# Special Functions and Their Symmetries

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# Contents

# Chapter 1

# Introduction

In the previous part of this course we learn the basic theory of special functions. Some principal elements of the theory are:

(i). Integral representations;

(ii). Recurrence formulae;

(iii). Addition theorems;

(iv). Differential equations;

The above elements are usually constructed for different kinds of special functions (Gamma, Beta, hypergeometric functions, orthogonal polynomials) by different means. It is desirable to put an order in the huge amount of such facts. One possibility is to use language of representation theory of Lie groups. The purpose of the present notes to give an introduction to the subject. We follow mainly the book [31], other closely related books are [26, 29], more references will be given in appropriate places.

The rôle of groups and symmetries in science is everlasting, see for example book [32] for inspiring reading. Particularly they make a bridge between geometry and analysis.

Due to large amount of material we chose to adopt the style of books [17, 18] which put many facts and examples into exercises for reader.

# Chapter 2

# Groups, Homogeneous Spaces, and Their Representations

# Chapter 3

# Groups and Homogeneous Spaces

The group theory and the representation theory are two enormous and interesting subjects themselves. However they are auxiliary in our consideration and we are forced to restrict ourselves only to brief and very dry overview.

Besides introduction to that areas presented in [26, 31] we recommend additionally the books [17, 30]. The representation theory intensively uses tools of functional analysis and on the other hand inspires its future development. We use the book [18] for references on functional analysis here and recommend it as a nice reading too.

## 3.1 Basics of Group Theory

We start from the definition of central object which formalizes the universal notion of symmetries.

**Definition 3.1.1** A *transformation group* $G$ is a nonvoid set of mappings of a certain set $X$ into itself with the following properties:

(i). if $g_1 \in G$ and $g_2 \in G$ then $g_1 g_2 \in G$;

(ii). if $g \in G$ then $g^{-1}$ exists and belongs to $G$.

**Exercise 3.1.2** List all transformation groups on a set of three elements.

**Exercise 3.1.3** Verify that the following are groups in fact:

(i). Group of permutations of $n$ elements;

(ii). Group of $n \times n$ matrixes with non zero determinant over a field $\mathbb{F}$ under matrix multiplications;

(iii). Group of rotations of the unit circle $\mathbb{T}$;

(iv). Groups of shifts of the real line $\mathbb{R}$ and plane $\mathbb{R}^2$;

(v). Group of linear fractional transformations of the extended complex plane.

**Definition 3.1.4** An *abstract group* (or simply *group*) is a nonvoid set $G$ on which there is a law of *group multiplication* (i.e. mapping $G \times G \to G$) with the properties

(i). *associativity*: $g_1(g_2g_3) = (g_1g_2)g_3$;

(ii). the existence of *identity*: $e \in G$ such that $eg = ge = g$ for all $g \in G$;

(iii). the existence of *inverse*: for every $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

**Exercise 3.1.5** Check that any transformation group is an abstract group.

**Exercise 3.1.6** Check that the following transformation groups (cf. Example 3.1.3) have the same law of multiplication, i.e. are equivalent as abstract groups:

(i). The group of isometric mapping of an equilateral triangle onto itself;

(ii). The group of all permutations of a set of free elements;

(iii). The group of invertible matrix of order 2 with coefficients in the field of integers modulo 2;

(iv). The group of linear fractional transformations of the extended complex plane generated by the mappings $z \mapsto z^{-1}$ and $z \mapsto 1 - z$.

**Exercise$^*$ 3.1.7** Expand the list in the above exercise.

It is simpler to study groups with the following additional property.

**Definition 3.1.8** A group $G$ is *commutative* if for all $g_1$, $g_2 \in G$, we have $g_1g_2 = g_2g_1$.

Most of interesting and important groups are *noncommutative*, however.

**Exercise 3.1.9** (i). Which groups among found in Exercise 3.1.2 are commutative?

(ii). Which groups among listed in Exercise 3.1.3 are noncommutative?

Groups could have some additional analytical structures, e.g. they could be a topological sets with a corresponding notion of limit. We always assume that our groups are *locally compact* [17, § 2.4].

**Definition 3.1.10** If for a group $G$ the group multiplication and the taking of inverse are continuous mappings then $G$ is *continuous group*.

Even a better structure could be found among *Lie groups* [17, § 6], e.g. groups with a differentiable law of multiplication. Investigating such groups we could employ the whole arsenal of analytical tools, thereafter most of groups studied in this notes will be Lie groups.

**Exercise 3.1.11** Check that the following are noncommutative Lie (and thus continuous) groups:

(i). [30, Chap. 7] The *$ax + b$ group*: set of elements $(a, b)$, $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ with the group law:

$$(a, b) * (a', b') = (aa', ab' + b).$$

The identity is $(1, 0)$, and $(a, b)^{-1} = (a^{-1}, -b/a)$.

(ii). The *Heisenberg group* [15], [30, Chap. 1]: a set of triples of real numbers $(s, x, y)$ with the group multiplication:

$$(s, x, y) * (s', x', y') = (s + s' + \frac{1}{2}(x'y - xy'), x + x', y + y'). \tag{3.1.1}$$

The identity is $(0, 0, 0)$, and $(s, x, y)^{-1} = (-s, -x, -y)$.

(iii). The $SL_2(\mathbb{R})$ group [16, 23]: a set of $2 \times 2$ matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries $a$, $b$, $c$, $d \in \mathbb{R}$, the determinant $\det = ad - bc$ equal to 1 and the group law coinciding with matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

The identity is the unit matrix and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The above three groups are behind many important results of real and complex analysis [15, 16, 23] and we meet them many times in these notes.

## 3.2 Homogeneous Spaces and Invariant Measures

While abstract group are a suitable language for investigation of their general properties we meet groups in applications as transformation groups acting on a set $X$.

Let $X$ be a set and let be defined an operation $G : X \to X$ of $G$ on $X$. There is an equivalence relation on $X$, say, $x_1 \sim x_2 \Leftrightarrow \exists g \in G : gx_1 = x_2$, with respect to which $X$ is a disjoint union of distinct *orbits* [22, § I.5].

**Exercise 3.2.1** Let action of $SL_2(\mathbb{R})$ group on $\mathbb{C}$ by means of *linear-fractional transformations*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

Show that there three orbits: the real axis $\mathbb{R}$, *upper (lower) half plane* $\mathbb{R}^n_\pm$:

$$\mathbb{R}^n_\pm = \{x \pm iy \mid x, y \in \mathbb{R},\ y > 0\}.$$

Thus from now on, without lost of a generality, we assume that the operation of $G$ on $X$ is *transitive*, i. e. for every $x \in X$ we have

$$Gx := \bigcup_{g \in G} g(x) = X.$$

In this case $X$ is *G-homogeneous space.*

**Exercise 3.2.2** Show that for any group $G$ we could define its action on $X = G$ as follows:

(i). The *conjugation* $g : x \mapsto gxg^{-1}$ (which is even a group homomorphism, but is trivial for all commutative groups).

(ii). The *left shift* $\lambda(g) : x \mapsto gx$ and the *right shift* $\rho(g) : x \mapsto xg^{-1}$.

If we fix a point $x \in X$ then the set of elements $G_x = \{g \in G \mid g(x) = x\}$ obviously forms the *isotropy (sub)group* of $x$ in $G$ [22, § I.5]. The set $X$ is in the bijection with the factor set $G/G_x$ for any $x \in X$.

**Exercise 3.2.3** Find a subgroup which correspond to the given action of $G$ on $X$:

(i). Action of $ax + b$ group on $\mathbb{R}$ by the formula: $(a, b) : x \mapsto ax + b$.

(ii). Action of $SL_2(\mathbb{R})$ group on one of three orbit from Exercise 3.2.1.

To do some analysis on groups we need suitably defined basic operation: differentiation and integration. The first operation is naturally defined for Lie group. If $G$ is a Lie group then the homogeneous space $G/G_x$ is a smooth manifold (and a *loop* as an algebraic object) for every $x \in X$. Therefore the one-to-one mapping $G/G_x \rightarrow X : g \mapsto g(x)$ induces a structure of $C^\infty$-manifold on $S$. Thus the class $C_0^\infty(X)$ of smooth functions with compact supports on $x$ has the evident definition.

In order to perform an integration we need a suitable *measure*. A smooth measure $d\mu$ on $X$ is called (left) *invariant measure* with respect to an operation of $G$ on $X$ if

$$\int_X f(x)\, d\mu(x) = \int_X f(g(x))\, d\mu(x), \quad \text{for all } g \in G,\ f(x) \in C_0^\infty(X). \tag{3.2.1}$$

**Exercise 3.2.4** Show that measure $y^{-2} dy\, dx$ on the upper half plane $\mathbb{R}^2_+$ is invariant under action from Exercise 3.2.1.

Left invariant measures on $X = G$ is called the *Haar measure*. It always exists and is uniquely defined up to a scalar multiplier [30, § 0.2]. An equivalent formulation of (3.2.1) is: *$G$ operates on $L_2(X, d\mu)$ by unitary operators.* We will transfer the Haar measure $d\mu$ from $G$ to $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \rightarrow G$ and will call it as the *invariant measure on a Lie algebra* $\mathfrak{g}$.

**Exercise 3.2.5** Check that the following are Haar measures for corresponding groups:

(i). The *Lebesgue measure* $dx$ on the real line $\mathbb{R}$.

(ii). The Lebesgue measure $d\phi$ on the unit circle $\mathbb{T}$.

(iii). $dx/x$ is a Haar measure on the multiplicative group $\mathbb{R}_+$;

(iv). $dx\, dy/(x^2 + y^2)$ is a Haar measure on the multiplicative group $\mathbb{C} \setminus \{0\}$, with coordinates $z = x + iy$.

(v). $a^{-2}\, da\, db$ and $a^{-1}\, da\, db$ are the left and right invariant measure on $ax+b$ group.

(vi). The Lebesgue measure $ds\, dx\, dy$ of $\mathbb{R}^3$ for the Heisenberg group $\mathbb{H}^1$.

In this notes we assume *all integrations on groups performed over the Haar measures.*

**Exercise 3.2.6** Show that invariant measure on a compact group $G$ is finite and thus may be normalized to total measure 1.

The above simple result has surprisingly important consequences.

**Definition 3.2.7** The left *convolution* $f_1 * f_2$ of two functions $f_1(g)$ and $f_2(g)$ defined on a group $G$ is

$$f_1 * f_2(g) = \int_G f_1(h)\, f_2(h^{-1}g)\, dh$$

**Exercise 3.2.8** Let $k(g) \in L_1(G, d\mu)$ and operator $K$ on $L_1(G, d\mu)$ is the left *convolution operator* with $k$, .i.e. $K : f \mapsto k * f$. Show that $K$ commutes with all right shifts on $G$.

The following Lemma characterizes *linear subspaces* of $L_2(G, d\mu)$ invariant under shifts in the term of *ideals of convolution algebra* $L_2(G, d\mu)$ and is of the separate interest.

**Lemma 3.2.9** *A closed linear subspace $H$ of $L_2(G, d\mu)$ is invariant under left (right) shifts if and only if $H$ is a left (right) ideal of the right group convolution algebra $L_2(G, d\mu)$.*

*A closed linear subspace $H$ of $L_2(G, d\mu)$ is invariant under left (right) shifts if and only if $H$ is a right (left) ideal of the left group convolution algebra $L_2(G, d\mu)$.*

Proof. Of course we consider only the “right-invariance and right-convolution” case. Then the other three cases are analogous. Let $H$ be a closed linear subspace of $L_2(G, d\mu)$ invariant under right shifts and $k(g) \in H$. We will show the inclusion

$$[f * k]_r(h) = \int_G f(g) k(hg)\, d\mu(g) \in H, \tag{3.2.2}$$

for any $f \in L_2(G, d\mu)$. Indeed, we can treat integral (3.2.2) as a limit of sums

$$\sum_{j=1}^{N} f(g_j) k(hg_j) \Delta_j. \tag{3.2.3}$$

But the last sum is simply a linear combination of vectors $k(hg_j) \in H$ (by the invariance of $H$) with coefficients $f(g_j)$. Therefore sum (3.2.3) belongs to $H$ and this is true for integral (3.2.2) by the closeness of $H$.

Otherwise, let $H$ be a right ideal in the group convolution algebra $L_2(G, d\mu)$ and let $\phi_j(g) \in L_2(G, d\mu)$ be an approximate unit of the algebra [12, § 13.2], i. e. for any $f \in L_2(G, d\mu)$ we have

$$[\phi_j * f]_r(h) = \int_G \phi_j(g) f(hg)\, d\mu(g) \rightarrow f(h), \text{ when } j \rightarrow \infty.$$

Then for $k(g) \in H$ and for any $h' \in G$ the right convolution

$$[\phi_j * k]_r(hh') = \int_G \phi_j(g) k(hh'g)\, d\mu(g) = \int_G \phi_j(h'^{-1}g') k(hg')\, d\mu(g'),\ g' = h'g,$$

from the first expression is tensing to $k(hh')$ and from the second one belongs to $H$ (as a right ideal). Again the closeness of $H$ implies $k(hh') \in H$ that proves the assertion. □

# Chapter 4

# Elements of the Representation Theory

## 4.1 Representations of Groups

Objects unveil their nature in actions. Groups act on other sets by means of *representations*. A representation of a group $G$ is a group homomorphism of $G$ in a transformation group of a set. It is a fundamental observation that *linear* objects are easer to study. Therefore we begin from linear representations of groups.

**Definition 4.1.1** A linear continuous *representation of a group* $G$ is a continuous function $T(g)$ on $G$ with values in the group of non-degenerate linear continuous transformation in a linear space $H$ (either finite or infinite dimensional) such that $T(g)$ satisfies to the functional identity:

$$T(g_1 g_2) = T(g_1)\, T(g_2). \tag{4.1.1}$$

**Exercise 4.1.2** Show that $T(g^{-1}) = T^{-1}(g)$ and $T(e) = I$, where $I$ is the identity operator on $B$.

**Exercise 4.1.3** Show that these are linear continuous representations of corresponding groups:

(i). Operators $T(x)$ such that $[T(x)\, f](t) = f(t + x)$ form a representation of $\mathbb{R}$ in $L_2(\mathbb{R})$.

(ii). Operators $T(n)$ such that $T(n)a_k = a_{k+n}$ form a representation of $\mathbb{Z}$ in $\ell_2$.

(iii). Operators $T(a,b)$ defined by

$$[T(a,b)\,f](x) = \sqrt{a}f(ax+b), \qquad a \in \mathbb{R}_+,\ b \in \mathbb{R} \tag{4.1.2}$$

form a representation of $ax+b$ group in $L_2(\mathbb{R})$.

(iv). Operators $T(s,x,y)$ defined by

$$[T(s,x,y)\,f](t) = e^{i(2s-\sqrt{2}yt+xy)}f(t-\sqrt{2}x) \tag{4.1.3}$$

form *Schrödinger representation* of the Heisenberg group $\mathbb{H}^1$ in $L_2(\mathbb{R})$.

(v). Operators $T(g)$ defined by

$$[T(g)f](t) = \frac{1}{ct+d}f\left(\frac{at+b}{ct+d}\right), \qquad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.1.4}$$

form a representation of $SL_2(\mathbb{R})$ in $L_2(\mathbb{R})$.

In the sequel a representation *always means* linear continuous representation. $T(g)$ is an *exact representation* (or *faithful representation* if $T(g) = I$ only for $g = e$. The opposite case when $T(g) = I$ for all $g \in G$ is a *trivial representation*. The space $H$ is *representation space* and in most cases will be a *Hilber space* [18, § III.5]. If dimensionality of $H$ is finite then $T$ is a *finite dimensional representation*, in the opposite case it is *infinite dimensional representation*.

We denote the *scalar product* on $H$ by $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{e}_j\}$ be an (finite or infinite) *orthonormal basis* in $H$, i.e.

$$\langle \mathbf{e}_j, \mathbf{e}_j \rangle = \delta_{jk},$$

where $\delta_{jk}$ is the *Kroneker delta*, and linear span of $\{\mathbf{e}_j\}$ is dense in $H$.

**Definition 4.1.4** The *matrix elements* $t_{jk}(g)$ of a representation $T$ of a group $G$ (with respect to a basis $\{\mathbf{e}_j\}$ in $H$) are complex valued functions on $G$ defined by

$$t_{jk}(g) = \langle T(g)\mathbf{e}_j, \mathbf{e}_k \rangle . \tag{4.1.5}$$

**Exercise 4.1.5** Show that [31, § 1.1.3]

(i). $T(g)\,\mathbf{e}_k = \sum_j t_{jk}(g)\,\mathbf{e}_j$.

(ii). $t_{jk}(g_1 g_2) = \sum_n t_{jn}(g_1)\,t_{nk}(g_2)$.

It is typical mathematical questions to determine identical objects which may have a different appearance. For representations it is solved in the following definition.

**Definition 4.1.6** Two representations $T_1$ and $T_2$ of the same group $G$ in spaces $H_1$ and $H_2$ correspondingly are *equivalent representations* if there exist a linear operator $A : H_1 \rightarrow H_2$ with the continuous inverse operator $A^{-1}$ such that:

$$T_2(g) = A\,T_1(g)\,A^{-1}, \qquad \forall g \in G.$$

**Exercise 4.1.7** Show that representation $T(a,b)$ of $ax+b$ group in $L_2(\mathbb{R})$ from Exercise 4.1.3.(iii) is equivalent to the representation

$$[T_1(a,b)\,f](x) = \frac{e^{i\frac{b}{a}}}{\sqrt{a}}\,f\left(\frac{x}{a}\right). \tag{4.1.6}$$

HINT. Use the Fourier transform. □

The *relation of equivalence* is reflexive, symmetric, and transitive. Thus it splits the set of all representations of a group $G$ into *classes of equivalent representations. In the sequel we study group representations up to their equivalence classes only.*

**Exercise 4.1.8** Show that equivalent representations have the same matrix elements in appropriate basis.

**Definition 4.1.9** Let $T$ is a representation of a group $G$ in $H$ The *adjoint representation* $T'(g)$ of $G$ in $H$ is defined by

$$T'(g) = \left(T(g^{-1})\right)^*,$$

where $^*$ denotes the adjoint operator in $H$.

**Exercise 4.1.10** Show that

(i). $T'$ is indeed a representation.

(ii). $t'_{jk}(g) = \bar{t}_{kj}(g^{-1})$.

Recall [18, § III.5.2] that a bijection $U : H \rightarrow H$ is a *unitary operator* if

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \qquad \forall x, y \in H.$$

**Exercise 4.1.11** Show that $UU^* = I$.

**Definition 4.1.12** $T$ is a *unitary representation* of a group $G$ in a space $H$ if $T(g)$ is a unitary operator for all $g \in G$. $T_1$ and $T_2$ are *unitary equivalent representations* if $T_2 = UT_2U^{-1}$ for a unitary operator $U$.

**Exercise 4.1.13** (i). Show that all representations from Exercises 4.1.3 are unitary.

(ii). Show that representations from Exercises 4.1.3.(iii) and 4.1.7 are unitary equivalent.

HINT. Take that the Fourier transform is unitary for granted. □

**Exercise 4.1.14** Show that if a Lie group $G$ is represented by unitary operators in $H$ then its Lie algebra $\mathfrak{g}$ is represented by self-adjoint (possibly unbounded) operators in $H$.

The following definition have a sense for *finite* dimensional representations.

**Definition 4.1.15** A *character of representation* $T$ is equal $\chi(g) = \operatorname{tr}(T(g))$, where $\operatorname{tr}$ is the *trace* [18, § III.5.2 (Probl.)] of operator.

**Exercise 4.1.16** Show that

(i). Characters of a representation $T$ are constant on the adjoint elements $g^{-1}hg$, for all $g \in G$.

(ii). Character is an algebra homomorphism from an algebra of representations with Kronecker's (tensor) multiplication [31, § 1.9] to complex numbers.

HINT. Use that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$, and $\operatorname{tr}(A \otimes B) = \operatorname{tr} A \operatorname{tr} B$. □

For *infinite* dimensional representation characters could be defined either as distributions [17, § 11.2] or in infinitesimal terms of Lie algebras [17, § 11.3].

The characters of a representation should not be confused with the following notion.

**Definition 4.1.17** A *character of a group* $G$ is a one-dimensional representation of $G$.

**Exercise 4.1.18** (i). Let $\chi$ be a character of a group $G$. Show that a character of representation $\chi$ coincides with it and thus is a character of $G$.

(ii). A matrix element of a group character $\chi$ coincides with $\chi$.

(iii). Let $\chi_1$ and $\chi_1$ be characters of a group $G$. Show that $\chi_1 \otimes \chi_2 = \chi_1 \chi_2$ and $\chi'(g) = \chi_1(g^{-1})$ are again characters of $G$. In other words *characters of a group form a group themselves.*

## 4.2 Decomposition of Representations

The important part of any mathematical theory is classification theorems on structural properties of objects. Very well known examples are:

(i). The main theorem of arithmetics on unique representation an integer as a product of powers of prime numbers.

(ii). Jordan's normal form of a matrix.

The similar structural results in the representation theory are very difficult. The easiest (but still rather difficult) questions are on classification of unitary representations up to unitary equivalence.

**Definition 4.2.1** Let $T$ be a representation of $G$ in $H$. A linear subspace $L \subset H$ is *invariant subspace* for $T$ if for any $\mathbf{x} \in L$ and any $g \in G$ the vector $T(g)\mathbf{x}$ again belong to $L$.

There are always two trivial invariant subspaces: the null and entire $H$. All other are *nontrivial invariant subspaces.*

**Definition 4.2.2** If there are only two trivial invariant subspaces then $T$ is *irreducible representation*. In the opposite case we have *reducible representation.*

For any nontrivial invariant subspace we could define the *restriction of representation* of $T$ on it. In this way we obtain a *subrepresentation* of $T$.

**Example 4.2.3** Let $T(a)$, $a \in \mathbb{R}_+$ be defined as follows: $[T(a)]f(x) = f(ax)$. Then spaces of even and odd functions are invariant.

**Definition 4.2.4** If the closure of liner span of all vectors $T(g)v$ is dense in $H$ then $v$ is called *cyclic vector* for $T$.

**Exercise 4.2.5** Show that for an irreducible representation any non zero vector is cyclic.

The important property of unitary representation is complete reducibility.

**Exercise 4.2.6** Let a unitary representation $T$ has an invariant subspace $L \subset H$, then its orthogonal completion $L^{\perp}$ is also invariant.

**Theorem 4.2.7** *[17, § 8.4] Any unitary representation $T$ of a locally compact group $G$ could be decomposed in a (continuous) direct sum irreducible representations: $T = \int_X T_x \, d\mu(x)$.*

The necessity of continuous sums appeared in very simple examples:

**Exercise 4.2.8** Let $T$ be a representation of $\mathbb{R}$ in $L_2(\mathbb{R})$ as follows: $[T(a)f](x) = e^{iax}f(x)$. Show that

(i). Any measurable set $E \subset \mathbb{R}$ define an invariant subspace of functions vanishing outside $E$.

(ii). $T$ does not have invariant irreducible subrepresentations.

**Definition 4.2.9** The set of equivalence classes of unitary irreducible representations of a group $G$ is denoted be $\hat{G}$ and called *dual object* (or *dual space*) of the group $G$.

**Definition 4.2.10** A left *regular representation* $\Lambda(g)$ of a group $G$ is the representation by left shifts in the space $L_2(G)$ of square-integrable function on $G$ with the left Haar measure

$$\Lambda g : f(h) \mapsto f(g^{-1}h). \tag{4.2.1}$$

The *main problem of representation theory* is to decompose a left regular representation $\Lambda(g)$ into irreducible components.

## 4.3 Invariant Operators and Schur's Lemma

It is a pleasant feature of an abstract theory that we obtain important general statements from simple observations. Finiteness of invariant measure on a compact group is one such example. Another example is Schur's Lemma presented here.

To find different classes of representations we need to compare them each other. This is done by *intertwining operators.*

**Definition 4.3.1** Let $T_1$ and $T_2$ are representations of a group $G$ in a spaces $H_1$ and $H_2$ correspondingly. An operator $A : H_1 \rightarrow H_2$ is called an *intertwining operator* if

$$A\,T_1(g) = T_2(g)\,A, \qquad \forall g \in G.$$

If $T_1 = T_2 = T$ then $A$ is *interntwinig operator* or *commuting operator* for $T$.

**Exercise 4.3.2** Let $G$, $H$, $T(g)$, and $A$ be as above. Show that [31, § 1.3.1]

(i). Let $\mathbf{x} \in H$ be an eigenvector for $A$ with eigenvalue $\lambda$. Then $T(g)\mathbf{x}$ for all $g \in G$ are eigenvectors of $A$ with the same eigenvalue $\lambda$.

(ii). All eigenvectors of $A$ with a fixed eigenvalue $\lambda$ for a linear subspace invariant under all $T(g)$, $g \in G$.

(iii). If an operator $A$ is commuting with irreducible representation $T$ then $A = \lambda I$.

HINT. Use the spectral decomposition of selfadjoint operators [18, § V.2.2]. □

The next result have very important applications.

**Lemma 4.3.3 (Schur)** *[17, § 8.2] If two representations $T_1$ and $T_2$ of a group $G$ are irreducible, then every intertwining operator between them either zero or is invertible.*

HINT. Consider subspaces $\ker A \subset H_1$ and $\operatorname{im} A \subset H_2$. □

**Exercise 4.3.4** Show that

(i). Two irreducible representations either equivalent or disjunctive.

(ii). All operators commuting with an irreducible representation form a field.

(iii). Irreducible representation of commutative group are one-dimensional.

(iv). If $T$ is unitary irreducible representation in $H$ and $B(\cdot,\cdot)$ is a bounded semi linear form in $H$ invariant under $T$: $B(T(g)\mathbf{x}, T(g)\mathbf{y}) = B(\mathbf{x},\mathbf{y})$ then $B(\cdot,\cdot) = \lambda\,\langle\cdot,\cdot\rangle$.

HINT. Use that $B(\cdot,\cdot) = \langle A\cdot,\cdot\rangle$ for some $A$ [18, § III.5.1]. □

# Chapter 5

# Group of Reals and Harmonic Analysis

As we know the exponential and trigonometric functions are special (hypergeometric) functions. Thus they will be our first examples of special functions arisen from representations of group according to the following definition.

**Definition 5.0.1** A *special function* associated with a representation $T$ of a group $G$ is a matrix element $t_{ij}(g)$ of $T$.

We recall that matrix elements of group characters coincide with themselves.

## 5.1 Exponential and Trigonometric Functions

The group of translations of the real line $\mathbb{R}$ is commutative thus by Exercise 4.3.4.(iii) all its irreducible representations are one dimensional, i.e. characters—functions $\mathbb{R} \rightarrow \mathbb{C}$ which satisfy to the functional equation:

$$f(x)\, f(y) = f(x+y).$$

Recall that characters form a group.

**Exercise 5.1.1** Show that from the above equation follow that

(i). $f(x)$ should be differentiable infinitely many times;

(ii). $f(x)$ satisfies to $f'(0)f(x) = f'(x)$ and thus is $f(x) = e^{ax}$ with $a = f'(0)$.

**Theorem 5.1.2** *All unitary irreducible representations of* $\mathbb{R}$ *are* $T(x) = e^{iax}$ *for an arbitrary* $a \in \mathbb{R}$*. In other words* $\hat{\mathbb{R}} = \mathbb{R}$.

**Exercise 5.1.3** Show that all irreducible representations of the multiplicative group $\mathbb{R}_+$ are of the form $T(t) = t^a$ for arbitrary $a \in \mathbb{C}$. They are unitary if $a = ib$, $b \in \mathbb{R}$.

HINT. Use the group homomorphism $\exp : \mathbb{R} \rightarrow \mathbb{R}_+$. □

The group of rotations $SO(2)$ of a unit circle $\mathbb{T}$ or Euclidean plane preserving quadratic form $x^2 + y^2$ is also commutative. A rotation by an angle $\phi$ described in Cartesian coordinates by the matrix:

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \qquad (5.1.1)$$

which could be considered as *definition* of trigonometric functions sin and cos. From the functional identity $g(\phi)\,g(\psi) = g(\phi + \psi)$ translated to the matrix multiplication follows *addition formulae*:

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \end{aligned}$$

This is the first occurrence of important formulae which we oftenly meet later. In general case addition formulae are realization of the property of matrix elements 4.1.5.(ii).

To find all irreducible unitary representations of $SO(2)$ we will use that $SO(2) = \mathbb{R}/\mathbb{Z}_{2\pi}$.

**Exercise 5.1.4** Show that all irreducible unitary representations of $SO(2)$ hev the form $g(\phi) = e^{in\phi}$, $n \in \mathbb{Z}$. In other words, $\hat{\mathbb{T}} = \mathbb{Z}$.

**Exercise 5.1.5** Decompose representation (5.1.1) into irreducible components.

HINT. Use the unitary transformation:

$$\frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

□

**Exercise$^*$ 5.1.6** Find

(i). All unitary irreducible representation of the group $SH(2)$ of hyperbolic rotations, i.e. preserving the quadratic form $x^2 - y^2$ on $\mathbb{R}^2$.

(ii). Corresponding addition formulae.

## 5.2 Duality and the Fourier Transform

**Exercise 5.2.1** (i). Show that $\hat{\mathbb{Z}} = \mathbb{T}$.

(ii). Collect the above result together with Theorem 5.1.2 and Exercise 5.1.4 to obtain that dual object of the Abelian group $\mathbb{R}^n \times \mathbb{Z}^k \times \mathbb{T}^l$ is the group $\mathbb{R}^n \times \mathbb{T}^k \times \mathbb{Z}^l$.

The above result is a particular case of the Pontrjagin's duality, which plays an exceptional rôle in the representation theory of Abelian groups

**Theorem 5.2.2 (Pontrjagin's duality)** *[17, § 12.1], [18, § IV.2.1] For an arbitrary locally compact abelian (*l.c.a.*) group $G$, the canonical mapping of $G$ into $\hat{\hat{G}}$ is an isomorphism of topological groups. Haar measures on $G = \hat{\hat{G}}$ and $\hat{G}$ can be normalized so that :*

$$\begin{aligned} \hat{f}(\chi) &= \int_G f(g)\,\chi(g)\,dg && (5.2.1) \\ f(g) &= \int_{\hat{G}} \hat{f}(\chi)\,\overline{\chi(g)}\,d\chi && (5.2.2) \\ \int_G |f(g)|^2\,dg &= \int_{\hat{G}} \left|\hat{f}(\chi)\right|^2\,d\chi && (5.2.3) \end{aligned}$$

*Particularly normalized to* 1 *measure on a compact group corresponds to a point measure* 1 *on the discrete dual.*

The formulas above deserve special names.

**Definition 5.2.3** The transformations (5.2.1) is called the *Fourier transform* from $G$ to $\hat{G}$, (5.2.2) is *inverse Fourier transform* from $\hat{G}$ to $G = \hat{\hat{G}}$, and (5.2.3) is known as *Plancherel's identity.*

The following is remarkable properties of the Fourier transform.

**Theorem 5.2.4** *Let $G$ is a l.c.a. group with an invariant measure $\mu$. The Fourier transform maps*

*(i). $L_1(G, \mu)$ into the space of continuous bounded functions on $\hat{G}$.*

*(ii). Convolutions into multiplication: $(f_1 * f_2)\hat{}(\chi) = \hat{f}_1(\chi) \cdot \hat{f}_2(\chi)$.*

*(iii). Shifts into multiplication by the character: $(\lambda(g)f)\hat{}(\chi) = \chi(g)\hat{f}(\chi)$.*

*(iv). Multiplication by a character $\chi_1 \in \hat{G}$ to the shifts on $\hat{G}$: $(\chi_1 \cdot f)\hat{}(\chi) = \hat{f}(\chi_1^{-1}\chi)$.*

## 5.3 Fourier Series

We consider $\mathbb{T} = SO(2)$ which is a simplest example of *compact group*. We state here without proofs main features of representation theory of compact groups.

**Theorem 5.3.1** *[17, § 9.2]*

*(i). Every topologically irreducible representation of a compact group $G$ is finite-dimensional and unitarizable.*

*(ii). If $T_1$ and $T_2$ are two inequivalent irreducible representations, then every matrix element of $T_1$ is orthogonal in $L_2(G)$ to every matrix element of $T_2$.*

*(iii). For a compact group $G$ its dual space $\hat{G}$ is discrete.*

Due to relation $SO(2) = \mathbb{R}/\mathbb{Z}_{2\pi}$ we can identify functions on $\mathbb{T}$ with periodic functions on $\mathbb{R}$ with a period $2\pi$. We define *invariant integration* on $SO(()2)$ by the invariant measure from Exercise 3.2.5.(ii):

$$\int_{SO(2)} f(g)\, dg = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi.$$

**Theorem 5.3.2** *Functions $\{e^{in\phi}\}$ form a complete orthonormal system in $L_2(SO(2))$:*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{im\phi}\, \overline{e^{in\phi}}\, d\phi = \delta_{nm}.$$

*Therefore any function $f(\phi) \in L_2(SO(2))$ can be represented by its* Fourier series *(cf.* (5.2.2)*):*

$$f(\phi) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi},$$

*where* Fourier coefficients *$c_n$ defined from (cf.* (5.2.1)*)*

$$c_n = \left\langle f(\phi), e^{in\phi} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, \overline{e^{in\phi}}\, d\phi$$

**Corollary 5.3.3 (Plancherel Identity)** *(cf.* (5.2.3)*)*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\phi)|^2\, d\phi = \sum_{n=-\infty}^{\infty} c_n^2.$$

The main problem of representation theory for $SO(2)$ has the following solution:

**Theorem 5.3.4** *The regular representation $R(\phi)$ of $G$ is a direct sum of one dimensional representations $T_n(\phi) = e^{in\phi}$ with the multiplicity 1:*

$$R(\phi) = \sum_{n=-\infty}^{\infty} T_n(\phi).$$

**Exercise 5.3.5** (i). Find the particular form of the general Theorem 5.2.4 for $G = SO(2)$.

(ii). The Fourier transform on $SO(2)$ sends the derivative to the operator of multiplication by the sequence $\{2\pi in\}_{n\in\mathbb{Z}}$.

As a corollary we could advance to the point-wise convergence of the Fourier series:

**Exercise 5.3.6** If $f(\phi)$ on $SO(2)$ is differentiable infinitely many times

(i). then its Fourier coefficients decrease rapidly:

$$\lim_{n\rightarrow\infty} n^k c_n = 0, \qquad \forall k \in \mathbb{N}.$$

(ii). then its Fourier series converges point-wise.

**Exercise 5.3.7** If Fourier coefficients of a function $f(\phi)$ decrease rapidly then it is differentiable infinitely many time.

## 5.4 Fourier Integral

According to general formula (5.2.1) and specific form of characters on $\mathbb{R}$ (Theorem 5.1.2) the Fourier transform and its inverse on $\mathbb{R}$ defined as follows:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i\xi x}\, dx \qquad f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i\xi x}\, d\xi$$

There are many alternative normalizations of the Haar measure on $\mathbb{R}$, for example:

$$\hat{f}(\xi) = \sqrt{2\pi} \int_{\mathbb{R}} f(x)\, e^{-i\xi x}\, dx \qquad f(x) = \sqrt{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)\, e^{i\xi x}\, d\xi$$

The decomposition of the left regular representation into irreducible components combines the Fourier transform and decomposition in Example 4.2.8.

**Exercise 5.4.1** (i). Find the particular form of the general Theorem 5.2.4 for $G = \mathbb{R}$.

(ii). The Fourier transform on $\mathbb{R}$ sends the derivative $\frac{d}{dx}$ to the operator of multiplication by $2\pi i\xi$.

It is turn to be that commutative harmonic analysis could be better understood with help of representation theory of the non-commutative Heisenberg group $\mathbb{H}^1$ [15].

**Exercise 5.4.2** Check that

(i). $\rho$ defined by

$$\rho(s,x,y)f(t) = e^{i(s+xt+xy/2)}f(t+y) \tag{5.4.1}$$

is a unitary representation of $\mathbb{H}^1$ in $L_2(\mathbb{R})$.

(ii). $\rho$ is irreducible (*Hint.* It may commute only with operators of multiplication by a constant function).

(iii). $r(s,x,y) = r(s,-y,x)$ is an automorphism of $\mathbb{H}^1$.

(iv). The Fourier transform intertwines two representations $\rho$ and $\rho \circ r$ of $\mathbb{H}^1$: $\hat{}\rho(h) = \rho(r(h))\hat{}$.

From the above Exercise and the Schur's Lemma we conclude that the Fourier transform is a unitary operator up to a scalar multiplier. By iteration we know that $\mathbb{H}^1$: $\hat{}\hat{}\rho(h) = \rho(r^2(h))\hat{}\hat{}$ and $r^2(s,x,y) = r(s,-x,-y)$. From that $\hat{}^2 = w(-1)$, where $w(-1)f(t) = f(-t)$.

**Lemma 5.4.3** *Show that the scalar multiplier is equal to* 1 *by demonstration that* $(e^{-\pi x^2})\hat{} = e^{-\pi x^2}$. *(This formula were stated in the first part of the course).*

Proof. The function $e^{-\pi x^2}$ solves the equation $(d/dx + 2\pi x)f = 0$ which is invariant under the Fourier transform due to Exercise 5.4.1.(ii). Then $e^{-\pi x^2}$ is the Fourier transform up to a scalar factor, which is equal to 1 from the formula:

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1.$$

(It was proven in the first part of the course). □

Remark 5.4.4 The above results of harmonic analysis could be extended for general l.c.a. group, particularly for $\mathbb{R}^n \times \mathbb{Z}^k \times \mathbb{T}^l$.

# Chapter 6

# Harmonic Analysis on the Sphere

## 6.1 Rotations of the Euclidean Space

Let us start from two problems [17, § 17.1]:

**Problem 6.1.1** A convex centrally symmetric body in $\mathbb{R}^n$ is uniquely determined by the area of its projections on all possible hyperplanes.

**Problem 6.1.2** A convex centrally symmetric body $K$ in $\mathbb{R}^n$ is uniquely determined by the areas of its sections by all possible hyperplanes.

In fact we have two faces of the same problem.

**Exercise 6.1.3** Show that Problems 6.1.1 and 6.1.2 are equivalent.

HINT. Use a norm in $\mathbb{R}^n$ defined by a convex centrally symmetric body. Show that projections and section define dual norms. □

We restrict ourself to the case $n = 3$. We could describe a convex body by an even function on $\mathbb{S} \subset \mathbb{R}^3$:

$$f(x) = \frac{1}{2} r_x^2, \qquad x \in \mathbb{S},$$

where $r_x$ distance from 0 to the boundary in direction of $x$.

**Exercise 6.1.4** Let $C$ be a great circle of $\mathbb{S}$ on a plane $P$. Then area $(K \cap \mathbb{P}) = \int_C f(x)\, dx$.

**Exercise 6.1.5** Problem 6.1.2 is equivalent to: *an even function on the sphere is uniquely determined by its integrals on all great circles.*

Let $L_2(\mathbb{S})$ be the space of square integrable function on $S$ and $L_2^+(\mathbb{S})$ its subspace of the even functions. We have natural representations $T$ and $T_+$ of the *group of isometric rotations* $SO(3)$ of Euclidean space $\mathbb{R}^3$ in $L_2(\mathbb{S})$ and $L_2^+(\mathbb{S})$ respectively. We define operator $J$

$$Jf(x) = \int_{C_x} f(y)\,dy, \tag{6.1.1}$$

where $C_x$ is the great circle with epicenter at the point $x \in \mathbb{S}$.

**Exercise 6.1.6** The operator $J$ intertwines $T$ and $T_+$.

**Exercise 6.1.7** Problem 6.1.2 is equivalent to $\ker J = 0$ in $L_2^+(\mathbb{S})$.

From Theorem 5.3.1 $L_2^+(\mathbb{S})$ is a direct sum of irreducible finite dimensional spaces in each of which the operator $J$ is scalar by the Schur's Lemma. Let $P_n$ be the space of all functions on $\mathbb{S}$ that are restriction to $\mathbb{S}$ of homogeneous polynomials of degree $n$ in $\mathbb{R}^3$.

**Exercise 6.1.8** Prove

(i). $P_n \subset P_{n+2}$ (use $x^2 + y^2 + z^2 = 1$ on $\mathbb{S}$).

(ii). $\dim \mathbb{P}^n = (n+1)(n+2)/2$ (use induction).

Let $H_n$ be the orthogonal completion of $P_{n-2}$ in $P_n$.

**Theorem 6.1.9** *The decomposition of the spaces $L_2(\mathbb{S})$ and $L_2^+(\mathbb{S})$ into irreducible subspaces for $T$ and $T_+$ of $SO(3)$ have the form:*

$$L_2(\mathbb{S}) = \sum_{n=0}^{\infty} H_n \qquad \text{and} \qquad L_2^+(\mathbb{S}) = \sum_{n=0}^{\infty} H_{2n},$$

*respectively.*

Proof. Decompositions are valid the standard functional analytical reasoning. The invariance of $H_n$ is also obvious. The remaining part is irreducibility of $H_n$ which is proven by the next two Exercises. □

**Exercise 6.1.10** $H_n$ contains exactly one function $L_n$ that is invariant under the subgroup of rotations about the $z$-axis.

HINT. Consider $[n/2]+1$ functions in $P_n$: $z^n$, $z^{n-2}(x^2+y^2)$, …, $z^{n-2[n/2]}(x^2+y^2)$. □

**Exercise 6.1.11** Prove that every irreducible subspace $V \subset L_2(\mathbb{S})$ contains at least one non-zero function that is invariant under rotations about the $z$-axis.

HINT. Use invariant integration. □

To conclude solution of Exercise 6.1.7 (and thus Problems 6.1.1 and 6.1.2) we will explicitly calculate eigenvalues of $J$ in $H_n$

**Exercise 6.1.12** Show that $z$-invariant function $L_n$ could be taken to be $n$-th *Legendre polynomial*:

$$L_n(z) = \frac{d^n}{dz^n}\left[(z^2-1)^n\right].$$

HINT. Prove for functions $f$ on $\mathbb{S}$ that depends only on the coordinate $z$ (use integration in cylindrical coordinates):

$$\int_{\mathbb{S}} f(x)\,dx = \pi \int_{-1}^{1} f(z)\,dz.$$

Use integration by parts to prove that $L_n$ orthogonal to all polynomials in $z$ of degree less than $n$ with respect to the following inner product:

$$(f_1, f_2) = \pi \int_{-1}^{1} f_1(z)\,\bar{f}_2(z)\,dz.$$

Show that $L_n$ uniquely defined up to scalar factor by the above properties. □

The Legendre polynomials are another example of the special functions, more precisely they are *orthogonal polynomials*.

Substituting $L_n$ instead of $f$ and $(0,0,1)$ instead of $x$ in (6.1.1) we have:

$$\lambda_n L_n(1) = 2\pi L_n(0).$$

Obviously:

$$\begin{aligned}
L_n(1) &= \left.\frac{d^n}{dz^n}\left[(z-1)^n(z+1)^n\right]\right|_{z=1} = n!(z+1)^n\big|_{z=1} = 2^n \cdot n!,\\
L_n(0) &= \begin{cases} 0 & \text{if } n \text{ is odd,}\\ (2k)!\binom{2k}{k} & \text{if } n=2k \text{ is even.}\end{cases}
\end{aligned}$$

As the result:

$$\lambda_n(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2\pi\frac{(2k-1)!!}{2k!!} & \text{if } n = 2k \text{ is even,} \end{cases}$$

where $n!! = n(n-2)(n-4)\cdots$. This finishes the consideration.

# Chapter 7

# Hermit Polynomials, Heisenberg Group, and Segal-Bargmann Spaces

## 7.1 Introduction

This lecture is based on the paper [8].

It is well known, by the celebrated Stone-von Neumann theorem, that all models for the canonical quantisation [24] are isomorphic and provide us with equivalent representations of the Heisenberg group [30, Chap. 1]. Nevertheless it is worthwhile to look for some models which can act as alternatives for the Schrödinger representation. In particular, the Segal-Bargmann representation [2, 28] serves to

- give a geometric representation of the dynamics of the harmonic oscillators;
- present a nice model for the creation and annihilation operators, which is important for quantum field theory;
- allow applying tools of analytic function theory.

The huge abilities of the Segal-Bargmann (or Fock [13]) model are not yet completely employed, see for example new ideas in a recent preprint [27].

We look for similar connections between nilpotent Lie groups and spaces of monogenic [5, 10] Clifford valued functions. Particularly we are interested in a third possible representation of the Heisenberg group, acting on monogenic functions on $\mathbb{R}^n$. There are several reasons why such a model can be of interest. First of all the theory of monogenic functions is (at least) as interesting as several complex variable theory, so the monogenic model should share many pleasant features with the Segal-Bargmann model. Moreover, monogenic functions take their value in a Clifford algebra, which is a natural environment in which to represent internal degrees of freedom of elementary particles such as spin. Thus from the very beginning it has a structure which in the Segal-Bargmann model has to be added, usually by means of the second quantization procedure [11]. So a monogenic representation can be even more relevant to quantum field theory than the Segal-Bargmann one (see Remark 7.3.2).

From the different aspects of the Segal-Bargmann space $F_2(\mathbb{C}^n)$ we select the one giving a unitary representation of the Heisenberg group $\mathbb{H}^n$. The representation is unitary equivalent to the Schrödinger representation on $L_2(\mathbb{R}^n)$ and the Segal-Bargmann transform is precisely the intertwining operator between these two representations (see subsection 7.3.2).

This lecture is closely related to [21], where connections between analytic function theories and group representations were described. Representations of another group ($SL_2(\mathbb{R})$) in spaces of monogenic functions can be found in [20]. We hope that the present lecture make only few first steps towards an interesting function theory and other steps will be done elsewhere.

## 7.2 Wavelets or Coherent States

In our approach we will need some basic facts on *wavelets* (or *coherent states*) and associated *wavelet transform.*

Let $G$ be a group which acts via transformation of a closed domain $\bar{\Omega}$. Moreover, let $G : \partial\Omega \rightarrow \partial\Omega$ and $G$ act on $\Omega$ and $\partial\Omega$ transitively. Let us fix a point $x_0 \in \Omega$ and let $H \subset G$ be a stationary subgroup of point $x_0$. Then domain $\Omega$ is naturally identified with the homogeneous space $G/H$. Till the moment we do not request anything untypical. Now let

- *there exist a $H$-invariant measure $d\mu$ on $\partial\Omega$.*

We consider the Hilbert space $L_2(\partial\Omega, d\mu)$. Then geometrical transformations of $\partial\Omega$ give us the representation $\pi$ of $G$ in $L_2(\partial\Omega, d\mu)$. Let $f_0(x) \equiv 1$ and $F_2(\partial\Omega, d\mu)$ be the closed liner subspace of $L_2(\partial\Omega, d\mu)$ with the properties:

(i). $f_0 \in F_2(\partial\Omega, d\mu)$;

(ii). $F_2(\partial\Omega, d\mu)$ is $G$-invariant;

(iii). $F_2(\partial\Omega, d\mu)$ is $G$-irreducible, or $f_0$ is cyclic in $F_2(\partial\Omega, d\mu)$.

The *standard wavelet transform* $W$ is defined by

$$W : F_2(\partial\Omega, d\mu) \to L_2(G) : f(x) \mapsto \widehat{f}(g) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(\partial\Omega, d\mu)}$$

Due to the property $[\pi(h) f_0](x) = f_0(x)$, $h \in H$ and identification $\Omega \sim G/H$ it could be translated to the embedding:

$$\widetilde{W} : F_2(\partial\Omega, d\mu) \to L_2(\Omega) : f(x) \mapsto \widehat{f}(y) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(\partial\Omega, d\mu)}, \tag{7.2.1}$$

We define the *inverse wavelet transform* $\mathcal{M}$ according to the formula:

$$[\mathcal{M}\widehat{f}](x) = \int_\Omega \widehat{f}(a) f_{s(a)}(x)\, da, \tag{7.2.2}$$

The following proposition explain the usage of the name for $\mathcal{M}$.

**Theorem 7.2.1** *The operator*

$$\mathcal{P} = \mathcal{M}\mathcal{W} : B \to B \tag{7.2.3}$$

*is a projection of $B$ to its linear subspace for which $b_0$ is cyclic. Particularly if $\pi$ is an irreducible representation then the inverse wavelet transform $\mathcal{M}$ is a* left inverse operator *on $B$ for the wavelet transform $\mathcal{W}$:*

$$\mathcal{M}\mathcal{W} = I.$$

## 7.3 The Heisenberg Group and Spaces of Analytic Functions

### 7.3.1 The Schrödinger Representation of the heisenberg Group

We recall here some basic facts on the Heisenberg group $\mathbb{H}^n$ and its Schrödinger representation, see [14, Chap. 1] and [30, Chap. 1] for details.

The Lie algebra of the Heisenberg group is generated by the $2n+1$ elements $p_1, \ldots, p_n$, $q_1, \ldots, q_n$, $e$, with the well-known Heisenberg commutator relations:

$$[p_i, q_j] = \delta_{ij} e. \tag{7.3.1}$$

All other commutators vanish. In the standard quantum mechanical interpretation the operators are momentum and coordinate operators [14, § 1.1].

It is common practice to switch between real and complex Lie algebras. Complexify $\mathfrak{h}^n$ to obtain the complex algebra $\mathbb{C}\mathfrak{h}^n$, and take four complex numbers $a$, $b$, $c$ and $d$ such that $ad - bc \neq 0$. The *real* $2n+1$-dimensional subspace spanned by

$$A_k = ap_k + bq_k \qquad B_k = cp_k + dq_k$$

and the commutator $[A_k, B_k] = (ad - bc)e$, where $e = [p_k, q_k]$ is of course isomorphic to $\mathfrak{h}^n$, and exponentiating will give a group isomorphic to the Heisenberg group.

An example of this procedure is obtained from the construction of the so-called creation and annihilation operators of Bose particles in the $k$-th state, $a_k^+$ and $a_k^-$ (see [14, § 1.1]). These are defined by:

$$a_k^{\pm} = \frac{q_k \mp \imath p_k}{\sqrt{2}}, \tag{7.3.2}$$

giving the commutators $[a_i^+, a_j^-] = (-\imath)\delta_{ij}e$. Putting $-\imath e = \ell$, the real algebra spanned by $a_k^{\pm}$ and $\ell$ is an alternative realization of $\mathfrak{h}^n$, $\mathfrak{h}_a^n$.

An element $g$ of the Heisenberg group $\mathbb{H}^n$ (for any positive integer $n$, cf. (3.1.1)) can be represented as $g = (t, \mathbf{z})$ with $t \in \mathbb{R}$, $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. The group law in coordinates $(t, \mathbf{z})$ is given by

$$g * g' = (t, \mathbf{z}) * (t', \mathbf{z}') = (t + t' + \frac{1}{2}\sum_{j=1}^{n} \Im(\bar{z}_j z'_j), \mathbf{z} + \mathbf{z}'), \tag{7.3.3}$$

where $\Im z$ denotes the imaginary part of the complex number $z$. Of course the Heisenberg group is non-commutative.

The relation between the Heisenberg group and its Lie algebra is given by the exponentiation $\exp : \mathfrak{h}_a^n \to \mathbb{H}^n$. We define the formal vector $\mathbf{a}^+$ as being $(a_1^+, \ldots, a_n^+)$ and $\mathbf{a}^-$ as $(a_1^-, \ldots, a_n^-)$, which allows us to use the formal inner products

$$\begin{aligned} \mathbf{u} \cdot \mathbf{a}^+ &= \sum_{k=1}^{n} u_k a_k^+ \\ \mathbf{v} \cdot \mathbf{a}^- &= \sum_{k=1}^{n} v_k a_k^-. \end{aligned}$$

With these we define, for real vectors $\mathbf{u}$ and $\mathbf{v}$, and real $s$

$$\exp(\mathbf{u} \cdot (\mathbf{a}^+ + \mathbf{a}^-)) = (0, \sqrt{2}\mathbf{u}) \tag{7.3.4}$$

$$\exp(\mathbf{v} \cdot (\mathbf{a}^- - \mathbf{a}^+) = (0, \imath\mathbf{v}) \tag{7.3.5}$$

$$\exp(s\ell) = (e^{-2s}, 0). \tag{7.3.6}$$

Possible Schrödinger representations (cf. (5.4.1)) are parameterized by the non-zero real number $\hbar$ (the Planck constant). As usual, for considerations where the correspondence principle between classic and quantum mechanics is irrelevant, we consider only the case $\hbar = 1$. The Hilbert space for the Schrödinger representation is $L_2(\mathbb{R}^n)$, where elements of the complex Lie algebra $\mathbb{C}\mathfrak{h}^n$ are represented by the unbounded operators

$$\sigma(a_k^\pm) = \frac{1}{\sqrt{2}}\left(x_k I \mp \frac{\partial}{\partial x_k}\right). \tag{7.3.7}$$

From which it follows, using any $j$, that

$$\sigma(\ell) = [a_j^+, a_j^-] = -2I.$$

The corresponding representation $\pi$ of the Heisenberg group is given by exponentiation of the $\sigma(a_k^+)$ and $\sigma(a_k^-)$, but this is most readily expressed by using $p_k$ and $q_k$, and so is generated by shifts and multiplications $s_\mathbf{c} : f(\mathbf{x}) \mapsto f(\mathbf{x}+\mathbf{c})$ and $m_\mathbf{b} : f(\mathbf{x}) \mapsto e^{\imath\mathbf{x}\cdot\mathbf{b}} f(\mathbf{x})$, with the Weyl commutation relation

$$s_\mathbf{c} m_\mathbf{b} = e^{\imath\mathbf{c}\cdot\mathbf{b}} m_\mathbf{b} s_\mathbf{c}.$$

There is an orthonormal basis of $L_2(\mathbb{R}^n)$ on which the operators $\sigma(a_k^\pm)$ act in an especially simple way. It consists of the functions:

$$\phi_m(\mathbf{y}) = [2^m m! \sqrt{\pi}]^{-1/2} e^{-\mathbf{x}\cdot\mathbf{x}/2} H_m(\mathbf{y}), \tag{7.3.8}$$

where $\mathbf{y} = (y_1, \ldots, y_n)$, $m = (m_1, \ldots, m_n)$, and $H_m(\mathbf{y})$ is the generalized Hermite polynomial

$$H_m(\mathbf{y}) = \prod_{i=1}^{n} H_{m_i}(y_i).$$

For these

$$a_k^+ \phi_m(\mathbf{y}) = \sqrt{m_k + 1}\, \phi_{m'}(\mathbf{y}), \qquad a_k^- \phi_m(\mathbf{y}) = \sqrt{m_k}\, \phi_{m''}(\mathbf{y})$$

where

$$\begin{aligned} m' &= (m_1, m_2, \ldots, m_{k-1}, m_k + 1, m_{k+1}, \ldots, m_n) \\ m'' &= (m_1, m_2, \ldots, m_{k-1}, m_k - 1, m_{k+1}, \ldots, m_n). \end{aligned}$$

This is the most straightforward way to express the creation or annihilation of a particle in the $k$-th state.

Let us now consider the generating function of the $\phi_m(\mathbf{x})$,

$$A(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{\infty} \frac{x^j}{\sqrt{j!}} \phi_k(\mathbf{y}) = \exp(-\frac{1}{2}(\mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y}) + \sqrt{2}\mathbf{x}\cdot\mathbf{y}). \tag{7.3.9}$$

We state the following elementary fact in Dirac's bra-ket notation.

**Lemma 7.3.1** *Let $H$ and $H'$ be two Hilbert spaces with orthonormal bases $\{\phi_k\}$ and $\{\phi'_k\}$ respectively. Then the sum*

$$U = \sum_{j=0}^{\infty} \left|\phi'_j\right\rangle \left\langle\phi_j\right| \qquad (7.3.10)$$

*defines a unitary operator $U : H \rightarrow H'$ with the following properties:*

*(i). $U\phi_k = \phi'_k$;*

*(ii). If an operator $T : H \rightarrow H$ is expressed, relative to the basis $\phi_k$, by the matrix $(a_{ij})$ then the operator $UTU^{-1} : H' \rightarrow H'$ is expressed relative to the basis $\phi'_k$ by the same matrix.*

Now, if we take the function $A(\mathbf{x}, \mathbf{y})$ from (7.3.9) as a kernel for an *integral transform*,

$$[Af](\mathbf{y}) = \int_{\mathbb{R}^n} A(\mathbf{y}, \mathbf{x}) f(\mathbf{x})\, dx$$

we can consider it subject to the Lemma above. However, for this we need to define the space $H'$ and an orthonormal basis $\{\phi'_k\}$ (we already identified $H$ with $L_2(\mathbb{R}^n)$ and the $\{\phi_k\}$ are given by (7.3.8)). There is some freedom in doing this.

For example it is possible to take the holomorphic extension $A(\mathbf{z}, \mathbf{y})$ of $A(\mathbf{x}, \mathbf{y})$ with respect to the first variable. Then

(i). $H'$ is the Segal-Bargmann space of analytic functions over $\mathbb{C}^n$ with scalar product defined by the integral with respect to Gaussian measure $e^{-|\mathbf{z}|^2}\, d\mathbf{z}$;

(ii). The Heisenberg group acts on the Segal-Bargmann space as follows:

$$[\beta_{(t,\mathbf{z})} f](\mathbf{u}) = f(\mathbf{u} + \mathbf{z}) e^{\imath t - \bar{\mathbf{z}} \cdot \mathbf{u} - |\mathbf{z}|^2/2}. \qquad (7.3.11)$$

This action generates the set of coherent states $f_{(0,\mathbf{v})}(\mathbf{u}) = e^{-\bar{\mathbf{v}}\mathbf{u} - |\mathbf{v}|^2/2}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^n$ from the vacuum vector $f_0(\mathbf{u}) \equiv 1$;

(iii). The operators of creation and annihilation are $a_k^+ = z_k I$, $a_k^- = \frac{\partial}{\partial z_k}$.

(iv). The Segal-Bargmann space is spanned by the orthonormal basis $\phi'_k = \frac{1}{\sqrt{m!}} z^n$ or by the set of coherent states $f_{(0,\mathbf{v})}(\mathbf{u}) = e^{-\bar{\mathbf{v}}\mathbf{u} - |\mathbf{v}|^2/2}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^n$

(v). The intertwining kernel for $\sigma_{(t,\mathbf{z})}$ (7.3.7) and $\beta_{(t,\mathbf{z})}$ (7.3.11) is

$$A(\mathbf{z}, \mathbf{y}) = e^{-(\mathbf{z} \cdot \mathbf{z} + \mathbf{x} \cdot \mathbf{x})/2 - \sqrt{2} \mathbf{z} \cdot \mathbf{x}} = \sum_{k=0}^{\infty} \frac{\mathbf{z}^m}{\sqrt{m!}} \cdot \frac{1}{\sqrt{2^m m! \sqrt[4]{\pi}}} e^{-\mathbf{x} \cdot \mathbf{x}/2} H_m(\mathbf{y})$$

(vi). The Segal-Bargmann space has a reproducing kernel

$$K(\mathbf{u},\mathbf{v}) = e^{\mathbf{u}\cdot\bar{\mathbf{v}}} = \sum_{k=1}^{\infty} \phi_k(\mathbf{u})\bar{\phi}_k(\mathbf{v}) = \int e^{\mathbf{u}\cdot\bar{\mathbf{z}}} e^{\mathbf{z}\cdot\bar{\mathbf{v}}} e^{-|\mathbf{z}|^2}\, d\mathbf{z}.$$

### 7.3.2 The Segal-Bargmann space

We consider a representation of the Heisenberg group $\mathbb{H}^n$ (see Section 7.3) on $L_2(\mathbb{R}^n)$ by shift and multiplication operators [30, § 1.1]:

$$g = (t,\mathbf{z}) : f(\mathbf{x}) \to [\pi_{(t,\mathbf{z})} f](\mathbf{x}) = e^{\imath(2t-\sqrt{2}\mathbf{q}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{p})} f(\mathbf{x}-\sqrt{2}\mathbf{p}), \qquad \mathbf{z} = \mathbf{p} + \imath q, \tag{7.3.12}$$

This is the Schrödinger representation with parameter $\hbar = 1$. As a subgroup $H$ we select the centre of $\mathbb{H}^n$ consisting of elements $(t, 0)$. It is non-compact but using the special form of representation (7.3.12) we can consider the cosets[1] $\widetilde{G}$ and $\widetilde{H}$ of $G$ and $H$ by the subgroup with elements $(\pi m, 0)$, $m \in \mathbb{Z}$. Then (7.3.12) also defines a representation of $\widetilde{G}$ and $\widetilde{H} \sim \Gamma$. We consider the Haar measure on $\widetilde{G}$ such that its restriction on $\widetilde{H}$ has total mass equal to 1.

As "vacuum vector" we will select the original *vacuum vector* of quantum mechanics—the Gauss function $f_0(\mathbf{x}) = e^{-\mathbf{x}\cdot\mathbf{x}/2}$. Its transformations are defined as follows:

$$\begin{aligned} w_g(\mathbf{x}) = \pi_{(t,\mathbf{z})} f_0(\mathbf{x}) &= e^{\imath(2t-\sqrt{2}\mathbf{q}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{p})}\, e^{-(\mathbf{x}-\sqrt{2}\mathbf{p})^2/2} \\ &= e^{2\imath t-(\mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q})/2} e^{-((\mathbf{p}-\imath\mathbf{q})^2+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}(\mathbf{p}-\imath\mathbf{q})\cdot\mathbf{x}} \\ &= e^{2\imath t-\mathbf{z}\cdot\bar{\mathbf{z}}/2} e^{-(\bar{\mathbf{z}}\cdot\bar{\mathbf{z}}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\bar{\mathbf{z}}\cdot\mathbf{x}}. \end{aligned}$$

In particular $w_{(t,0)}(\mathbf{x}) = e^{-2it} f_0(\mathbf{x})$, i.e. it really is a vacuum vector with respect to $\widetilde{H}$ in the sense of our definition. Of course $\widetilde{G}/\widetilde{H}$ is isomorphic to $\mathbb{C}^n$. Embedding $\mathbb{C}^n$ in $G$ by the identification of $(0,\mathbf{z})$ with $\mathbf{z}$, the mapping $s : \widetilde{G} \to \widetilde{G}$ is defined simply by $s((t,\mathbf{z})) = (0,\mathbf{z}) = \mathbf{z}$; $\Omega$ then is identical with $\mathbb{C}^n$.

The Haar measure on $\mathbb{H}^n$ coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ [30, § 1.1] and so the invariant measure on $\Omega$ also coincides with Lebesgue measure on $\mathbb{C}^n$. Note also that the composition law sending $\mathbf{z}_1\ \mathbf{z}_2$ to $s((0,\mathbf{z}_1)(0,\mathbf{z}_2))$ reduces to Euclidean shifts on $\mathbb{C}^n$. We also find $s((0,\mathbf{z}_1)^{-1}\cdot(0,\mathbf{z}_2)) = \mathbf{z}_2 - \mathbf{z}_1$ and $r((0,\mathbf{z}_1)^{-1}\cdot(0,\mathbf{z}_2)) = (\frac{1}{2}\Im\bar{\mathbf{z}}_1\cdot\mathbf{z}_2, 0)$.

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[1] $\widetilde{G}$ is sometimes called the *reduced Heisenberg group*Heisenberg group!reduced. It seems that $\widetilde{G}$ is a virtual object, which is important in connection with a selected representation of $G$.

The reduced wavelet transform takes the form of a mapping $L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{C}^n)$ and is given by the formula

$$\begin{aligned}\widehat{\mathcal{W}}f(\mathbf{z}) &= \left\langle f, w_{(0,\mathbf{z})} \right\rangle \\ &= \pi^{-n/4} \int_{\mathbb{R}^n} f(\mathbf{x})\, e^{-\mathbf{z}\cdot\bar{\mathbf{z}}/2}\, e^{-(\mathbf{z}\cdot\mathbf{z}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\mathbf{z}\cdot\mathbf{x}}\, dx \\ &= e^{-|\mathbf{z}|^2/2}\pi^{-n/4} \int_{\mathbb{R}^n} f(\mathbf{x})\, e^{-(\mathbf{z}\cdot\mathbf{z}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\mathbf{z}\cdot\mathbf{x}}\, dx, \qquad (7.3.13)\end{aligned}$$

where $\mathbf{z} = \mathbf{p} + \imath\mathbf{q}$. Then $\widehat{\mathcal{W}}f$ belongs to $L_2(\mathbb{C}^n, dg)$. This can better be expressed by saying that the function $\breve{f}(\mathbf{z}) = e^{|\mathbf{z}|^2/2}\widehat{\mathcal{W}}f(\mathbf{z})$ belongs to $L_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2}dg)$ because $\breve{f}(\mathbf{z})$ is analytic in $\mathbf{z}$. These functions constitute the *Segal-Bargmann space* [2, 28] $F_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2}dg)$ of functions analytic in $\mathbf{z}$ and square-integrable with respect the Gaussian measure $e^{-|\mathbf{z}|^2}d\mathbf{z}$. Analyticity of $\breve{f}(\mathbf{z})$ is equivalent to the condition that $(\frac{\partial}{\partial\bar{\mathbf{z}}_j} + \frac{1}{2}\mathbf{z}_j I)\mathcal{W}f(\mathbf{z})$ equals zero.

The integral in (7.3.13) is the well-known Segal-Bargmann transform [2, 28]. Its inverse is given by a realization of (7.2.2):

$$\begin{aligned}f(\mathbf{x}) &= \int_{\mathbb{C}^n} \widehat{\mathcal{W}f(\mathbf{z})} w_{(0,\mathbf{z})}(\mathbf{x})\, d\mathbf{z} \\ &= \int_{\mathbb{C}^n} \breve{f}(\mathbf{z}) e^{-(\bar{\mathbf{z}}^2+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\bar{\mathbf{z}}x}\, e^{-|\mathbf{z}|^2}\, d\mathbf{z}. \qquad (7.3.14)\end{aligned}$$

This gives (7.2.2) the name of Segal-Bargmann inverse. The corresponding operator $\mathcal{P}$ (7.2.3) is the identity operator $L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$ and (7.2.3) gives an integral presentation of the Dirac delta.

Meanwhile the orthoprojection $L_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2}dg) \rightarrow F_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2}dg)$ is of interest and is a principal ingredient in Berezin quantisation [3, 9]. We can easy find its kernel. Indeed, $\widehat{\mathcal{W}}f_0(\mathbf{z}) = e^{-|\mathbf{z}|^2}$, and the kernel is

$$\begin{aligned}K(\mathbf{z}, \mathbf{w}) &= \widehat{\mathcal{W}}f_0(\mathbf{z}^{-1}\cdot\mathbf{w})\bar{\chi}(r(\mathbf{z}^{-1}\cdot\mathbf{w})) \\ &= \widehat{\mathcal{W}}f_0(\mathbf{w}-\mathbf{z})\exp(\imath\Im(\bar{\mathbf{z}}\cdot\mathbf{w}) \\ &= \exp(\frac{1}{2}(-\left|\mathbf{w}-\mathbf{z}\right|^2 + \mathbf{w}\cdot\bar{\mathbf{z}} - \mathbf{z}\cdot\bar{\mathbf{w}})) \\ &= \exp(\frac{1}{2}(-\left|\mathbf{z}\right|^2 - \left|\mathbf{w}\right|^2) + \mathbf{w}\cdot\bar{\mathbf{z}}).\end{aligned}$$

To obtain the reproducing kernel for functions $\breve{f}(\mathbf{z}) = e^{|\mathbf{z}|^2}\widehat{\mathcal{W}}f(\mathbf{z})$ in the Segal-Bargmann space we multiply $K(\mathbf{z}, \mathbf{w})$ by $e^{(-|\mathbf{z}|^2+|\mathbf{w}|^2)/2}$ which gives the standard reproducing kernel, $\exp(-\left|\mathbf{z}\right|^2 + \mathbf{w}\cdot\bar{\mathbf{z}})$ [2, (1.10)].

The Segal-Bargmann space is an interesting and important object, but there are also other options. In particular we can consider an alternative representation of the Heisenberg group, this time acting on monogenic functions, an action we introduce in the next subparagraph.

### 7.3.3 Representation of $\mathbb{H}^n$ in Spaces of Monogenic Functions

We consider the real Clifford algebra $\mathcal{C}\ell(n)$, i.e. the algebra generated by $e_0 = 1$, $e_j$, $1 \leq j \leq n$, using the identities:

$$e_i e_j + e_j e_i = -2\delta_{ij}, \qquad 1 \leq i, j \leq n.$$

For a function $f$ with values in $\mathcal{C}\ell(n)$, the action of the Dirac operator of $\mathbb{R}^{n+1}$ is defined by (here $x = x_0 + \mathbf{x}$ is the $n+1$ dimensional variable)

$$Df(x) = \sum_{i=0}^{n} \partial_i f(x).$$

A function $f$ satisfying $Df = 0$ in a certain domain is called monogenic there; later on we shall use the term ‘monogenic’ for solutions of more general Dirac operators. Obviously the notion of monogenicity is closely related to the one of holomorphy on the complex plane. As a matter of fact $D^2 = -\Delta$, and monogenic functions are solutions of the Laplacian. The Clifford algebra is not commutative, and so it is necessary to introduce a symmetrized product. For $k$ elements $a_i$, $1 \leq i \leq k$ of the algebra it is defined by

$$a_1 \times a_2 \times \ldots \times a_k = \frac{1}{k!} \sum_{\sigma} a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)},$$

where the sum is taken over all possible permutations of $k$ elements. If the same element appears several times, we use an exponent notation, e.g. $a^2 \times b^3 = a \times a \times b \times b \times b$.

Let now $V_k$ be the symmetric power monomial defined by the expression

$$V_k(\mathbf{x}) = \frac{1}{\sqrt{k!}} (e_1 x_0 - e_0 x_1)^{k_1} \times (e_2 x_0 - e_0 x_2)^{k_2} \times \cdots \times (e_n x_0 - e_0 x_n)^{k_n}. \quad (7.3.15)$$

It can be proved that these monomials are all monogenic (see e.g. [25]), and even that they constitute a basis for the space of monogenic polynomials (as a module over $\mathcal{C}\ell(n)$). In general the symmetrized product is not associative, and manipulating it can become quite formal. However, if we restrict the monomials defined above to the hyperplane $x_0 = 0$, we obtain

$$V_k(x) = \frac{1}{\sqrt{k!}} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n},$$

and so we have the multiplicative property

$$\sqrt{\frac{k!k'!}{(k+k)!}} V_k V_{k'} = V_{k+k'}, \quad x_0 = 0.$$

Another important function is the monogenic exponential function which is defined by

$$E(\mathbf{u}, x) = \exp(\mathbf{u} \cdot \mathbf{x}) \left( \cos(\|\mathbf{u}\| \, x_0) - \frac{\mathbf{u}}{\|\mathbf{u}\|} \sin(\mathbf{u} x_0) \right).$$

It is not hard to check [5, § 14] that this function is monogenic, and of course its restriction to the hyperplane $x_0 = 0$ is simply the exponential function, $E(\mathbf{u}, \mathbf{x}) = \exp(\mathbf{u} \cdot \mathbf{x})$.

We can therefore extend the symmetric product by the so-called Cauchy-Kovalevskaya product [5, § 14]: If $f$ and $g$ are monogenic in $\mathbb{R}^{n+1}$, then $f \times g$ is the monogenic function equal to $fg$ on $\mathbb{R}^n$. Introducing the monogenic functions $\mathbf{x}_i = e_i x_0 - e_0 x_i$ we can then write

$$V_k(x) = \frac{1}{\sqrt{k!}} x_1^{k_1} \times x_2^{k_2} \times \dots \times x_n^{k_n}.$$

It is fairly easy to check the $V_k$ form an orthonormal set with respect to the following inner product (see [6, § 3.1] on Clifford valued inner products):

$$\langle V_k, V_{k'} \rangle = \int_{\mathbb{R}^{n+1}} \bar{V}_k(x) V_{k'}(x) \, e^{-|x|^2} \, dx. \tag{7.3.16}$$

Let $M_2$ be closure of the linear span of $\{V_k\}$, using complex coefficients.

The creation and annihilation operators $a_k^+$ and $a_k^-$ can be represented by symmetric multiplication (see [25]) with the monogenic variable $\mathbf{x}_j$, which will be written $\mathbf{x}_k I_\times$, and by the (classical) partial derivative $\frac{\partial}{\partial \mathbf{x}_j} = \frac{\partial}{\partial x_j}$ with respect to $\mathbf{x}_j$, which appear in the definition of hypercomplex differentiability. On basis elements they act as follows:

$$\begin{aligned} \mathbf{x}_j I_\times V_{(k_1, \dots, k_j, \dots, k_n)} &= \sqrt{k_j + 1} V_{(k_1, \dots, k_j + 1, \dots, k_n)}, \\ \frac{\partial}{\partial \mathbf{x}_j} V_{(k_1, \dots, k_j, \dots, k_n)} &= \sqrt{k_j} V_{(k_1, \dots, k_j - 1, \dots, k_n)}, \end{aligned}$$

It can be checked that this really is a representation of $a_k^\pm$, and that $a_k^+$ and $a_k^-$ are each other's adjoint. We use the equalities $a_j^- = \frac{1}{\sqrt{2}}(a_j^+ + a_j^-)$ and $a_j^+ = \frac{\imath}{\sqrt{2}}(a_j^- - a_j^+)$, and the commutation relations $[a_i^+, a_j^-] = e\delta_{ij}$ to obtain a representation of the Heisenberg group. Thus an element $(t, \mathbf{z})$, $\mathbf{z} = \mathbf{u} + \imath\mathbf{v}$ of the Heisenberg group can be written as

$$\begin{aligned}(t,\mathbf{z}) &= \left(t + \frac{\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}}{4}, 0\right)\left(0, \frac{(1+\imath)(\mathbf{u}+\mathbf{v})}{2}\right)\left(0, \frac{(1-\imath)(\mathbf{u}-\mathbf{v})}{2}\right)\\ &= \exp\left(\left(t + \frac{\mathbf{u}^2 - \mathbf{v}^2}{4}\right)e\right)\exp\left(\frac{(\mathbf{u}+\mathbf{v})q}{\sqrt{2}}\right)\exp\left(\frac{(\mathbf{u}-\mathbf{v})\imath p}{\sqrt{2}}\right).\end{aligned}$$

It is therefore represented by the operator

$$\begin{aligned}\pi_{(t,\mathbf{z})} &= \exp\left(-\left(t + \frac{\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}}{4}\right)\right)\\ &\quad\exp\left(\frac{((\mathbf{u}+\mathbf{v})\cdot\mathbf{x})I_\times}{\sqrt{2}}\right)\exp\left(\frac{(\mathbf{u}-\mathbf{v})\cdot(\partial_\mathbf{x})}{\sqrt{2}}\right), \qquad (7.3.17)\end{aligned}$$

where obviously for a monogenic function $f$ we have

$$\begin{aligned}\exp\left(\frac{(\mathbf{u}-\mathbf{v})\imath p}{\sqrt{2}}\right)f(x) &= f\left(x + \frac{\mathbf{u}-\mathbf{v})}{\sqrt{2}}\right)\\ \exp\left(\frac{((\mathbf{u}+\mathbf{v})\cdot\mathbf{x})I_\times}{\sqrt{2}}\right)f(x) &= E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}}, \cdot\right) \times f(x)\end{aligned}$$

Therefore it is easy to calculate the image of the constant function $f_0(\mathbf{x}) = V_0(\mathbf{x}) \equiv 1$, and we obtain the set of functions

$$\begin{aligned}f_{(t,\mathbf{z})}(\mathbf{x}) &= \pi_{(t,\mathbf{z})}f_0(\mathbf{x})\\ &= \exp\left(-\left(t + \frac{\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}}{4}\right)\right)E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}}, \cdot\right) \times f_0(x)\\ &\quad\exp\left(-\left(t + \frac{\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}}{4}\right)\right)E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}}, x\right). \qquad (7.3.18)\end{aligned}$$

In the language of quantum physics $f_0(\mathbf{x})$ is the *vacuum vector* and functions $f_{(t,\mathbf{z})}(\mathbf{x})$ are *coherent states* (or *wavelets*) for the representation of $\mathbb{H}^n$ we described. We can summarize the properties of the representation:

(i). All functions in $M_2$ are complex-vector valued, monogenic in $\mathbb{R}^{n+1}$, and square integrable with respect to the measure $e^{-|x|^2}dx$.

(ii). The representation of the Heisenberg group is given by (7.3.17). This representation generates a set of coherent states $f_{(0,z)}(\mathbf{x})$ (7.3.18) as shifts of the vacuum vector $f_0(\mathbf{x}) \equiv 1$.

(iii). The creation and annihilation operators $a_k^+$ and $a_k^-$ are represented by symmetric (Cauchy-Kovalevskaya) multiplication by $\mathbf{x}_j$ and by derivation of monogenic functions. They are adjoint with respect to the inner product (7.3.16).

(iv). $M_2$ is generated as a closed linear space by the orthonormal basis $V_k(\mathbf{x}) = \frac{1}{\sqrt{k!}}(e_1x_0 - e_0x_1)^{k_1} \times (e_2x_0 - e_0x_2)^{k_2} \times \cdots \times (e_nx_0 - e_0x_n)^{k_n}$, and also by the set of coherent states $f_{(t,\mathbf{z})}(\mathbf{x})$ of (7.3.18).

(v). The kernel of the operator intertwining the model constructed here and the Segal-Bargmann one is given by

$$B(\mathbf{z},x)\sum_{j=0}^{\infty} V_j(x)\frac{\mathbf{z}^j}{\sqrt{j!}} = \exp(\sum_{k=1}^{n}\mathbf{x}_k\bar{z}_k),$$

which is the holomorphic extension in $\mathbf{z} = \mathbf{u} + \imath\mathbf{v}$ of $E(\mathbf{u},x)$. The transformation pair is given by

$$\begin{aligned}\mathcal{B}f(x) &= \int_{\mathbb{C}^n} B(\mathbf{z},x)f(\mathbf{z})\exp\left(\frac{-|\mathbf{z}|^2}{2}\right)d\mathbf{z}\\ \mathcal{B}^{-1}\phi(\mathbf{z}) &= \int_{\mathbb{R}^{n+1}} \overline{B(\mathbf{z},x)}\phi(x)\exp\left(\frac{-|x|^2}{2}\right)dx\end{aligned}$$

(vi). The space $M_2$ has a reproducing kernel

$$K(x,y) = \sum_{k=0}^{\infty} V_k(x)\bar{V}_k(y) = \int_{\mathbb{C}^n} B(\mathbf{z},x)\overline{B(\mathbf{z},y)}\,e^{-|z|^2}dz.$$

Notice that $\overline{K(x,y)}$ is monogenic in $y$; it is the monogenic extension of $\overline{E(\mathbf{y},x)}$.

One can see that some properties of $M_2$ are closer to those of the Segal-Bargmann space than to those of the space $L_2(\mathbb{R}^n)$ it replaces. It should be noted that the representation of the Heisenberg group we obtained here is new and quite unexpected.

REMARK 7.3.2 We construct $M_2$ as a space of complex-vector valued functions. We can also consider an extended space $\widetilde{M}_2$ being generated by the orthonormal basis $V_k(\mathbf{x})$ or coherent states $f_{(0,z)}(\mathbf{x})$ with Clifford valued coefficients multiplied from the right hand side. Such a space will share many properties of $\mathbb{M}^2$ and have an additional structure: there is a natural representation $s : f(\mathbf{x}) \mapsto s^*f(s\mathbf{x}s^*)s$ of $\mathrm{Spin}\,(n)$ group in $\widetilde{M}_2$. Thus this space provides us with a representation of two main symmetries in quantum field theory: the Heisenberg group of quantized coordinate and momentum (external degrees of freedom) and Spin$\,(n)$ group of quantified inner degrees of freedom. Another composition of the Heisenberg group and Clifford algebras can be found in [19].

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