw_0, w_0 \rangle = \langle \mathcal{M}\mathcal{W}w_0, w_0 \rangle = \langle w_0, w_0 \rangle$  and therefore  $c = 1$ .  $\square$

We have similar

**THEOREM 2.18.** *Operator  $\mathcal{W}\mathcal{M}$  is a projection of  $L_1(X)$  to  $W(X)$ .*

**COROLLARY 2.19.** *In the space  $W(X)$  the strong convergence implies point-wise convergence.*

**PROOF.** From the definition of the wavelet transform:

$$|\hat{f}(x)| = |\langle f, \rho(x)w_0 \rangle| \leq \|f\| \|w_0\|.$$

Since the wavelet transform is an isometry we conclude that  $|\hat{f}(x)| \leq c\|f\|$  for  $c = \|w_0\|$ , which implies the assertion about two types of convergence.  $\square$

**EXAMPLE 2.20.** The corresponding operator for the Segal-Bargmann space  $\mathcal{P}$  (2.9) is an identity operator  $L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$  and (2.9) gives an integral representation of the Dirac delta.

While the orthoprojection  $L_2(\mathbb{C}^n, e^{-|z|^2} dg) \rightarrow F_2(\mathbb{C}^n, e^{-|z|^2} dg)$  is of a separate interest and is a principal ingredient in Berezin quantization [9, 16]. We could easily find its kernel from (2.12). Indeed,  $\hat{f}_0(z) = e^{-|z|^2}$ , then the kernel is

$$\begin{aligned} K(z, w) &= \hat{f}_0(z^{-1} \cdot w) \bar{\chi}(r(z^{-1} \cdot w)) \\ &= \hat{f}_0(w - z) e^{i\mathcal{J}(\bar{z}w)} \\ &= \exp\left(\frac{1}{2}(-|w - z|^2 + w\bar{z} - z\bar{w})\right) \\ &= \exp\left(\frac{1}{2}(-|z|^2 - |w|^2 + w\bar{z})\right). \end{aligned}$$

To receive the reproducing kernel for functions  $\check{f}(z) = e^{|z|^2} \hat{f}(z)$  in the Segal-Bargmann space we should multiply  $K(z, w)$  by  $e^{(-|z|^2 + |w|^2)/2}$  which gives the standard reproducing kernel  $= \exp(-|z|^2 + w\bar{z})$  [6, (1.10)].

We denote by  $\mathcal{W}^* : W^*(X) \rightarrow H$  and  $\mathcal{M}^* : H \rightarrow W^*(X)$  the adjoint (in the standard sense) operators to  $\mathcal{W}$  and  $\mathcal{M}$  respectively.

**COROLLARY 2.21.** *We have the following identity:*

$$(2.10) \quad \langle \mathcal{W}v, \mathcal{M}^*l \rangle_{W(X)} = \langle v, l \rangle_H, \quad \forall v, l \in H,$$

or equivalently

$$(2.11) \quad \int_X \langle \rho(x^{-1})v, w_0 \rangle \langle \rho(x)w_0, l \rangle \, d\mu(x) = \langle v, l \rangle.$$

**PROOF.** We show the equality in the first form (2.11) (but we will apply it often in the second one):

$$\langle \mathcal{W}v, \mathcal{M}^*l \rangle_{W(X)} = \langle \mathcal{M}\mathcal{W}v, l \rangle_H = \langle v, l \rangle_H.$$

$\square$

COROLLARY 2.22. *The space  $W(X)$  has the reproducing formula*

$$(2.12) \quad \hat{v}(y) = \int_X \hat{v}(x) \hat{b}_0(x^{-1} \cdot y) d\mu(x),$$

where  $\hat{b}_0(y) = [Ww_0](y)$  is the wavelet transform of the vacuum vector  $w_0$ .

PROOF. Again we have a simple application of the previous formulas:

$$(2.13) \quad \begin{aligned} \hat{v}(y) &= \langle \rho(y^{-1})v, w_0 \rangle \\ &= \int_X \langle \rho(x^{-1})\rho(y^{-1})v, w_0 \rangle \langle \rho(x)w_0, w_0 \rangle d\mu(x) \\ &= \int_X \langle \rho(s(y \cdot x)^{-1})v, w_0 \rangle \langle \rho(x)w_0, w_0 \rangle d\mu(x) \\ &= \int_X \hat{v}(y \cdot x) \hat{b}_0(x^{-1}) d\mu(x) \\ &= \int_X \hat{v}(x) \hat{b}_0(x^{-1}y) d\mu(x), \end{aligned}$$

where transformation (2.13) is due to (2.11).  $\square$

### 3. Advanced Properties

We make the following simple but nice observation about the integral kernel of wavelet transform:

PROPOSITION 3.1. *Let  $e_j, j \in \mathbb{N}$  be an orthonormal basis in  $H$ ,  $\hat{e}_j$  be their images under a wavelet transform  $W$  then the kernel  $\langle \cdot, \rho(x)w_0 \rangle$  of the wavelet transform  $W : v \mapsto \langle x, \rho(x)w_0 \rangle$  has the following decomposition in the Dirac bra-ket notations:*

$$\langle \cdot, \rho(x)w_0 \rangle = \sum_{j=1}^{\infty} |\hat{e}_j\rangle \langle e_j| = \sum_{j=1}^{\infty} \langle \cdot, e_j \rangle \hat{e}_j.$$

Particularly if  $e_j$  are orthogonal polynomials and  $\hat{e}_j(x)$  are just powers of  $x$  then the kernel of the wavelet transform  $W$  is a generating function for  $e_j$ .

EXERCISE 3.2. Give a proof.

EXERCISE 3.3. (i) Let  $H_n(q)$  be the [Hermite polynomials](#), show that functions  $H_n(q)e^{-q^2/2}$  form an orthonormal basis in  $L_2(\mathbb{R}^n)$ .

(ii) Show that functions  $z^n/\sqrt{n!}$  are images under the Segal-Bargmann transform  $\check{W}$  of functions  $H_n(q)e^{-q^2/2}$ . (*Hint* use that the Hermite polynomial obtained from the Gaussian by derivatives which are infinitesimals of the Schödinger representation of the Heisenberg group).

(iii) Show that the Segal-Bargmann kernel is [the generating function of Hermite polynomials](#).

Another example of this type is given by Bargmann in [6, § 2g]. It links representations of  $SL_2(\mathbb{R})$  in the Berman space and Laguerre polynomials, we will consider it later.

PROPOSITION 3.4. *Let  $A$  be an operator  $H \rightarrow H$ . Then the wavelet transform  $W$  intertwines  $A$  with an operator  $\hat{A}$  on  $W(X)$  given by the integral kernel  $\hat{a}(g, g')$ :*

$$(3.1) \quad \hat{A}\hat{v}(x) = \int_X \hat{a}(x, x')\hat{v}(x') dx', \quad \text{where } \hat{a}(x, x') = \langle Aw_{x'}, w_x \rangle.$$

EXAMPLE 3.5. [25, (1.81)] For the Segal-Bargmann space: if an operator  $A$  on  $L_2(\mathbb{R}^n)$  is given as an integral operator with a kernel  $a(q, q')$  then  $\hat{a}(z, z')$  is its double Segal-Bargmann transform.

EXAMPLE 3.6 (Harmonic Oscillator). Let

$$H = \frac{1}{2} \sum_{k=1}^n (p_k^2 + q_k^2 - 1) = \sum_{k=1}^n a_k^+ a_k^-,$$

be the Hamiltonian of a harmonic oscillator—the simplest non-trivial system in classic and quantum mechanics (creation  $a^+$  and annihilation  $a^-$  operators were defined in (1.7)). The dynamics of quantum oscillator is governed by the *Schrödinger equation*:

$$\frac{d}{dt} \phi(t) = iH\phi(t),$$

and its solution  $\phi(t) = e^{iHt} \phi(0)$  is given by means of the evolution operator  $e^{iHt}$ . It is easier to construct the exponent (as any other function) of  $H$  if we could diagonalise  $H$  and that is done in the Segal-Bargmann representation. Indeed,  $H = \sum_{k=1}^n z_k \frac{\partial}{\partial z_k}$ —the Euler operator. Its eigenvectors are  $z^m$  with eigenvalues  $|m|$ . Consequently the evolution of the harmonic oscillator is given by

$$e^{iHt} f(z) = f(e^{it} z),$$

which is in a nice resemblance with geometrical dynamic of classic harmonic oscillator. In the contrast, the picture in  $L_2(\mathbb{R}^n)$  is not as simple. The eigenvectors of  $H$  are constructed from the Hermite polynomials (see Exercise 3.3) and dynamic is given by the complicated Mehler's formula [55, Chap. 1, (7.15)].

## Wavelets in Banach Spaces and Functional Calculus

### 1. Coherent States for Banach Spaces

**1.1. Abstract Nonsense.** Let  $G$  be a group and  $G_0$  be its closed normal subgroup. Let  $X = G/G_0$  be the corresponding homogeneous space with an invariant measure  $d\mu$  and  $s : X \rightarrow G$  be a Borel section in the principal bundle  $G \rightarrow G/G_0$ . Let  $\pi$  be a continuous representation of a group  $G$  by invertible isometry operators  $\rho_g, g \in G$  in a (complex) Banach space  $B$ .

The following definition simulates ones from the Hilbert space case [1, § 3.1].

**DEFINITION 1.1.** Let  $G, G_0, X = G/G_0, s : X \rightarrow G, \pi : G \rightarrow \mathcal{L}(B)$  be as above. We say that  $b_0 \in B$  is a *vacuum vector* if for all  $h \in G_0$

$$(1.1) \quad \rho(h)b_0 = \chi(h)b_0, \quad \chi(h) \in \mathbb{C}.$$

We will say that set of vectors  $b_x = \rho(x)b_0, x \in X$  form a family of *coherent states* if there exists a continuous non-zero linear functional  $l_0 \in B^*$  such that

- (i)  $\|b_0\| = 1, \|l_0\| = 1, \langle b_0, l_0 \rangle \neq 0$ ;
- (ii)  $\rho(h)^*l_0 = \bar{\chi}(h)l_0$ , where  $\rho(h)^*$  is the adjoint operator to  $\rho(h)$ ;
- (iii) The following equality holds

$$(1.2) \quad \int_X \langle \rho(x^{-1})b_0, l_0 \rangle \langle \rho(x)b_0, l_0 \rangle d\mu(x) = \langle b_0, l_0 \rangle.$$

The functional  $l_0$  is called the *test functional*. According to the strong tradition we call the set  $(G, G_0, \pi, B, b_0, l_0)$  *admissible* if it satisfies to the above conditions.

We note that mapping  $h \rightarrow \chi(h)$  from (1.1) defines a character of the subgroup  $G_0$ . The following Lemma demonstrates that condition (1.2) could be relaxed.

**LEMMA 1.2.** *For the existence of a vacuum vector  $b_0$  and a test functional  $l_0$  it is sufficient that there exists a vector  $b'_0$  and continuous linear functional  $l'_0$  satisfying to (1.1) and 1.1ii correspondingly such that the constant*

$$(1.3) \quad c = \int_X \langle \rho(x^{-1})b'_0, l'_0 \rangle \langle \rho(x)b'_0, l'_0 \rangle d\mu(x)$$

*is non-zero and finite.*

**PROOF.** There exist a  $x_0 \in X$  such that  $\langle \rho(x_0^{-1})b'_0, l'_0 \rangle \neq 0$ , otherwise one has  $c = 0$ . Let  $b_0 = \rho(x_0^{-1})b'_0 \|\rho(x_0^{-1})b'_0\|^{-1}$  and  $l_0 = l'_0 \|l'_0\|^{-1}$ . For such  $b_0$  and  $l_0$  we have 1.1i already fulfilled. To obtain (1.2) we change the measure  $d\mu(x)$ . Let  $c_0 = \langle b_0, l_0 \rangle \neq 0$  then  $d\mu' = \|\rho(x^{-1})b'_0\| \|l'_0\| c_0 c^{-1} d\mu$  is the desired measure.  $\square$

**REMARK 1.3.** Conditions (1.2) and (1.3) are known for unitary representations in Hilbert spaces as *square integrability* (with respect to a subgroup  $G_0$ ). Thus our definition describes an analog of square integrable representations for Banach spaces. Note that in Hilbert space case  $b_0$  and  $l_0$  are often the same function, thus condition 1.1ii is exactly (1.1). In the particular but still important case of trivial  $G_0 = \{e\}$  (and thus  $X = G$ ) all our results take simpler forms.

CONVENTION 1.4. In that follow we will usually write  $x \in X$  and  $x^{-1}$  instead of  $s(x) \in G$  and  $s(x)^{-1}$  correspondingly. The right meaning of “ $x$ ” could be easily found from the context (whether an element of  $X$  or  $G$  is expected there).

The wavelet transform (similarly to the Hilbert space case) could be defined as a mapping from  $B$  to a space of bounded continuous functions over  $G$  via representational coefficients

$$v \mapsto \widehat{v}(g) = \langle \rho(g^{-1})v, l_0 \rangle = \langle v, \pi(g)^* l_0 \rangle.$$

Due to 1.1ii such functions have simple transformation properties along orbits  $gG_0$ , i.e.  $\widehat{v}(gh) = \bar{\chi}(h)\widehat{v}(g)$ ,  $g \in G$ ,  $h \in G_0$ . Thus they are completely defined by their values indexed by points of  $X = G/G_0$ . Therefore we prefer to consider so-called reduced wavelet transform.

DEFINITION 1.5. The *reduced wavelet transform*  $\mathcal{W}$  from a Banach space  $B$  to a space of function  $F(X)$  on a homogeneous space  $X = G/G_0$  defined by a representation  $\pi$  of  $G$  on  $B$ , a vacuum vector  $b_0$  and a test functional  $l_0$  is given by the formula

$$(1.4) \quad \mathcal{W} : B \rightarrow F(X) : v \mapsto \widehat{v}(x) = [\mathcal{W}v](x) = \langle \rho(x^{-1})v, l_0 \rangle = \langle v, \rho^*(x)l_0 \rangle.$$

There is a natural representation of  $G$  in  $F(X)$ . For any  $g \in G$  there is a unique decomposition of the form  $g = s(x)h$ ,  $h \in G_0$ ,  $x \in X$ . We will define  $r : G \rightarrow G_0 : r(g) = h = (s^{-1}(g))^{-1}g$  from the previous equality and write a formal notation  $x = s^{-1}(g)$ . Then there is a geometric action of  $G$  on  $X \rightarrow X$  defined as follows

$$g : x \mapsto g^{-1} \cdot x = s^{-1}(g^{-1}s(x)).$$

We define a representation  $\lambda(g) : F(X) \rightarrow F(X)$  as follow

$$(1.5) \quad [\lambda(g)f](x) = \chi(r(g^{-1} \cdot x))f(g^{-1} \cdot x).$$

We recall that  $\chi(h)$  is a character of  $G_0$  defined in (1.1) by the vacuum vector  $b_0$ . For the case of trivial  $G_0 = \{e\}$  (1.5) becomes the left regular representation  $\rho_l(g)$  of  $G$ .

PROPOSITION 1.6. *The reduced wavelet transform  $\mathcal{W}$  intertwines  $\pi$  and the representation  $\lambda$  (1.5) on  $F(X)$ :*

$$\mathcal{W}\rho(g) = \lambda(g)\mathcal{W}.$$

PROOF. We have:

$$\begin{aligned} [\mathcal{W}(\rho(g)v)](x) &= \langle \rho(x^{-1})\rho(g)v, l_0 \rangle \\ &= \langle \rho((g^{-1}s(x))^{-1})v, l_0 \rangle \\ &= \langle \rho(r(g^{-1} \cdot x)^{-1})\rho(s(g^{-1} \cdot x)^{-1})v, l_0 \rangle \\ &= \langle \rho(s(g^{-1} \cdot x)^{-1})v, \rho^*(r(g^{-1} \cdot x)^{-1})l_0 \rangle \\ &= \chi(r(g^{-1} \cdot x)^{-1})[\mathcal{W}v](g^{-1}x) \\ &= \lambda(g)[\mathcal{W}v](x). \end{aligned}$$

□

COROLLARY 1.7. *The function space  $F(X)$  is invariant under the representation  $\lambda$  of  $G$ .*

We will see that  $F(X)$  possesses many properties of the *Hardy space*. The duality between  $l_0$  and  $b_0$  generates a transform dual to  $\mathcal{W}$ .

DEFINITION 1.8. The *inverse wavelet transform*  $\mathcal{M}$  from  $F(X)$  to  $B$  is given by the formula:

$$(1.6) \quad \begin{aligned} \mathcal{M} : F(X) \rightarrow B : \widehat{v}(x) &\mapsto \mathcal{M}[\widehat{v}(x)] = \int_X \widehat{v}(x) b_x \, d\mu(x) \\ &= \int_X \widehat{v}(x) \rho(x) \, d\mu(x) b_0. \end{aligned}$$

PROPOSITION 1.9. The *inverse wavelet transform*  $\mathcal{M}$  intertwines the representation  $\lambda$  on  $F(X)$  and  $\pi$  on  $B$ :

$$\mathcal{M}\lambda(g) = \rho(g)\mathcal{M}.$$

PROOF. We have:

$$\begin{aligned} \mathcal{M}[\lambda(g)\widehat{v}(x)] &= \mathcal{M}[\chi(r(g^{-1} \cdot x))\widehat{v}(g^{-1} \cdot x)] \\ &= \int_X \chi(r(g^{-1} \cdot x))\widehat{v}(g^{-1} \cdot x) b_x \, d\mu(x) \\ &= \chi(r(g^{-1} \cdot x)) \int_X \widehat{v}(x') b_{g \cdot x'} \, d\mu(x') \\ &= \rho_g \int_X \widehat{v}(x') b_{x'} \, d\mu(x') \\ &= \rho_g \mathcal{M}[\widehat{v}(x')], \end{aligned}$$

where  $x' = g^{-1} \cdot x$ . □

COROLLARY 1.10. The *image*  $\mathcal{M}(F(X)) \subset B$  of *subspace*  $F(X)$  under the *inverse wavelet transform*  $\mathcal{M}$  is *invariant* under the *representation*  $\pi$ .

The following proposition explain the usage of the name for  $\mathcal{M}$ .

THEOREM 1.11. The *operator*

$$(1.7) \quad \mathcal{P} = \mathcal{M}\mathcal{W} : B \rightarrow B$$

is a *projection* of  $B$  to its *linear subspace* for which  $b_0$  is *cyclic*. Particularly if  $\pi$  is an *irreducible representation* then the *inverse wavelet transform*  $\mathcal{M}$  is a *left inverse operator* on  $B$  for the *wavelet transform*  $\mathcal{W}$ :

$$\mathcal{M}\mathcal{W} = I.$$

PROOF. It follows from Propositions 1.6 and 1.9 that operator  $\mathcal{M}\mathcal{W} : B \rightarrow B$  intertwines  $\pi$  with itself. Then Corollaries 1.7 and 1.10 imply that the image  $\mathcal{M}\mathcal{W}$  is a  $\pi$ -invariant subspace of  $B$  containing  $b_0$ . Because  $\mathcal{M}\mathcal{W}b_0 = b_0$  we conclude that  $\mathcal{M}\mathcal{W}$  is a *projection*.

From irreducibility of  $\pi$  by Schur's Lemma [31, § 8.2] one concludes that  $\mathcal{M}\mathcal{W} = cI$  on  $B$  for a constant  $c \in \mathbb{C}$ . Particularly

$$\mathcal{M}\mathcal{W}b_0 = \int_X \langle \rho(x^{-1})b_0, l_0 \rangle \rho(x)b_0 \, d\mu(x) = cb_0.$$

From the condition (1.2) it follows that  $\langle cb_0, l_0 \rangle = \langle \mathcal{M}\mathcal{W}b_0, l_0 \rangle = \langle b_0, l_0 \rangle$  and therefore  $c = 1$ . □

We have similar

THEOREM 1.12. *Operator*  $\mathcal{W}\mathcal{M}$  is a *projection* of  $L_1(X)$  to  $F(X)$ .

We denote by  $\mathcal{W}^* : F^*(X) \rightarrow B^*$  and  $\mathcal{M}^* : B^* \rightarrow F^*(X)$  the *adjoint* (in the standard sense) operators to  $\mathcal{W}$  and  $\mathcal{M}$  respectively.

COROLLARY 1.13. *We have the following identity:*

$$(1.8) \quad \langle \mathcal{W}v, \mathcal{M}^*l \rangle_{F(X)} = \langle v, l \rangle_B, \quad \forall v \in B, \quad l \in B^*$$

or equivalently

$$(1.9) \quad \int_X \langle \rho(x^{-1})v, l_0 \rangle \langle \rho(x)b_0, l \rangle d\mu(x) = \langle v, l \rangle.$$

PROOF. We show the equality in the first form (1.9) (but will apply it often in the second one):

$$\langle \mathcal{W}v, \mathcal{M}^*l \rangle_{F(X)} = \langle \mathcal{M}\mathcal{W}v, l \rangle_B = \langle v, l \rangle_B.$$

□

COROLLARY 1.14. *The space  $F(X)$  has the reproducing formula*

$$(1.10) \quad \widehat{v}(y) = \int_X \widehat{v}(x) \widehat{b}_0(x^{-1} \cdot y) d\mu(x),$$

where  $\widehat{b}_0(y) = [\mathcal{W}b_0](y)$  is the wavelet transform of the vacuum vector  $b_0$ .

PROOF. Again we have a simple application of the previous formulas:

$$(1.11) \quad \begin{aligned} \widehat{v}(y) &= \langle \rho(y^{-1})v, l_0 \rangle \\ &= \int_X \langle \rho(x^{-1})\rho(y^{-1})v, l_0 \rangle \langle \rho(x)b_0, l_0 \rangle d\mu(x) \\ &= \int_X \langle \rho(s(y \cdot x)^{-1})v, l_0 \rangle \langle \rho(x)b_0, l_0 \rangle d\mu(x) \\ &= \int_X \widehat{v}(y \cdot x) \widehat{b}_0(x^{-1}) d\mu(x) \\ &= \int_X \widehat{v}(x) \widehat{b}_0(x^{-1}y) d\mu(x), \end{aligned}$$

where transformation (1.11) is due to (1.9). □

REMARK 1.15. To possess a reproducing kernel—is a well-known property of spaces of analytic functions. The space  $F(X)$  shares also another important property of analytic functions: it belongs to a kernel of a certain first order differential operator with Clifford coefficients (the Dirac operator) and a second order operator with scalar coefficients (the Laplace operator) [5, 35, 34, 36].

Let us now assume that there are two representations  $\pi'$  and  $\pi''$  of the same group  $G$  in two different spaces  $B'$  and  $B''$  such that two admissible sets  $(G, G_0, \pi', B', b'_0, l'_0)$  and  $(G, G_0, \pi'', B'', b''_0, l''_0)$  could be constructed for the same normal subgroup  $G_0 \subset G$ .

PROPOSITION 1.16. *In the above situation if  $F'(X) \subset F''(X)$  then the composition  $\mathcal{T} = \mathcal{M}''\mathcal{W}'$  of the wavelet transform  $\mathcal{W}'$  for  $\pi'$  and the inverse wavelet transform  $\mathcal{M}''$  for  $\pi''$  is an intertwining operator between  $\pi'$  and  $\pi''$ :*

$$\mathcal{T}\pi' = \pi''\mathcal{T}.$$

$\mathcal{T}$  is defined as follows

$$(1.12) \quad \mathcal{T} : b \mapsto \int_X \langle \pi'(x^{-1})b, l'_0 \rangle \pi''(x)b''_0 d\mu(x).$$

This transformation defines a  $B''$ -valued linear functional (a distribution for function spaces) on  $B'$ .

The Proposition has an obvious proof. This simple result is a base for an alternative approach to functional calculus of operators [33, 35] and will be used in Subsection 2.2. Note also that formulas (1.4) and (1.6) are particular cases of (1.12) because  $\mathcal{W}$  and  $\mathcal{M}$  intertwine  $\pi$  and  $\lambda$ .

**1.2. Wavelets and a Positive Cone.** The above results are true for wavelets in general. In applications a Banach space  $B$  is usually equipped with additional structures and wavelets are interplay with them. We consider an example of such interaction.

We recall [37], [30, Chap. X] the notion of positivity in Banach spaces. Let  $C \subset B$  be a sharp cone, i.e.  $x \in C$  implies that  $\lambda x \in C$  and  $-\lambda x \notin C$  for  $\lambda > 0$ . We call such elements  $x \in C$  *positive vectors*. We say also that  $x \geq y$  iff  $x - y$  is positive. There is the *dual cone*  $C^* \subset B^*$  defined by the condition

$$C^* = \{f \mid f \in B^*, \langle b, f \rangle \geq 0 \ \forall x \in C\}.$$

An operator  $A : B \rightarrow B$  is called *positive* if  $Ab \geq 0$  for all  $b \geq 0$ . If  $A$  is positive with respect to  $C$  then  $A^*$  is positive with respect to  $C^*$ .

DEFINITION 1.17. We call a representation  $\rho(g)$  *positive* if there exists a vector  $b_0 \in C$  such that  $\rho(x)b_0 \in C$  for all  $x \in X$ . A linear functional  $f \in B^*$  is *positive* ( $f > 0$ ) with respect to a vacuum vector  $b_0$  if  $\langle \rho(x)b_0, f \rangle \geq 0$  for all  $x \in X$  and  $\langle \rho(x)b_0, f \rangle$  is not identically equal to 0.

LEMMA 1.18. For any positive representation  $\rho(g)$  and vacuum vector  $b_0$  there exists a positive test functional.

PROOF. Obvious. □

We consider an estimation of positive linear functionals.

PROPOSITION 1.19. Let  $b \in B$  be a vector such that  $b = \int_X \widehat{b}(x)b_x \, d\mu(x)$ . Let

- (i)  $D(b) = \{\langle \rho(x^{-1})b, b_0 \rangle \mid x \in X\}$  be the set of value of reduced wavelets transform;
- (ii)  $\check{D}(b)$  be a convex shell of the values of  $\widehat{b}(x)$ ;
- (iii)  $\hat{D}(b) = \{\langle b, f \rangle \mid f \in C^*, \|f\| = 1, f \geq 0\}$ .

Then

$$D(b) \subset \hat{D}(b) \subset \check{D}(b).$$

PROOF. The first inclusion is obvious. The second could be easily checked:

$$\langle b, f \rangle = \left\langle \int_X \widehat{b}(x)b_x \, d\mu(x), f \right\rangle = \int_X \widehat{b}(x) \langle b_x, f \rangle \, d\mu(x).$$

□

**1.3. Singular Vacuum Vectors.** In many important cases the above general scheme could not be carried out because the representation  $\pi$  of  $G$  is not square-integrable or even not square-integrable modulo a subgroup  $G_0$ . Thereafter the vacuum vector  $b_0$  could not be selected within the original space  $B$  which the representation  $\pi$  acts on. The simplest mathematical example is the Fourier transform (see Example 3.1). In physics this is the well-known problem of *absence of vacuum state* in the constructive algebraic quantum field theory [47, 48, 49]. The absence of the vacuum within the linear space of system's states is another illustration to the old thesis *Natura abhorret vacuum*<sup>1</sup> or even more specifically *Natura abhorret vectorem vacui*<sup>2</sup>.

<sup>1</sup>Nature is horrified by (any) vacuum (Lat.).

<sup>2</sup>Nature is horrified by a carrier of nothingness (Lat.). This illustrates how far a humane beings deviated from Nature.

We will present a modification of our construction which works in such a situation. For a singular vacuum vector the algebraic structure of group representations could not describe the situation alone and requires an essential assistance from analytical structures.

DEFINITION 1.20. Let  $G, G_0, X = G/G_0, s : X \rightarrow G, \pi : G \rightarrow \mathcal{L}(B)$  be as in Definition 1.1. We assume that there exist a topological linear space  $\widehat{B} \supset B$  such that

- (i)  $B$  is dense in  $\widehat{B}$  (in topology of  $\widehat{B}$ ) and representation  $\pi$  could be uniquely extended to the continuous representation  $\widehat{\pi}$  on  $\widehat{B}$ .
  - (ii) There exists  $b_0 \in \widehat{B}$  be such that for all  $h \in G_0$
- $$(1.13) \quad \widehat{\pi}(h)b_0 = \chi(h)b_0, \quad \chi(h) \in \mathbb{C}.$$
- (iii) There exists a continuous non-zero linear functional  $l_0 \in B^*$  such that  $\rho(h)^*l_0 = \bar{\chi}(h)l_0$ , where  $\rho(h)^*$  is the adjoint operator to  $\rho(h)$ ;
  - (iv) The composition  $\mathcal{M}\mathcal{W} : B \rightarrow \widehat{B}$  of the wavelet transform (1.4) and the inverse wavelet transform (1.6) maps  $B$  to  $B$ .
  - (v) For a vector  $p_0 \in B$  the following equality holds

$$(1.14) \quad \left\langle \int_X \langle \rho(x^{-1})p_0, l_0 \rangle \rho(x)b_0 \, d\mu(x), l_0 \right\rangle = \langle p_0, l_0 \rangle,$$

where the integral converges in the weak topology of  $\widehat{B}$ .

As before we call the set of vectors  $b_x = \rho(x)b_0, x \in X$  by *coherent states*; the vector  $b_0$ —a *vacuum vector*; the functional  $l_0$  is called the *test functional* and finally  $p_0$  is the *probe vector*.

This Definition is more complicated than Definition 1.1. The equation (1.14) is a substitution for (1.2) if the linear functional  $l_0$  is not continuous in the topology of  $\widehat{B}$ . Example 3.1 shows that the Definition does not describe an empty set. The function theory in  $\mathbb{R}^{1,1}$  constructed in [34] provides a more exotic example of a singular vacuum vector.

We shall show that 1.20v could be satisfied by an adjustment of other components.

LEMMA 1.21. *For the existence of a vacuum vector  $b_0$ , a test functional  $l_0$ , and a probe vector  $p_0$  it is sufficient that there exists a vector  $b'_0$  and continuous linear functional  $l'_0$  satisfying to 1.20i–1.20iv and a vector  $p'_0 \in B$  such that the constant*

$$c = \left\langle \int_X \langle \rho(x^{-1})p_0, l_0 \rangle \rho(x)b_0 \, d\mu(x), l_0 \right\rangle$$

*is non-zero and finite.*

The proof follows the path for Lemma 1.2. The following Proposition summarizes results which could be obtained in this case.

PROPOSITION 1.22. *Let the wavelet transform  $\mathcal{W}$  (1.4), its inverse  $\mathcal{M}$  (1.6), the representation  $\lambda(g)$  (1.5), and functional space  $F(X)$  be adjusted accordingly to Definition 1.20. Then*

- (i)  $\mathcal{W}$  intertwines  $\rho(g)$  and  $\lambda(g)$  and the image of  $F(X) = \mathcal{W}(B)$  is invariant under  $\lambda(g)$ .
- (ii)  $\mathcal{M}$  intertwines  $\lambda(g)$  and  $\widehat{\pi}(g)$  and the image of  $\mathcal{M}(F(B)) = \mathcal{M}\mathcal{W}(B) \subset B$  is invariant under  $\rho(g)$ .

(iii) If  $\mathcal{M}(F(X)) = B$  (particularly if  $\rho(g)$  is irreducible) then  $\mathcal{M}\mathcal{W} = I$  otherwise  $\mathcal{M}\mathcal{W}$  is a projection  $B \rightarrow \mathcal{M}(F(X))$ . In both cases  $\mathcal{M}\mathcal{W}$  is an operator defined by integral

$$(1.15) \quad \mathbf{b} \mapsto \int_X \langle \rho(x^{-1})\mathbf{b}, \mathbf{l}_0 \rangle \rho(x)\mathbf{b}_0 \, d\mu(x),$$

(iv) Space  $F(X)$  has a reproducing formula

$$(1.16) \quad \widehat{\mathbf{v}}(\mathbf{y}) = \left\langle \int_X \widehat{\mathbf{v}}(x) \rho(x^{-1}\mathbf{y})\mathbf{b}_0 \, dx, \mathbf{l}_0 \right\rangle$$

which could be rewritten as a singular convolution

$$\widehat{\mathbf{v}}(\mathbf{y}) = \int_X \widehat{\mathbf{v}}(x) \widehat{\mathbf{b}}(x^{-1}\mathbf{y}) \, dx$$

with a distribution  $\mathbf{b}(\mathbf{y}) = \langle \rho(\mathbf{y}^{-1})\mathbf{b}_0, \mathbf{l}_0 \rangle$  defined by (1.16).

The proof is algebraic and completely similar to Subsection 1.1.

## 2. Wavelets in Operator Algebras

We are going to apply the above abstract scheme to special spaces, which our main targets—wavelets on operator algebras. This gives a possibility to study operators by means of functions—symbols of operators.

**2.1. Co- and Contravariant Symbols of Operators.** We construct a realization of the wavelet transform as co- and contravariant symbols (also known as Wick and anti-Wick symbols) of operators. These symbols and their connections with wavelets in Hilbert spaces are known for a while [7, 8, 10, 9]. However their realization (described bellow) as wavelets in Banach algebras seems to be new.

Let  $\rho(g)$  be a representation of a group  $G$  in a Banach space  $B$  by isometry operators. Then we could define two new representations for groups  $G$  and  $G \times G$  correspondingly in the space  $\mathcal{L}(B)$  of bounded linear operators  $B \rightarrow B$ :

$$(2.1) \quad \widehat{\pi} : G \rightarrow \mathcal{L}(\mathcal{L}(B)) : A \mapsto \rho(g)^{-1}A\rho(g),$$

$$(2.2) \quad \widetilde{\pi} : G \times G \rightarrow \mathcal{L}(\mathcal{L}(B)) : A \mapsto \rho(g_1)^{-1}A\rho(g_2),$$

where  $A \in \mathcal{L}(B)$ . Note that  $\widehat{\pi}(g)$  are algebra automorphisms of  $\mathcal{L}(B)$  for all  $g$ . Representation  $\widetilde{\pi}(g_1, g_2)$  is an algebra homomorphism from  $\mathcal{L}(B)$  to the algebra  $L_{(g_1, g_2)}(B)$  of linear operators on  $B$  equipped with a composition

$$A_1 \circ A_2 = A_1 \rho(g_1)^{-1} \rho(g_2) A_2$$

with the usual multiplication of operators in the right-hand side. The rôle of such algebra homomorphisms in a symbolical calculus of operators was explained in [27]. It is also obvious that  $\widehat{\pi}(g)$  is the restriction of  $\widetilde{\pi}(g_1, g_2)$  to the diagonal of  $G \times G$ .

Let there are selected a vacuum vector  $\mathbf{b}_0 \in B$  and a test functional  $\mathbf{l}_0 \in B^*$  for  $\pi$ . Then there are the canonically associated vacuum vector  $P_0 \in \mathcal{L}(B)$  and test functional  $f_0 \in L^*(B)$  defined as follows:

$$(2.3) \quad P_0 : B \rightarrow B : \mathbf{b} \mapsto P_0\mathbf{b} = \langle \mathbf{b}, \mathbf{l}_0 \rangle \mathbf{b}_0;$$

$$(2.4) \quad f_0 : \mathcal{L}(B) \rightarrow \mathbb{C} : A \mapsto \langle A\mathbf{b}_0, \mathbf{l}_0 \rangle.$$

They define the following coherent states and transformations of the test functional

$$\begin{aligned} P_g &= \widehat{\pi}(g)P_0 = \langle \cdot, \mathbf{l}_g \rangle \mathbf{b}_g, & P_{(g_1, g_2)} &= \widetilde{\pi}(g_1, g_2)P_0 = \langle \cdot, \mathbf{l}_{g_1} \rangle \mathbf{b}_{g_2}, \\ f_g &= \widehat{\pi}^*(g)f_0 = \langle \cdot, \mathbf{b}_g, \mathbf{l}_g \rangle, & f_{(g_1, g_2)} &= \widetilde{\pi}^*(g_1, g_2)f_0 = \langle \cdot, \mathbf{b}_{g_1}, \mathbf{l}_{g_2} \rangle \end{aligned}$$

where as usually we denote  $b_g = \rho(g)b_0$ ,  $l_g = \rho^*(g)l_0$ . All these formulas take simpler forms for Hilbert spaces if  $l_0 = b_0$ .

DEFINITION 2.1. The *covariant (pre-)symbol*  $A(x)$  ( $A(x_1, x_2)$ ) of an operator  $A$  acting on a Banach space  $B$  defined by  $b_0 \in B$  and  $l_0 \in B^*$  is its wavelet transform with respect to representation  $\widehat{\pi}(g)$  (2.1) ( $\widetilde{\pi}(g_1, g_2)$  (2.2)) and the functional  $f_0$  (2.4), i.e. they are defined by the formulas

$$(2.5) \quad \check{A}(x) = (\widehat{\pi}(x)A, f_0) = \langle \rho(x)^{-1}A\rho(x)b_0, l_0 \rangle = \langle Ab_x, l_x \rangle,$$

$$(2.6) \quad A(x_1, x_2) = (\widetilde{\pi}(x_1, x_2)A, f_0) = \langle \rho(x_1)^{-1}A\rho(x_2)b_0, l_0 \rangle = \langle Ab_{x_2}, l_{x_1} \rangle.$$

The *contravariant (pre-)symbol* of an operator  $A$  is a function  $\check{\check{A}}(x)$  (a function  $\check{\check{A}}(x_1, x_2)$  correspondingly) such that  $A$  is the inverse wavelet transform of  $\check{\check{A}}(x)$  (of  $\check{\check{A}}(x_1, x_2)$  correspondingly) with respect to  $\widehat{\pi}(g)$  ( $\widetilde{\pi}(g_1, g_2)$ ), i.e.

$$(2.7) \quad A = \int_X \check{\check{A}}(x)\widehat{\pi}(x)P_0 \, d\mu(x) = \int_X \check{\check{A}}(x)P_x \, d\mu(x),$$

$$(2.8) \quad \begin{aligned} A &= \int_X \int_X \check{\check{A}}(x_1, x_2)\widetilde{\pi}(x_1, x_2)P_0 \, d\mu(x_1) \, d\mu(x_2) \\ &= \int_X \int_X \check{\check{A}}(x_1, x_2)P_{(x_1, x_2)} \, d\mu(x_1) \, d\mu(x_2), \end{aligned}$$

where the integral is defined in the weak sense.

Obviously the covariant symbol  $\check{A}(x)$  is the restriction of the covariant pre-symbol  $\check{A}(x_1, x_2)$  to the diagonal of  $G \times G$ .

PROPOSITION 2.2. A mapping  $\sigma : A \mapsto \sigma_A(x_1, x_2)$  of operators to their covariant symbols is the algebra homomorphism from algebra of operators on  $B$  to algebra of integral operators on  $F(G)$ , i.e.

$$(2.9) \quad \sigma_{A_1 A_2}(x_1, x_3) = \int_X \sigma_{A_1}(x_1, x_2)\sigma_{A_2}(x_2, x_3) \, d\mu(x_2).$$

PROOF. One could easily see that:

$$(2.10) \quad \begin{aligned} &\int_X \sigma_{A_1}(x_1, x_2)\sigma_{A_2}(x_2, x_3) \, d\mu(x_2) \\ &= \int_X \langle \rho(x_1)A_1\rho(x_2^{-1})b_0, l_0 \rangle \langle \rho(x_2)A_2\rho(x_3^{-1})b_0, l_0 \rangle \, d\mu(x_2) \\ &= \int_X \langle \rho(x_2^{-1})b_0, A_1^*\rho^*(x_1)l_0 \rangle \langle \rho(x_2)A_2\rho(x_3^{-1})b_0, l_0 \rangle \, d\mu(x_2) \\ &= \langle A_2\rho(x_3^{-1})b_0, A_1^*\rho^*(x_1)l_0 \rangle \\ &= \langle \rho(x_1)A_1A_2\rho(x_3^{-1})b_0, l_0 \rangle \\ &= \sigma_{A_1 A_2}(x_1, x_3), \end{aligned}$$

where transformation (2.10) is due to (1.9). □

The following proposition is obvious.

PROPOSITION 2.3. An operator  $A$  could be reconstructed from its covariant presymbol  $A(g_1, g_2)$  by the formula

$$Av = \int_G \int_G A(g_1, g_2)\widehat{v}(g_2) \, d\mu(g_2)b_{g_1} \, d\mu(g_1).$$

We have a particular interest in operators closely connected with the representation  $\rho_g$ .

PROPOSITION 2.4. *Let an operator  $A$  on  $B$  is defined by the formula*

$$Av = \int_G \alpha(g) \rho_g v \, d\mu(g)$$

for a function  $\alpha(g)$  on  $G$ . Then  $A'W = WA$  where  $A'$  is a two-sided convolution on  $G$  defined by the formula

$$[A'\hat{v}](h) = \int_G \int_G \alpha(g_1) \hat{b}_0(g_2) \hat{v}(g_1^{-1} h g_2) \, d\mu(g_1) \, d\mu(g_2).$$

For operator algebras there are the standard notions of positivity: any operator of the form  $A^*A$  is positive; if algebra is realized as operators on a Hilbert space  $H$  then  $b \in H$  defines a positive functional  $f_b(A) = \langle Ab, b \rangle$ . Thus the following proposition is a direct consequence of the Proposition 1.19.

PROPOSITION 2.5. [7, Thm. 1] *Let  $A$  be an operator, let  $D(A)$  be the set of values of the covariant symbol  $A(x)$ , let  $\check{D}(A)$  be a convex shell of the values of contravariant symbol  $\check{A}(x)$ . Let  $\hat{D}(A)$  be the set of values of the quadratic form  $\langle Ab, b \rangle$  for all vectors  $\|b\| = 1$ . Then*

$$D(b) \subset \hat{D}(b) \subset \check{D}(b).$$

EXAMPLE 2.6. There are at least two very important realizations of symbolical calculus of operators. The theory of pseudodifferential operators (PDO) [18, 50, 54] is based on the Schrödinger representation of the Heisenberg group  $\mathbb{H}^n$  (see Subsection 3.1) on the spaces of functions  $L_p(\mathbb{R}^n)$  [27]. The Wick and anti-Wick symbolical calculi [7, 9] arise from the Segal-Bargmann representation [46, 6] (see Subsection 2.1) of the same group  $\mathbb{H}^n$ . Connections (intertwining operators) between these two representations were exploited in [27] to obtain fundamentals of the theory of PDO.

**2.2. Functional Calculus and Group Representations.** This Subsection illustrates a new approach to functional calculus of operators outlined in [33, 35]. The approach uses the intertwining property for two representations instead of an algebraic homomorphism.

Let  $\mathfrak{B}$  be a Banach algebra and  $\mathbf{T} \subset \mathfrak{B}$  be its subset of elements. Let  $G$  be a group,  $G_0$  be its normal subgroup and  $X = G/G_0$ —the corresponding homogeneous space. We assume that there is a representation  $\tau$  depending from  $\mathbf{T} \subset \mathfrak{B}$  defined on measurable functions from  $L(X, \mathfrak{B})$  by the formula

$$(2.11) \quad \tau(g)f(x) = t(g, x)f(g^{-1} \cdot x), \quad f(x) \in L(X, \mathfrak{B}),$$

where  $t(g, x) : \mathfrak{B} \rightarrow \mathfrak{B}$  depends from  $x \in X$  and  $g \in G$ . It is convenient to use a linear functional  $l \in \mathfrak{B}'$  to make the situation more tractable by reducing it to the scalar case. Using  $l$  we could define a representation  $\tau_l(x)$  on  $F(X)$  by the following formula

$$(2.12) \quad \tau_l(x) : f_l(y) = \langle f(y), l \rangle \mapsto [\tau_l(x)f_l](y) = \langle \tau(x)f(y), l \rangle,$$

where  $f(y) \in L(X, \mathfrak{B})$ ,  $l \in \mathfrak{B}'$ . We will understand convergence of all integrals involving  $\tau$  in a weak sense, i.e. as convergence of all corresponding integrals with  $\tau_l$ ,  $l \in \mathfrak{B}'$ . We also say that  $\tau$  is *irreducible* if all  $\tau_l$  are irreducible.

REMARK 2.7. If  $\mathfrak{B}$  is realized as an algebra of operators on a Banach space  $B$  then  $l \in \mathfrak{B}'$  could be realized as an element of  $B \otimes B'$ . In this case the formula (2.12) looks like (2.5). The important difference is the following. In (2.5) the representation in the operator algebra  $\mathfrak{B}$  arises from a representation in Banach space  $B$  and is the same for all elements of  $\mathfrak{B}$ . Representation  $\tau_l$  in (2.12) is defined via the representation  $\tau$  which depends in its turn from a set  $\mathbf{T} \subset \mathfrak{B}$ . Such representations are usually connected with some (non-linear) geometric actions of a group

directly on operator algebra. Examples of these geometric actions are the representation of the Heisenberg group (3.10) leading to the Weyl functional calculus, fractional-linear transformations of operators leading [35] to Dunford-Riesz functional calculus and monogenic functional calculus [33]. Thus such representations contain important information on  $\mathbf{T}$ .

We also assume that there is a representation  $\pi$  of  $G$  in  $F(X)$  with a vacuum vector  $b_0$ , a test functional  $l_0$  and the system of wavelets (coherent states)  $b_x, x \in X$ , which were main actors in the previous Section. Let  $\rho^*(g) = \rho(g^{-1})^*$  be the adjoint representation of  $\rho(g)$  in  $F(X)$ .

We need a preselected element  $T_0(x) \in L(X, \mathfrak{B})$  which plays a rôle of a *vacuum vector* for the representation  $\tau$ , it is defined by the condition:

$$(2.13) \quad \int_X \widehat{b}_0(x') \tau(x') T_0(x) dx' = T_0(x),$$

where  $\widehat{b}_0(x) = \langle \rho(x^{-1})b_0, l_0 \rangle$  is the wavelet transform of the vacuum vector  $b_0 \in F(X)$  for  $\pi$ .

LEMMA 2.8. *A vacuum vector for  $\tau$  always exists and is given by the formula*

$$(2.14) \quad T_0(x) = \int_X \widehat{b}_0(x') \tau(x') T(x) dx',$$

where  $T(x) \in L(X, \mathfrak{B})$  is an arbitrary element which the integral (2.14) converges for. If  $\tau$  is irreducible (i.e. all  $\tau_l$  (2.12) are irreducible) in the linear span of  $\tau(x')T(x), x' \in X$  then  $T_0(x)$  does not depend from a particular chose of  $T(x)$ .

PROOF. First we could easily verify condition (2.13) for the  $T_0$  defined by (2.14):

$$(2.15) \quad \begin{aligned} \int_X \widehat{b}_0(x) \tau(x) T_0(x_1) dx &= \int_X \widehat{b}_0(x) \tau(x) \int_X \widehat{b}_0(x') \tau(x') T(x_1) dx' dx \\ &= \int_X \int_X \widehat{b}_0(x) \widehat{b}_0(x') \tau(x) \tau(x') T(x_1) dx' dx \\ &= \int_X \left( \int_X \widehat{b}_0(x) \widehat{b}_0(x^{-1}x'') dx \right) \tau(x'') T(x_1) dx'' \\ (2.16) \quad &= \int_X \widehat{b}_0(x'') \tau(x'') T(x_1) dx'' \\ &= T_0(x_1). \end{aligned}$$

Here we use the change of variables  $x'' = x \cdot x'$  in (2.15) and reproducing property (1.10) of  $\widehat{b}_0(x)$  in (2.16).

To prove that for any admissible  $T(x)$  we will receive the same  $T_0(x)$  is enough to pass from the representation  $\tau$  to representations  $\tau_l$  (2.12) defined by  $l \in \mathfrak{B}'$ . Then we deal with scalar valued (not operator valued) functions and knew that one could use any admissible vector  $T_l(x) = \langle T(x), l \rangle$  as a vacuum vector in the reconstruction formula (1.6).  $\square$

Now we could specify the Definition 1.1 from [33] as follows.

DEFINITION 2.9. Let  $G, G_0, X = G/G_0, \mathfrak{B}, \mathbf{T}, \tau, \pi, F, b_0, T_0$  be as described above. One says that a continuous linear  $\mathfrak{B}$ -valued functional  $\Phi_T(\cdot, x) : F(X) \rightarrow \mathfrak{B}$ , parametrized by a point  $x \in X$  and depending from  $\mathbf{T} \subset \mathfrak{B}$ :

$$\Phi_T(\cdot, x) : f(y) \mapsto [\Phi_T f](x) = \int_X f(y) \Phi_T(y, x) dy$$

is a *functional calculus* if

(i)  $\Phi_{\mathbf{T}}$  is an intertwining operator between  $\rho(g)$  and  $\tau(g)$ , namely

$$(2.17) \quad [\Phi_{\mathbf{T}}\rho(g)f(\mathbf{y})](\mathbf{x}) = \tau(g)[\Phi_{\mathbf{T}}f(\mathbf{y})](\mathbf{x}),$$

for all  $g \in G$  and  $f(\mathbf{y}) \in F(X)$

(ii)  $\Phi_{\mathbf{T}}$  maps the vacuum vector  $b_0(\mathbf{y})$  for the representation  $\pi$  to the vacuum vector  $T_0(\mathbf{x})$  for the representation  $\tau$ :

$$(2.18) \quad [\Phi_{\mathbf{T}}b_0(\mathbf{y})](\mathbf{x}) = T_0(\mathbf{x}).$$

$\mathfrak{B}$ -valued distribution  $\Phi_{\mathbf{T}}(\mathbf{y}, x_0)$ ,  $s(x_0) = e \in G$  associated with  $\mathfrak{B}$ -valued linear functional on  $F(X)$  is called a *spectral decomposition* of operators  $\mathbf{T}$ .

Representation  $\tau$  in (2.17) is defined by (2.11):

$$\tau(g)[\Phi_{\mathbf{T}}f(\mathbf{y})](\mathbf{x}) = t(g, \mathbf{x})[\Phi_{\mathbf{T}}f(\mathbf{y})](g^{-1} \cdot \mathbf{x}).$$

We could state (2.17) equivalently as

$$[I_{\mathbf{y}} \otimes \tau_x(g)]\Phi(\mathbf{y}, \mathbf{x}) = [\rho_{\mathbf{y}}^*(g^{-1}) \otimes I_{\mathbf{x}}]\Phi(\mathbf{y}, \mathbf{x}).$$

REMARK 2.10. The functional calculus  $\Phi_{\mathbf{T}}(\mathbf{y}, \mathbf{x})$  as defined here has the explicit covariant property with respect to variable  $\mathbf{x}$ . Thus it could be restored by the representation  $\tau$  from a single value, e.g.  $\Phi_{\mathbf{T}}(\mathbf{y}, s^{-1}(e))$ , where  $e$  is the identity of  $G$ . We particularly will calculate only  $[\Phi_{\mathbf{T}}f](s^{-1}(e))$  in Subsection 3.3 as the value of a functional calculus. This value is usually denoted by  $f(\mathbf{T})$  and is exactly the functional calculus of operators in the traditional meaning.

In particular cases different characteristics of the spectral decomposition could give relevant information on the set of operators  $\mathbf{T}$ , e.g. the support  $\text{supp}_{\mathbf{y}}\Phi_{\mathbf{T}}(\mathbf{y}, x_0)$  of  $\Phi_{\mathbf{T}}(\mathbf{y}, x_0)$

$$f(\mathbf{y}) = 0 \forall \mathbf{y} \in \text{supp}_{\mathbf{y}}\Phi_{\mathbf{T}}(\mathbf{y}, x_0) \Rightarrow [\Phi_{\mathbf{T}}f](x_0) = 0$$

is called (*joint*) *spectrum* of set  $\mathbf{T} \subset \mathfrak{B}$ . This definition of the spectrum is connected with the *Arveson-Connes spectral theory* [4, 17, 52] while there are several important differences mentioned in [33, Rem. 4.4].

In the paper [33] the approach was illustrated by a newly developed functional calculus for several non-commuting operators based on Möbius transformations of the unit ball in  $\mathbb{R}^n$ . It was shown in [35, § 7] that the classic Dunford-Riesz functional calculus is generated by a representation of  $SL(2, \mathbb{R})$  within this procedure. However an abstract scheme of the approach was not presented yet. We give some its elements here.

From Proposition 1.16 we know a general form of an intertwining operator of two related representations of a group, which could be employed here. Let  $l_0(\mathbf{y})$  be the distribution corresponding to a test functional  $l_0$  for the representation  $\pi$  on  $F(X)$  such that we could write

$$\langle f(\mathbf{y}), l_0 \rangle_{F(X)} = \int_X f(\mathbf{y})l_0(\mathbf{y}) \, d\mathbf{y}.$$

We denote also by  $\rho^*(\mathbf{x})l_0(\mathbf{y})$ ,  $\mathbf{x} \in X$ ,  $\mathbf{y} \in X$  distributions corresponding to linear functionals  $\rho^*(\mathbf{x})l_0$ , where  $\rho^*(\mathbf{x})$  is the adjoint representation to  $\pi$  on the space  $F(X)$ .

PROPOSITION 2.11 (Spectral syntesis). *Under assumption of Proposition 1.16 the functional calculus exists and is unique. The spectral decomposition  $\Phi_{\mathbf{T}}(\mathbf{y}, \mathbf{x})$  as a distribution on  $X$  is given by the formula*

$$(2.19) \quad \Phi_{\mathbf{T}}(\mathbf{y}, \mathbf{x}) = \int_X \rho^*(\mathbf{x})l_0(\mathbf{y}) \tau(\mathbf{x})T_0(\mathbf{x}) \, d\mathbf{x}.$$

The functional calculus  $\Phi_{\mathbf{T}}(\cdot, x)$  as a mapping  $F(X) \rightarrow \mathfrak{B}$  is given correspondingly

$$(2.20) \quad \Phi_{\mathbf{T}}(\cdot, x) : f(\mathbf{y}) \mapsto [\Phi_{\mathbf{T}}f(\mathbf{y})](x) = \int_{\mathbf{X}} \int_{\mathbf{X}} \langle f(\mathbf{y}), \rho^*(x')l_0(\mathbf{y}) \rangle \tau(x')T_0(x) \, d\mathbf{y} \, dx'.$$

PROOF. Obviously (2.19) and (2.20) are equivalent. Thus we will prove (2.20) only. For an arbitrary  $f(\mathbf{y}) \in F(X)$  we could write

$$(2.21) \quad [\Phi_{\mathbf{T}}f(\mathbf{y})](x) = \left[ \Phi_{\mathbf{T}} \int_{\mathbf{X}} \widehat{f}(x')\rho(x')b_0(\mathbf{y}) \, dx' \right] (x)$$

$$(2.22) \quad = \int_{\mathbf{X}} \widehat{f}(x')[\Phi_{\mathbf{T}}\rho(x')b_0(\mathbf{y})](x) \, dx'$$

$$(2.23) \quad = \int_{\mathbf{X}} \widehat{f}(x')\tau(x')[\Phi_{\mathbf{T}}b_0(\mathbf{y})](x) \, dx'$$

$$(2.24) \quad = \int_{\mathbf{X}} \widehat{f}(x')\tau(x')T_0(x) \, dx'$$

$$(2.25) \quad = \int_{\mathbf{X}} \langle f(\mathbf{y}), \rho^*(x')l_0 \rangle \tau(x')T_0(x) \, dx'$$

We use in (2.21) that functions in  $F(X)$  are superpositions of coherent states, transformation (2.22) is made by linearity and continuity of  $\Phi_{\mathbf{T}}$ , step (2.23) is due to condition (2.17) and we finally apply (2.18) to receive (2.24). Thus it is proven that the functional calculus which is continuous, linear, and satisfies to (2.17) and (2.18) (if exists) is unique and given by (2.25). Now we should check that (2.25) really gives the right answer.

We will check first that (2.25) satisfies to (2.17):

$$(2.26) \quad \begin{aligned} \tau(g)\Phi(\mathbf{y}, x) &= \tau(x_1) \int_{\mathbf{X}} \rho^*(x')l_0(\mathbf{y}) \tau(x')T_0(x) \, dx' \\ &= \int_{\mathbf{X}} \rho^*(x')l_0(\mathbf{y}) \tau(g \cdot x')T_0(x) \, dx' \\ &= \int_{\mathbf{X}} \rho^*(g^{-1} \cdot x'')l_0(\mathbf{y}) \tau(x'')T_0(x) \, dx'' \\ &= \rho^*(g^{-1}) \int_{\mathbf{X}} \rho^*(x'')l_0(\mathbf{y}) \tau(x'')T_0(x) \, dx'' \\ &= \rho^*(g^{-1})\Phi(\mathbf{y}, x), \end{aligned}$$

where we made substitution  $x'' = g \cdot x'$  in (2.26). Finally (2.18) directly follows from the condition (2.13).  $\square$

Let there exists  $L_0(x) \in L'(X, \mathfrak{B})$ —a test functional for a vacuum vector  $T_0(x)$  and representation  $\tau$ , i.e.

$$\langle T_0, L_0 \rangle_{L(X, \mathfrak{B})} = \int_{\mathbf{X}} \langle \tau(x^{-1}T_0), L_0 \rangle_{L(X, \mathfrak{B})} \langle \tau(x)T_0, L_0 \rangle_{L(X, \mathfrak{B})} \, dx,$$

where

$$\langle T_0, L_0 \rangle_{L(X, \mathfrak{B})} = \int_{\mathbf{X}} \langle T_0(x), L_0(x) \rangle_{\mathfrak{B}} \, dx$$

and  $\langle T_0(x), L_0(x) \rangle_{\mathfrak{B}}$  is the pairing between  $\mathfrak{B}$  and  $\mathfrak{B}'$ .

PROPOSITION 2.12 (Spectral analysis). *If a  $\mathfrak{B}$ -valued function  $F(x)$  from  $L(X, \mathfrak{B})$  belongs to the closer of the linear span of  $\tau(x')T_0(x)$ ,  $x' \in X$  then*

$$F(x) = [\Phi_{\mathbf{T}}f(\mathbf{y})](x),$$

where

$$(2.27) \quad f(\mathbf{y}) = \int_{\mathbf{X}} \langle \tau(x^{-1}F), L_0 \rangle_{L(X, \mathfrak{B})} \rho(x)b_0(\mathbf{y}) \, dx.$$

PROOF. The formula (2.27) is just another realization of intertwining operator (1.12) from Proposition 1.16.  $\square$

Let  $K : F(X_1) \rightarrow F(X_2)$  be an intertwining mapping between two representations  $\rho_1$  and  $\rho_2$  of groups  $G_1$  and  $G_2$  in spaces  $F(X_1)$  and  $F(X_2)$  respectively. Let  $K(z, y)$ ,  $z \in X_1, y \in X_2$  be the Schwartz kernel of  $K$ .

THEOREM 2.13 (Mapping of spectral decompositions). *Let*

$$\begin{aligned} f_1(z) = [Kf_2](z) &= \int_{X_2} f_2(y)K(z, y) dy \\ \Phi_{T_1}(z, x) &= \int_{X_2} K(z, y)\Phi_{T_2}(y, x) dy, \end{aligned}$$

where a functional calculus  $\Phi_{T_2}$  is defined by representations  $\rho_2$  and  $\tau$ . Then  $\Phi_{T_2}(z, x)$  is a functional calculus for  $\rho_1$  and  $\tau$  and we have an identity:

$$(2.28) \quad [\Phi_{T_1} f_1(z)](x) = [\Phi_{T_2} f_2(y)](x).$$

PROOF. The intertwining property for  $\Phi_{T_2}(z, x)$  follows from transitivity. The identity (2.28) is a simple application of the Fubini theorem:

$$\begin{aligned} [\Phi_S g(z)](x) &= \int_{X_1} g(z)\Phi_S(z, x) dz \\ &= \int_{X_1} g(z) \int_{X_2} K(z, y)\Phi_T(y, x) dy dz \\ &= \int_{X_2} \int_{X_1} g(z)K(z, y) dz \Phi_T(y, x) dy \\ &= \int_{X_2} g(f(y))\Phi_T(y, x) dy \\ &= [\Phi_T g(f(y))](x). \end{aligned}$$

$\square$

This Theorem could be turned in the spectral mapping theorem under suitable conditions [33, Thm. 3.19].

### 3. Examples

We are going to demonstrate that the above construction is not only algebraically attractive but also belongs to the heart of analysis. More examples could be found in [15, 35, 34] and will be given elsewhere.

**3.1. The Heisenberg Group and Schrödinger Representation.** We will consider a realization of the previous results in a particular cases of the Fourier transform and Segal-Bargmann [6, 46] type spaces  $F_p(\mathbb{C}^n)$ . They arise from representations of the Heisenberg group  $\mathbb{H}^n$  [25, 26, 55] on  $L_p(\mathbb{R}^n)$ .

The Lie algebra  $\mathfrak{h}_n$  of  $\mathbb{H}^n$  spanned by  $\{T, P_j, Q_j\}$ ,  $n = 1, \dots, n$  is defined by the commutation relations:

$$(3.1) \quad [P_i, Q_j] = T\delta_{ij}.$$

They are known from quantum mechanics as the canonical commutation relations of coordinates and momentum operators. An element  $g \in \mathbb{H}^n$  could be represented as  $g = (t, z)$  with  $t \in \mathbb{R}, z = (z_1, \dots, z_n) \in \mathbb{C}^n$  and the group law is given by

$$(3.2) \quad g * g' = (t, z) * (t', z') = (t + t' + \frac{1}{2} \sum_{j=1}^n \Im(\bar{z}_j z'_j), z + z'),$$

where  $\Im z$  denotes the imaginary part of a complex number  $z$ . The Heisenberg group is (non-commutative) nilpotent step 2 Lie group.

We take a representation of  $\mathbb{H}^n$  in  $L_p(\mathbb{R}^n)$ ,  $1 < p < \infty$  by operators of shift and multiplication [55, § 1.1]:

$$(3.3) \quad g = (t, z) : f(y) \rightarrow [\sigma_{(t,z)} f](y) = e^{i(2t - \sqrt{2}vy + uv)} f(y - \sqrt{2}u), \quad z = u + iv,$$

i.e., this is the Schrödinger type representation with parameter  $\hbar = 1$ . These operators are isometries in  $L_p(\mathbb{R}^n)$  and the adjoint representation  $\rho_{(t,z)}^* = \rho_{(-t,-z)}$  in  $L_q(\mathbb{R}^n)$ ,  $p^{-1} + q^{-1} = 1$  is given by a formula similar to (3.3).

### 3.2. Wavelet Transforms for the Heisenberg Group in Function Spaces.

EXAMPLE 3.1. We start from the subgroup  $G_0 = \mathbb{R}^{n+1} = \{(t, z) \mid \Im(z) = 0\}$ . Then  $X = G/G_0 = \mathbb{R}^n$  and an invariant measure coincides with the Lebesgue measure. Mappings  $s : \mathbb{R}^n \rightarrow \mathbb{H}^n$  and  $r : \mathbb{H}^n \rightarrow \mathbb{H}$  are defined by the identities  $s(x) = (0, ix)$ ,  $s^{-1}(t, z) = \Im z$ ,  $r(t, u + iv) = (t, u)$ . The composition law  $s^{-1}((t, z) \cdot s(x)) = x + u$  reduces to Euclidean shifts on  $\mathbb{R}^n$ . We also find  $s^{-1}((s(x_1))^{-1} \cdot s(x_2)) = x_2 - x_1$  and  $r((s(x_1))^{-1} \cdot s(x_2)) = 0$ .

We consider the representation  $\sigma(g)$  of  $\mathbb{H}^n$  in the space of smooth rapidly decreasing functions  $B = \mathcal{S}(\mathbb{R}^n)$ . As a character of  $G_0 = \mathbb{R}^{n+1}$  we take the  $\chi(t, u) = e^{2it}$ . The corresponding test functional  $l_0$  satisfying to 1.20iii is the integration  $l_0(f) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) dy$ . Thus the wavelet transform is as follows

$$(3.4) \quad \hat{f}(x) = \int_{\mathbb{R}^n} \sigma(s(x)^{-1}) f(y) dy = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\sqrt{2}xy} f(y) dy$$

and is nothing else but the Fourier transform<sup>3</sup>.

Now we arrive to the absence of a vacuum vector in  $B$ , indeed there is no a  $f(x) \in \mathcal{S}(\mathbb{R}^n)$  such that

$$[\sigma(t, u) f](y) = \chi(t, u) f(y) \iff e^{i2t} f(y - \sqrt{2}u) = e^{i2t} f(y).$$

There is a way out accordingly to Subsection 1.3. We take  $B' = L_\infty(\mathbb{R}^n) \supset B$  and the vacuum vector  $b_0(y) \equiv (2\pi)^{-n/2} \in B'$ . Then coherent states are  $b_x(y) = (2\pi)^{-n/2} e^{-i\sqrt{2}xy}$  and the inverse wavelet transform is defined by the inverse Fourier transform

$$f(y) = \int_{\mathbb{R}^n} \hat{f}(y) b_x(y) dx = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(y) e^{-i\sqrt{2}xy} dx.$$

The condition 1.20iv  $\mathcal{M}\mathcal{W} : B \rightarrow B$  follows from the composition of two facts  $\mathcal{W} : B \rightarrow B$  and almost identical to it  $\mathcal{M} : B \rightarrow B$ , which are proved in standard analysis textbooks (see for example [32, § IV.2.3]). To check scaling (1.14) according to the tradition in analysis [26] we take a probe vector  $p_0 = e^{-y^2/2} \in B$ . Due to well known formula  $\int_{-\infty}^{+\infty} e^{-y^2/2} dy = (2\pi)^{1/2}$  of real analysis we have

$$\begin{aligned} \left\langle \int_X \langle \hat{p}_0(x), l_0 \rangle b_x dx, l_0 \right\rangle &= (2\pi)^{-n} \iiint e^{i\sqrt{2}xy} e^{-y^2/2} dy e^{-i\sqrt{2}xw} dx dw \\ &= (2\pi)^{n/2} \int_{\mathbb{R}^n} e^{-y^2/2} dy \\ &= \langle p_0, l_0 \rangle. \end{aligned}$$

<sup>3</sup>The *inverse* Fourier transform in fact. In our case the signs selection is opposite to the standard one, but we will neglect this difference.

Thus our scaling is correct.  $\mathcal{W}$  and  $\mathcal{M}$  intertwine the left regular representation — multiplication by  $e^{i\sqrt{2}y\nu}$  with operators

$$\begin{aligned} [\lambda(g)f](x) &= \chi(r(g^{-1} \cdot x))f(g^{-1} \cdot x) \\ &= e^{i\sqrt{2} \cdot 0} f(x - \sqrt{2}u) = f(x - \sqrt{2}u), \end{aligned}$$

i.e. with Euclidean shifts. From the identity  $\langle \mathcal{W}v, \mathcal{M}^*l \rangle_{F(X)} = \langle v, l \rangle_B$  (1.8) follows the Plancherel's identity:

$$\int_{\mathbb{R}^n} \widehat{v}(y) \widehat{l}(y) dy = \int_{\mathbb{R}^n} v(x) l(x) dx.$$

These are basic and important properties of the Fourier transform.

The Schrödinger representation is irreducible on  $\mathcal{S}(\mathbb{R}^n)$  thus  $\mathcal{M} = \mathcal{W}^{-1}$ . Thereafter integral formulas (1.15) and (1.16) representing operators  $\mathcal{M}\mathcal{W} = \mathcal{W}\mathcal{M} = 1$  correspondingly give an integral resolution for a convolution with the Dirac delta  $\delta(x)$ . We have integral resolution for the Dirac delta

$$\delta(x - y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi(x-y)} d\xi.$$

All described results on the Fourier transform are a part of any graduate curriculum. What is a reason for a reinvention of a bicycle here? First, the same path works with minor modifications for a function theory in  $\mathbb{R}^{1,1}$  described in [34]. Second we will use this interpretation of the Fourier transform in Example 3.5 for a demonstration how the Weyl functional calculus fits in the scheme outlined in [33, 35] and Subsection 2.2.

REMARK 3.2. Of course, the Heisenberg group is not the only possible source for the Fourier transform. We could consider the “ $ax + b$ ” group [55, Chap. 7] of the affine transformations of Euclidean space  $\mathbb{R}^n$ . The normal subgroup  $G_0 = \mathbb{R}$  of dilations generates the homogeneous space  $X = \mathbb{R}^n$  on which shifts act simply transitively. The Fourier transform deduced from this setting will naturally exhibit scaling properties. We could alternatively consider a group  $\mathbb{M}^n$  of Möbius transformation [13, Chap. 2] in  $\mathbb{R}^{n+1}$  which map upper half plane to itself. Then there is an induced action of  $\mathbb{M}^n$  on  $\mathbb{R}^n$ —the boundary of upper half plane.  $\mathbb{M}^n$  generated by composition of the affine transformations and the Kelvin inverse [13, Chap. 2]. If we take the normal subgroup  $G_0$  generated by dilations and the Kelvin inverse then the quotient space  $X$  will again coincide with  $\mathbb{R}^n$  and we immediately arrive to the above case. On the other hand the Fourier transform derived in such a way could be easily connected with the plane wave decomposition [51] in Clifford analysis [12, 20].

EXAMPLE 3.3. As a subgroup  $G_0$  we select now the center of  $\mathbb{H}^n$  consisting of elements  $(t, 0)$ . Of course  $X = G/G_0$  isomorphic to  $\mathbb{C}^n$  and mapping  $s : \mathbb{C}^n \rightarrow G$  simply is defined as  $s(z) = (0, z)$ . The Haar measure on  $\mathbb{H}^n$  coincides with the standard Lebesgue measure on  $\mathbb{R}^{2n+1}$  [55, § 1.1] thus the invariant measure on  $X$  also coincides with the Lebesgue measure on  $\mathbb{C}^n$ . Note also that composition law  $s^{-1}(g \cdot s(z))$  reduces to Euclidean shifts on  $\mathbb{C}^n$ . We also find  $s^{-1}((s(z_1))^{-1} \cdot s(z_2)) = z_2 - z_1$  and  $r((s(z_1))^{-1} \cdot s(z_2)) = \frac{1}{2} \mathcal{J} \bar{z}_1 z_2$ .

As a “vacuum vector” we will select the original *vacuum vector* of quantum mechanics—the Gauss function  $f_0(x) = e^{-x^2/2}$  which belongs to all  $L_p(\mathbb{R}^n)$ . Its transformations are defined as follow:

$$\begin{aligned} f_g(x) = [\rho_{(t,z)} f_0](x) &= e^{i(2t - \sqrt{2}vx + uv)} e^{-(x - \sqrt{2}u)^2/2} \\ &= e^{2it - (u^2 + v^2)/2} e^{-((u - iv)^2 + x^2)/2 + \sqrt{2}(u - iv)x} \\ &= e^{2it - z\bar{z}/2} e^{-(\bar{z}^2 + x^2)/2 + \sqrt{2}\bar{z}x}. \end{aligned}$$

Particularly  $[\rho_{(t,0)}f_0](x) = e^{-2it}f_0(x)$ , i.e., it really is a vacuum vector in the sense of our definition with respect to  $G_0$ . For the same reasons we could take  $l_0(x) = e^{-x^2/2} \in L_q(\mathbb{R}^n)$ ,  $p^{-1} + q^{-1} = 1$  as the test functional.

It could be shown that  $[\rho_{(0,z)}f_0](x)$  belongs to  $L_q(\mathbb{R}^n) \otimes L_p(\mathbb{C}^n)$  for all  $p > 1$  and  $q > 1$ ,  $p^{-1} + q^{-1} = 1$ . Thus transformation (1.4) with the kernel  $[\rho_{(0,z)}f_0](x)$  is an embedding  $L_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{C}^n)$  and is given by the formula

$$\begin{aligned} \widehat{f}(z) &= \langle f, \rho_{s(z)}f_0 \rangle \\ &= \pi^{-n/4} \int_{\mathbb{R}^n} f(x) e^{-z\bar{z}/2} e^{-(z^2+x^2)/2+\sqrt{2}zx} dx \\ (3.5) \quad &= e^{-z\bar{z}/2} \pi^{-n/4} \int_{\mathbb{R}^n} f(x) e^{-(z^2+x^2)/2+\sqrt{2}zx} dx. \end{aligned}$$

Then  $\widehat{f}(g)$  belongs to  $L_p(\mathbb{C}^n, dg)$  or its preferably to say that function  $\check{f}(z) = e^{z\bar{z}/2}\widehat{f}(t_0, z)$  belongs to space  $L_p(\mathbb{C}^n, e^{-|z|^2} dg)$  because  $\check{f}(z)$  is analytic in  $z$ . Such functions for  $p = 2$  form the *Segal-Bargmann space*  $F_2(\mathbb{C}^n, e^{-|z|^2} dg)$  of functions [6, 46], which are analytic by  $z$  and square-integrable with respect the Gaussian measure  $e^{-|z|^2} dz$ . For this reason we call the image of the transformation (3.5) by *Segal-Bargmann type space*  $F_p(\mathbb{C}^n, e^{-|z|^2} dg)$ . Analyticity of  $\check{f}(z)$  is equivalent to condition  $(\frac{\partial}{\partial \bar{z}_i} + \frac{1}{2}z_i I)\widehat{f}(z) = 0$ .

The integral in (3.5) is the well-known Segal-Bargmann transform [6, 46]. Inverse to it is given by a realization of (1.6):

$$\begin{aligned} f(x) &= \int_{\mathbb{C}^n} \widehat{f}(z) f_{s(z)}(x) dz \\ (3.6) \quad &= \int_{\mathbb{C}^n} \widehat{f}(u, v) e^{iv(u-\sqrt{2}x)} e^{-(x-\sqrt{2}u)^2/2} du dv \\ &= \int_{\mathbb{C}^n} \check{f}(z) e^{-(\bar{z}^2+x^2)/2+\sqrt{2}\bar{z}x} e^{-|z|^2} dz. \end{aligned}$$

The corresponding operator  $\mathcal{P}$  (1.7) is an identity operator  $L_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)$  and (1.7) gives an integral presentation of the Dirac delta.

Integral transformations (3.5) and (3.6) intertwines the Schrödinger representation (3.3) with the following realization of representation (1.5):

$$\begin{aligned} (3.7) \quad \lambda(t, z)f(w) &= \widehat{f}_0(z^{-1} \cdot w) \bar{\chi}(t + r(z^{-1} \cdot w)) \\ (3.8) \quad &= \widehat{f}_0(w - z) e^{it+i\mathcal{J}(\bar{z}w)} \end{aligned}$$

Meanwhile the orthoprojection  $L_2(\mathbb{C}^n, e^{-|z|^2} dg) \rightarrow F_2(\mathbb{C}^n, e^{-|z|^2} dg)$  is of a separate interest and is a principal ingredient in Berezin quantization [9, 16]. We could easy find its kernel from (1.10). Indeed,  $\widehat{f}_0(z) = e^{-|z|^2}$ , then the kernel is

$$\begin{aligned} K(z, w) &= \widehat{f}_0(z^{-1} \cdot w) \bar{\chi}(r(z^{-1} \cdot w)) \\ &= \widehat{f}_0(w - z) e^{i\mathcal{J}(\bar{z}w)} \\ &= \exp\left(\frac{1}{2}(-|w - z|^2 + w\bar{z} - z\bar{w})\right) \\ &= \exp\left(\frac{1}{2}(-|z|^2 - |w|^2) + w\bar{z}\right). \end{aligned}$$

To receive the reproducing kernel for functions  $\check{f}(z) = e^{|z|^2}\widehat{f}(z)$  in the Segal-Bargmann space we should multiply  $K(z, w)$  by  $e^{(-|z|^2+|w|^2)/2}$  which gives the standard reproducing kernel  $= \exp(-|z|^2 + w\bar{z})$  [6, (1.10)].

**3.3. Operator Valued Representations of the Heisenberg Group.** We proceed now with our main targets: wavelets in operator algebras. We shall show that well-known and new functional calculi are realizations of the scheme from Subsection 2.2.

CONVENTION 3.4. [3] Let  $B$  be a Banach space. We will say that an operator  $A : B \rightarrow B$  is *unitary* if  $A$  is invertible and  $\|Ab\| = \|b\|$  for all  $b \in B$ . An operator  $A : B \rightarrow B$  is called *self-adjoint* if the operator  $\exp(iA)$  is unitary. In the Hilbert space case this convention coincides with the standard definition.

Let  $T_1, \dots, T_n$  be an  $n$ -tuples of selfadjoint linear operators on a Banach space  $B$ . We put for our convenience  $T_0 = I$ —the identical operator. It follows from the Trotter-Daletskii<sup>4</sup> formula [45, Thm. VIII.31] that any linear combination  $\sum_{j=0}^n \alpha_j T_j$  is again a selfadjoint operator. We will consider a set of unitary operators

$$(3.9) \quad T(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n) = \exp \left( i \sum_{j=0}^n \alpha_j T_j \right)$$

parametrized by vectors  $(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{R}^{n+1}$ . Particularly  $T(0, 0, \dots, 0) = I$ . A family of their transformations  $\omega(t, z)$ ,  $t \in \mathbb{R}$ ,  $z \in \mathbb{C}^n$  is defined by the rule

$$(3.10) \quad \omega(t, z)T(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n) = T \left( \mathbf{a}_0 + t + \sum_{j=1}^n (u_j v_j - \sqrt{2} \alpha_j u_j), \right. \\ \left. \mathbf{a}_1 - \sqrt{2} v_1, \dots, \mathbf{a}_n - \sqrt{2} v_n \right),$$

where  $z_j = u_j + iv_j$ . A direct calculation shows that  $\omega(t', z')\omega(t'', z'') = \omega(t' + t'' + \frac{1}{2}\mathcal{I}(z'z''), z' + z'')$ —this is a non-linear geometric representation of the Heisenberg group  $\mathbb{H}^n$ . We could observe that

$$T(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n) = \omega(\mathbf{a}_0, \mathbf{a})T(0, 0, \dots, 0) = \omega(\mathbf{a}_0, \mathbf{a})T_0 = \omega(\mathbf{a}_0, \mathbf{a})I,$$

where  $\mathbf{a} = (ia_1, \dots, ia_n)$ . Obviously all transformations  $\omega(t, z)$  are isometries if the norm of elements  $T(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n)$  is defined as their operator norm.

The representation  $\omega$  (3.10) is not linear and we would like to use the procedure outlined in Remark 1.2. We construct the linear space of operator valued functions  $L(\mathbb{R}^n, \mathfrak{B})$  for a  $\mathbb{H}^n$ -homogeneous space  $X$  as follows

$$(3.11) \quad [\mathcal{I}f](t) = \int_X f(x) \omega(s(x)) dx T(t), \quad t \in \mathbb{R}^n, s(x) \in \mathbb{H}^n.$$

We also extend the representation  $\omega$  to  $L(X, \mathfrak{B})$  as follows:

$$(3.12) \quad \omega : [\mathcal{I}f](t) \mapsto \omega(g)[\mathcal{I}f](t) = \int_X f(g \cdot x) \omega(s(x)) dx T(t),$$

where  $[\mathcal{I}f](t) \in L_1(\mathbb{H}^n)$ ,  $g \in \mathbb{H}^n$ .

We will go on with coherent states defined by such a representation. In the notations of Subsection 2.2 operators  $T_1, \dots, T_n$  form a set  $\mathbf{T}$  defining the representation  $\tau = \omega$  in (3.10) with  $T_0(x) = I$  being a vacuum vector.

EXAMPLE 3.5. We are ready to demonstrate that the Weyl functional calculus is an application of Definition 2.9 and Example 3.1 as was announced in [33, Remark 4.3]. Consider again the subgroup  $G_0 = \mathbb{R}^{n+1} = \{(t, z) \mid \mathcal{I}(z) = 0\}$  and a realization of scheme from Subsection 1.3 for this subgroup. Then the first paragraph of Example 3.1 is applicable here.

<sup>4</sup>The formula is usually attributed to Trotter alone. It is widely unknown that the result appeared in [19] also.

We could easily see that  $\omega(t, u_1, \dots, u_n)I = e^{it+i\sum_1^n u_j}I$  and we select the identity operator  $I$  times  $(2\pi)^{-n/2}$  as the vacuum vector  $T_0(x)$  of the representation  $\omega$ . Thereafter the transformation  $\mathcal{T} : S(\mathbb{R}^n) \rightarrow L(\mathbb{R}^n, \mathfrak{B})$  (3.11) is exactly the inverse wavelet transform for the representation  $\omega$ . This transformation is defined at least for all  $f \in S(\mathbb{R}^n)$ . The space  $S(\mathbb{R}^n)$  is the image of the wavelet (Fourier) transform (3.4). Thus as outlined in Proposition 2.11 we could construct an intertwining operator  $\mathcal{F}$  between  $\sigma$  (3.3) and  $\omega$  (3.10) from the formula (2.20) as follow (see Remark 2.10):

$$\begin{aligned} [\Phi_{\mathcal{T}}f](0) &= \mathcal{M}_{\omega} \mathcal{W}_{\sigma} f = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(x) \omega(0, x_1, \dots, x_n) I \, dx \\ (3.13) \quad &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(x) e^{i\sum_1^n x_j T_j} \, dx. \end{aligned}$$

This formula is exactly the integral formula for the Weyl functional calculus [3, 44, 53]. As an example one could define a function

$$(3.14) \quad e^{-\sum_1^n T_j^2/2} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2} e^{i\sum_1^n x_j T_j} \, dx.$$

As we have seen (2.19) one could formally write the integral kernel from (3.13) as convolution of the integral kernels for  $\mathcal{W}_{\sigma}$  and  $\mathcal{M}_{\omega}$ :

$$(3.15) \quad \Phi(y, 0) = \int_{\mathbb{R}^n} e^{-i\sum_1^n y_j x_j} e^{i\sum_1^n x_j T_j} \, dx.$$

This expression looks very formal, but it is possible to give it a precise mathematical meaning as an operator valued distribution. Such an approach was explored by ANDERSON in [3]. The support of this distribution was defined as the *Weyl joint spectrum* for  $n$ -tuple of non-commuting operators  $T_1, \dots, T_n$  and studied in [3].

REMARK 3.6. As was mentioned in Remark 3.2 one could construct the Fourier transform from representations of  $ax + b$  group or the group  $\mathbb{M}^n$  of Möbius transformations of the upper half plane. Analogously one could deduce the Weyl functional calculus as an intertwining operator between two representation of this group. The Cauchy kernel  $G(x)$  [12, § 9] in Clifford analysis is the kernel of the Cauchy integral transform

$$f(y) = \int_{\partial X} G_y(x) \vec{n}(x) f(x) \, d\sigma(x), \quad y \in X,$$

where  $\vec{n}(x)$  is the outer unit vector orthogonal to  $\partial X$  and  $d\sigma(x)$  is the surface element at a point  $x$ . The Cauchy integral formula (as any wavelet transform) intertwines two representations acting on  $X$  and  $\partial X$  of  $\mathbb{M}^n$  similarly to the case of complex analysis [35, § 6]. Thus we could apply here Theorem 2.13 on a mapping of spectral distributions. The formula (2.28) take the form

$$\Phi_{\mathcal{W}} f = \int_{\partial X} \Phi_{\mathcal{W}}(G)(x) \vec{n}(x) f(x) \, d\sigma(x),$$

where  $\Phi_{\mathcal{W}}$  stands for the Weyl functional calculus. This gives another interpretation for the main result of the paper [29, Thm. 5.4].

There is no a reason to restrict ourselves only to the case of subgroup  $G_0 = \mathbb{R}^{n+1} = \{(t, z) \mid \mathcal{I}(z) = 0\}$ . Thus we proceed with the next example.

EXAMPLE 3.7. In an analogy with Example 3.3 let us consider now the wavelet theory associated to the subgroup  $G_0 = \mathbb{R}^1 = \{(t, 0)\}$  and the representation  $\omega$ . The first paragraph of Example 3.3 depends only on  $G = \mathbb{H}^n$  and  $G_0 = \mathbb{R}^1$  and thus is applicable in our case.

It is easy to see from formula (3.10) that any operator valued function  $[\mathcal{J}f](x)$  (3.11) is an eigen vector for  $\omega(h)$ ,  $h \in G_0$ . To be concise with function models we select as a vacuum vector the operator  $\exp(-\sum_1^n T_j^2)$  (3.14). Then the condition (2.13) immediately follows from (3.14). Thus we could define a functional calculus  $\Phi : F_p(\mathbb{C}^n) \rightarrow L(\mathbb{C}^n, \mathfrak{B})$  by the formula (see Remark 2.10):

$$\begin{aligned}
[\Phi_{\mathbf{T}}f](0) &= \int_{\mathbb{C}^n} f(z)\omega(0, z) \exp(-\sum_1^n T_j^2/2) dz \\
&= \int_{\mathbb{C}^n} f(z)\omega(0, z)(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2} e^{i\sum_1^n x_j T_j} dx dz \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{C}^n} f(z) \int_{\mathbb{R}^n} e^{-\sum_1^n x_j^2/2} \omega(0, z) e^{i\sum_1^n x_j T_j} dx dz \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{3n}} \exp \sum_{j=1}^n \left( -\frac{x_j^2}{2} + i(v_j - \sqrt{2}x_j)u_j + i(x_j - \sqrt{2}v_j)T_j \right) \\
(3.16) \quad &\quad \times f(z) dx du dv.
\end{aligned}$$

The last formula could be rewritten for mutually commuting operators  $T_j$  as follows:

$$\begin{aligned}
[\Phi_{\mathbf{T}}f](0) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} \exp \sum_{j=1}^n \left( -\frac{x_j^2}{2} + ix_j(T_j - \sqrt{2}u_j) \right) dx \\
&\quad \times \exp \sum_{j=1}^n i(v_j u_j - \sqrt{2}v_j T_j) f(z) du dv \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{C}^n} \exp \sum_{j=1}^n \left( iv_j(u_j - \sqrt{2}T_j) - \frac{(T_j - \sqrt{2}u_j)^2}{2} \right) f(z) dz
\end{aligned}$$

where the exponent of operator is defined in the standard sense, e.g. via the Weyl functional calculus (3.13) or the Taylor expansion. The last formula is similar to (3.6). This is very natural for commuting operators as well as that for non-commuting operators formula (3.16) is more complicated.

The spectral distribution

$$\Phi_{\mathbf{T}}(z, 0) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp \sum_{j=1}^n \left( -\frac{x_j^2}{2} + i(v_j - \sqrt{2}x_j)u_j + i(x_j - \sqrt{2}v_j)T_j \right) dx$$

derived from (3.16) contains at least as much information on operators  $T_1, \dots, T_n$  as the Weyl distribution (3.15) and deserves a careful separate investigation. We will just mention in conclusion that the Segal-Bargmann space is an example of the Fock space—space of second quantization for bosonic fields. Thus the functional calculus based on the Segal-Bargmann model sketched here seems to be an appropriate model for quantized bosonic fields.

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# Index

- $SB_2(\mathbb{C})$ , 8
- $\Lambda(\mathfrak{g})$ , 22
- $SL_2(\mathbb{R})$ , 14
- $\mathbb{R}^{\pm}$ , 15
- $\delta_{jk}$ , 20
- $\text{tr}$ , 21
- $\langle \cdot, \cdot \rangle$ , 20
- $\mathfrak{ax} + \mathfrak{b}$  group, 14
- abstract group, 13
- action
  - transitive, 15
- adjoint representation, 20
- admissible vector, 27
- annihilation operator, 38
- annihilation operator, 32
- associativity, 13
- Bargmann projection, *see also* Segal-Bargmann projection
- Bergman projection, 8
- Bergman space, 7
- bra and ket, 6
- calculus
  - functional, *see also* functional calculus
- Cauchy integral formula, 7, 25
- character
  - of group, 21
  - of representation, 21
- character of a group, 21
- character of representation, 21
- classes of equivalent representations, 20
- coherent states, 3, 11, 25
- commutation relations, 31
- commutative, 14
- commuting operator, 22
- configuration space, 31
- conjugation, 15
- continuous group, 14
- convolution, 16
- convolution operator, 16
- creation operator, 32, 38
- cyclic vector, 22
- discrete series, 27
- Dirac
  - bra and ket, 6
- dual object, 22
- dual space, 22
- equation Schrödinger, 38
- equivalent representations, 20
- exact representation, 20
- faithful representation, 20
- fiducial vector, 25
- finite dimensional representation, 20
- functional calculus
  - Riesz-Dunford, 9
- generating function, 37
- ground state, 25
- group, 13
  - $SL_2(\mathbb{R})$ , 14
  - $\mathfrak{ax} + \mathfrak{b}$ , 14
  - abstract, 13
  - commutative, 14
  - continuous, 14
  - Heisenberg, 14
  - Lie, 14
  - locally compact, 14
  - noncommutative, 14
  - of characters, 21
  - representation, 19
  - subgroup
    - isotropy, 15
  - transformation, 13
- group multiplication, 13
- Haar measure, 15
- Hardy space, 7
- Heisenberg commutation relations, 31
- Heisenberg group, 14
  - Schrödinger representation, 19
- Hilbert space, 5
  - orthonormal base, 5
- homogeneous space, 15
- identity, 13
  - Parseval(-Pythagoras), 5
  - Plancherel, 6
  - Pythagoras, 5
- infinite dimensional representation, 20
- intertwining operator, 22
- invariant measure, 15
- invariant subspace, 21
- inverse, 13
- inverse wavelet transform, 26
- irreducible representation, 22
- isotropy (sub)group, 15

- kernel
  - reproducing, 28
- Kroneker delta, 20
- Lebesgue measure, 15
- left shift, 15
- lemma
  - Schur's, 23
- Lie groups, 14
- linear-fractional transformations, 15
- linearization procedure, 19
- locally compact, 14
- loop, 15
- lower (upper) half plane, 15
- main problem of representation theory, 22
- matrix elements, 20
- measure, 15
  - Haar, 15
  - Lebesgue, 15
  - left invariant, 15
- measurement, 31
- momentum space, 31
- mother wavelet, 11, 25
- noncommutative, 14
- nontrivial invariant subspaces, 22
- observables, 31
- operator
  - annihilation, 32, 38
  - commuting, 22
  - convolution, 16
  - creation, 32, 38
  - intertwining, 22
  - Töplitz, 8
  - unitary, 20
- orbits, 14
- orthogonal polynomials, 37
- orthonormal base, 5
- orthonormal basis, 20
- Parseval(-Pythagoras) identity, 5
- Plancherel identity, 6
- projection
  - Bergman, 8
  - Segal-Bargmann, 9
- reducible representation, 22
- regular representation, 22
- relation of equivalence, 20
- relations
  - commutation, 31
- representaion
  - linearization, 19
- representation, 19
  - adjoint, 20
  - continuous, 19
  - exact, 20
  - faithful, 20
  - finite dimensional, 20
  - infinite dimensional, 20
  - irreducible, 22
  - linear, 19
  - reducible, 22
  - regular, 22
  - Schrödinger, 19
  - series
    - discrete, 27
    - square integrable, 25, 27
    - modulo subgroup, 31
  - trivial, 20
  - unitary, 21
- representation of a group, 19
- representation space, 20
- representations, 19
  - equivalent, 20
  - unitary, 21
- reproducing kernel, 6, 28
- restriction of representation, 22
- Riesz-Dunford functional calculus, 9
- right shift, 15
- scalar product, 20
- Schrödinger equation, 38
- Schrödinger representation, 19, 32
- Schur's lemma, 23
- Segal-Bargmann projection, 9
- Segal-Bargmann space, 8
- set
  - total, 25
- shift
  - left, 15
  - right, 15
- signal processing, 9
- singular integral operator, 7
- singular integral operator
  - Szegö, 7
- space
  - Bergman, 7
  - Hardy, 7
  - Hilbert, 5
  - Segal-Bargmann, 8
- square integrable modulo subgroup, 31
- square integrable representation, 25, 27
- state
  - ground, 25
- states, 31
- Stone-von Neumann theorem, 32
- subrepresentation, 22
- subspace
  - invariant, 21
- Szegö singular integral operator, 7
- Töplitz operator, 8, 9
- theorem
  - Stone-von Neumann, 32
- total set, 25
- trace, 21
- transform
  - wavelet, 25
- transformation
  - linear-fractional, 15
- transformation group, 13
- transitive, 15
- trivial representation, 20
- unit circle, 7
- unit disk, 7

- unitary equivalent representations, 21
- unitary operator, 20
- unitary representation, 21
- upper (lower) half plane, 15
  
- vacuum vector, 11, 25
- vacuum vector of quantum mechanics, 33
- vector
  - admissible, 27
  - cyclic, 22
  - fiducial, 25
  - vacuum, 11, 25
  
- wave functions, 31
- wavelet
  - mother, 11, 25
- wavelet transform, 25
  - inverse, 26
- wavelets, 3, 5, 11, 25
- WFT, *see also* Windowed Fourier Transform
- Windowed Fourier Transform, 9