Exercise

1. Without using a calculator evaluate:

(a) $6+(-3)$ (b) $6-(-3)$ (c) $16+(-5)$ (d) $-16-(-5)$

(e) $27-(-3)$ (f) $27-(-29)$ (g) $-16+3$ (h) $-16+(-3)$

(i) $-23+52$ (j) $-23+(-52)$

(k) $3\times(-8)$ (l) $(-4)\times 8$ (m) $15\times(-2)$ (n) $(-2)\times(-8)$

(o) $\frac{15}{-3}$ (p) $\frac{21}{7}$ (r) $\frac{-21}{7}$ (s) $\frac{21}{-7}$ (t) $\frac{-12}{2}$ (u) $\frac{-12}{-2}$

2. Evaluate the following expressions:

(a) $6-2\times 2$ (b) $(6-2)\times 2$ (c) $6-(2+3)\times 2$ (d) $6-2+3\times 2$

(e) $6\div 2-2$ (f) $(6\div 2)-2$ (g) $(6-2)+3\times 2$ (h) $-6\times(-2)\times(-3)$

3. Place brackets in the following expressions to make them correct:

(a) $6\times 12-3+1=55$ (b) $6\times 12-3+1=68$ (c) $6\times 12-3+1=60$

(d) $5\times 4-3+2=7$ (e) $5\times 4-3+2=15$ (f) $5\times 4-3+2=-5$

4. Evaluate:

(a) $6\div 2+1$ (b) $6\div(2+1)$ (e) $6-2+4\div 2$ (f) $6-(2+4)\div 2$

(c) $12+4\div 4$ (d) $(12+4)\div 4$ (g) $2\times 6\div(3-1)$ (h) $2\times(6\div 3-1)$

Exercise

In the following calculations digits are replaced by letters. Within each calculation any digit is always replaced by the same letter, no two different digits are covered by the same letter.

Recover original calculations:

1.
$$\begin{array}{r} \texttt{THREE} \\ +\ \texttt{FOUR} \\ \hline \texttt{SEVEN} \end{array} \qquad \begin{array}{r} \texttt{SEVEN} \\ +\ \texttt{SEVEN} \\ \hline \texttt{TWENTY} \end{array} \qquad \begin{array}{r} \texttt{NINE} \\ -\ \texttt{FOUR} \\ \hline \texttt{FIVE} \end{array}$$

2.
$$\begin{array}{r} \texttt{FIVE} \\ -\ \texttt{FOUR} \\ \hline \texttt{ONE} \\ +\ \texttt{ONE} \\ \hline \texttt{TWO} \end{array}$$

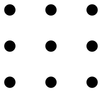
3. $\texttt{TWO} \times \texttt{TWO} = \texttt{THREE} \qquad \texttt{TOC} \times \texttt{TOC} = \texttt{ENTRE}$

Exercise

1. There are some chickens and rabbits on a farm. In total they have 18 heads and 58 legs.
   How many chickens and how many rabbits on the farm?
2. A car left Leeds at 9:00 and arrived to London at 14:15 on the same day. Assume that the distance is 320 kilometres.
   What was the car's speed? What was the car's average speed?
3. A car left Leeds towards London at 9:00 and drives with a constant speed 105 km/hour. Another car left London to Leeds twenty minutes later with a constant speed 95 km/hour.
   What time cars will meet each other? Where is the meeting point?
4. There are 13 pupils in a class. Boys have as many teeth as girls have fingers (a child has 32 teeth and 20 fingers).
   How many boys and how many girls are in the class?

Exercise

1. Nine points are arranged into a square. Cross all nine points by a *single* stroke made out of *only four* segments of *straight* lines.



2. The distance between Leeds and Bradford is 18 km. At 9am two students started walk along the same road but in the opposite directions: Lionel walked from Leeds to Bradford at a constant speed 4 km/h and Brian went from Bradford to Leeds with the constant speed 5 km/h.
   At the same time (9am) a drake start to fly from Lionel to Brian with the constant speed 10 km/h. As soon as it reached Brian it turned around and flied to Lionel (with the same speed), when it reached Lionel it turned to Brian and so on till Lionel met Brian on the road.
   Which distance the drake flied?

Exercise

1. Let two lines are given by equations

$$a_1x + b_1y + c_1 = 0 \qquad \text{and} \qquad a_2x + b_2y + c_2 = 0 \tag{1}$$

Show that they are *parallel* if and only if:

$$a_1b_2 - a_2b_1 = 0. \tag{2}$$

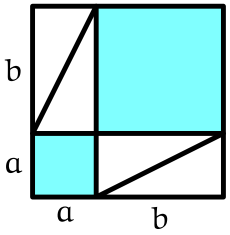
2. Let two lines are given by the above equations (1). Show that they are *perpendicular* if and only if:

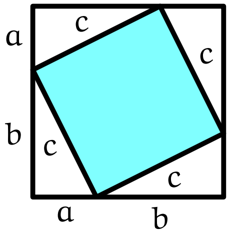
$$a_1a_2 + b_1b_2 = 0. \tag{3}$$

3. Use algebra and equations (2) and (3) to show that *if two lines are perpendicular to a third line then they are parallel to each other*.
4. Let three lines *passing the same point* are given by the equations: $y = m_1x + d_1$, $y = m_2x + d_2$, and $y = m_3x + d_3$. Express d_3 through m_1, m_2, m_3, d_1, and d_2.

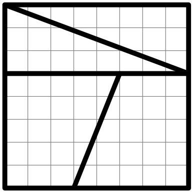
Exercise

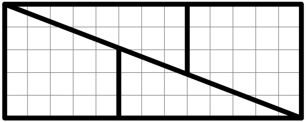
1. The following pictures are usually used for a visual proof of the Pythagoras Theorem. Reconstruct this proof.





2. Count areas of the two rectangles made by an rearrangement of the "same" four pieces. Explain the difference.





Are visual proofs of the Pythagorean theorem convincing you now?

Exercise

1. We know that $3^2 + 4^2 = 5^2$. Check also that $3^3 + 4^3 + 5^3 = 6^3$. Can it be extended further?
2. A *Pythagorean triple* consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$. Show that
   - 2.1 If m and n are two positive integers with $m > n$, then $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$ is a Pythagorean triple.
   - 2.2 Exactly one of a, b is odd; c is odd. The area $A = ab/2$ is an integer.
   - 2.3 Exactly one of a, b is divisible by 3. Exactly one of a, b is divisible by 4. Exactly one of a, b, c is divisible by 5.
   - 2.4 For any Pythagorean triple, ab is divisible by 12, and abc is divisible by 60.
   - 2.5 Exactly one of a, b, $a + b$, $a \cdot b$ is divisible by 7. At most one of a, b is a square.
   - 2.6 Every integer greater than 2 is part of a Pythagorean triple.

Exercise

1. Calculate and spot the pattern of results:

(a)
$$\begin{aligned} 12 \times 9 + 3 &= \\ 123 \times 9 + 4 &= \\ 1234 \times 9 + 5 &= \\ 12345 \times 9 + 6 &= \\ \ldots\ldots & \ldots \end{aligned}$$

(b)
$$\begin{aligned} 9 \times 9 + 7 &= \\ 97 \times 9 + 8 &= \\ 978 \times 9 + 6 &= \\ 9786 \times 9 + 5 &= \\ \ldots\ldots & \ldots \end{aligned}$$

Can you *explain* the pattern?

2. 2.1 Is there a digit such that any number finished by this digits has a square finished by the same digit?

   2.2 Which digit is at the end of cube of the digit described above? Fourth power? Fifth?

   2.3 If a number finished by 25, what are two last digits of its square? Cube?

   2.4 Can you found another two digits which has the same property as 25 from above?

   2.5 Are there triples of digits with the similar property?

3. Write statements "*neither a or b is zero*" and "*either a or b is not zero*" as formulae. [Chose from these: $a + b \neq 0$, $a^2 + b^2 \neq 0$, $ab \neq 0$.]

In the following questions you should *not* use any signs of arithmetic operations, braces, etc.

Example

How to write the biggest possible number using only three digits 9?
Answer: 9^{9^9} (this is really a hu-uuu-ge number!)

Exercise

1. 1.1 How to write the biggest possible number using only three digits 2? (*Hint:* it is bigger than 16).
   1.2 How to write the biggest possible number using only three digits 3?
   1.3 How to write the biggest possible number using only three digits 4?
   1.4 Let a be a *digit*. For which values of a the biggest possible number formed by these three digits is a^{a^a}?
2. 2.1 How to write the biggest possible number using only *four* digits 1?
   2.2 Which number is bigger 222^2, 22^{22} or 2^{222}?
   2.3 How to write the biggest possible number using only *four* digits 2?

Exercise

1. On the last meeting of the *Foundation Year Math Soc* members shacked hands with each other. It was 66 handshakes in total.
   How many member in FYMS?
2. A ball is thrown upward with the speed 25 meters per second. *When it will be* 20 *meters above the initial position?*
3. (L.Euler's problem) Two peasants brought apples to a local market. There were 100 apples in total. By the end of the day peasants sold all their apples and received equal amounts of money. First peasant told to second one: "I could get 15 escudos in total if I had your apples for sale as well". The second peasant replied: "However I would obtain $6\dfrac{2}{3}$ escudos for your apples alone".
   How many apples had each peasant?
4. There are two different loudspeakers: 400 Watts and 900 Watts.
   Where between them is the position to get the proper stereo sound?
5. Check the identity: $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.
   Are their other five consecutive integers having the same property?

Exercise

1. Give an example of two functions $f(x)$ and $g(x)$ such that $f \circ g \neq g \circ f$.
2. Give an example of two functions $f(x)$ and $g(x)$ such that $f \circ g = g \circ f$.
3. A function $f(x)$ is called one-to-one function if an identity $f(x_1) = f(x_2)$ always implies the identity $x_1 = x_2$.
   - 3.1 Give an example of one-to-one function.
   - 3.2 Give an example of function which is not one-to-one.
   - 3.3 Show that the composition of one-to-one functions is again a one-to-one function.
4. Let function $f(x)$ have the property

$$f(x + y) = f(x) \cdot f(y), \qquad \text{for all reals } x, y.$$

Show that its inverse function $g(x)$ has the property:

$$g(x \cdot y) = g(x) + g(y), \qquad \text{for all reals } x, y.$$

Exercise

1. Find the gradient of the functions:

   (a) $y = t^{3/2}$; (b) $y = t^{-5/11}$; (c) $y = \sqrt[3]{t^2}$; (d) $y = \dfrac{t^{2/7} + \sqrt{}}{\sqrt[5]{t^3}}$

2. Find the gradient of the graph of the function $y = f(x)$ at the given point x_0:

   (a) $y = x^2$, $x_0 = 1$; (b) $y = x^{2/3}$, $x_0 = \dfrac{7}{11}$; (c) $y = \sqrt[3]{x}$, $x_0 = 0$

3. Find the equation of the straight line touching the graph of the function $y = f(x)$ at the given point x_0:

   (a) $y = x^2$, $x_0 = 1$; (b) $y = x^{2/3}$, $x_0 = \dfrac{7}{11}$; (c) $y = \sqrt[3]{x}$, $x_0 = 0$

4. A particle moving along a straight line such that its distance S (in meters) from the origin is depend from time t as follows:

$$S = 2t^3 - 21t^2 + 120t + 49.$$

   What is the distance from the reference point to the particle when its

Exercise

1. Giving that $(e^x)' = e^x$ find derivatives of the functions:

   (a) $y = e^{kx}$, (b) $y = 2^x$, (c) $y = a^x$ (use that $e^{\ln a} = a$) (d) $y = x^x$

2. 2.1 From graphs of $\sin x$ and $\cos x$ verify that their slops at $x = 0$ are 1 and 0 correspondingly. Find values of $\sin' 0$ and $\cos' 0$.

   2.2 Using the previous results and trigonometric identities:
   $\sin(a+b) = \sin a \cos b + \cos a \sin b$, $\cos(a+b) = \cos a \cos b - \sin a \sin b$
   find derivatives of the functions $\sin x$ and $\cos x$.

   2.3 Find derivatives of $y = \tan x$ and $y = \cot x$.

3. 3.1 Differentiate the identity $g(x) \cdot \frac{1}{g(x)} = 1$ and show $\left[\frac{1}{g(x)}\right]' = -\frac{g'(x)}{g^2(x)}$.

   3.2 Derive the quotient rule: $\left[\frac{u}{v}\right]' = \frac{u' \cdot v - u \cdot v'}{v^2}$.

4. 4.1 Let $x = g(y)$ be the *inverse function* of $y = f(x)$, i.e. $g(f(x)) = x$ for all x. Using the chain rule show that $g'(f(x)) = \frac{1}{f'(x)}$.

   4.2 Using the above property and the fact that $\ln x$ is the inverse function of e^x show that $(\ln x)' = \frac{1}{x}$.

   4.3 Find derivatives of functions arcsin, arccos and arctan.

Exercise

1. You are given 100 meters of a rope.
    1.1 What is the maximal area of a *rectangular shape* which can be bounded by this rope?
    1.2 The same for *triangular shape*? (Hint: use Heron's formula)
2. You are again given 100 meters of a rope and need to make a maximal area as in the previous exercise, however you can now use a straight river's bank as one side of your area which do not need a rope. What would be answer to both questions above?
3. What would be the maximal area bounded by the rope if you are not limited in the shape?
    3.1 Have you got physical evidence to support the answer?
    3.2 Can you prove it mathematically?
4. You are permitted to make a garden of 100 square meters and wish to chose it shape to minimise the length of fences.
    4.1 What are sides of a rectangular garden? A triangular one?
    4.2 What if one side is river's bank which do not need a fence?
    4.3 Garden of an arbitrary shape?

Exercise

1. Show that:

$$\begin{aligned}
\sin\alpha \pm \sin\beta &= 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)\\
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\cos\alpha - \cos\beta &= 2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)\\
\cos^4\alpha - \sin^4\alpha &= \cos 2\alpha
\end{aligned}$$

2. Express $\sin\alpha \pm \cos\beta$ as a product of two trigonometric functions.
3. Solve the equation for $0^\circ \leq \alpha \leq 360^\circ$:
   - 3.1 $2\tan^2\alpha - 5\sec\alpha - 1 = 0$.
   - 3.2 $\sin 2\alpha = -0.6$
   - 3.3 $\cos(\alpha + 40^\circ) = -0.25$