

Math 1060 Linear Algebra Homework Exercises 1

1. Find the complete solutions (if any!) to each of the following systems of simultaneous equations:

$$(i) \quad \begin{aligned} x - 4y + 3z &= 2 \\ 3x - 11y + 13z &= 3 \\ 2x - 9y + 2z &= 7 \\ x - 2y + 6z &= -2 \end{aligned}$$

$$(ii) \quad \begin{aligned} x - 4y + 3z &= 2 \\ 3x - 11y + 13z &= 3 \\ 2x - 9y + 2z &= 7 \\ x - 2y + 11z &= -4 \end{aligned}$$

$$(iii) \quad \begin{aligned} x - 4y + 3z &= 2 \\ 3x - 11y + 13z &= 3 \\ 2x - 9y + 2z &= 7 \\ x - 2y + 11z &= -2 \end{aligned}$$

$$(iv) \quad \begin{aligned} -4x + 3y + z &= 2 \\ -11x + 13y + 3z &= 3 \\ -9x + 2y + 2z &= 7 \\ -2x + 6y + z &= -2 \end{aligned}$$

2. Solve the linear system

$$\begin{aligned} \frac{1}{29}x + \frac{1}{42}y &= 1 \\ \frac{1}{67}x + \frac{1}{97}y &= 1 \end{aligned}$$

3. Solve the system

$$\begin{aligned} -7x + 11y + 2z &= -7 \\ 12x + 18y + 2z &= -20 \\ -13x + 2y + z &= 3 \end{aligned}$$

Do you *have* to eliminate x first?

4. (i) Find the general solution of:

$$\begin{aligned} 3x + 2y - 2z + 2u &= 1 \\ 2x + 3y - z + u &= -1 \\ 5x - 4z + 4u &= 5 \end{aligned}$$

(ii) Is there a particular solution with $x = -7$, $u = 100$?

5. Apply elementary row operations to change the following matrices first to echelon form and then to *reduced* echelon form

$$(i) \quad \begin{pmatrix} 1 & 2 & 2 & -1 \\ 1 & 2 & 0 & 3 \\ 2 & 4 & 6 & -6 \end{pmatrix} \quad (ii) \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

6. Suppose that

$$\begin{aligned} x + 2y + 3z &= a \\ 2x + 3y + 4z &= b \\ 3x + 4y + kz &= c, \quad \text{where } k, a, b, c \text{ are constant.} \end{aligned}$$

Show that these equations are consistent for all values of k except, perhaps, $k = 5$ and find the relation between a , b , and c if the equations *are* consistent when $k = 5$.

7. Find the value(s) of k (if any) for which the linear system

$$\begin{aligned} x + y + kz &= 1 \\ x + ky + z &= 1 \\ kx + y + z &= 1 \end{aligned}$$

has (i) a unique solution; (ii) no solution; (iii) infinitely many solutions. Write down the complete solution in cases (i) and (iii).

Math 1060 Linear Algebra Homework Exercises 2

1. Solve for x, y, z and w : $\begin{pmatrix} x & y \\ z & w \end{pmatrix} + \begin{pmatrix} z & 1 \\ y & x \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 1 & 7 & 2 \\ 0 & -5 & 3 \end{pmatrix}$, $B = \begin{pmatrix} a & X \\ b & Y \\ c & Z \end{pmatrix}$, $C = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 2 & 2 \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$, $D = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

Find, if possible, $AB, BA, BD, DB, CD, DC, C^{10}, (AB)C, A(BC)$. Are $(AB)^2$ and A^2B^2 equal?

3. Give, if possible, examples of 2×2 matrices $A, B, C, D, E, F, G, H, J$ such that:

(i) $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $AB = AC$ and yet $B \neq C$. (This means that cancellation by non-zero matrices is, unlike the case with non-zero numbers, not always possible).

(ii) $D \neq 0_{2 \times 2}$ and $E \neq 0_{2 \times 2}$ and yet $DE = 0_{2 \times 2}$. (Again, unlike real numbers: If two numbers a and b have product 0 then at least one of a and b *must* be 0.)

(iii) $(FG)^2 \neq F^2G^2$.

(iv) $HJ \neq JH$ and yet $(HJ)^2$ is *unequal* to H^2J^2 .

4. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$. Find (using only pencil and paper) the matrix A^8 .

Hint: $A^8 = ((A^2)^2)^2$. It's nice to get A^8 using only three multiplications!

5. Let A and B be matrices, of sizes $m \times n$ and $n \times r$ respectively. If one *column* of B contains *only zeros*, what can be said about the matrix AB ? If one *row* of B contains *only zeros*, what can be said about the matrix AB ?

6. Find the inverse, if it exists, for each of the following matrices:

(i) $\begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix}$; (ii) $\begin{pmatrix} 5 & -3 \\ 2 & -4 \end{pmatrix}$; (iii) $\begin{pmatrix} 12 & 28 \\ 27 & 63 \end{pmatrix}$; (iv) $\begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}$.

7. Find the inverses, if they exist, of:

(i) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$; (ii) $\begin{pmatrix} 1 & 3 & 5 \\ 0 & 7 & 9 \\ 0 & 0 & 11 \end{pmatrix}$; (iii) $\begin{pmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{pmatrix}$; (iv) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

8. (i) Let A and P be 2×2 matrices, P being invertible. By writing the product out in full and then cancelling neighbouring P and P^{-1} show that $(P^{-1}AP)^3 = P^{-1}A^3P$. In fact one can prove, by mathematical induction, that, for all positive integers n , $(P^{-1}AP)^n = P^{-1}A^nP$.

(ii) Given that

$$\begin{pmatrix} -34 & -63 \\ 20 & 37 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 5 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 5 & 9 \end{pmatrix}$$

find (by hand, not by computer) $\begin{pmatrix} -34 & -63 \\ 20 & 37 \end{pmatrix}^{10}$.

9. Use exercise 7(iii) above to solve the system of equations:

$$\begin{aligned} 3x + 6y + 7z &= 1 \\ 2y + z &= 2 \\ 2x + 3y + 4z &= -3 \end{aligned}$$

Hint: from $Ax = b$ obtain $x = A^{-1}b$.

Math 1060 Linear Algebra Homework Exercises 3

1. Evaluate the determinant $\begin{vmatrix} 19 & -11 \\ 46 & 79 \end{vmatrix}$.

2. (i) Evaluate the determinant $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{vmatrix}$;

(ii) Find the value of x for which $\begin{vmatrix} 1 & 2 & 3 \\ 8 & x & 4 \\ 7 & 6 & 5 \end{vmatrix} = 0$.

3. Find x such that (i) $\begin{vmatrix} 1-x & -2 \\ 3 & 4-x \end{vmatrix} = 0$; (ii) $\begin{vmatrix} 1-x & 3 & 5 \\ 2 & 4-x & 6 \\ 3 & 5 & 7-x \end{vmatrix} = 0$.

4. Evaluate the determinants (i) $\begin{vmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$; (ii) $\begin{vmatrix} 0 & b & c & d \\ a & 0 & c & d \\ a & b & 0 & d \\ a & b & c & 0 \end{vmatrix}$;

(iii) $\begin{vmatrix} 0 & 0 & 0 & 0 & 4 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \end{vmatrix}$; (iv) $\begin{vmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{vmatrix}$.

5. (i) Arrange seven 1s and two 0s to form a 3×3 matrix with non-zero determinant.

(ii) A 4×4 matrix has 14 entries equal to 1, the remaining two entries are 2 and 3. Find value of the determinant of this matrix.

6. Prove that the determinant of an elementary matrix is non-zero.

7. Write the so-called **Vandermonde determinant** $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ as a product of linear factors (i.e. factors of degree 1).

8. (i) We know that there are $n \times n$ matrices A, B such that $AB \neq BA$. Are there matrices A, B such that $\det(AB) \neq \det(BA)$?

(ii) Find two non-zero 2×2 matrices A and B such that $\det(A+B) = \det A + \det B$.

Math 1060 Linear Algebra Homework Exercises 4

1. For each of the following, write down its additive inverse (see axioms 4.2iv and 4.2v in lecture notes):

(i) $(e^2, 0, -2\pi, -3, 5)$ in \mathbb{R}^5 ;

(ii) $\begin{pmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{pmatrix}$ in $M_{3,2}$.

2. State, without proof, whether or not the following sets are real vector spaces (where addition and (scalar) multiplication are the usual ones).

For those which are *not* vector spaces give the first of the ten axioms in the axioms list which is not satisfied.

(Later a theorem on subspaces will show how the somewhat tedious checking of all ten axioms — in those cases which *are* vector spaces — can frequently be avoided.)

(i) The subset $\{(x, y, z) : 2x + y - 3z + 2 = 0\}$ of \mathbb{R}^3 .

(ii) The subset $\{(x, y, z) : 2x + y - 3z = 0\}$ of \mathbb{R}^3 .

(iii) The set of all skew symmetric matrices in $M_2(\mathbb{R})$.

A (square) matrix is *skew symmetric* if $a_{i,j} = -a_{j,i}$ for all i, j .

(iv) The set of all quadratic polynomials of the form $t^2 + at + b$ in t with real coefficients a and b (together with the zero polynomial).

(v) The subset of $M_{2,3}$ comprising the *single element* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3. On \mathbb{R}^2 define \oplus as usual but $\lambda \circ (x, y)$ to be (x, y) for any $\lambda \in \mathbb{R}$.

By choosing particular value for λ , x , and y show that, with this definition of addition and scalar multiplication, \mathbb{R}^2 does not satisfy axiom 4.3iii.

In fact this version of \mathbb{R}^2 satisfies all Axioms of vector spaces except 4.3iii. So we would have to include this silly example in our list of vectors spaces if it were not for axiom 4.3iii.

4. In \mathbb{R}^2 define a “new” addition, namely

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1, x_2 + y_2).$$

Does \mathbb{R}^2 form a vector space with respect to this new addition and usual (scalar) multiplication?

5. Prove, directly from the vector space axioms that:

If λ and \mathbf{v} are such that $\lambda \circ \mathbf{v} = \mathbf{0}$, then either $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$ (or both!)

Hint: If $\lambda \neq 0$ then

$$\mathbf{v} = 1 \circ \mathbf{v} = (\lambda^{-1}\lambda) \circ \mathbf{v} = \lambda^{-1} \circ (\lambda \circ \mathbf{v}) = \lambda^{-1} \circ (\mathbf{0}) = \dots$$

Finish up the proof and give the explanation for the *each above* transformation.

Math 1060 Linear Algebra Homework Exercises 5

1. For each of the following either verify that the given set W is a subspace of \mathbb{R}^3 or explain why it is not:
 - (i) $W = \{(x, y, z) : y + z \geq 0\}$;
 - (ii) $W = \{(x, y, z) : 3x + y - 2z = 0\}$;
 - (iii) $W = \{(x, y, z) : 3x + y - 2z = 1\}$;
 - (iv) $W = \{(x, y, z) : xy + yz + zx = 0\}$;
 - (v) $W = \{(0, 0, 0)\}$;
 - (vi) $W = \{(x, y, z) : x^2 + y^2 + z^2 = 0\}$.
2. Let P and Q be subspaces of the vector space \mathbb{R}^7 and let $T = P \cap Q$ be their *intersection*. (That is T is the set of all 7-tuples which lie in both P and Q .) Show that T is a subspace of \mathbb{R}^7 . What significance has the number 7?
3. Express $(4, 9, 6)$ as a linear combination of the three vectors $(1, 2, 3)$, $(2, 4, 1)$, $(0, 1, -1)$ and also as a linear combination of the three vectors $(1, 2, 3)$, $(2, 4, 1)$, $(0, 1, 1)$.
4. Show that *every* vector in \mathbb{R}^4 is a linear combination of the four vectors $(1, 2, 3, 4)$, $(0, 1, 2, 3)$, $(0, 0, 1, 2)$ and $(0, 0, 0, 1)$.
5. Is $\begin{pmatrix} 2 & 1 & -1 \\ 3 & -1 & -4 \\ 1 & -2 & -3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$?
6. Show directly from the definition that the set $\{(2, 3, 1, 4), (1, -2, 0, 2), (3, 0, 1, 5)\}$ of vectors is a linearly independent set of vectors in \mathbb{R}^4 .
7. Are the following sets of vectors linearly dependent or are they linearly independent?
 - (i) In \mathbb{R}^3 : $\{(0, 2, 2), (1, 0, 2), (1, 1, 0)\}$;
 - (ii) In \mathbb{R}^3 : $\{(0, -2, 2), (1, 0, -2), (-1, 2, 0)\}$;
 - (iii) In \mathbb{R}^5 : $\{(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 0), (1, 1, 0, 0, 0), (1, 0, 0, 0, 0)\}$.
8. For which scalars k is $\{(1, k, 0), (k, 0, 1), (0, 1, k)\}$ a linearly independent set in \mathbb{R}^3 ?
9. Is the subset $\left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ of $M_3(\mathbb{R})$ linearly independent?
10.
 - (i) Show that the vectors $\mathbf{v}_1 = (2, 3, 1, 4)$, $\mathbf{v}_2 = (1, -2, 0, 2)$ and $\mathbf{v}_3 = (3, 0, 1, 5)$ form a linearly independent set in \mathbb{R}^4 .
 - (ii) Find a fourth vector $\mathbf{v}_4 = (a, b, c, d)$ in \mathbb{R}^4 such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set in \mathbb{R}^4 .
11. Show, by putting the vectors into a matrix and then using Theorem 4.21, that at least one of the following vectors is a linear combination of its predecessors: $(1, 2, 1, 2, 3)$, $(4, 6, 3, 10, 12)$, $(2, 0, 1, 7, 4)$, $(1, 4, 1, 1, 5)$.
12.
 - (i) Show, by putting the vectors into a matrix, that the set $\{(1, 3, -1, 0, 2), (1, 5, -2, 1, 2), (1, 7, -2, 4, 3)\}$ is a linearly independent set in \mathbb{R}^5 .
 - (ii) Write down two vectors in \mathbb{R}^5 which, together with the given three, span \mathbb{R}^5 .

Math 1060 Linear Algebra Homework Exercises 6

1. Show that the sets

$$S = \{(1, 2, 1, -1, 2), (2, 6, 4, -2, 5), (3, 2, -1, -2, 7)\}$$

$$T = \{(1, -4, -5, -1, -1), (1, 8, 7, -1, 5), (7, -4, -11, -7, 5), (2, 6, 4, -5, -4)\}$$

of vectors span the same subspace of \mathbb{R}^5 .

Hint: Change each echelon matrix you obtain into row reduced echelon form.

2. By reducing the matrix $A = \begin{pmatrix} 1 & 4 & 1 & -1 & 4 \\ 2 & 2 & 2 & 1 & -1 \\ 0 & 2 & 0 & 0 & 4 \\ 1 & 6 & 1 & 1 & 0 \end{pmatrix}$ to echelon form show that $(1, 4, 2, 0, 7)$ is *not* in the row space of the matrix A .

3. (i) Find a linearly independent set of four vectors in the *column space* of the matrix A in the previous exercise.

(ii) Can you find a linearly independent set of *five* vectors in that column space?

4. (i) Find a basis for the row space of the matrix $A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ -2 & 1 & 3 & 4 \\ 1 & 1 & 9 & -2 \\ -2 & 0 & -4 & 5 \\ -1 & 1 & 5 & 3 \end{pmatrix}$.

(ii) Extend this basis to a basis for \mathbb{R}^4 .

5. Are the following bases for \mathbb{R}^4 ?

(i) $\{(1, 2, 3, 4), (2, 3, 4, 5), (3, 4, 5, 1), (4, 5, 1, 2), (5, 1, 2, 3)\}$?

(ii) $\{(501, 499, 452, 443), (10, 18, 20, 26), (1018, 887, 830, 819)\}$?

(iii) $\{(12, 23, 34), (22, 33, 41), (21, 32, 43), (22, 33, 14)\}$?

6. (i) Find a basis for the solution space of the system of equations $Ax = \mathbf{0}$ where $x = (x, y, z, w, t)^T$,

$$A = \begin{pmatrix} 1 & 1 & 9 & 1 & 1 \\ 2 & -1 & -3 & -4 & 2 \\ 2 & 0 & 4 & -1 & 1 \end{pmatrix}. \text{ Hence determine the dimension of this solution space.}$$

(ii) Determine whether or not the vectors $(-34, 1, 17, -60, -60)^T$ and $(1, 2, 3, 4, 5)^T$ are in this solution space.

(iii) Is there a solution with $x = 100$, $y = 99$ and with z, w and t also integers?

7. What is the dimension of the subspace

$$\text{sp}\{(1, 3, 1, -2), (2, 5, 1, -7), (3, 11, 5, 0), (2, 4, 0, 1), (4, 13, 5, -5)\} \text{ of } \mathbb{R}^4?$$

8. Is $\left\{ \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}, \begin{pmatrix} 8 & 13 \\ 13 & 21 \end{pmatrix} \right\}$ a basis for $M_2(\mathbb{R})$?

9. Is the set $\{1 + 3x - 2x^2 + x^3, 2 + 5x - 5x^2 + x^3, 1 + 4x - 2x^2 + 2x^3, 2 + 4x - 5x^2 + x^3\}$ a basis for the vector space of all real polynomials of degree at most 3?

10. For which α , if any, is $\text{sp}\{(1, 1, 1, \alpha), (1, 1, \alpha, \alpha), (1, \alpha, \alpha, \alpha), (\alpha, \alpha, \alpha, \alpha)\}$

(i) a two dimensional subspace of \mathbb{R}^4 ;

(ii) a three dimensional subspace of \mathbb{R}^4 ?

11. Find the row rank and the column rank of the matrix $\begin{pmatrix} -1 & 7 & -6 & 8 & -7 & -5 & 7 \\ 1 & 3 & -2 & 4 & -1 & -1 & 3 \\ 2 & 1 & 0 & 2 & 2 & 1 & 1 \\ 2 & 8 & -8 & 9 & -6 & 5 & 8 \end{pmatrix}$

Math 1060 Linear Algebra Homework Exercises 7

1. (i) Check that $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 10 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -4 \end{pmatrix}$ are eigenvectors for $\begin{pmatrix} -9 & -28 \\ 8 & 21 \end{pmatrix}$ and find the corresponding eigenvalues.
 (ii) Is $\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 10 \\ -5 \end{pmatrix}$ an eigenvector for $\begin{pmatrix} -9 & -28 \\ 8 & 21 \end{pmatrix}$? What about $\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \end{pmatrix}$?

2. For each of the following matrices A :

- Find the eigenvalues;
- Find, where possible, an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ and
- A^n , the elements of A^n are to be expressed in terms of n^{th} powers of the eigenvalues of A .
- For fun, find, for the first two matrices, the least power of A which contains an entry greater than 1000 or less than -1000 .

(i) $\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$; (ii) $\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$; (iii) $\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix}$; (iv) $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$.

3. (i) You are told that λ is an eigenvalue for the $n \times n$ matrix A . Show that λ^4 is an eigenvalue for the matrix A^4 .
 (ii) If $A^4 = 9A^2$ what are the possible eigenvalues of A ?

4. Given the simultaneous equations

$$\begin{aligned} \frac{dx}{dt} &= -x + 2y \\ \frac{dy}{dt} &= \frac{1}{2}x - y \end{aligned}$$

and given that, $x = 600$ and $y = 1300$ when $t = 0$ determine what happens to the sizes of $x(t)$ and $y(t)$ as t increases. (This is a so-called *symbiotic* (cooperative) system. If x and y represent the populations of two species at time t it will be found that the species can live side by side with neither species dying out.)

5. Find the general solution of the following system of differential equations.

$$\begin{aligned} \dot{x} &= -x + 2y + 2z \\ \dot{y} &= 2x + 2y + 2z \\ \dot{z} &= -3x - 6y - 6z. \end{aligned}$$

Given that $x = y = z = 0$ when $t = 0$ find the three functions $x(t)$, $y(t)$ and $z(t)$ which solve this system of equations.

6. At the end of year n the number r_n of rats and c_n of cats in a certain area is given, in terms of the numbers in the year $n - 1$, by the matrix formula

$$\begin{pmatrix} r_n \\ c_n \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_{n-1} \\ c_{n-1} \end{pmatrix}.$$

- By diagonalizing $\begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix}$ determine a formula for r_n and c_n in terms of r_0 and c_0 .
- Given that $r_0 = 1000$ and $c_0 = 300$ determine whether or not the rat population is killed off. (This kind of equation is of the *predator-prey* type.)