Analysis

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CHAPTER 1

Limits

The notion of *limits* is central for calculus almost any other notion of calculus (continuity, convergence of a sequence, derivative, integral, etc.) is based on limits. Thus it is very important to be command with limits.

1.1. Introduction to Limits

The following is a quote from a letter of Donald E. Knuth to "Notices of AMS".

The most important of these changes would be to introduce the O notation and related ideas at an early stage. This notation, first used by Bachmann in 1894 and later popularized by Landau, has the great virtue that it makes calculations simpler, so it simplifies many parts of the subject, yet it is highly intuitive and easily learned. The key idea is to be able to deal with quantities that are only partly specified, and to use them in the midst of formulas.

I would begin my ideal calculus course by introducing a simpler " A *notation*," which means "absolutely at most." For example, $A(2)$ stands for a quantity whose absolute value is less than or equal to 2. This notation has a natural connection with decimal numbers: Saying that π is approximately 3.14 is equivalent to saying that $\pi = 3.14 + A(.005)$. Students will easily discover how to calculate with A:

$$\begin{aligned} 10^{A(2)} &= A(100)\,; \\ \big(3.14 + A(.005)\big)\big(1 + A(0.01)\big) &= 3.14 + A(.005) + A(0.0314) + A(.00005) \\ &= 3.14 + A(0.3645) = 3.14 + A(.04)\,. \end{aligned}$$

I would of course explain that the equality sign *is not symmetric* with respect to such notations; we have $3 = A(5)$ and $4 = A(5)$ but not $3 = 4$, nor can we say that $A(5) = 4$. We can, however, say that $A(0) = 0$. As de Bruijn points out in [1, §1.2], mathematicians customarily use the $=$ sign as they use the word "is" in English: Aristotle is a man, but a man isn't necessarily Aristotle.

The A notation applies to variable quantities as well as to constant ones. For example,

$$\begin{aligned} \sin x &= A(1)\,; \\ x &= A(x)\,; \\ A(x) &= xA(1)\,; \\ A(x) + A(y) &= A(x+y) \text{ if } x \geqslant 0 \text{ and } y \geqslant 0\,; \\ \big(1 + A(t)\big)^2 &= 1 + 3A(t) \text{ if } t = A(1)\,. \end{aligned}$$

Once students have caught on to the idea of A notation, they are ready for O *notation*, which is even less specific. In its simplest form, $O(x)$ stands for something that is $CA(x)$

for some constant C, but we don't say what C is. We also define side conditions on the variables that appear in the formulas. For example, if n is a positive integer we can say that any quadratic polynomial in n is $O(n^2)$. If n is sufficiently large, we can deduce that

$$\begin{aligned}(n+O(\sqrt{n}\,))&(\ln n+\gamma+O(1/n))\\ &= n\ln n+\gamma n+O(1)+O(\sqrt{n}\ln n)+O(\sqrt{n}\,)+O(1/\sqrt{n}\,)\\ &= n\ln n+\gamma n+O(\sqrt{n}\ln n)\,.\end{aligned}$$

I'm sure it would be a pleasure for both students and teacher if calculus were taught in this way. The extra time needed to introduce O notation is amply repaid by the simplifications that occur later. In fact, there probably will be time to introduce the "o *notation*," which is equivalent to the taking of limits, and to give the general definition of a not-necessarily-strong derivative:

$$f(x+\epsilon)=f(x)+f'(x)\epsilon+o(\epsilon)\,.$$

The function f is continuous at x if

$$f(x+\epsilon)=f(x)+o(1)\,;$$

and so on. But I would not mind leaving a full exploration of such things to a more advanced course, when it will easily be picked up by anyone who has learned the basics with O alone. Indeed, I have not needed to use "o" in 2200 pages of *The Art of Computer Programming*, although many techniques of advanced calculus are applied throughout those books to a great variety of problems.

Students will be motivated to use O notation for two important reasons. First, it significantly simplifies calculations because it allows us to be sloppy—but in a satisfactorily controlled way. Second, it appears in the power series calculations of symbolic algebra systems like *Maple* and *Mathematica*, which today's students will surely be using.

[1]: N. G. de Bruijn, *Asymptotic Methods in Analysis* (Amsterdam: North-Holland, 1958). [2]: R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics* (Reading, Mass.: Addison–Wesley, 1989).

1.2. Definition of Limit

DEFINITION 1.2.1. Let a function f be defined on an open interval containing a, except possible at a itself, and let L be a real number. The statement

$$\lim_{x\to a} f(x) = L \qquad (1)$$

(L is the *limit* of function f at a) means that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x)-L| < \epsilon$.

We introduce a special kind of limits:

DEFINITION 1.2.2. We say that variable y is $o(z)$ (o *notation*) in the neighborhood of a point a if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|y| < \epsilon\,|z|$.

Then we could define limit of a function as follows

DEFINITION 1.2.3. A function $f(x)$ has a limit L at point a if for $x \neq 0$

$$f(a+x) = L + o(1).$$

THEOREM 1.2.4. *If*

$$\lim_{x\to a} f(x) = L$$

and $L > 0$*, then there is an open interval* $(a-\delta, a+\delta)$ *containing* a *such that* $f(x) > 0$ *for every* x *in* $(a-\delta, a+\delta)$*, except possibly* $x = a$*.*

EXERCISE 1.2.5. Verify limits using Definition

$$\begin{aligned}\lim_{x\to 3}(5x+3) &= 18;\\ \lim_{x\to 2}(x^2+1) &= 5.\end{aligned}$$

1.3. Techniques for Finding Limits

THEOREM 1.3.1. *The following basic limits are:*

$$\lim_{x\to a} c = c; \tag{2}$$

$$\lim_{x\to a} x = a.. \tag{3}$$

THEOREM 1.3.2. *If both limits*

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exist, then

$$\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x); \tag{4}$$

$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x). \tag{5}$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \tag{6}$$

provided

$$\lim_{x\to a} g(x) \neq 0.$$

COROLLARY 1.3.3. (1) *From formulas* (2) *and* (5) *follows*

$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x).$$

(2) *If* a, m, b *are real numbers then*

$$\lim_{x\to a}(mx+b) = ma+b.$$

(3) *If* n *is a positive integer then*

$$\begin{aligned}\lim_{x\to a} x^n &= a^n.\\ \lim_{x\to a}[f(x)]^n &= \left[\lim_{x\to a} f(x)\right]^n,\end{aligned}$$

provided there exists the limit

$$\lim_{x\to a} f(x).$$

(4) *If* $f(x)$ *is a polynomial function and* a *is a real number, then*

$$\lim_{x\to a} f(x) = f(a).$$

(5) *If* $q(x)$ *is a rational function and* a *is in the domain of* q*, then*

$$\lim_{x\to a} q(x) = q(a).$$

EXERCISE 1.3.4. Find limits

$$\lim_{x\to 2}\sqrt{3}; \qquad \lim_{x\to -3} x;$$
$$\lim_{x\to 4}(3x+1); \qquad \lim_{x\to -2}(2x-1)^15;$$
$$\lim_{x\to 1/2}\frac{2x^2+5x-3}{6x^2-7x+2}; \qquad \lim_{x\to -2}\frac{x^2+2x-3}{x^2+5x+6};$$
$$\lim_{x\to \pi}\sqrt[5]{\frac{x-\pi}{x+\pi}}; \qquad \lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{1}{\sqrt{1+h}}-1\right)$$

THEOREM 1.3.5. *If* $a > 0$ *and* n *is a positive integer, or if* $a \leqslant 0$ *and* n *is an odd positive integer, then*

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

THEOREM 1.3.6 (Sandwich Theorem). *Suppose* $f(x) \leqslant h(x) \leqslant g(x)$ *for every* x *in an open interval containing* a*, except possibly at* a*. If*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} g(x),$$

then

$$\lim_{x\to a} h(x) = L.$$

EXERCISE 1.3.7. Find limits

$$\lim_{x\to 0} x^2 \sin\frac{1}{x^2}; \qquad \lim_{x\to \pi/2}(x-\frac{\pi}{2})\cos x.$$

REMARK 1.3.8. All such type of result could be modified for *one-sided* limits.

1.4. Limits Involving Infinity

DEFINITION 1.4.1. Let a function f be defined on an infinite interval (c,∞) (respectively $(-\infty, c)$) for a real number c, and let L be a real number. The statement

$$\lim_{x\to\infty} f(x) = L \qquad (\lim_{x\to -\infty} f(x) = L)$$

means that for every $\epsilon > 0$ there is a number M such that if $x > M$ ($x < M$), then $|f(x) - L| < \epsilon$.

THEOREM 1.4.2. *If* k *is a positive rational number and* c*, then*

$$\lim_{x\to\infty}\frac{c}{x^k} = 0 \textit{ and } \lim_{x\to -\infty}\frac{c}{x^k} = 0$$

DEFINITION 1.4.3. Let a function f be defined on an open interval containing a, except possibly at a itself. The statement

$$\lim_{x\to a} f(x) = \infty$$

means that for every $M > 0$, there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $f(x) > M$.

EXERCISE 1.4.4. Find limits

$$\lim_{x\to -1} \frac{2x^2}{x^2 - x - 1} \qquad \lim_{x\to 9/2} \frac{3x^2}{(2x-9)^2} \tag{7}$$

$$\lim_{x\to \infty} \frac{-x^3 + 2x}{2x^2 - 3} \qquad \lim_{x\to -\infty} \frac{2x^2 - x + 3}{x^3 + 1} \tag{8}$$

1.5. Continuous Functions

The notion of *continuity* is absorbed from our every day life. Here is its mathematical definition

DEFINITION 1.5.1. A function f is *continuous* at a point c if

$$\lim_{x\to c} f(x) = f(c).$$

If function is not continuous at c then it is *discontinuous* at c, or that f has *discontinuity* at c. We give names to the following types of discontinuities:

(1) *Removable* discontinuity:

$$\lim_{x\to c} f(x) \neq f(c).$$

(2) *Jump* discontinuity:

$$\lim_{x\to +c} f(x) \neq \lim_{x\to -c} f(x).$$

(3) *Infinite* discontinuity:

$$\lim_{x\to \pm c} f(x) = \pm\infty.$$

EXERCISE 1.5.2. Classify discontinuities of

$$f(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ (x-2)^{-1} & \text{if } x > 1. \end{cases}$$

THEOREM 1.5.3. (1) *A polynomial function f is continuous at every real point c.*
(2) *A rational function $q = f/g$ is continuous at every number except the numbers c such that $g(c) = 0$.*

PROOF. The proof follows directly from Corollary 1.3.3. □

EXERCISE 1.5.4. Find all points at which f is discontinuous

$$f(x) = \frac{x-1}{x^2 + x - 2}.$$

DEFINITION 1.5.5. If a function f is continuous at every number in an open interval (a, b) we say that f is continuous on the *interval* (a, b). We say also that f is continuous on the *interval* $[a, b]$ if it is continuous on (a, b) and

$$\lim_{x\to \pm + a} f(x) = a \qquad \lim_{x\to \pm - b} f(x) = b.$$

THEOREM 1.5.6. *If two functions f and g are continuous at a real point c, the following functions are also continuous at c:*

(1) *the sum $f + g$.*

(2) *the difference* $f - g$.
(3) *the product* fg.
(4) *the quotient* f/g, *provided* $g(c) \neq 0$.

PROOF. Proof follows directly from the Theorem 1.3.2. □

EXERCISE 1.5.7. Find all points at which f is continuous

$$f(x) = \frac{\sqrt{9-x}}{\sqrt{x-6}} \qquad f(x) = \frac{x-1}{\sqrt{x^2-1}}.$$

THEOREM 1.5.8. (1) *If*

$$\lim_{x \to c} g(x) = b$$

and f *is continuous at* b, *then*

$$\lim_{x \to c} f(g(x)) = f(b) = f(\lim_{x \to c} g(x)).$$

(2) *If* g *is continuous at* c *and if* f *is continuous at* $g(c)$, *then the composite function* $f \circ g$ *is continuous at* c.

EXERCISE 1.5.9. Suppose that

$$f(x) = \begin{cases} c^2x, & \text{if} x < 1 \\ 3cx - 2 & \text{if } x \geqslant 1. \end{cases}$$

Determine all c such that f is continuous on $\mathbb{R}$.

THEOREM 1.5.10 (Intermediate Value Theorem). *If* f *is continuous on a closed interval* $[a, b]$ *and if* w *is any number between* $f(a)$ *and* $f(b)$, *then there is at least one point* $c \in [a, b]$ *such that* $f(c) = w$.

COROLLARY 1.5.11. *If* $f(a)$ *and* $f(b)$ *have opposite signs, then there is a number* c *between* a *and* b *such that* $f(x) = 0$.

EXERCISE 1.5.12. Let $f(x) = x^7 + 3x + 2$ and $g(x) = -10x^6 + 3x^2 - 1$. Show that there is a solution of the equation $f(x) = g(x)$ on the interval $(-1, 0)$.

CHAPTER 2

Derivative

2.1. Tangent Lines and Rates of Changes

Let we construct a secant line to a graph of a function $f(x)$ through the points $(a, f(a))$ and $(a+h, f(a+h))$. Then from the formula (79) it will have a slope (see (79))

$$m = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}. \tag{9}$$

If $h \to 0$ than secant line became a tangent and we obtain a formula for its slope

$$m_t = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}. \tag{10}$$

Thus the *equation of tangent line* could be written as follows (see page 62)

$$y - f(a) = m_t(x-a) \text{ or } y = f(a) + m_t(x-a). \tag{11}$$

If a body pass a distance d within time t then *average velocity* is defined as

$$v_{av} = \frac{d}{t}.$$

If the time interval $t \to 0$ then we obtain *instantaneous velocity*

$$v_a = \lim_{t\to 0} \frac{s(a+t)-s(a)}{t}. \tag{12}$$

These and many other examples lead to the notion of *derivative*.

2.2. Definition of Derivative

The following is a quote from a letter of Donald E. Knuth to "Notices of AMS".

I would define the derivative by first defining what might be called a "strong derivative": The function f has a strong derivative $f'(x)$ at point x if

$$f(x+\epsilon) = f(x) + f'(x)\epsilon + O(\epsilon^2)$$

whenever ϵ is sufficiently small. The vast majority of all functions that arise in practical work have strong derivatives, so I believe this definition best captures the intuition I want students to have about derivatives.

DEFINITION 2.2.1. The *derivative* of a function f is the function f' whose value at x is given by

$$f'(x) = \lim_{x\to 0} \frac{f(x+h)-f(x)}{h}, \tag{13}$$

provided the limit exists.

2.3. Techniques of Differentiation

We see immediately, for example, that if $f(x) = x^2$ we have

$$(x+\epsilon)^2 = x^2 + 2x\epsilon + \epsilon^2,$$

so the derivative of x^2 is $2x$. And if the derivative of x^n is $d_n(x)$, we have

$$\begin{aligned}(x+\epsilon)^{n+1} &= (x+\epsilon)\big(x^n + d_n(x)\epsilon + O(\epsilon^2)\big)\\ &= x^{n+1} + \big(xd_n(x) + x^n\big)\epsilon + O(\epsilon^2);\end{aligned} \tag{14}$$

hence the derivative of x^{n+1} is $xd_n(x) + x^n$ and we find by induction that $d_n(x) = nx^{n-1}$. Similarly if f and g have strong derivatives $f'(x)$ and $g'(x)$, we readily find

$$f(x+\epsilon)g(x+\epsilon) = f(x)g(x) + \big(f'(x)g(x) + f(x)g'(x)\big)\epsilon + O(\epsilon^2)$$

and this gives the strong derivative of the product. The *chain rule*

$$f\big(g(x+\epsilon)\big) = f\big(g(x)\big) + f'\big(g(x)\big)g'(x)\epsilon + O(\epsilon^2) \tag{15}$$

also follows when f has a strong derivative at point $g(x)$ and g has a strong derivative at x.

It is also follows that

THEOREM 2.3.1. *If a function f is differentiable at a, then f is continuous at a.*

EXERCISE 2.3.2. Give an example of a function f which is continuous at point $x = 0$ but is not differentiable there.

It could be similarly proven the following

THEOREM 2.3.3. *Let f and g be differentiable functions at point c then the following functions are differentiable at c also and derivative could be calculated as follows:*

(1) *sum and difference* $(f \pm g)'(x) = f'(x) \pm g'(x)$.
(2) *product* $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
(3) *fraction*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

2.4. Derivatives of the Trigonometric Functions

Before find derivatives of trigonometric functions we will need the following:

THEOREM 2.4.1. *The following limits are:*

$$\lim_{\phi\to 0} \sin\phi = 0; \tag{16}$$

$$\lim_{\phi\to 0} \cos\phi = 1; \tag{17}$$

$$\lim_{\phi\to 0} \frac{\sin\phi}{\phi} = 1. \tag{18}$$

PROOF. Only the last limit is non-trivial. It follows from obvious inequalities

$$\sin\phi < \phi < \tan\phi$$

and the Sandwich Theorem 1.3.6. □

COROLLARY 2.4.2. *The following limit is*

$$\lim_{x\to 0} \frac{1-\cos\phi}{\phi} = 0.$$

EXERCISE 2.4.3. Find following limits if they exist:

$$\lim_{t\to 0} \frac{4t^2+3t\sin t}{t^2}; \qquad \lim_{t\to 0} \frac{\cos t}{1-\sin t};$$
$$\lim_{x\to 0} x\cot x; \qquad \lim_{x\to 0} \frac{\sin 3x}{\sin 5x}.$$

We are able now to calculate derivatives of trigonometric functions

THEOREM 2.4.4.

$$\sin' x = \cos x; \qquad \cos' x = \sin x; \tag{19}$$
$$\tan' x = \sec^2 x; \qquad \cot' x = \csc^2 x; \tag{20}$$
$$\sec' x = \sec x \tan x; \qquad \csc' x = -\csc x \cot x. \tag{21}$$

PROOF. The proof easily follows from the Theorem 2.4.1 and trigonometric identities on page 71. □

EXERCISE 2.4.5. Find derivatives of functions

$$y = \frac{\sin x}{x}; \qquad y = \frac{1-\cos z}{1+\cos z};$$
$$y = \frac{\tan x}{1+x^2}; \qquad y = \csc t \sin t.$$

2.5. The Chain Rule

The *chain rule*

$$(f\circ g)'(x) = f'(g(x))g'(x) \tag{22}$$

was proven above 15.

EXERCISE 2.5.1. Find derivatives

$$y = \left(z^2 - \frac{1}{z^2}\right)^3; \qquad y = \frac{x^4-3x^2+1}{(2x+3)^4};$$
$$y = \sqrt[3]{8r^3+27}; \qquad y = (7x+\sqrt{x^2+3})^6.$$

2.6. Implicit Differentiation

If a function $f(x)$ is given by formula like $f(x) = 2x^7 + 3x - 1$ then we will say that it is an *explicit function*. In contrast an identity like

$$y^4 + 3y - 4x^3 = 5x + 1$$

define an *implicit function*. The derivative of implicit function could be found from an equation which it is defined. Usually it is a function both x and y. This procedure is called *implicit differentiation*.

EXERCISE 2.6.1. Find the slope of the tangent lines at given points

(1) $x^2y + \sin y = 2\pi$ at $P(1, 2\pi)$.
(2) $2x^3 - x^2y + y^3 - 1 = 0$ at $P(2, -3)$.

2.7. Related Rates

If two variables x and y satisfy to some relationship then we could found their *related rates* by the implicit differentiation.

EXERCISE 2.7.1. (1) If $S = z^3$ and $dz/dt = -2$ when $z = 3$, find dS/dt.
(2) If $x^2 + 3y^2 + 2y = 10$ and $dx/dt = 2$ when $x = 3$ and $y = -1$, find dy/dt.

EXERCISE 2.7.2. Suppose a spherical snowball is melting and the radius is decreasing at a constant rate, changing from 12 in. to 8 in. in 45 min. How fast was the volume changing when the radius was 10 in.?

2.8. Linear Approximations and Differentials

It is known from geometry that a tangent line is closest to a curve at given point among all lines. Thus equation of a tangent line (11)

$$y = f(a) + f'(a)(x - a) \tag{23}$$

gives the best approximation to a given graph of $f(x)$. We could use this *linear approximation* in order to estimate value of $f(x)$ in a vicinity of a. We denote an increment of the independent variable x by Δx and

$$\Delta y = f(x_0 + \Delta x) - f(x)$$

is the increment of dependent variable. Therefore

$$\Delta y \approx f'(x)\Delta x \text{ if } \Delta x \approx 0; \tag{24}$$

$$f(x) \approx f(x) + f'(x)\Delta x; \tag{25}$$

$$f(x) \approx f(x) + dy, \tag{26}$$

where $dy = f'(x)\Delta x$ is defined to be *differential* of $f(x)$.

An application of this formulas connected with estimation of errors of measurements:

	Exact value	Approximate value
Absolute error	$\Delta y = y - y_0$	$dy = f'(x_0)\Delta x$
Relative error	$\frac{\Delta y}{y_0}$	$\frac{dy}{y_0}$
Percentage error	$\frac{\Delta y}{y_0} \times 100\%$	$\frac{dy}{y_0} \times 100\%$

EXERCISE 2.8.1. Use linear approximation to estimate $f(b)$:
(1) $f(x) = -3x^2 + 8x - 7$; $a = 4$, $b = 3.96$.
(2) $f(\phi) = \csc\phi + \cot\phi$, $a = 45°$, $b = 46°$.

EXERCISE 2.8.2. Find Δy, dy, $dy - \Delta y$ for $y = 3x^2 + 5x - 2$.

CHAPTER 3

Appliactions of Derivative

3.1. Extrema of Functions

We defined *increasing* and *decreasing* functions in Section C.9. There are more definitions

DEFINITION 3.1.1. Let f is defined on $S \subset \mathbb{R}$ and $c \in S$.

(1) $f(c)$ is the *maximum* value of f on S if $f(x) \leqslant f(c)$ for every $x \in S$.
(2) $f(c)$ is the *minimum* value of f on S if $f(x) \geqslant f(c)$ for every $x \in S$.

Maximum and minimum are called *extreme values*, or *extrema* of f. If S is the domain of f then maximum and minimum are called *global* or *absolute*.

EXERCISE 3.1.2. Give an examples of functions which do not have minimum or maximum values.

The important property of continuous functions is given by the following

THEOREM 3.1.3. *If f is continuous on $[a, b]$, then f takes on a maximum and minimum values at least once in $[a, b]$.*

Sometimes the following notions are of great importance

DEFINITION 3.1.4. Let c be a number in domain of f.

(1) $f(c)$ is the *local maximum* f if there is an open interval (a, b) such that $c \in (a, b)$ and $f(x) \leqslant f(c)$ for every $x \in (a, b)$ in the domain of f.
(2) $f(c)$ is the *local minimum* f if there is an open interval (a, b) such that $c \in (a, b)$ and $f(x) \geqslant f(c)$ for every $x \in (a, b)$ in the domain of f.

The local extrema could be determined from values of derivative:

THEOREM 3.1.5. *If f has a local extremum at a number c in an open interval, then either $f'(c) = 0$ or $f'(c)$ do not exist.*

PROOF. The proof follows from the linear approximation of Δf by $f'(c)\Delta x$: if $f'(c)$ exists and $f'(c) \neq 0$ then in an open interval around c there is values of f whose greater and less than $f(c)$. □

The direct consequence is:

COROLLARY 3.1.6. *if $f'(c)$ exists and $f'(c) \neq 0$ then $f(c)$ is not a local extremum of c.*

Critical numbers of f are whose points c in the domain of f where either $f'(c) = 0$ or $f'(c)$ does not exist.

THEOREM 3.1.7. *If a function* f *is continuous on a* $[a, b]$ *and has its maximum or minimum values at a number* $c \in (a, b)$*, then either* $f'(c) = 0$ *or* $f'(c)$ *does not exist.*

So to determine maximum and minimum values of f one should accomplish the following steps

(1) Find all critical points of f on (a, b).
(2) Calculate values of f in all critical points from step 1.
(3) Calculate the endpoint values $f(a)$ and $f(b)$.
(4) The maximal and minimal values of f on $[a, b]$ are the largest and smallest values calculated in 2 and 3.

EXERCISE 3.1.8. Find extrema of f on the interval

(1) $y = x^4 - 5x + 4$; $[0, 2]$.
(2) $y = (x - 1)^{2/3} - 4$; $[0, 9]$.

EXERCISE 3.1.9. Find the critical numbers of f

$$\begin{aligned} y &= 4x^3 + 5x^2 - 42x + 7; \\ y &= \sqrt[3]{x^2 - x - 2}; \\ y &= (4z + 1)\sqrt{z^2 - 16}; \\ y &= 8\cos^3 t - 3\sin 2x - 6x. \end{aligned}$$

3.2. The Mean Value Theorem

THEOREM 3.2.1 (Rolle's Theorem). *If* f *is continuous on a closed interval* $[a, b]$ *and differentiable on the open interval* (a, b) *and if* $f(a) = f(b)$*, then* $f'(c) = 0$ *for at least one number* c *in* (a, b).

PROOF. There is two possibilities

(1) f is constant on $[a, b]$ then $f'(x) = 0$ everywhere.
(2) $f(x)$ is not constant then it has at least one extremum point c (Theorem 3.1.3) which is not the end point of $[a, b]$, then $f'(c) = 0$ (Theorem 3.1.5).

□

EXERCISE 3.2.2. Shows that f satisfy to the above theorem and find c:

(1) $f(x) = 3x^2 - 12x + 11$; $[0, 4]$.
(2) $f(x) = x^3 - x$; $[-1, 1]$.

Rolle's Therem is the principal step to the next

THEOREM 3.2.3 (Mean Value Theorem or Lagrange's Theorem). *If* f *is continuous on a closed interval* $[a, b]$ *and differentiable on the open interval* (a, b)*, then there exists a number* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

PROOF. The proof follows from applivcation of the Rolle's Theorem 3.2.1 to the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

□

We start applications of Mean Value Theorem by two corrolaries:

COROLLARY 3.2.4. *If* $f'(x) = 0$ *for all* x *in some interval* I, *then there is a constant* C *such that* $f(x) = C$ *for all* x *in* I.

COROLLARY 3.2.5. *If* $f'(x) = g'(x)$ *foar all* $x \in I$, *then there is a constant* C *such that* $f(x) = g(x) + C$.

EXERCISE 3.2.6. Shows that f satisfy to the above theorem and find c:

(1) $f(x) = 5x^2 - 3x + 1$; $[1, 3]$.
(2) $f(x) = x^{2/3}$; $[-8, 8]$.
(3) $f(x) = x^3 + 4x$; $[-3, 6]$.

EXERCISE 3.2.7. Prove: if f continuous on $[a, b]$ and if $f'(x) = c$ there, then $f(x) = cx + d$ for a $d \in \mathbb{R}$.

EXERCISE 3.2.8. Prove:

$$|\sin u - \sin v| \leqslant |u - v|.$$

3.3. The First Derivative Test

Derivative of function could provide future information on its behavior:

THEOREM 3.3.1. *Let* f *be continuous on* $[a, b]$ *and differentiable on* (a, b).

(1) *If* $f'(x) > 0$ *for every* x *in* (a, b), *then* f *is increasing on* $[a, b]$.
(2) *If* $f'(x) < 0$ *for every* x *in* (a, b), *then* f *is decreasing on* $[a, b]$.

PROOF. For any numbers x_1 and x_2 in (a, b) we could write using the Mean Value Theorem:

$$f(x_1) - f(x_2) = f'(c)(x_1 - x_2).$$

Then for $x_1 > x_2$ we will have $f(x_1) > f(x_2)$ if $f'(c)$ is positive and $f(x_1) < f(x_2)$ if $f'(c)$ is negative. □

To check the sign of continuous derivative in an interval $[a, b]$ which does not contain critical points it is enough to verify it for a single point $k \in (a, b)$ (See Intermediate Value Theorem). We shall call $f'(k)$ a *test value*.

TEST 3.3.2 (First Derivative Test). Let c is a critical number for f, f is continuous in an open interval I containing c and differentiable in I, except possibly at c itself. Then from the above theorem it follows that

(1) If f' changes from positive to negative at c, then $f(c)$ is a local maximum of f.
(2) If f' changes from negative to positive at c, then $f(c)$ is a local minimum of f.
(3) If $f'(x) > 0$ or $f'(x) < 0$ for all $x \in I$, $x \neq c$, then $f(c)$ is not a local extremum of of c.

EXERCISE 3.3.3. Find the local extrema of f and intervals of monotonicity, sketch the graph

(1) $y = 2x^3 + x^2 - 20x + 1$.
(2) $y = 10x^3(x-1)^2$.
(3) $y = x(x^2-9)^{1/2}$.
(4) $y = x/2 - \sin x$.
(5) $y = 2\cos x + \cos 2x$.

EXERCISE 3.3.4. Find local extrema of f on the given interval

(1) $y = \cot^2 x + 2\cot x$ $[\pi/6, 5\pi/6]$.
(2) $y = \tan x - 2\sec x$ $[-\pi/4, \pi/4]$.

3.4. Concavity and the Second Derivative Test

DEFINITION 3.4.1. Let f be differentiable on an open interval I. The graph of f is

(1) *concave upward* on I if f' is increasing on I;
(2) *concave downward* on I if f' is decreasing on I.

If a graph is concave upward then it lies above any tangent line and for downward concavity it lies below every tangent line.

TEST 3.4.2 (Test for Concavity). If the second derivative f'' of f exists on an open interval I, then the graph of f is

(1) *concave upward* on I if $f''(x) > 0$ on I;
(2) *concave downward* on I if $f''(x) < 0$ on I.

DEFINITION 3.4.3. A point $(c, f(c))$ on the graph of f is a *point of inflection* if the following conditions are satisfied:

(1) f is continuous at c.
(2) There is an open interval (a, b) containing c such that the graph has different types of concavity on (a, c) and (c, b).

TEST 3.4.4 (Second Derivative Test). Suppose that f is differentiable on an open interval containing c and $f'(c) = 0$.

(1) If $f''(c) < 0$, then f has a local maximum at c.
(2) If $f''(c) > 0$, then f has a local minimum at c.

EXERCISE 3.4.5. Find the local extrema of f, intervals of concavity and points of inflections.

(1) $y = 2x^6 - 6x^4$.
(2) $y = x^{1/5} - 1$.
(3) $y = 6x^{1/2} - x^{3/2}$.
(4) $y = \cos x - \sin x$.
(5) $y = x + 2\cos x$.

3.5. Summary of Graphical Methods

In order to sketch a graph the following steps should be performed

(1) Find domain of f.

(2) Estimate range of f and determine region where f is negative and positive.
(3) Find region of continuity and classify discontinuity (if any).
(4) Find all x- and y-intercepts.
(5) Find symmetries of f.
(6) Find critical numbers and local extrema (using the First of Second Derivative Test), region of monotonicity of f.
(7) Determine concavity and points of inflections.
(8) Find asymptotes.

EXERCISE 3.5.1. Sketch the graphs:

$$\begin{aligned} f(x) &= \frac{2x^2 - x - 3}{x - 2}; \\ f(x) &= \frac{8 - x^3}{2x^2}; \\ f(x) &= \frac{-3x}{\sqrt{x^2 + 4}}; \\ f(x) &= x^3 + \frac{3}{x}; \\ f(x) &= \frac{-4}{x^2 + 1}; \\ f(x) &= \left|x^3 - x\right|; \\ f(x) &= |\cos x| + 2. \end{aligned}$$

3.6. Optimization Problems. Review

To solve optimization problem one need to translate the problem to a question on extrema of a function of one variable.

EXERCISE 3.6.1. (1) Find the minimum value of A if $A = 4y + x^2$, where $(x^2 + 1)y = 324$.
(2) Find the minimum value of C if $C = (x^2 + y^2)^{1/2}$, where $xy = 9$.

EXERCISE 3.6.2. A metal cylindrical container with an open top is to hold 1 m^3. If there is no waste in construction, find the dimension that require the least amount of material.

EXERCISE 3.6.3. Find the points of the graph of $y = x^3$ that is closest to the point $(4, 0)$.

CHAPTER 4

Integrals

4.1. Antiderivatives and Indefinite Integrals

DEFINITION 4.1.1. A function F is an *antiderivative* of the function f on an interval I if $F'(x) = f(x)$ for every $x \in I$.

It is obvious that if F is an antiderivative of f then $F(x) + C$ is also antiderivative of f for any real constant C. C is called an *arbitrary constant*. It follows from Mean Value Theorem that *every* antiderivative is of this form.

THEOREM 4.1.2. *Let F be an antiderivative of f on an interval I. If G is any antiderivative of f on I, then*

$$G(x) = F(x) + C$$

for some constant C and every $x \in I$.

EXERCISE 4.1.3. The above Theorem may be false if the domain of f is different from an interval I. Give an example.

DEFINITION 4.1.4. The notation

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$ and C is an arbitrary constant, denotes the family of all antiderivatives of $f(x)$ on an interval I and is called *indefinite integral*.

THEOREM 4.1.5.

$$\int \frac{d}{dx}(f(x))\,dx = f(x) + C; \tag{27}$$

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x). \tag{28}$$

The above Theorem allows us to construct a primitive table of antiderivatives from the tables of derivatives:

(29)

Derivative	**Indefinite Integral**
$\frac{d}{dx}(x) = 1$	$\int 1\,dx = \int dx = x + C$
$\frac{d}{dx}(\frac{x^{r+1}}{r+1}) = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}(-\cot x) = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}(-\csc x) = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

THEOREM 4.1.6.

$$\int cf(x)\,dx = c\int f(x)\,dx \tag{30}$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx \tag{31}$$

EXERCISE 4.1.7. Find antiderivatives of the following functions

$$9t^2 - 4t + 3; \quad 4x^2 - 8x + 1; \quad \frac{1}{z^3} - \frac{3}{z^2};$$
$$\sqrt{u^3} - \frac{1}{2}u^{-2} + 5; \quad (3x-1)^2; \quad \frac{(t^2+3)^2}{t^6};$$
$$\frac{7}{\csc x}; \quad -\tfrac{1}{5}\sin x; \quad \frac{1}{\sin^2 t}.$$

A *Differential equation* is an equation that involves derivatives or differentials of an unknown function. Additional values of f or its derivatives are called *initial conditions*.

EXERCISE 4.1.8. Solve the differential equations subject to the given conditions

$$f'(x) = 12x^2 - 6x + 1, \qquad f(1) = 5; \tag{32}$$

$$f''(x) = 4x - 1, \qquad f'(2) = -2, \quad f(1) = 3; \tag{33}$$

$$\frac{d^2y}{dx^2} = 3\sin x - 4\cos x, \qquad y = 7, y' = 2 \text{ if } x = 0. \tag{34}$$

4.2. Change of Variables in Indefinite Integrals

One more important formula for indefinite integral could be obtained from the rules of differentiation. The chain rule implies:

THEOREM 4.2.1. *If* F *is an antiderivative of* f*, then*

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$ *and* $du = g'(x)\,dx$*, then*

$$\int f(u)\,du = F(u) + C.$$

EXERCISE 4.2.2. Find the integrals

$$\int x(2x^2+3)^{10}\,dx; \qquad \int x^2\sqrt[3]{3x^3+7}\,dx; \tag{35}$$

$$\int \left(1+\frac{1}{x}\right)^{-3}\left(\frac{1}{x^2}\right)dx; \qquad \int \frac{t^2+t}{(4-3t^2-2t^3)^4}\,dt; \tag{36}$$

$$\int \frac{\sin 2x}{\sqrt{1-\cos 2x}}\,dx; \qquad \int \sin^3 x\cos x\,dx. \tag{37}$$

4.3. Summation Notation and Area

DEFINITION 4.3.1. We use the following *summation notation*:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

THEOREM 4.3.2.

$$\sum_{k=1}^{n} c = cn; \tag{38}$$

$$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k; \tag{39}$$

$$\sum_{k=1}^{n} c a_k = c\left(\sum_{k=1}^{n} a_k\right). \tag{40}$$

THEOREM 4.3.3.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \tag{41}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \tag{42}$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2. \tag{43}$$

This sum will help us to find inscribed rectangular polygon and circumscribed rectangular polygon.

EXERCISE 4.3.4. Find the area under the graph of the following functions:

(1) $y = 2x + 3$, from 2 to 4.
(2) $y = x^2 + 1$, from 0 to 3.

4.4. The Definite Integral

There is a way to calculate an area under the graph of a function $y = f(x)$. We could approximate it by a sum of the form

$$\sum_{k=1}^{n} f(w_k)\Delta x_k, \qquad w_k \in \Delta x_k.$$

It is a *Riemann sum* The approximation will be precise if will come to the *limit of Riemann sums*:

$$\lim_{\delta x_k \to 0} \sum_{k=1}^{n} f(w_k)\delta x_k = L.$$

If this limit exists it called *definite integral* of f from a to b and denoted by:

$$\int_a^b f(x)\,dx = \lim_{\delta x_k \to 0} \sum_{k=1}^{n} f(w_k)\delta x_k = L.$$

If the limit exist we say that f is *integrable function* on $[a, b]$.

4.5. Properties of the Definite Integral

THEOREM 4.5.1. *If* c *is a real number, then*

$$\int_a^b c\,dx = c(b-a).$$

THEOREM 4.5.2. *If* f *is integrable on* $[a, b]$ *and* c *is any real number, then* cf *is integrable on* $[a, b]$ *and*

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

THEOREM 4.5.3. *If* f *and* g *are integrable on* $[a, b]$, *then* $f \pm g$ *is also integrable on* $[a, b]$ *(*$a > b$*) and*

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b [f(x) \pm g(x)]\,dx.$$

THEOREM 4.5.4. *If* f *is integrable on* $[a, b]$ *and* $f(x) \geqslant 0$ *for all* $x \in [a, b]$ *then*

$$\int_a^b f(x)\,dx \geqslant 0.$$

COROLLARY 4.5.5. *If* f *and* g *are integrable on* $[a, b]$ *and* $f(x) \geqslant g(x)$ *for all* $x \in [a, b]$ *then*

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx.$$

THEOREM 4.5.6 (Mean Value Theorem for Definite Integrals). *If* f *is continuous on a closed interval* $[a, b]$, *then there is a number z in the open interval* (a, b) *such that*

$$\int_a^b f(x)\,dx = f(z)(b-a).$$

DEFINITION 4.5.7. Let f be continuous on $[a, b]$. The *average value* f_{av} of f on $[a, b]$ is

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Once it is known that integration is the inverse of differentiation and related to the area under a curve, we can observe, for example, that if f and f' both have strong derivatives at x, then

$$\begin{aligned} f(x+\epsilon) - f(x) &= \int_0^\epsilon f'(x+t)\,dt \\ &= \int_0^\epsilon \left(f'(x) + f''(x)\,t + O(t^2)\right) dt \\ &= f'(x)\epsilon + f''(x)\epsilon^2/2 + O(\epsilon^3)\,. \end{aligned} \tag{44}$$

4.6. The Fundamental Theorem of Calculus

There is an unanswered question from the previous section: *Why undefined and defined integrals shared their names and notations*? The answer is given by the following

THEOREM 4.6.1 (Fundamental Theorem of Calculus). *Suppose* f *is continuous on a closed interval* $[a, b]$.

(1) *If the function* G *is defined by*

$$G(x) = \int_a^x f(t)\,dt$$

for every x *in* $[a, b]$, *then* G *is an antiderivative of* f *on* $[a, b]$.

(2) *If* F *is any antiderivative of* f *on* $[a, b]$, *then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF. The proof of the first statement follows from the Mean Value Theorem for Definite Integral. End the second part follows from the first and initial condition

$$\int_a^a f(x)\,dx = 0.$$

□

COROLLARY 4.6.2. (1) *If* f *is continuous on* $[a, b]$ *and* F *is any antiderivative of* f, *then*

$$\int_a^b f(x)\,dx = F(x)]_a^b = F(b) - F(a).$$

(2)

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b.$$

(3) *Let* f *be continuous on* $[a, b]$. *If* $a \leqslant c \leqslant b$, *then for every* x *in* $[a, b]$

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x).$$

THEOREM 4.6.3. *If* $u = g(x)$, *then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

THEOREM 4.6.4. *Let* f *be continuous on* $[-a, a]$.

(1) *If* f *is an even function,*

$$\int_{-a}^a f(x)\,dx = \int_{-a}^a f(x)\,dx.$$

(2) *If* f *is an odd function,*

$$\int_{-a}^a f(x)\,dx = 0.$$

EXERCISE 4.6.5. Calculate integrals

$$\int_{-2}^{-1}\left(x-\frac{1}{x}\right)^2\,dx; \qquad \int_1^4\sqrt{5-x}\,dx; \qquad \int_{-1}^{1}(v^2-1)^3 v\,dv;$$

$$\int_0^{\pi/2} 3\sin(\frac{1}{2}x)\,dx; \quad \int -\pi/6^{\pi/6}(x+\sin 5x)\,dx; \quad \int_0^{\pi/3}\frac{\sin x}{\cos^2 x}\,dx;$$

$$\int_{-1}^{5}|2x-3|\,dx;$$

CHAPTER 5

Applications of the Definite Integral

5.1. Area

We know that the geometric meaning of the definite integral of a positive function is the area under the graph. We could calculate areas of more complicated figures by combining several definite integrals.

EXERCISE 5.1.1. Find areas bounded by the graphs:

(1) $x = y^2, x - y = 2$.
(2) $y = x^3, y = x^2$.
(3) $y = x^{2/3}, x = y^2$.
(4) $y = x^3 - x, y = 0$.
(5) $x = y^3 + 2y^2 - 3y, x = 0$.
(6) $y = 4 + \cos 2x, y = 3 \sin \frac{1}{2} x$.

EXERCISE 5.1.2. Express via sums of integrals areas:

(1) $y = \sqrt{x}, y = -x, x = 1, x = 4$.

5.2. Solids of Revolution

THEOREM 5.2.1. *Let* f *be continuous on* $[a, b]$*, and let* R *be the region bounded by the graph of* f*, the* x*-axis, and the vertical lines* $x = a$ *and* $x = b$*. The volume* V *of the solid of revolution generated by revolving* R *about the* x*-axis is*

$$V = \int_a^b \pi [f(x)]^2 \, dx.$$

THEOREM 5.2.2 (Volume of a Washer).

$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) \, dx.$$

EXERCISE 5.2.3. (1) $y = 1/x, x = 1, x = 3, y = 0$; x-axis;
(2) $y = x^3, x = -2, y = 0$, x-axis;
(3) $y = x^2 - 4x, y = 0$; x-axis;
(4) $y = (x - 1)^2 + 1, y = -(x - 1)^2 + 3$; x-axis;

EXERCISE 5.2.4. Find volume of revolution for $y = x^3$, $y = 4x$ rotated around $x = 4$.

5.3. Volumes by Cylindrical Shells

Let a cylindrical shell has outer and inner radiuses as r_1 and r_2 then and altitude h. We introduce the average radius $r = (r_1 + r_2)/2$ and the thickness $\Delta r = r_2 - r_1$. Then its volume is:

$$V = \pi r_1^2 h - \pi r_2^2 h = 2\pi r \Delta r h.$$

Let a region bounded by a function $f(x)$ and x-axis. If we rotate it around the y-axis then it is an easy to observe that the volume of the solid will be as follow:

$$V = \int_a^b 2\pi x f(x)\, dx.$$

EXERCISE 5.3.1. Find volumes:

(1) $y = \sqrt{x}$, $x = 4$, $y = 0$, y-axis.
(2) $y = x^2$, $y^2 = 8x$, y-axis.
(3) $y = 2x$, $y = 6$, $x = 0$, x-axis.
(4) $y = \sqrt{x+4}$, $y = 0$, $x = 0$, x-axis.

5.4. Volumes by Cross Section

If a plane intersects a solid, then the region common to the plane and the solid is a *cross section* of the solid. There is a simple formula to calculate volumes by cross sections:

THEOREM 5.4.1 (Volumes by Cross Sections). *Let S be a solid bounded by planes that are perpendicular to the x-axis at a and b. If, for every x in $[a, b]$, the cross-sectional area of S is given by $A(x)$, there A is continuous on $[a, b]$, then the volume S is*

$$V = \int_a^b A(x)\, dx.$$

COROLLARY 5.4.2 (Cavalieri's theorem). *If two solids have equal altitudes and if all cross sections by planes parallel to their bases and at the same distances from their bases have equal areas, then the solids have the same volume.*

EXERCISE 5.4.3. Let R be the region bounded by the graphs of $x = y^2$ and $x = 9$. Find the volume of the solid that has R as its base if every cross section by a plane perpendicular to the x-axis has the given shape.

(1) Rectangle of height 2.
(2) A quartercircle.

EXERCISE 5.4.4. Find volume of a pyramid if its altitude is h and the base is a rectangle of dimensions a and $2a$.

EXERCISE 5.4.5. A solid has as its base the region in xy-plane bounded by the graph of $y^2 = 4x$ and $x = 4$. Find the volume of the solid if every cross section by a plane perpendicular to the y-axis is semicircle.

5.5. Arc Length and Surfaces of Revolution

We say that a function f is *smooth* on an interval if it has a derivative f' that is continuous throughout the interval.

THEOREM 5.5.1. *Let f be smooth on $[a, b]$. The arc length of the graph of f from $A(a, f(a))$ to $B(b, f(b))$ is*

$$L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

We could introduce the *arc length function* s for the graph of f on $[a, b]$ is defined by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt.$$

THEOREM 5.5.2. *Let f be smooth on $[a, b]$, and let s be the arc length function for the graph of $y = f(x)$ on $[a, b]$. if Δx is an increment in the variable x, then*

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2}; \tag{45}$$

$$ds = \sqrt{1 + [f'(x)]^2}\Delta x. \tag{46}$$

EXERCISE 5.5.3. Find the arc length:

(1) $y = 2/3x^{2/3}$; $A(1, 2/3)$, $B(8, 8/3)$;
(2) $y = 6\sqrt[3]{x^2} + 1$; $A(-1, 7)$, $B(-8, 25)$;
(3) $30xy^3 - y^8 = 15$; $A(8/15, 1)$, $B(\frac{271}{240}, 2)$;

EXERCISE 5.5.4. Find the length of the graph $x^{2/3} + y^{2/3} = 1$.

THEOREM 5.5.5. *If f is smooth and $f(x) \geqslant> 0$ on $[a, b]$, then the area S of the surface generated by revolving the graph of f about the x-axis is*

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2} dx.$$

EXERCISE 5.5.6. Find the area of the surface generated by revolving of the graph

(1) $y = x^3$; $A(1, 1)$, $B(2, 8)$;
(2) $8y = 2x^4 + x^{-2}$, $A(1, 3/8)$, $B(2, 129/32)$;
(3) $x = 4\sqrt{y}$; $A(4, 1)$, $B(12, 9)$.

CHAPTER 6

Trancendential Functions

There is a special sort of functions which have a strange name *transcendental*. We will explore the important rôle of these functions in calculus and mathematics in general.

6.1. The Derivative of the Inverse Function

We define *one-to-one functions* in Section C.6. For such function we could give the following definition.

DEFINITION 6.1.1. Let f be a one-to-one function with domain D and range R. A function g with domain R and range D is the *inverse function* of f, if for all $x \in D$ and $y \in R$ $y = f(x)$ iff $x = g(y)$.

THEOREM 6.1.2. *Let f be a one-to-one function with domain D and range R. If g is a function with domain R and range D, then g is the inverse function of f iff both the following conditions are true:*

(1) $g(f(x)) = x$ *for every* $x \in D$.
(2) $f(g(x)) = x$ *for every* $y \in R$.

EXERCISE 6.1.3. Find inverse function.

(1) $f(x) = \frac{3x+2}{2x-5}$;
(2) $f(x) = 5x^2 + 2$, $x \geqslant 0$;
(3) $f(x) = \sqrt{4 - x^2}$.

THEOREM 6.1.4. *If f is continuous and increasing on $[a, b]$, then f has an inverse function f^{-1} that is continuous and increasing on $[f(a), f(b)]$.*

THEOREM 6.1.5. *If a differentiable function f has an inverse function $g = f^{-1}$ and if $f'(g(c)) \neq 0$, then g is differentiable at c and*

(47)
$$g'(c) = \frac{1}{f'(g(c))}.$$

PROOF. The formula follows directly from differentiation by the Chain rule the identity $f(g(x)) = x$ (see Theorem 6.1.2). □

EXERCISE 6.1.6. Find domain and derivative of the inverse function.

(1) $f(x) = \sqrt{2x + 3}$;
(2) $f(x) = 4 - x^2$, $x \geqslant 0$;
(3) $f(x) = \sqrt{9 - x^2}$, $0 \leqslant x \leqslant 3$.

EXERCISE 6.1.7. Prove that inverse function exists and find slope of tangent line to the inverse function in the given point.

(1) $f(x) = x^5 + 3x^3 + 2x - 1$, $P(5, 1)$;
(2) $f(x) = 4x^5 - (1/x^3)$, $P(3, 1)$;
(3) $f(x) = x^5 + x$, $P(2, 1)$.

6.2. The Natural Logarithm Function

We know that antiderivative for a function x^n is $x^{n+1}/(n+1)$. This expression is defined for all $n \neq -1$. This case deserves a special name

DEFINITION 6.2.1. The *natural logarithm function*, denoted by ln, is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt,$$

for every $x > 0$.

From the properties of definite integral follows that

$$\begin{aligned} \ln x > 0 \quad &\text{if} \quad x > 1; \\ \ln x < 0 \quad &\text{if} \quad x < 1; \\ \ln x = 0 \quad &\text{if} \quad x = 1. \end{aligned}$$

THEOREM 6.2.2.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

THEOREM 6.2.3. *If $u = g(x)$ and g is differentiable, then*

$$\begin{aligned} \frac{d}{dx}(\ln u) &= \frac{1}{u}\frac{du}{dx}, \quad &\textit{if } u > 0; \\ \frac{d}{dx}(\ln |u|) &= \frac{1}{u}\frac{du}{dx}, \quad &\textit{if } u \neq 0. \end{aligned}$$

COROLLARY 6.2.4. *The natural logarithm is an increasing function.*

This gives a new way to prove the principal laws of logarithms.

EXERCISE 6.2.5. Prove using laws of logarithms that

$$\lim_{x \to +\infty} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} = -\infty.$$

From this Exercise and Corollary 6.2.4 follows

COROLLARY 6.2.6. *To every real number x there corresponds exactly one positive real number y such that $\ln y = x$.*

EXERCISE 6.2.7. Find implicit derivatives:

(1) $3y - x^2 + \ln xy = 2$.
(2) $y^3 + x^2 \ln y = 5x + 3$.

Another useful application is *logarithmic differentiation* which is given by the formula:

$$\frac{d}{dx} f(x) = f(x) \frac{d}{dx} \ln(f(x)). \tag{48}$$

It is useful for functions consisting from products and powers of elemntary functions.

EXERCISE 6.2.8. Find derivative of functions using logarithmic differentiation:

(49) $$f(x) = (5x+2)^3(6x+1)^2;$$

(50) $$f(x) = \sqrt{(3x^2+2)\sqrt{6x-7}};$$

(51) $$f(x) = \frac{(x^2+3)^5}{\sqrt{x+1}}.$$

6.3. The Exponential Function

Corollary 6.2.6 justify the following

DEFINITION 6.3.1. The *natural exponential function*, denoted by $\exp x = e^x$, is the inverse of the natural logarithm function. The letter e ($= 2.718281828\ldots$) denotes the positive real number such that $\ln e = 1$.

By the definition

(52) $$\ln e^x = x, \qquad x \in \mathbb{R}$$

(53) $$e^{\ln x} = x, \qquad x > 0.$$

By the same definition we could derive laws of exponents from the laws of logarithms.

THEOREM 6.3.2.

(54) $$\frac{d}{dx}(e^x) = e^x$$

(55) $$\frac{d}{dx}(e^{g(x)}) = e^{g(x)}\frac{dg(x)}{dx}$$

(56)

PROOF. The proof of the first identity follows from differentiation of 52 by the chain rule. The second identity follows from the first one and the chain rule. □

EXERCISE 6.3.3. Find implicit derivatives

(1) $xe^y + 2x - \ln(y+1) = 3;$
(2) $e^x \cos y = xe^y.$

EXERCISE 6.3.4. Find extrema and regions of monotonicity:

(1) $f(x) = x^2 e^{-2x};$
(2) $f(x) = e^{1/x}.$

6.4. Integration Using Natural Logarithm and Exponential Functions

The following formulas are direct consequences of change of variables in definite integral and definition of logarithmic and exponential functions:

(57) $$\int \frac{1}{g(x)} g'(x)\,dx = \ln|g(x)| + C;$$

(58) $$\int e^{g(x)} g'(x)\,dx = e^{g(x)} + C.$$

From here we could easily derive

THEOREM 6.4.1.

$$\int \tan u\, du = -\ln|\cos u| + C; \tag{59}$$

$$\int \cot u\, du = \ln|\sin u| + C; \tag{60}$$

$$\int \sec u\, du = \ln|\sec u + \tan u| + C; \tag{61}$$

$$\int \csc u\, du = \ln|\csc u - \cot u| + C. \tag{62}$$

EXERCISE 6.4.2. Evaluate integrals

$$\int \frac{x^3}{x^4-5}\,dx; \qquad \int \frac{(2+\ln x)^{10}}{x}\,dx;$$

$$\int \frac{e^x}{(e^x+2)^2}\,dx; \qquad \int \frac{\cot \sqrt[3]{x}}{\sqrt[3]{x^2}}\,dx.$$

6.5. General Exponential and Logarithmic Functions

Using laws of logarithms we could make

DEFINITION 6.5.1. The *exponential function with base* a is defined as follows:

$$f(x) = a^x = e^{\ln a^x} = e^{x \ln a}. \tag{63}$$

From this definition the following properties follows

THEOREM 6.5.2.

$$\frac{d}{dx}(a^x) = a^x \ln a;$$

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx};$$

$$\int a^x\, dx = \left(\frac{1}{\ln a}\right) a^x + C;$$

$$\int a^u\, du = \left(\frac{1}{\ln a}\right) a^u + C.$$

EXERCISE 6.5.3. Evaluate integrals

$$\int 5^{-5x}\, dx; \qquad \int \frac{(2^x+1)^2}{2^x}\, dx;$$

$$\int e^e\, dx; \qquad \int x^5\, dx;$$

$$\int x^{\sqrt{5}}\, dx; \qquad \int (\sqrt{5})^x\, dx;$$

EXERCISE 6.5.4. The region under the graph of $y = 3^{-x}$ from $x = 1$ to $x = 2$ is revolved about the x-axis. Find the volume of the resulting solid.

Having a^x already defined we could give the following

DEFINITION 6.5.5. The *logarithmic function with base* a) $f(x) = \log_a x$ is defined by the condition $y = \log_a x$ iff $x = a^y$.

The following properties follows directly from the definition

THEOREM 6.5.6.

$$\begin{aligned} \frac{d}{dx}(\log_a x) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{1}{x}; \\ \frac{d}{dx}(\log_a |u|) &= \frac{d}{dx}\left(\frac{\ln|u|}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{ddx}. \end{aligned}$$

EXERCISE 6.5.7. Find derivatives of functions:

$$\begin{aligned} f(x) &= \log_{\sqrt{3}} \cos 5x; \\ f(x) &= \ln \log x. \end{aligned}$$

6.7. Inverse Trigonometric Functions

We would like now to define inverse trigonometric functions. But there is a problem: inverse functions exist only for one-to-one functions and trigonometric functions are not the such.

EXERCISE 6.7.1. Prove that a periodic function could not be a one-to-one function.

A way out could be as follows: we restrict a trigonometric function f to an interval I in such a way that f is one-to-one on I and there is no a bigger interval $I' \supset I$ that f is one-to-one on I'.

DEFINITION 6.7.2.
(1) The *arcsine* (*inverse sine function*) denoted arcsin is defined by the condition $y = \arcsin x$ iff $x = \sin y$ for $-1 \leqslant x \leqslant 1$ and $-\pi/2 \leqslant y \leqslant \pi/2$.
(2) The *arccosine* (*inverse cosine function*) denoted arccos is defined by the condition $y = \arccos x$ iff $x = \cos y$ for $-1 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant \pi$.
(3) The *arctangent* (*inverse tangent function*) denoted arctan is defined by the condition $y = \arctan x$ iff $x = \tan y$ for $x \in \mathbb{R}$ and $-\pi/2 \leqslant y \leqslant \pi/2$.
(4) The *arcsecant* (*inverse secant function*) denoted arcsec is defined by the condition $y = \operatorname{arcsec} x$ iff $x = \sec y$ for $|x| > 1$ and $y \in [0, \pi/2)$ or $y \in [\pi, 3\pi/2)$.

EXERCISE* 6.7.3. Why there is no a much need to introduce inverse functions for cotangent and cosecant?

Applying Theorem on derivative of an inverse function we could conclude that

THEOREM 6.7.4.

$$\begin{aligned} \frac{d}{dx}(\arcsin u) &= \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}; \\ \frac{d}{dx}(\arccos u) &= -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}; \\ \frac{d}{dx}(\arctan u) &= \frac{1}{1+u^2}\frac{du}{dx}; \\ \frac{d}{dx}(\operatorname{arcsec} u) &= \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}. \end{aligned}$$

As usually we could invert these formulas for taking antiderivatives:

THEOREM 6.7.5.

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-u^2}}\,du &= \arcsin\frac{u}{a}+C = -\arccos\frac{u}{a}+C;\\ \int \frac{1}{a^2+u^2}\,du &= \frac{1}{a}\arctan\frac{u}{a}+C;\\ \int \frac{1}{u\sqrt{u^2-a^2}}\,du &= \frac{1}{a}\operatorname{arcsec}\frac{u}{a}+C.\end{aligned}$$

And following formulas could be verified by differentiation:

THEOREM 6.7.6.

$$\begin{aligned}\int \arcsin u\,du &= u\arcsin u+\sqrt{1-u^2}+C;\\ \int \arccos u\,du &= u\arccos u-\sqrt{1-u^2}+C;\\ \int \arctan u\,du &= u\arctan u-\frac{1}{2}\ln(1+u^2)+C;\\ \int \operatorname{arcsec} u\,du &= u\operatorname{arcsec} u\ln\left|u\sqrt{u^2-1}\right|+C.\end{aligned}$$

6.8. Hyperbolic Functions

The following functions arise in many areas of mathematics and applications.

DEFINITION 6.8.1.

$$\begin{aligned}\textit{hyperbolic sine function}: &\quad \sinh x = \frac{e^x-e^{-x}}{2};\\ \textit{hyperbolic cosine function}: &\quad \cosh x = \frac{e^x+e^{-x}}{2};\\ \textit{hyperbolic tangent function}: &\quad \tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}};\\ \textit{hyperbolic cotangent function}: &\quad \coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}}.\end{aligned}$$

There are a lot of identities involving hyperbolic functions which are similar to the trigonometric ones. We will mentions only few most important of them

THEOREM 6.8.2.

$$\begin{aligned}\cosh^2 x-\sinh^2 x &= 1;\\ 1-\tanh^2 x &= (\cosh x)^{-2};\\ \coth^2 x-1 &= (\sinh x)^{-2}.\end{aligned}$$

From formula $(e^x)' = e^x$ easily follows the following formulas of differentiation:

THEOREM 6.8.3.

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \cosh u \frac{du}{dx};\\ \frac{d}{dx}(\cosh x) &= \sinh u \frac{du}{dx};\\ \frac{d}{dx}(\tanh x) &= (\cosh u)^{-2}\frac{du}{dx};\\ \frac{d}{dx}(\coth x) &= -(\sinh u)^{-2}\frac{du}{dx}.\end{aligned}$$

EXERCISE 6.8.4. Find derivative of functions

$$f(x) = \frac{1+\cosh x}{1+\sinh x}; \qquad f(x) = \ln|\tanh x|.$$

We again could rewrite these formulas for indefinite integral case:

THEOREM 6.8.5.

$$\begin{aligned}\int \sinh u\, du &= \cosh u + C;\\ \int \cosh u\, du &= \sinh u + C;\\ \int (\cosh u)^{-2}\, du &= \tanh u + C;\\ \int (\sinh u)^{-2}\, du &= \coth u + C.\end{aligned}$$

EXERCISE 6.8.6. Evaluate integrals

$$\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx; \qquad \int \frac{1}{\cosh^2 3x}\,dx.$$

6.9. Indeterminate Forms and l'Hospital's Rule

In this section we describe a general tool which simplifies evaluation of limits.

THEOREM 6.9.1 (Cauchy's Formula). *If* f *and* g *are continuous on* $[a,b]$ *and differentiable on* (a,b) *and if* $g'(x)\neq 0$ *for all* $x\in(a,b)$, *then there is a number* $w\in(a,b)$ *such that*

$$\frac{f(a)-f(b)}{g(a)-g(b)} = \frac{f'(w)}{g'(w)}.$$

PROOF. The proof follows from the application of Rolle's Theorem to the function

$$h(x) = [f(b)-f(a)]g(x) - [g(b)-g(a)]f(b).$$

□

THEOREM 6.9.2 (l'Hospital's Rule). *Suppose that* f *and* g *are differentiable on an open interval* (a,b) *containing* c, *except possibly at* c *itself. If* $f(x)/g(x)$ *has the indeterminate form* $0/0$ *or* ∞/∞ *then*

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided

$$\lim_{x\to c}\frac{f'(x)}{g'(x)} \quad \textit{exists or} \lim_{x\to c}\frac{f'(x)}{g'(x)}=\infty.$$

EXERCISE 6.9.3. Find the following limits

$$\lim_{x\to 1}\frac{x^3-3x+2}{x^2-2x-1}; \qquad \lim_{x\to 0}\frac{\sin x - x}{\tan x - x};$$
$$\lim_{x\to 0}\frac{x+1-e^x}{x^2}; \qquad \lim_{x\to 0^+}\frac{\ln\sin x}{\ln\sin 2x};$$
$$\lim_{x\to 0}\frac{e^x-e^{-x}-2\sin x}{x\sin x}; \qquad \lim_{x\to\infty}\frac{x\ln x}{x+\ln x}.$$

There are more indeterminant forms which could be transformed to the case $0/0$ or ∞/∞:

(1) $0\cdot\infty$: write $f(x)g(x)$ as
$$\frac{f(x)}{1/g(x)} \quad \text{or} \quad \frac{g(x)}{1/f(x)}.$$
(2) $0^0, 1^\infty, \infty^0$: instead of $f(x)^{g}(x)$ look for the limit L of $g(x)\ln f(x)$. Then $f(x)^{g}(x) = e^L$.
(3) $\infty-\infty$: try to pass to a quotient or a product.

EXERCISE 6.9.4. Find limits if exist.

$$\lim_{x\to 0^+}(e^x-1)^x; \qquad \lim_{x\to\infty}x^{1/x};$$
$$\lim_{x\to -3^-}\left(\frac{x}{x^2+2x-3}-\frac{4}{x+3}\right) \qquad \lim_{x\to\infty}\left(\frac{x^2}{x-1}-\frac{x^2}{x+1}\right);$$
$$\lim_{x\to 0}(\cot^2 x-\csc^2 x); \qquad \lim_{x\to 0^+}(1+3x)^{\csc x}.$$

CHAPTER 7

Techniques of Integration

We will study more advanced technique of integration.

7.1. Integration by Parts

Among different formulae of differentiation there is one which was not converted to the formulae of integration yet. This is derivative of a product of two functions. We will use it as follows:

THEOREM 7.1.1. *If* $u = f(x)$ *and* $v = g(x)$ *and if* f' *and* g' *are continuous, then*

$$\int u\,dv = uv - \int v\,du.$$

EXERCISE 7.1.2. Evaluate integrals.

$$\int xe^{-x}\,dx; \qquad \int x\sec x\tan x\,dx;$$
$$\int x\csc^2 3x\,dx; \qquad \int x^2\sin 4x\,dx;$$
$$\int e^x\cos x\,dx; \qquad \int \sin\ln x\,dx;$$
$$\int \cos\sqrt{x}\,dx; \qquad \int \sin^n x\,dx..$$

7.2. Trigonometric Integrals

To evaluate $\int \sin^m x\cos^n x\,dx$ we use the following procedure:

(1) **If m is an odd integer**: use the change of variable $u = \cos x$ and express $\sin^2 x = 1 - \cos^2 x$.
(2) **If n is an odd integer**: use the change of variable $u = \sin x$ and express $\cos^2 x = 1 - \sin^2 x$.
(3) **If m and n are even**: Use half-angle formulas for $\sin^2 x$ and $\cos^2 x$ to reduce the exponents by one-half.

EXERCISE 7.2.1. Evaluate integrals.

$$\int \sin^3 x\cos^2 x\,dx; \qquad \int \sin^4 x\cos^2 x\,dx.$$

To evaluate $\int \tan^m x\sec^n x\,dx$ we use the following procedure:

(1) **If m is an odd integer**: use the change of variable $u = \sec x$ and express $\tan^2 x = \sec^2 x - 1$.

(2) **If n is an even integer**: use the change of variable $u = \tan x$ and express $\sec^2 x = 1 + \tan^2 x$.
(3) **If m is an even and n is odd numbers**: There is no a standard method, try the integration by parts.

EXERCISE 7.2.2. Evaluate integrals.

$$\int \cot^4 x \, dx; \qquad \int \sin 4x \cos 3x \, dx.$$

7.3. Trigonometric Substitution

The following trigonometric substitution applicable if integral contains one of the following expression cases

Expression	**Substitution**
$\sqrt{a^2 - x^2}$	$x = a \sin\theta;$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta;$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta.$

EXERCISE 7.3.1. Evaluate integrals

$$\int \frac{1}{x^3\sqrt{x^2-25}} dx; \qquad \int \frac{1}{x^2\sqrt{x^2+9}} dx;$$
$$\int \frac{1}{(16-x^2)^{5/2}} dx; \qquad \int \frac{(4+x^2)^2}{x^3} dx.$$

7.4. Integrals of Rational Functions

To integrate a rational function $f(x)/g(x)$ we acomplish the following steps:
(1) If the degree of $f(x)$ is not lower than the degree of $g(x)$, use long division to obtain the proper form.
(2) Express $g(x)$ as a product of linear factors $ax + b$ or irreducible quadratic factors $cx^2 + dx + e$, and collect repeated factors so that $g(x)$ is a product of *different* factors of the form $(ax + b)^n$ or $(cx^2 + dx + e)^m$ for a nonnegative n.
(3) Find real coefficients A_{ij}, B_{ij}, C_{ij}, D_{ij} such that

$$\frac{f(x)}{g(x)} = \sum_{k=1}^{n} \left(\frac{A_{1k}}{a_k x + b_k} + \frac{A_{n_k k}}{(a_k x + b_k)^2} + \cdots + \frac{A_{n_k k}}{(a_k x + b_k)^n} \right)$$
$$+ \sum_{k=1}^{n} \left(\frac{C_{1k}x + D_{1k}}{c_k x^2 + d_k x + e_k} + \frac{C_{2k}x + D_{2k}}{(c_k x^2 + d_k x + e_k)^2} + \cdots + \frac{C_{1k}x + D_{1k}}{(c_k x^2 + d_k x + e_k)^n} \right).$$

EXERCISE 7.4.1. Evaluate integrals:

$$\int \frac{11x+2}{2x^2-5x-3} \, dx \qquad \int \frac{4x}{(x^2+1)^3} \, dx$$
$$\int \frac{x^4+2x^2+3}{x^3-4x} \, dx$$

7.5. Quadratic Expressions and Miscellaneous Substitutions

There are a lot of different substitutions which could be useful in particular cases. Particularly, if the integrand is a rational expression in $\sin x$, $\cos x$, the following substitution will produce a rational expression in u:

$$\sin x = \frac{2u}{1+u^2}, \qquad \cos x = \frac{1-u^2}{1+u^2}, \qquad dx = \frac{2}{1+u^2}\,du,$$

where $u = \tan\frac{x}{2}$ for $-\pi < x < \pi$.

EXERCISE 7.5.1. Evaluate integrals

$$\int \frac{1}{\sqrt{7+6x-x^2}}\,dx; \qquad \int \frac{1}{(x^2-6x+34)^{3/2}}\,dx;$$
$$\int \frac{1}{x(\ln^2 x + 3\ln x + 2)}\,dx.$$

7.6. Improper Integrals

We could extend the notion of integral for the following *integrals with with infinite limits* or *improper integral*

DEFINITION 7.6.1. (1) If f is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx,$$

provided the limit exists.

(2) If f is continuous on $(-\infty, a]$, then

$$\int_\infty^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx,$$

provided the limit exists.

(3) Let f be continuous for every x. If a is any real number, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx,$$

provided both of the improper integrals on the right converge.

EXERCISE 7.6.2. Determine if improper integrals converge and find the its value if so.

$$\int_0^\infty xe^{-x}\,dx; \qquad \int_{-\infty} \infty\cos^2 x\,dx;$$
$$\int_1^\infty \frac{x}{(1+x^2)^2}\,dx.$$

DEFINITION 7.6.3. (1) If f is discontinuous on $[a,b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx,$$

provided the limit exists.

(2) If f is discontinuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx,$$

provided the limit exists.

(3) If f has a discontinuity at c in the open interval (a, b) but is continuous elsewhere on $[a, b]$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx,$$

provided both of the improper integrals on the right converge.

EXERCISE 7.6.4. Determine if improper integrals converge and find the its value if so.

$$\int_0^4 \frac{1}{(4-x)^{2/3}}\, dx; \qquad \int_1^2 \frac{x}{x^2-1}\, dx;$$
$$\int_0^{\pi/2} \tan x\, dx; \qquad \int_{1/e}^{e} \frac{1}{x(\ln x)^2}\, dx.$$

CHAPTER 8

Infinite Series

8.1. Sequences

DEFINITION 8.1.1. A *sequence* is a function f whose domain is the set of positive integers.

EXAMPLE 8.1.2. (1) Sequense of even numbers: $a_1 = 2$, $a_2 = 4$, $a_3 = 6, \dots$
(2) Sequence of prime numbers: $a_1 = 2$, $a_2 = 3$, $a_3 = 5, \dots$

DEFINITION 8.1.3. A sequence $\{a_n\}$ *has the limit* L, or *convereges to* L, denoted by

$$\lim_{n\to\infty} a_n = L, \qquad \text{or} \qquad a_n \to L \text{ when } n \to \infty.$$

if for every $\epsilon > 0$ there exists a positive number N such that

$$|a_n - L| < \epsilon \quad \text{whenever} \quad n > N.$$

If such a number L does not exist, the sequence *has no limit*, or *diverges*.

DEFINITION 8.1.4. The notation

$$\lim_{n\to\infty} a_n = \infty, \quad \text{or} \quad a_n \to \infty \text{ or } n \to \infty.$$

means that for every positive real number P there exists a positive number N such that $a_n > P$ whenever $n > N$.

THEOREM 8.1.5. *Let $\{a_n\}$ be a sequence, let $f(n) = a_n$, and suppose that $f(x)$ exists for all real numbers $x > 1$.*

(1) *If* $\lim_{x\to\infty} f(x) = L$, *then* $\lim_{n\to\infty} a_n = L$.
(2) *If* $\lim_{x\to\infty} f(x) = \infty$, *then* $\lim_{n\to\infty} a_n = \infty$.

THEOREM 8.1.6.

$$\lim_{n\to\infty} r^n = 0 \quad \textit{if} \quad |r| < 1$$
$$\lim_{n\to\infty} r^n = \infty \quad \textit{if} \quad |r| > 1$$

EXERCISE 8.1.7. Check if the sequences are convergent

$$\left\{\frac{n^2}{\ln n + 1}\right\}; \quad \left\{\frac{\cos n}{n}\right\}; \quad \left\{\frac{e^n}{n^4}\right\};$$

THEOREM 8.1.8 (The Squeeze Rule, or theorem about two policemen). *Suppose that (a_n) and (b_n) are two sequences which tend to the same limit l as $n \to \infty$. Suppose that (c_n) is another sequence such that there exists $n_0 \in \mathbb{N}$ such that $a_n \leqslant c_n \leqslant b_n$ for each $n \geqslant n_0$. Then $c_n \to l$ as $n \to \infty$.*

EXERCISE 8.1.9. Show that $c_n = \sin(n^2)/n$ converge to 0.

8.2. Convergent or Divergent Series

DEFINITION 8.2.1. An *infinite series* (or *series*) is an expression of the form

$$\sum_{1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots.$$

Here a_n is nth *term* of the series.

DEFINITION 8.2.2. (1) The kth *partial sum* of the series is $S_k = \sum_{n=1}^{k} a_n$.
(2) The *sequence of partial sums* of the series is $S_1, S_2, S_3, \dots$

DEFINITION 8.2.3. A series is *convergent* or *divergent* iff the sequense of partial sums is correspondingly convergent or divergent. If limit of partial sum exists then it is the *sum* of series. A divergent series has no sum.

EXAMPLE 8.2.4. (1) Series $\sum \frac{1}{n(n+1)}$ is convergent with sum 1.
(2) Series $\sum (-1)^k$ is divergent.
(3) The *harmonic series* $\sum \frac{1}{n}$ is divergent.
(4) The *geometric series* $\sum ar^n$ is convergent with sum $\frac{a}{1-r}$ if $|r| < 1$ and divergent otherwise.

THEOREM 8.2.5. *If a series* $\sum a_n$ *is convergent, then* $\lim_{x\to\infty} a_n = 0$.

EXERCISE 8.2.6. Determine whether the series converges or diverges

$$\sum (\sqrt{2})^{n-1}; \qquad \sum (\sqrt{3})^{1-n};$$
$$\sum \frac{-1}{(n+1)(n+2)}; \qquad \sum (n+3)^{-1}.$$

8.3. Positive-Term Series

We will investigate first *positive-term series*—that is, series $\sum a_n$ such that $a_n > 0$ for all n—and will use these result for series of general type.

THEOREM 8.3.1. *If* $\sum a_n$ *is a positive-term series and if there exists a number* M *such that*

$$S_n = a_1 + a_2 + \ldots + a_n < M$$

for every n*, then the series converges and has a sum* $S \leqslant M$*. If no such* M *exists, the series diverges.*

THEOREM 8.3.2. *If* $\sum a_n$ *is a series, let* $f(n) = a_n$ *and let* f *be the function obtained by replacing* n *with* x*. If* f *is positive-valued, continuous, and decreasing for every real number* $x \geqslant 1$*, then the series* $\sum a_n$
(1) *converges if* $\int_1^\infty f(x)\,dx$ *converges;*
(2) *diverges if* $\int_1^\infty f(x)\,dx$ *diverges.*

DEFINITION 8.3.3. A p-*series*, or a *hyperharmonic series*, is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

where p is a positive real number.

THEOREM 8.3.4. *The* p*-series* $\sum \frac{1}{n^p}$

(1) *converges if* $p > 1$;
(2) *diverges if* $p \leqslant 1$.

PROOF. The proof is a direct application of the Theorem 8.3.2. □

THEOREM 8.3.5 (Basic Comparison Tests). *Let* $\sum a_n$ *and* $\sum b_n$ *be positive-term series.*

(1) *If* $\sum b_n$ *converges and* $a_n \leqslant b_n$ *for every positive integer* n, *then* $\sum a_n$ *converges.*
(2) *If* $\sum b_n$ *diverges and* $a_n \geqslant b_n$ *for every positive integer* n, *then* $\sum a_n$ *diverges.*

THEOREM 8.3.6 (Limit Comparison Test). *Let* $\sum a_n$ *and* $\sum b_n$ *be positive-term series. If there is a positive number c such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge or both series diverge.

EXERCISE 8.3.7. Determine convergency

$$\sum \frac{\ln n}{n}; \qquad \sum \frac{1}{1+16n^2};$$
$$\sum \sin n^4 e^{-n^5}; \qquad \sum \frac{1}{n^n};$$
$$\sum \frac{3n+5}{n2^n}; \qquad \sum \frac{n^2}{n^3+1};$$
$$\sum \tan \frac{1}{n}; \qquad \sum \frac{\sin n + 2^n}{n+5^n}.$$

8.4. The Ratio and Root Tests

The following two test of divergency are very important. Yet there several cases then they are not inconclusive (see the third clause).

THEOREM 8.4.1 (Ratio Test). *Let* $\sum a_n$ *be a positive-term series, and suppose that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

(1) *If* $L < 1$, *the series convergent.*
(2) *If* $L > 1$ *the series divergent.*
(3) *If* $L = 1$, *apply a different test; the series may be convergent or divergent..*

EXERCISE 8.4.2. Determine convergency

$$\sum \frac{100^n}{n!}; \qquad \sum \frac{3n}{\sqrt{n^3+1}}.$$

THEOREM 8.4.3 (Root Test). *Let* $\sum a_n$ *be a positive-term series, and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L.$$

(1) *If* $L < 1$, *the series convergent.*
(2) *If* $L > 1$ *the series divergent.*
(3) *If* $L = 1$, *apply a different test; the series may be convergent or divergent..*

EXERCISE 8.4.4. Determine convergency

$$\sum \frac{2^n}{n^2}; \qquad \sum \left(\frac{n}{\ln n}\right)^n;$$
$$\sum \frac{n!}{n^n}; \qquad \sum \frac{1}{(\ln n)^n}.$$

8.5. Alternating Series and Absolute Convergence

The simplest but still important case of non positive-term series are *alternating series*, in which the terms are alternately positive and negative..

THEOREM 8.5.1 (Alternating Series Test). *The alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n + \cdots$$

is convergent if the following two conditions are satisfied:

(1) $a_n \geqslant a_{k+1} \geqslant 0$ *for every* k;
(2) $\lim a_n = 0$.

THEOREM 8.5.2. *Let* $\sum (-1)^{n-1} a_n$ *be an alternating series that satisfies conditions* (i) *and* (ii) *of the alternating series test. If* S *is the sum of the series and* S_n *is a partial sum, then*

$$|S - S_n| \leqslant a_{n+1};$$

that is, the error involved in approximating S *by* S_n *is less that or equal to* a_{n+1}.

DEFINITION 8.5.3. A series $\sum a_n$ is *absolutely convergent* if the series

$$\sum |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots$$

is convergent.

DEFINITION 8.5.4. A series $\sum a_n$ is *conditionally convergent* if $\sum a_n$ is convergent and $\sum |a_n|$ is divergent.

THEOREM 8.5.5. *If a series* $\sum a_n$ *is absolutely convergent, then* $\sum a_n$ *is convergent.*

EXERCISE 8.5.6. Determine convergency

$$\sum \frac{(-1)^n 2}{n^2 + n}; \qquad \sum (-1)^n \frac{\sqrt[3]{n}}{n+1};$$
$$\sum (-1)^n \frac{\arctan n}{n^2}; \qquad \sum \frac{1}{n} \sin \frac{(2n-1)\pi}{2}.$$

APPENDIX A

Number Systems

We briefly recall some basic notion and results from algebra.

A.1. Numbers: Filling the Gaps on the Real Line

A.1.1. Natural Numbers. Natural numbers are $1, 2, 3, 4, \ldots$. They are used for counting of similar objects in real life. The set of natural numbers denoted by $\mathbb{N}$.

(1) Operations with natural numbers: addition, multiplication, subtraction, division;
(2) Order of operations and brackets.
(3) Rules: commutativity, associativity, distribution law.
(4) Prime, composite and co-prime numbers.

A.1.2. Number 0. Zero is the first example of mathematical abstraction and is *not* a natural number.

It is true and *natural* to say that the English alphabet contains 26 letters. It is still true but is *not* very natural to say that English alphabet contains 0 hieroglyphs.

DEFINITION A.1.1. A zero can be characterised by the property that its addition to any natural number does not change it:

$$0 + n = n. \tag{64}$$

The following properties of zero can be proved:

THEOREM A.1.2. (1) *Zero is unique, i.e. any two numbers satisfying (64) are equal.*
(2) *For any natural number n we have $0 \cdot n = 0$.*

A.1.3. Integers. Integers or entire or whole numbers appear as solutions of equations like $x + 3 = 5$. This equation (and many others) has the solution among natural numbers. However many other equations, e.g. $x + 5 = 3$ does not have a natural root.

Integers, which include positive, negative numbers and 0, obey the same rules as natural number. Their set is denoted by $\mathbb{Z}$. The *integer* numbers are: $\ldots, -2, -1, 0, 1, 2, \ldots$

A.1.4. Rationals. $\mathbb{Q}$—*rational* numbers of the form $\frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and m should be co-prime with n.

They are required to provide solutions (roots) to the equations of the sort $3x = 5$. Rational numbers can be also represented by decimal fractions: either finite or periodic. By the way finite decimal fraction can be considered like a periodic with single-digit period of zeros.

EXERCISE A.1.3. Show that a rational numbers are in one-to-one correspondence with periodic decimal fractions (if we exclude single-digit periods of nines).

Rationals are sufficient for any practical activity (measurement) since we use them up to an arbitrary precision.

A.1.5. Reals.

THEOREM A.1.4. *The equation* $x^2 = 2$ *does not have a rational root.*

PROOF. (Known for more than 2500 years) Let $x = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ and m shall be co-prime with n. Then:

$$\begin{aligned} \frac{m^2}{n^2} &= 2 \\ m^2 &= 2n^2, \end{aligned}$$

where m shall be even, i.e. $m = 2k$ for some $k \in \mathbb{Z}$. Therefore:

$$\begin{aligned} (2k)^2 &= 2n^2 \\ 4k^2 &= 2n^2 \\ 2k^2 &= n^2, \end{aligned}$$

thus n shall be even as well. But then m and n are not co-prime—contradiction. Our initial assumption about existence of a rational root shall be false. □

Such numbers are called *irrationals*. Other examples are $\sqrt{3}$, π, e. They are sitting in between of real numbers and "filling gaps" among them. Irrational numbers can be approximated with any desirable precision by rational numbers.

EXERCISE A.1.5. Show that irrational numbers are in one-to-one correspondence with aperiodic decimal fractions.

Rational and irrational numbers together (that is all decimal fractions—periodic and aperiodic) form *reals* or *real numbers*. The notation is $\mathbb{R}$.

A.1.6. Real Axis. We will be mainly interested in real numbers $\mathbb{R}$ which could be represented by the *coordinate line* (or *real axis*). This gives *one-to-one* correspondence between sets of real numbers and points of the real line.

A.1.7. Absolute Value. The *absolute value* $|a|$ (or *modulus*) of a real number a is defined as follows

$$|a| = \begin{cases} a & \text{if } a \geqslant 0, \\ -a & \text{if } a < 0. \end{cases}$$

It has the following properties (for $b > 0$)

(1) $|a| < b$ iff[1] $-b < a < b$.
(2) $|a| > b$ iff either $a > b$ or $a < -b$.
(3) $|a| = b$ iff $a = b$ or $a = -b$.

EXERCISE A.1.6. Prove the following properties of absolute value:

(1) $|a + b| \leqslant |a| + |b|$.
(2) $|ab| = |a|\,|b|$.

[1]The notation *iff* is used for an abbreviation of *if and only if*.

An *equation* is an equality that involves variables, e.g. $x^3 + 5x^2 - x + 10 = 0$. A *solution* of an equation (or *root* of an equation) is a number b that produces a true statement after substitution $x = b$ into equation. Equation could be solved by either *analytic* or *computational* means.

A.2. Complex Numbers

Our presentation of complex numbers follows V.I. Arnold (2002) approach.

A.2.1. Geometric representation. Consider an orthonormal (Cartesian) coordinates on the Euclidean plane. We denote basis vectors by 1 and i. Then any point on the plane can be represented as

$$z = a + b \cdot i. \tag{65}$$

Here i also called *imaginary unit* and such two-dimensional vectors are known as *complex numbers*. Such geometric representation of complex numbers is known as *Argand diagram*, see Fig. A.1. The expression (65) is known as the *standard form* of a complex number.

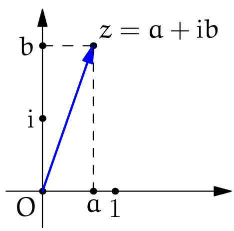

FIGURE A.1. Argand diagram for a complex number

DEFINITION A.2.1. For a complex number $z = a + b \cdot i$ a is called *real part* of z and b is called *imaginary part* of z.

We identify numbers $a + 0 \cdot i$ with reals and will call a number $0 + b \cdot i = b \cdot i$ *purely imaginary* one. A complex number is zero if and only if both its real and imaginary parts are equal to zero.

Any two such vectors can be added by the known rule of component-wise addition:

$$\begin{array}{rcc} z_1 = & a_1 + & b_1 \cdot i \\ z_1 = & a_1 + & b_1 \cdot i \\ \hline z_1 + z_2 = & (a_1 + a_2) + & (b_1 + b_2) \cdot i \end{array}$$

Geometrically this corresponds to the addition of vectors by parallelogram rule, see Fig. A.2.

We also define multiplication of two complex numbers to be a commutative and associative operations defined by the multiplication table:

$$1 \cdot 1 = 1, \qquad 1 \cdot i = i \cdot 1 = i,$$

and the most important multiplication:

$$i \cdot i = -1.$$

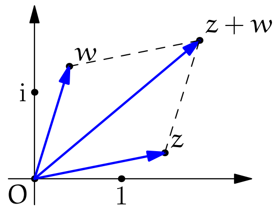

FIGURE A.2. Addition of complex numbers coinsides with vector addion

Thus it is clear that the imaginary unit i is not a real number. From the above definition the product of two complex numbers has the following algebraic expression:

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i. \tag{66}$$

EXERCISE A.2.2. Check that the above multiplication satisfies to distributive law with respect to addition.

A.2.2. Transformations of a plane. Complex numbers are useful to represent isometric maps of Euclidean plane.

DEFINITION A.2.3. Complex number

$$\bar{z} = a - b \cdot i$$

is called *complex conjugated* of $z = a + b \cdot i$.

Geometrically complex conjugation corresponds to the reflection in the horizontal (real) axis, see Fig. A.3. Note that the second conjugation coincides with the initial number: $\bar{\bar{z}} = z$.

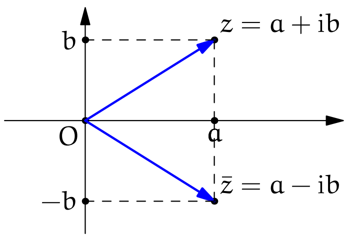

FIGURE A.3. Geometrically complex conjugation is the reflection in the real line

THEOREM A.2.4. (1) *Conjugated of a sum is the sum of conjugated:*

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

(2) *Conjugated of a product is the product of conjugated:*

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

PROOF. The first statement is obvious from its geometric interpretation. To verify second we can make complex conjugation of both side in the identity (66):

$$\bar{z}_1\bar{z}_2 = (a_1a_2 - b_1b_2) - (a_1b_2 + a_2b_1)i = \overline{z_1z_2}.$$

□

DEFINITION A.2.5. The product of a complex number and its conjugated is called the square of its *absolute value* (or *modulus*):

$$|z|^2 = z\bar{z}.$$

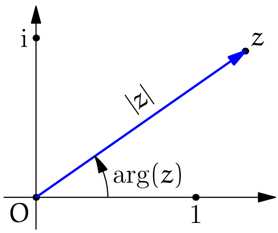

FIGURE A.4. Modulus of a complex number is the length of the corresponding vector.
Argument is the angle from the real axis to the direction of the vector.

LEMMA A.2.6. (1) $|z|^2 = a^2 + b^2$.
(2) *Geometrically modulus represent the magnitude of the vector z, see Fig. A.4.*
(3) *Absolute value is non-negative real number. It is zero only for $z = 0$.*

PROOF. The first statement is a direct calculation. Then the second statement follows from the Pythagoras Theorem. For the last statement: the square of the modulus is not changed by the conjugation:

$$\overline{z\bar{z}} = \bar{z}\bar{\bar{z}} = \bar{z}z = z\bar{z}.$$

Therefor this is a real number. Since $a^2 + b^2 \geqslant 0$ the modulus is a non-negative real as well. □

We can reduce division of complex numbers to their multiplication using the formula:

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2}.$$

From this formula become clear that we can divide by any complex number with non-zero modulus, that means by any non-zero complex number.

DEFINITION A.2.7. The *argument* of a non-zero complex number is the rotational angle from the real line to the direction of this complex number taken towards the imaginary axis, see Fig. A.4. Notation: $\arg(z)$.

Note that argument is defined up to a integer multiple of 2π. Thus we give the following definition as well

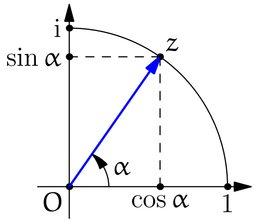

FIGURE A.5. Real and imaginary parts of a unimodular complex number and trigonometric functions

DEFINITION A.2.8. The *principal value of argument* of a non-zero complex number is the value of its argument from the interval from $-\pi$ to π. Notation: $\operatorname{Arg}(z)$.

We may define functions sin and cos as real and imaginary parts of a complex number with unit modulus and argument α, see Fig. A.5:

$$a = \cos\alpha, \qquad b = \sin\alpha, \qquad \text{where } \alpha = \arg(z).$$

We call a complex number with unit modulus *unimodular*.

THEOREM A.2.9. *Multiplication by a complex number with unit modulus is a rotation of the Euclidean plane.*

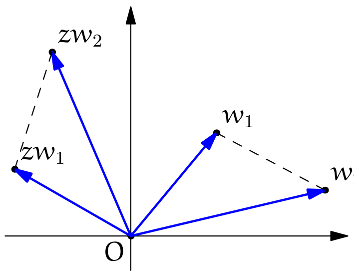

FIGURE A.6. Multiplication by unimodular number as a rotation

PROOF. We can easily see that multiplication by a unimodular number preserves modulus:

$$|zw|^2 = zw\overline{zw} = z\bar{z}w\bar{w} = |z|^2\,|w|^2 = |w|^2\,.$$

Distance between any two vectors is preserved as well, see Fig. A.6:

$$|z(w_1 - w_2)| = |w_1 - w_2|\,.$$

We also see that the multiplication maps 0 to 0, therefore multiplication is an isometry of Euclidean plane with a *fixed point*.

Moreover, if $z \neq 1$ this fixed point is unique. Indeed assume $zw = w$, then $(z-1)w = 0$ for $z \neq 1$ this implies $w = 0$.

Since the only isometries of the plane with a single fixed point are rotations, the proof is finished. □

We also have the following easy Lemma:

LEMMA A.2.10. *Multiplication of complex numbers by a real number* x *preserve their arguments and multiply their absolute values by* x.

REMARK A.2.11. If non-identical isometry fixes two different points, then it fixes the entire straight line passing through these two points and is a reflection in this line.

The important difference between rotations and reflections is that former preserve *orientation* of the plane and later (e.g. complex conjugation) revert it.

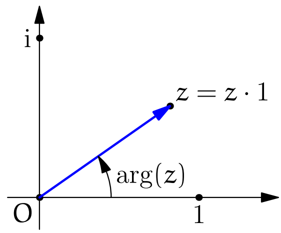

FIGURE A.7. The angle of rotation may be found by transformation of the number 1

REMARK A.2.12. Considering the product $z = 1 \cdot z$ we get the angle of the rotation: $\arg(z)$, since the initial $\arg(1) = 0$, see Fig. A.7.

THEOREM A.2.13. *Under multiplication of complex numbers their moduli are multiplied and their arguments are added:*

$$|zw| = |z| \cdot |w|, \tag{67}$$

$$\arg(zw) = \arg(z) + \arg(w). \tag{68}$$

PROOF. The first identity is simple:

$$|zw|^2 = zw \cdot \overline{zw} = z\bar{z} \cdot w\bar{w} = |z|^2 |w|^2 .$$

To obtain the second identity we introduce a complex number $z' = \frac{z}{|z|}$. Check that $|z'| = 1$ and $\arg(z') = \arg(z)$. Moreover:

$$zw = |z| \frac{z}{|z|} w = |z| z' w.$$

Therefore $\arg(zw) = \arg(z'w)$ by Lemma A.2.10.

Then (68) follows from the previous Theorem since z' is unimodular and the vector $z'w$ is obtained from the vector w by the rotation by the angle $\arg(z)$. □

EXERCISE A.2.14. Explain why the identity (68) would be wrong if we replace all arguments by their principal values.

Besides the standard form of a complex number $z = a + ib$ we can use the polar representation:

$$z = r(\cos\alpha + i\sin\alpha), \qquad \text{where } r = |z|, \quad \alpha = \arg(z). \tag{69}$$

It can be rewritten with the help of the *Euler identity*:

(70) $$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

as follows:

$$z = re^{i\alpha}, \qquad \text{where } r = |z|, \quad \alpha = \arg(z).$$

From the geometrical meaning of multiplication by a unimodular complex number as rotation we get the identity:

(71) $$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}.$$

It links the expression $e^{i\alpha}$ in the Euler identity (70) with the exponent e^x of the real variable.

Geometrically representations (65) and (69) correspond to the Cartesian and polar coordinates on the Euclidean plane respectively.

From the polar representation we get trigonometric identities for sum of arguments. For example, take the identity (71), which is presented in the polar form. If written in the standard form it produces the following equality of the real parts for the left-hand and right-hand sides:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

We also can obtain the following corollary from the Theorem A.2.13

COROLLARY A.2.15 (De Moivre's Theorem). *For any $n \in \mathbb{N}$ we have:*

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

Reverting this formula we can find that there are exactly n different complex n-th roots of a complex number:

$$w_k = r^{1/n}\left(\cos\frac{\alpha+2\pi k}{n} + i\sin\frac{\alpha+2\pi k}{n}\right),$$

where $k = 0, 1, \ldots, n-1$ and $w_k^n = r(\cos\alpha + i\sin\alpha)$. Complex roots are at vertices of a regular polygon. Fig. A.8 shows several examples. It is easy to see a pattern there.en

A.3. Mathematical Induction

The method of proof which we consider now is based on the fundamental properties of the set of natural numbers. There is the following description of mathematical induction in Wikipedia:

The simplest and most common form of mathematical induction proves that a statement involving a natural number n holds for all values of n. The proof consists of two stages:

(1) The basis (base case): showing that the statement holds when $n = 0$.
(2) The inductive step: showing that if the statement holds for some k, then the statement also holds when $k+1$ is substituted for n.

See the Wikipedia or `http://www.tech.plym.ac.uk/maths/resources/PDFLaTeX/inductio` `pdf` for further details.

EXAMPLE A.3.1. Prove by mathematical induction:

(1) For $n \geqslant 0$, $n^3 - n$ is divisible by 3.
(2) For $n \geqslant 0$, $5^n - 1$ is divisible by 4.

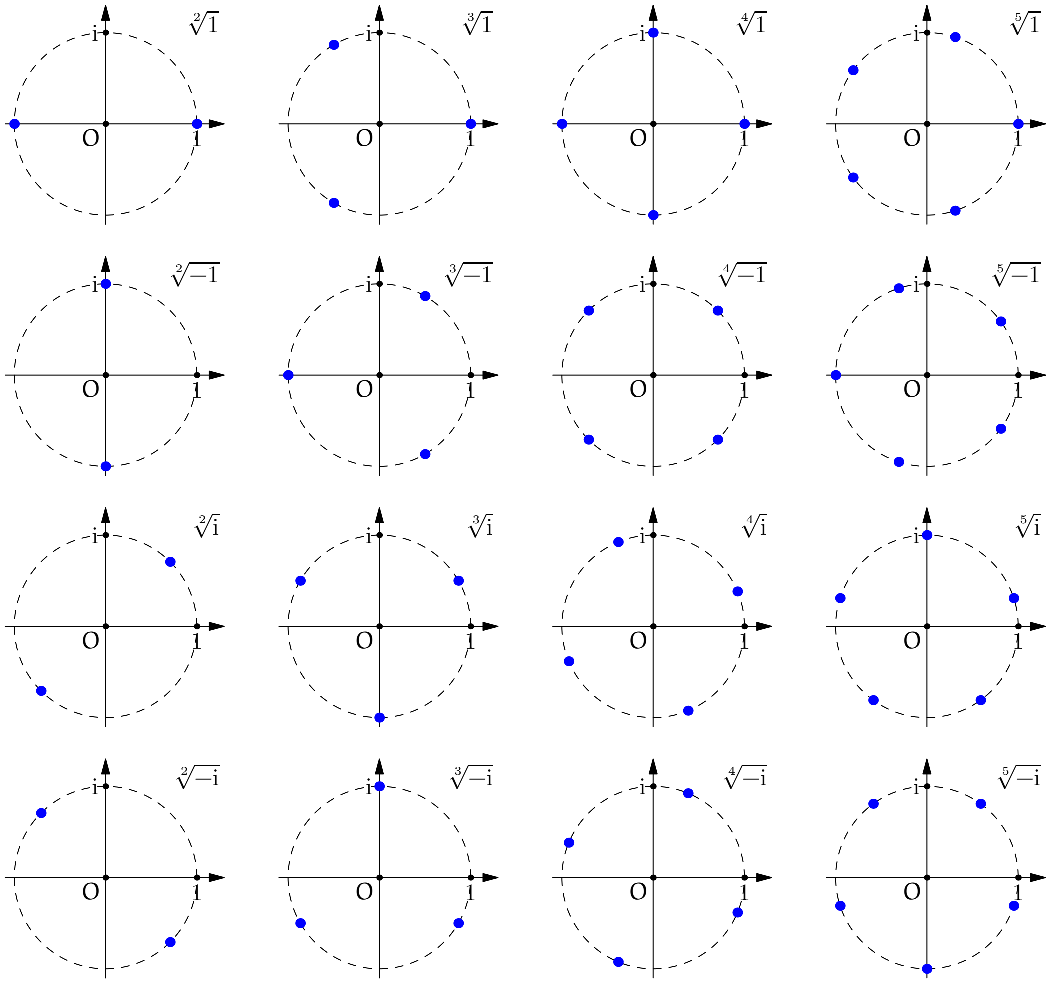

FIGURE A.8. Examples of complex roots

(3) For $n \geqslant 0$, $3^{2n} + 4^{n+1}$ is divisible by 5.
(4) For $n \geqslant 0$, 6 divides $n^3 - n$.
(5) For $n \geqslant 8$, $2^n \geqslant 3n^2 + 5$.
(6) For $n \geqslant 1$, $2^n > n$.
(7) Find expression from first several terms and prove them by the mathematical induction $\sum\limits_{r=1}^{n} r$, $\sum\limits_{r=1}^{n} r(r+1)$, $\sum_{r=1}^{n} r(r+1)(r+2)$.
(8) $(1+\frac{1}{3})^n \geqslant 1 + \frac{n}{3}$.
(9) Prove by induction a given formula for $\sum\limits_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2} - \frac{1}{n+2}$.

(10) $\sum_{j=1}^{n} j = 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.

(11) $\sum_{j=1}^{n} q^j = 1 + q + q^2 + \ldots + q^n = \dfrac{q^{n+1} - 1}{q - 1}$

(12) $\sum_{j=2}^{n} \dfrac{1}{(j-1)j} = \dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdot + \dfrac{1}{(n-1)n} = 1 - \dfrac{1}{n}$.

(13) $\sum_{j=1}^{n} \dfrac{1}{4j^2 - 1} = \dfrac{1}{3} + \dfrac{1}{15} + \dfrac{1}{35} + \cdots + \dfrac{1}{4n^2 - 1} = \dfrac{n}{2n+1}$.

(14) Consider

$$\begin{aligned} 1 &= 1 \\ 1 - 4 &= -(1+2) \\ 1 - 4 + 9 &= 1 + 2 + 3 \\ 1 - 4 + 9 - 16 &= -(1 + 2 + 3 + 4). \end{aligned}$$

Spot the pattern, formulate the statement and prove it by mathematical induction.

(15) $\prod_{j=2}^{n} \left(1 - \dfrac{1}{j^2}\right) = \left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right) \cdots \left(1 - \dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$.

(16) $\sum_{j=0}^{n} \dfrac{2^j x^{2^j}}{1 + x^{2^j}} = \dfrac{x}{1-x} - \dfrac{2^{n+1} x^{2^{n+1}}}{1 + x^{2^{n+1}}}$

(17) $\sum_{j=1}^{n} (j-1)j^2 = 1 \cdot 2 + 2 \cdot 3^2 + \ldots + (n-1)n^2 = \dfrac{n(n^2-1)(3n+2)}{12}$.

(18) $(1 - a_1)(1 - a_2) \cdots (1 - a_n) > 1 - a_1 - a_2 - \ldots a_n$, if $0 < a_i < 1$ for all i.

(19) $\dfrac{1}{2} \cdot \dfrac{3}{4} \cdot \dfrac{5}{6} \cdots \dfrac{2n-1}{2n} < \dfrac{1}{\sqrt{3n+1}}$.

(20) $\dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n} > \dfrac{13}{14}$.

(21) Let a convex polygon U be inside in a convex polygon V. Then the perimeter of U is less than the perimeter V.

(22) Find an error in the following "proof" of the statement: any set of straight lines has a common point.

- **Base of induction:** for one line it is trivial and for two lines is clear if we agree that two parallel line have intersection point at infinity.
- **Inductive step:** Let any k lines have a common point. If we ad one more line and consider k lines including the new one and excluding an old one. Then the new set has a common point as well. It is the same as the common point of first k lines since $k-1$ lines are common in both sets.

A.4. Polynomial. Factorisation of Polynomials

A *polynomial* $p(x)$ (in a variable x) is a function on real line defined by an expression of the form:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0. \tag{72}$$

Here a_i are fixed real numbers, $a_n \neq 0$ and n is the degree of polynomial $p(x)$.

According to the Main Theorem of algebra every polynomial $p(x)$ could be represented as a product of linear binomials and quadratic terms as follows:

$$p(x) = (b_1 x + c_1)\cdots(b_j x + c_j)(d_1 x^2 + f_1 x + g_1)\cdots(d_k x^2 + f_k x^2 + g + k),$$

moreover $n = 2k + j$, where n is the degree of $p(x)$.

EXERCISE A.4.1. Decompose to products:

(1) $p(x) = x^{16} - 1$.
(2) $p(x) = x^4 + 16y^4$.

A.5. Binomial Formula

Binomial formula of Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Here $n! = 1 \cdot 2 \cdots n$. These coefficients can be determined from the *Pascal triangle*.

EXERCISE* A.5.1. Prove by induction the following properties of the binomial coefficients

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k+1}.$$

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k+1}.$$

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n.$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

A.6. Inequalities

Order relations between numbers are given by $>, <, \leqslant, \geqslant, =$.

They have the following properties:

(1) If $a > b$ and $b > c$, then $a > c$ (*transitivity property*).
(2) If $a > b$, then $a \pm c > b \pm c$.
(3) If $a > b$ and $c > 0$, then $ac > bc$.
(4) If $a > b$ and $c < 0$, then $ac < bc$.
(5) If $a > b$ and $b > 0$, then $a^2 > b^2$.

An *inequality* is a statement involves variables and at least one of symbols $>$, $<$, $\leqslant$, $\geqslant$, e.g. $x^3 > 2x^2 - 5x + 1$. *Solution* of an inequality is similar for the case of equations (see Section A.1.7). They are often given by unions of intervals.

Intervals on real line are the following sets:

$$[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}; \qquad (a, b) = \{x \in \mathbb{R} \mid a < x < b\};$$
$$(a, b] = \{x \in \mathbb{R} \mid a < x \leqslant b\}; \qquad [a, b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}.$$

Particularly a can be $-\infty$ and $b = \infty$.

APPENDIX B

Sets and Relations

B.1. Set Theory

B.1.1. Sets and elements. A *set* S is a collection of elements of any nature. The main property: for an object x it is meaningful to discuss either it is in the set S (denoted by $x \in S$) or it is not ($x \notin S$). We often define a set by describing a property such that any object possessing the property is in the set and is not otherwise. A small set can be presented by the list of its elements.

It is possible that a set does not contain any element, then it is an *empty set*. We denote by $\emptyset$.

Two sets are equal if they contain exactly the same elements. We denote it by $A = B$.

EXAMPLE B.1.1. For intervals on the real line we use notations $[a, b]$ if both endpoints are included and $(a, b]$, $[a, b)$, (a, b) if the respective endpoints are excluded. We also consider infinite intervals of the type $[a, \infty)$.

B.1.2. Subset. Sets have a partial order defined by an inclusion.

DEFINITION B.1.2. If any element of a set A belongs to a set B then we call A a *subset* of B. Denoted: $A \subset B$ or $B \supset A$.

If there is at least one element of A which is *not* in B then A is not a subset of B: $A \not\subset B$. From the definition we always have $A \subset A$. We also agree by definition that $\emptyset \subset A$ for any set A.

EXERCISE B.1.3. List all subsets of the set $A = \{a, b, c\}$.

B.1.3. Intersection. We can define several operations on sets.

DEFINITION B.1.4. Let A and B be two sets. Their *intersection* is the set of all elements which belong to both A and B. Denote: $A \cap B$.

We can also define intersection as the biggest set which is a subset to both A and B.

EXERCISE B.1.5. Describe the intersection of the set of all rectangles with the set of all rhombuses.

If two sets do not have any common element we can say that they do not intersect or that their intersection is the empty set: $A \cap B = \emptyset$.

EXERCISE B.1.6. What is the intersection of the empty set with a set A?

We can define intersection of several sets in a similar way, it will obey the *associative law*:

$$(A \cap B) \cap C = A \cap (B \cap C).$$

B.1.4. Union. This is another important operation on sets.

DEFINITION B.1.7. Let A and B be two sets. Their *union* is the set of all elements which belong at least to one of A or B. Denote: $A \cup B$.

We can also define union as the smallest set such that both A and B are its subsets.

EXERCISE B.1.8. What is the union of the empty set with a set A?

Again union can be taken for several sets due to associativity of this operation.
We can represents sets and their relations by means of the Euler diagrams.

B.1.5. Some identities. There are some easy-to-check identities for the set operations. Some of them are in a clear resemblance to well-known properties of the numbers:

$$\begin{aligned}
A \cup B &= B \cup A, & a+b &= b+a & &\text{(commutativity)}\\
A \cap B &= B \cap A, & ab &= ba & &\text{(commutativity)}\\
(A \cup B) \cup C &= (A \cup B) \cup C, & (a+b)+c &= a+(b+c) & &\text{(associativity)}\\
(A \cap B) \cap C &= (A \cap B) \cap C, & (ab)c &= a(bc) & &\text{(associativity)}\\
(A \cup B) \cap C &= (A \cap C) \cup (B \cap C), & (a+b)c &= a(bc) & &\text{(distributivity)}
\end{aligned}$$

However this is not true for all identities. For example, the following identities do not correspond to properties of numbers:

$$\begin{aligned}
(A \cup C) \cap (B \cup C) &= (A \cap B) \cup C\\
A \cup A &= A\\
A \cap A &= A
\end{aligned}$$

We will distinguish sets with a finite number of elements and infinite one. The former will be called a *finite set*.

DEFINITION B.1.9. For a finite set A we denote by $m(A)$ the number of its elements.

We have a formula:

$$m(A \cap B) + m(A \cap B) = m(A) + m(B).$$

It is true because if we count elements of A first and then elements of B then we have counted all elements of $A \cup B$ with elements $A \cap B$ being counted twice.

B.1.6. Completion and the Base Set. It is often that we are focused on sets which are all subsets of a certain large set E. We will call it the *base set*. For example, we can consider sets of real solutions for algebraic equations, they all are subsets of $\mathbb{R}$. It is clear that for any subset A of the base set E we have:

$$A \cap E = A, \qquad A \cup E = E.$$

DEFINITION B.1.10. All elements of B which are not in a set A are called the *compliment* of A in B. We denote it $B \backslash A$. Complements of a set A in the base set E is denoted $\bar{A}$.

EXAMPLE B.1.11. A complement of $\mathbb{Q}$ to $\mathbb{R}$ is the set of irrational numbers.

We have the following simple identities:

$$A \cap \bar{A} = \emptyset, \qquad A \cup \bar{A} = E.$$

Moreover the following properties can be visualised by Euler diagrams:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}, \qquad \overline{A \cap B} = \bar{A} \cup \bar{B}.$$

B.2. Binary Relation

B.2.1. Cartesian Product of Sets. This is a very important operation and we have already seen its usage when introduced a complex number as a pair of reals.

DEFINITION B.2.1. Let A and B be two sets. Their *Cartesian product* $A \times B$ is the set of all pairs of the form (a, b), there $a \in A$ and $b \in B$.

EXAMPLE B.2.2.
(1) Let $F = \{\text{John}, \text{Adam}, \ldots\}$ be the collection of first names, and $L = \{\text{Smith}, \text{Brown}, \ldots\}$be the collection of last names. Then $F \times L$ makes the set of full names, e.g. $(\text{Adam}, \text{Smith})$.
(2) The product of real axis $\mathbb{R}$ with itself, denoted $\mathbb{R}$ can be identified with complex numbers $\mathbb{C} = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$: $(a, b) = a + ib$.

B.2.2. Binary Relation.

DEFINITION B.2.3. A *binary relation* R on a set X is a subset of its Cartesian square $X \times X$. We say that x is R-related to y if the pair (x, y) is in $R \subset X \times X$. We also denote it xRy.

EXAMPLE B.2.4. There are many relations already known to us: "is greater than", "is equal to", and "divides" in arithmetic, "is congruent to" in geometry, "is adjacent to" in graph theory, $|x - y| < \varepsilon$ is another example.

We may specify relations with certain properties:

- *reflexive*: for all x in X it holds that xRx. For example, "greater than or equal to" is a reflexive relation but "greater than" is not.
- *irreflexive*: for all x in X it holds that not xRx. "Greater than" is an example of an irreflexive relation.
- *coreflexive*: for all x and y in X it holds that if xRy then $x = y$.
- *symmetric*: for all x and y in X it holds that if xRy then yRx. "Is a blood relative of" is a symmetric relation, because x is a blood relative of y if and only if y is a blood relative of x.
- *antisymmetric*: for all x and y in X it holds that if xRy and yRx then $x = y$. "Greater than or equal to" is an antisymmetric relation, because if xy and yx, then $x = y$.
- *asymmetric*: for all x and y in X it holds that if xRy then not yRx. "Greater than" is an asymmetric relation, because if $x > y$ then not $y > x$.
- *transitive*: for all x, y and z in X it holds that if xRy and yRz then xRz. "Is an ancestor of" is a transitive relation, because if x is an ancestor of y and y is an ancestor of z, then x is an ancestor of z.

B.2.3. Equivalence Relation.

DEFINITION B.2.5. Let R be a reflexive, symmetric and transitive relation on a set X. Then R is called *equivalence relation* on X. We equivalence relation by $\sim$.

The set of all $y \in X$ such that $x \sim y$ is called the *equivalence class* of x and is denoted $[x]$.

PROPOSITION B.2.6. *Let* R *is an equivalence relation on* X. *Two equivalent classes* $[x]$ *and* $[y]$ *are either coincide or disjoint.*

PROOF. Let two equivalence classes $[x]$ and $[y]$ have a common element z. This means $x \sim z$ and $y \sim z$. By symmetry we also have $z \sim y$ then from transitivity $x \sim y$.

For any other element $w \in [y]$, that is $y \sim w$, we obtain from transitivity $x \sim w$, that is $w \in [x]$. In other words $[y] \subset [x]$ and from symmetry $[x] \subset [y]$.

Thus we have demonstrated that any two classes which have a common element shall coincide. □

REMARK B.2.7. The proof of this result is using only symmetry and transitivity of an equivalence relation. We need reflexivity for the next proposition.

PROPOSITION B.2.8. *Let* R *is an equivalence relation on* X. *Then* X *is partitioned into the set of disjoint equivalence classes.*

PROOF. By the reflexivity $x \in [x]$, so each element of X belong to an equivalence class. Different classes are disjoint by the previous proposition. □

COROLLARY B.2.9. *All elements of* X *equivalent to each other are also elements of the same equivalence class: for any* $a, b \in X$ $a \sim b$ *if and only if* $[a] = [b]$.

DEFINITION B.2.10. The set of all possible equivalence classes of X by $\sim$, denoted $X/\sim =: [x] : x \in X$, is the *quotient set* of X by $\sim$.

EXAMPLE B.2.11. (1) "Has the same birthday as" on the set of all people, given naive set theory.
(2) "Is similar to" or "congruent to" on the set of all triangles.
(3) "Is congruent to modulo n" on the integers.
(4) "Has the same image under a function" on the elements of the domain of the function.
(5) Logical equivalence of logical sentences.
(6) Let (r_n) and (s_n) be any two Cauchy sequences of rational numbers. The real numbers are the equivalence classes of the relation $(r_n) \sim (s_n)$, if the sequence $(r_n\ s_n)$ has limit 0.
(7) "Is parallel to" on the set of subspaces of an affine space.

DEFINITION B.2.12. We say that a binary operation $*$ on a set X is compatible with an equivalence relation $\sim$ if for any two classes $[x]$ and $[y]$ and any their representatives $x' \in [x]$ and $y' \in [y]$ all products $x' * y'$ belong to the same class $[z]$. This defines a binary operation on the quotient set: $[x] \cdot [y] = [x * y]$.

PROPOSITION B.2.13. *Let* a, b, c, d *be natural numbers, and let* (a, b) *and* (c, d) *be ordered pairs of such numbers. Then the equivalence classes under the relation* $(a, b) \sim (c, d)$ *are the:*

(1) *Integers if* $a + d = b + c$;

(2) *Positive rational numbers if* $ad = bc$.

PROOF. (1) We can identify:

a natural number n	with the class $[(1, n+1)]$,
the number 0	with the class $[(1,1)]$,
a negative integer m	with the class $[(-m+1, 1)]$.

A compatible addition is given as addition of vectors:

$$(a, b) + (c, d) = (a + c, b + d).$$

A compatible multiplication is

$$(a, b) \cdot (c, d) = (ac + bd, bc + ad).$$

(2) We can identify a positive rational number $\frac{a}{b}$ with a class generated by a pair (a, b). A compatible addition and multiplications are:

$$(a, b) + (c, d) = (ad + bc, bd), (a, b) \cdot (c, d) = (ac, bd).$$

□

B.3. Mathematical Propositions and Quantifiers

B.3.1. Propositions. Proposition is a statement such that it is meaningful to discuss is it *true* or *false*.

EXAMPLE B.3.1. The following are statements:

(1) London will host Olympic games in 2012.
(2) 221 is a prime number.
(3) $13 < 17$.

EXAMPLE B.3.2. The following are **not** statements:

(1) It is a fun to study Analysis at Leeds.
(2) $x > 0$.

We denote propositions by capital Latin letters, e.g.:

$$A = [6 < 7], \qquad B = [6 \text{ is a prime number}].$$

In the Boolean logic we denote the value of a true statement by 1, and false—by 0.

We sometimes use the following symbols, and you may use them in homeworks. But be cautious: they are often used wrongly, and tutors will be watching for sloppy usages. Remember that you must write in sentences, with proper punctuation, even if you include logical symbols in your writing.

There is an important operation on propositions:

$$\neg P \text{ or } \bar{P} \text{ — the negation of the proposition } P.$$

The negation is true if and only if the initial proposition is false. We can express it by the trueness table :

P	$\neg P$
0	1
1	0

We also use the following notation:

$$k \mid n \text{ — } k \text{ divides } n.$$

B.3.2. Quantifiers. We can convert statement like 2 to propositions by means of *quantifiers*:

$$\forall \text{ — for all}$$
$$\exists \text{ — there exists}$$

For example the following are propositions (both true):

$$(\forall x \in \mathbb{N})[x > 0], \qquad (\exists x \in \mathbb{Z})[x > 0].$$

They can be translated in plain English like this:

- Any natural number is greater than zero.
- There is an integer which is greater than zero.

It is important to use correct *negation of quantifiers*:

(73) $$\neg(\forall x \in X)[A(x)] = (\exists x \in X)[\neg A(x)]$$

(74) $$\neg(\exists x \in X)[A(x)] = (\forall x \in X)[\neg A(x)]$$

See Section B.3.4.4 for its usage.

B.3.3. Logical Operations. We can create new statements out of old ones by the following binary logical operations.

$P \vee Q$ — P or Q
$P \wedge Q$ — P and Q
$P \Rightarrow Q$ — proposition P implies proposition Q
$P \Leftrightarrow Q$ — propositions P and Q imply each other (i.e., P if and only if Q)

The compound statement is true or false depending from values of the elementary statements P and Q. This is illustrated by the following tables:

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

P	Q	$P \Leftrightarrow Q$
0	0	1
0	1	0
1	0	0
1	1	1

REMARK B.3.3. Note the interesting property of implication $P \Rightarrow Q$: it is always true unless P is true *and* Q is false. In particular it always true if P is false regardless of trueness of Q. It is always codified as "a lie implies anything".

EXERCISE B.3.4. Considering all four possible combinations of values for P and Q show that:

(75) $$(P \Rightarrow Q) = ((\neg P) \vee Q).$$

EXERCISE B.3.5. Using tables or otherwise demonstrate the logical identities below.

(1) Identities which are similar with the binary operation on numbers:

$$A \vee B = B \vee A, \qquad A \wedge B = B \wedge A,$$
$$(A \vee B) \vee C = A \vee (B \vee C), \qquad (A \wedge B) \wedge C = A \wedge (B \wedge C),$$
$$(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C),$$

(2) Identities which do not correspond to any operations on numbers:

$$(A \vee B) \wedge (A \vee C) = A \vee (B \wedge C),$$

$$\neg(A \vee B) = \neg A \wedge \neg B, \qquad \neg(A \wedge B) = \neg A \vee \neg B,$$

$$\neg(\neg A) = A, \qquad A \vee A = A \wedge A = A,$$

$$A \vee \neg A = 1, \qquad A \wedge \neg A = 0,$$

$$A \vee 1 = 1, \qquad A \wedge 1 = A,$$

$$A \vee 0 = A, \qquad A \wedge 0 = 0.$$

Can you notice a similarity to set-theoretical operations from Section B.1?

EXERCISE B.3.6. Translate to English the following propositions:

(1) $(\forall n \in \mathbb{N})[\{2 \mid n\} \Rightarrow \{4 \mid n^2\}]$.
(2) $(\forall n \in \mathbb{N})(\exists p \in \mathbb{N})[\{n < p\} \wedge \{p \text{ is a prime number}\}]$.

B.3.4. Common Methods of Proofs. We shall repeatedly meet several typical methods of proofs of mathematical statements.

B.3.4.1. *Contraposition.* It is often that instead of a theorem in the form

THEOREM. P implies Q.

we may prefer to prove its contraposition which is

THEOREM. Not Q implies not P.

Comparing trueness tables for both statements we may establish the logical equivalence

$$(P \Rightarrow Q) = ((\neg Q) \Rightarrow (\neg P)).$$

Thus these two propositions are logically equivalent.

B.3.4.2. *Sufficiency and Necessity.* We oftenly need to prove a logical equivalence of two statements of the type:

THEOREM. P if and only if Q.

The proof commonly consists of two parts which are known as necessary and sufficient conditions:

PROOF.
($\Rightarrow$) P implies Q (necessity).
($\Leftarrow$) Q implies P (sufficiency).

On the language of logical identities this is denoted by the following equivalence:

$$(P \Leftrightarrow Q) = ((P \Rightarrow Q) \wedge (Q \Rightarrow P)).$$

B.3.4.3. *Reduction to the Absurd.* Having a theorem

THEOREM. P is true.

we may proceed as follows:

PROOF. Assume P is false, then Q, then ..., then R, which is obviously false. Thus P have to be true.

This method of proof is known as reductio ad absurdum ("reduction to the absurd"–Lat.). The logical foundation for this method is as follow: the implication $(\neg P) \Rightarrow 0$ can be true only in the case of $\neg P$ is false, that is P is true.

B.3.4.4. *Counterexample.* We oftenly wish to disprove a statement

$$(\forall x \in X)[A(x)].$$

This means we need to demonstrate its negation, which is, cf. (73):

$$(\exists x \in X)[\neg A(x)].$$

To demonstrate the last statement it is enough just to present a *single* $x \in X$ such that $A(x)$ is false. It is known as counterexample.

If we wish to disprove a theorem

THEOREM. $P(x)$ implies $Q(x)$.

then we need to present a counterexample x which will show a trueness of its negation, cf. (75):

$$\neg(P(x) \Rightarrow Q(x)) = \neg((\neg P(x)) \vee Q(x)) = P(x) \wedge (\neg Q(x)).$$

This means we need to present x such that $P(x)$ is true *and* $Q(x)$ is false.

APPENDIX C

Function and Their Graph

C.1. Rectangular (Cartesian) Coordinates

Considering real axis in Section A.1.6 we introduce one-to one correspondence between real numbers and points of a line. This connection between numbers and geometric objects may be extended for other objects as well.

A *rectangular coordinate system* (or *Cartesian coordinates*) is an assignment of *ordered pairs* (a, b) to points in a plane, see [1, Fig. 6, p. 10].

REMARK C.1.1. It is also possible to introduce Cartesian coordinates in our three-dimensional world by means of triples of real numbers (x, y, z). This construction could be extended to arbitrary number of dimensions.

THEOREM C.1.2. *The* distance between two points $P_1 = (x_1, y_1)$ *and* $P_2 = (x_2, y_2)$ *is*

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{76}$$

This theorem is a direct consequence of Pythagorean theorem.

THEOREM C.1.3. *The* midpoint M *of segment* P_1P_2 *is*

$$M(P_1P_2) = M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \tag{77}$$

PROOF. The theorem follows from two observations:

(1) $d(P_1, M) = d(M, P_2)$;
(2) $d(P_1, M) + d(M, P_2) = d(P_1, P_2)$.

□

EXERCISE C.1.4. Prove the above two observation (Hint: use the distance formula (76)).

C.2. Graph of an Equation

An *equation in* x *and* y is an equality such as

$$2x + 3y = 5, \qquad y = x^2 + 3x - 6, \qquad y^x + \sin xy = 8.$$

A *solution* is an ordered pair (a, b) that produced a true statement when $x = a$ and $y = b$. The *graph of the equation* consists of all points (a, b) in a plane that corresponds to the solutions.

C.3. Line Equations

The general *equation* of a line in a plane is given by the formula

$$ax + by + c = 0. \tag{78}$$

This equation connect different geometric objects:

(1) *Slope* m:

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{79}$$

(2) *Point-Slope form* $y - y_1 = m(x - x_1)$.
(3) *Slope-Intercept form* $y = mx + b$ or $y = m(x - c)$.

Special lines

(1) Vertical: m undefined; horizontal: $m = 0$.
(2) Parallel: $m_1 = m_2$.
(3) Perpendicular $m_1 m_2 = -1$.

EXERCISE C.3.1. Prove the above geometric properties.

C.4. Symmetries and Shifts

We will say that a graph of an equation possesses a *symmetry* if there is a transformation of a plane such that it maps the graph to itself.

EXAMPLE C.4.1. There several examples of elementary symmetries:

(1) y-axis: substitution $x \to (-x)$, e.g.[1] equation $y = |x|$.
(2) x-axis: substitution $y \to (-y)$, e.g. $|y| = x$.
(3) Central symmetry: substitution both $x \to (-x)$ and $y \to (-y)$, e.g. $y = x$ or $|y| = |x|$.
(4) x-shifts: substitution $x \to (x + a)$ for $a \neq 0$, e.g. $y = \{x\}$ with $a = 1$. Here $\{x\}$ denotes the *fractional part* of x, i.e.[2] it is defined by two conditions: $0 \leqslant \{x\} < 1$ and $x - \{x\} \in \mathbb{Z}$.
(5) y-shifts: substitution $y \to (y + b)$ for $b \neq 0$, e.g. $\{y\} = x$ with $b = 1$.
(6) General shifts: substitution both $x \to (x + a)$ and $y \to (y + b)$ for $a \neq 0$, $b \neq 0$, e.g. $y = [x]$ with $a = b = 1$. Here $[x]$ denotes the *entire part* of x, i.e. $[x] \in \mathbb{Z}$ and $[x] \leqslant x < [x] + 1$.

EXERCISE* C.4.2. (1) Is there an equation with y-axis symmetry 1 and x-axis symmetry 2 but without central symmetry 3?
(2) Is there an equation with x-shift symmetry 4 with some $a \neq 0$ and y-shift symmetry 5 with some $b \neq 0$ but without general shift symmetry 6?

Symmetries are important because they allow us to reconstruct a whole picture from its parts.

[1] The abbreviation *e.g.* denotes *for example*.
[2] The abbreviation *i.e.* denotes *namely*.

C.5. Definition of Functions and Examples

The main object of calculus is *function*. We recall basic notations and definitions.

DEFINITION C.5.1. Let X and Y be two sets. A *function* $f : X \to Y$ is a subset of $X \times Y$ such that for any $x \in X$ there is exactly one pair with the first element x. X is called *domain* of the function and Y is *range*.

In other words A *function* f from a set X to a set Y is a correspondence that assigns to each element x of the set X exactly one element y of the set Y.

We write

$$f : x \mapsto f(x), \quad X \to Y.$$

It is common to call x *argument* or *independent variable*, $y = f(x)$ is called *value of function* or dependent variable. Here x is *independent variable* and y is*dependent variable*.

The set X is the *domain* of f, and Y is the *codomain*.

We will also denote by $f(S)$ the image of a subset $S \subset X$, that is $f(S) \subset Y$ consisting of $f(s)$ for all $s \in S$.

DEFINITION C.5.2. The *range* of f is the subset of codomain Y consisting of all possible function values $f(x)$ for x in X:

$$f(X) = \{f(x) : x \in X\}.$$

DEFINITION C.5.3. For numerically defined functions like $y = \sqrt{x-2}$ the *natural domain* is assumed to be all x that f is *is defined at* x, or $f(x)$ exists.

DEFINITION C.5.4. A *binary operation* on X is a function $X \times X \to X$.

It is often assumed that there is a rule how to evaluate the value of function for a specific argument. Alternatively function can be defined by a graph, formula or algorithm.

EXAMPLE C.5.5. (1) Function defined by a formula: $f(x) = 2x + 3$.

(2) We define the *Dirichlet function* as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

(3) The nth *Fibonacci number* F_n defines a function on $\mathbb{N}$ which can be evaluated by the recursive algorithm:

$$F_{n+2} = F_{n+1} + F_n,$$

and two initial values $F_1 = 1$ and $F_2 = 1$.

There are natural operation on functions

(1) *sum*: $(f + g)(x) = f(x) + g(x)$.
(2) *difference*: $(f - g)(x) = f(x) - g(x)$.
(3) *product*: $(fg)(x) = f(x)g(x)$.
(4) *quotient*: $(f/g)(x) = f(x)/g(x)$.

EXAMPLE C.5.6. There are some special functions with standard names.

(1) For any set X we can define the *identity function* $\mathrm{Id} : X \to X$ by the rule $\mathrm{Id}(x) = x$ for all $x \in X$.

(2) A *constant function* is such that for a fixed element $y \in Y$ we have $f(x) = y$ for all $x \in X$.
(3) A *linear function* $\mathbb{R} \to \mathbb{R}$ is given by $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.
(4) A *polynomial function* $\mathbb{R} \to \mathbb{R}$ of order $n \in \mathbb{N}$ is given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

for some $a_0, a_1, a_n \in \mathbb{R}$.
(5) A *rational function* is defined as a fraction of two polynomials:

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m},$$

where n and m are arbitrary natural numbers and $a_i, b_i \in \mathbb{R}$. The natural domain is a subset of $\mathbb{R}$ such that the denominator is not equal to zero.
(6) Function which are obtained from polynomials by four algebraic operations 1 and taking rational powers are *algebraic*. All other function (e.g $\sin x$, $\exp x$) are *transcendental*.
(7) A function from $\mathbb{N}$ to a set Y is called *sequence* (with values in Y).

EXERCISE C.5.7. Is it possible that:
(1) The identity function would be constant?
(2) A linear function would be the identity function?
(3) A linear function would be a constant function?

DEFINITION C.5.8. The *restriction* of a function f to a subset R of S is denoted by $f \mid R$.

C.6. Surjection, Injection, Bijection

DEFINITION C.6.1. The function f is *surjective* (or *onto*) if $f(X) = Y$.

DEFINITION C.6.2. The function f is *injective* (or *one-to-one*) if, for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

DEFINITION C.6.3. The function f is *bijective*, or a *bijection*, if it is both injective and surjective.

DEFINITION C.6.4. A bijective function f has an *inverse* $f^{-1} : Y \to X$, defined by the rule that $f^{-1}(y) = x$ if and only if $f(x) = y$ (for $x \in X$ and $y \in Y$).

Be sure you know the difference between an inverse and the *reciprocal* of a function with values in $\mathbb{R}$.

EXERCISE C.6.5. Show from the definition that if a function f has the inverse f^{-1}, then f^{-1} definitely has the inverse $(f^{-1})^{-1}$ and moreover $(f^{-1})^{-1} = f$.

EXERCISE C.6.6. Check which functions from Example C.5.6 can be surjection, injection and bijection.

C.7. Composition of Functions

DEFINITION C.7.1. The *composition* of $f : X \to Y$ and $g : Y \to Z$ is $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x)) \quad (x \in X)$.

EXAMPLE C.7.2. (1) Let $\arg : \mathbb{C} \to (-\pi, \pi]$ be the function of argument of a complex number, $\tan : (-\pi, \pi] \to \mathbb{R}$ is the tangent function. Then $\tan \circ \arg : \mathbb{C} \to \mathbb{R}$ is the function which produces the slope $\frac{b}{a}$ of vector $z = a + ib$.

(2) Let W be the function from the set of Saturdays to the set of numbers, which make the winning number for this Saturday. Let P be a function from the set of numbers to the set of humans producing the person owning a ticket with a particular number. The composition $P \circ W$ produces a winner for each Saturday.

EXERCISE C.7.3. That are composition $f \circ f^{-1}$ and $f^{-1} \circ f$ of a bijective function $f : X \to Y$ and its inverse f^{-1}. [Answer: They are identity maps $Y \to Y$ and $X \to X$ respectively, note which set they are defined on.]

C.8. Graph of a Function

The *graph of the a function* f with domain D is the graph of the equation $y = f(x)$ for x in D. The x-intercept of the graph are solutions of the equation $f(x) = 0$ and called *zeros*.

The following transformation are useful for sketch of graphs:

(1) *horizontal shift*: $y = f(x)$ to $y f(x - a)$;
(2) *vertical shift*: $y = f(x)$ to $y = f(x) + b$;
(3) *horizontal stretch/compression*: $y = f(x)$ to $y = f(cx)$;
(4) *vertical stretch/compression*: $y = f(x)$ to $y = cf(x)$;
(5) *horizontal reflections*: $y = f(x)$ to $y = -f(x)$;
(6) *vertical reflections*: $y = f(x)$ to $y = f(-x)$;

C.9. Increasing and Decreasing Functions. Odd and Even Functions

A function $f(x)$ is *increasing* if $f(x) > f(y)$ for all $x > y$. A function $f(x)$ is *decreasing* if $f(x) > f(y)$ for all $x > y$.

A function $f(x)$ is *even* if $f(-x) = f(x)$ and $f(x)$ is *odd* if $f(-x) = -f(x)$.

EXERCISE C.9.1. Which type of symmetries listed in Example (C.4.1) have to or may even and odd functions posses?

C.10. Countable Sets

DEFINITION C.10.1. An set S is said to be *countable* if there is a surjection $\mathbb{N} \to S$. In other word there is sequence a_n with values in S which contains all elements of S.

LEMMA C.10.2. *If a countable S is infinite then there is a bijection $\mathbb{N} \to S$.*

EXAMPLE C.10.3. (1) Any finite set is countable.
(2) The set of natural numbers is countable (by the identity map).
(3) A subset of any countable set is countable as well.

THEOREM C.10.4. *If two sets are countable then their union is countable as well.*

COROLLARY C.10.5. *The set of integers $\mathbb{Z}$ is countable.*

PROOF. Obviously $\mathbb{Z}$ is the union of two countable sets of natural numbers and their inverses (and 0). □

THEOREM C.10.6. *If two sets are countable then their Cartesian product is countable as well.*

COROLLARY C.10.7. *The set of rational numbers is countable.*

PROOF. The rationals form a subset (more exactly a quotient space, see 2) of the Cartesian product $\mathbb{Z} \times \mathbb{N}$ of two countable sets. □

THEOREM C.10.8. *The set of reals from* $[0, 1]$ *is uncountable.*

APPENDIX D

Conic Section

D.1. Circle

The beautiful and important objects arise by intersection of planes and cones, i.e. *conic sections*.

The simplest conic section is circle.

DEFINITION D.1.1. A *circle* with center (x_1, y_1) and radius r consists of points on the distance r from (x_1, y_1).

By the distance formula (76) the circle is defined by an *equation*:

$$(x - x_1)^2 + (y - y_1)^2 = r^2. \tag{80}$$

Circles are obtained as intersections of cones with planes orthogonal to their axes.

EXERCISE D.1.2.
(1) Write an equation of a circle which is tangent to a circle $x^2 - 6x + y^2 + 4y - 12 = 0$ and has the origin $(3, 0)$.
(2) Write an equation of a circle, which has a center at $(1, 2)$ and contains the center of the circle given by the equation $x^2 - 7x + y^2 + 8y - 17 = 0$.
(3) Write equations of all circles with a given radius r which are tangent to both axes.

D.2. Parabola

DEFINITION D.2.1. A *parabola* is the set of all points in a plane equidistant from a fixed point F (the *focus* of the parabola) and a fixed line l (the *directrix*) that lie in the plane.

The *axis* of the parabola is the line through F that is perpendicular to the directrix. The *vertex* of the parabola is the point V on the axis halfway from F to l.

A parabola with axis coinciding with y axis and the vertex at the origin with focus $F = (0, p)$ has an *equation*

$$x^2 = 4py. \tag{81}$$

EXERCISE* D.2.2. Verify the above equation of a parabola. (Hint: use distance formula (76).

EXERCISE D.2.3. List all symmetries of a parabola.

For a parabola with the vertex (h, k) and a horizontal directrix an equation takes the form

$$(x - h)^2 = 4p(y - k). \tag{82}$$

In general any equation of the form $y = ax^2 + bx + c$ defines a parabola with horizontal directrix.

EXERCISE D.2.4. (1) Find the vertex, the focus, and the directrix of the parabolas:
   (a) $3y^2 = -5x$.
   (b) $x^2 = 3y$.
   (c) $y^2 + 14y + 4x + 45 = 0$.
   (d) $y = 8x^2 + 16x + 10$.
(2) Find an equation of the parabola with properties:
   (a) vertex $V(-3, 4)$; directrix $y = 6$.
   (b) vertex $V(1, 1)$; focus $F(-2, 1)$.
   (c) focus $F(1, -3)$; directrix $y = 5$.

D.3. Ellipse

DEFINITION D.3.1. An *ellipse* is the set of all points in a plane. the sum of whose distances from two fixed points F and F' (the *foci*) in the plane is constant. The midpoint of the segment FF' is the *center* of the ellipse.

Let $F(-c, 0)$ and $F'(c, 0)$, $2a$ be the constant sum of distances, and $b = (a^2 - c^2)^{1/2}$. Then the ellipse has an *equation*

(83)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

EXERCISE* D.3.2. Verify the above equation of the ellipse. (Hint: use distance formula (76).

The ellipse intercepts x-axis in points $V(-a, 0)$ and $V'(a, 0)$—*vertices* of the ellipse. The line segment VV' is the *major axis* of the ellipse. Similarly the ellipse intercepts y-axis in points $M(-b, 0)$ and $M'(b, 0)$ and the line segment MM' is the *minor axis* of the ellipse.

EXERCISE D.3.3. List all symmetries of an ellipse.

For the ellipse with center in a point (h, k) an equation is given as

(84)
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

DEFINITION D.3.4. The *eccentricity* e of an ellipse is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$$

EXERCISE D.3.5. (1) Find the vertices and the foci of the ellipse
   (a) $4x^2 + 2y^2 = 8$.
   (b) $x^2/3 + 3y^2 = 9$
(2) Find an equation for the ellipse with center at the origin and
   (a) Vertices $V(\pm 9, 0)$; foci $F(\pm 6, 0)$.
   (b) Foci $F(\pm 6, 0)$; minor axis of length 4.
   (c) Eccentricity $3/4$; vertices $V(0, \pm 5)$.

D.4. Hyperbola

DEFINITION D.4.1. A *hyperbola* is the set of all points in a plane, the difference of whose distances from two fixed points F and F' (the *foci*) in the plane is a positive constant. The midpoint of the segment FF' is the *center* of the hyperbola.

Let a hyperbola has foci $F(\pm c, 0)$, $2a$ denote the constant difference, and let $b^2 = c^2 - a^2$. Then the hyperbola has an *equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{85}$$

EXERCISE* D.4.2. Verify the above equation of the hyperbola. (Hint: use distance formula (76).

Points $V(a, 0)$ and $V'(-a, 0)$ of interception of the hyperbola with x-axis are *vertices* and the line segment VV' is the *transverse axis* of the hyperbola. Points $W(0, b)$ and $W'(0, -b)$ span *conjugate axis* of the hyperbola. This two segments intercept in the *center* of the hyperbola.

EXERCISE D.4.3. Find all symmetries of a hyperbola.

If a graph approaches a line as the absolute value of x gets increasingly large, then the line is called an *asymptote* for the graph. It could be shown that lines $y = (b/a)x$ and $y = -(b/a)x$ are asymptotes for the hyperbola.

For the hyperbola with the center in a point (h, k) an equation is given as

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1. \tag{86}$$

EXERCISE D.4.4.
(1) Find vertices and foci of the hyperbola, sketch its graph.
   (a) $x^2/49 - y^2/16 = 1$.
   (b) $y^2 - 4x^2 - 12y - 16x + 16 = 0$.
   (c) $9y^2 - x^2 - 36y + 12x - 36 = 0$.
(2) Find an equation for the hyperbola that has its center at the origin and satisfies to the given conditions
   (a) foci $F(0, \pm 4)$; vertices $V(0, \pm 1)$.
   (b) foci $F(0, \pm 5)$; conjugate axis of length 4.
   (c) vertices $V(\pm 3, 0)$; asymptotes $y = \pm 2x$.

D.5. Conclusion

The graph of every quadratic equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of conic section

(1) Circle;
(2) Ellipse;
(3) Parabola;
(4) Hyperbola;

or a degenerated case

(1) A point;
(2) Two crossed lines;
(3) Two parallel lines;

(4) One line;
(5) The empty set.

APPENDIX E

Trigonometic Functions

An *angle* is determined by two rays having the same initial point O (*vertex*). Angles are measured either by *degree measure* 1° or *radian measure*. The complete counterclockwise revolution is 360° or 2π radians.

We consider six *trigonometric functions*

Name	Notation	Expression	Name	Notation	Expression
sine	sin	y/r	*cosecant*	csc	r/y
cosine	cos	x/r	*secant*	sec	r/x
tangent	tan	y/x	*cotangent*	cot	x/y

There are a lot of useful identities between trigonometric functions:

(1) *Reciprocal and Ratio Identities*:
  (a) $\csc\phi = (\sin\phi)^{-1}$, $\sec\phi = (\cos\phi)^{-1}$;
  (b) $\tan\phi = \sin\phi/\cos\phi$, $\cot = \cos\phi/\sin\phi$;
  (c) $\cot\phi = (\tan\phi)^{-1}$

(2) *Pythagorean Identities*
  (a) $\sin^2\phi + \cos^2\phi = 1$;
  (b) $1 + \tan^2\phi = \sec^2\phi$;
  (c) $1 + \cot^2\phi = \csc^2\phi$;

(3) *Law of Sines* and *Law of Cosine*
  (a) $\sin\alpha/a = \sin\beta/b = \sin\gamma/c = 2R$;
  (b) $a^2 = b^2 + c^2 - 2bc\cos\alpha$;

(4) Additional identities
  (a) Cosine and secant are even functions; sine, tangent, cosecant, cotangent are odd functions;
  (b) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$;
  (c) $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$;

APPENDIX F

Exponential and Logarithmic Functions

DEFINITION F.0.1. The *exponential function* with a base a is defined by $f(x) = a^x$, where $a > 0$, $a \neq 1$, and x is any real number.

It is increasing if $a > 1$ and decreasing if $0 < a < 1$. It is also one-to-one function. This allow us to solve equations and inequalities.

EXERCISE F.0.2. Find solutions

(1) $5^{3x} = 5^{x^2-1}$;
(2) $2^{|x-3|} > 2^2$;
(3) $(0.5)^{x^2} > (0.5)^{5x-6}$.

There are *laws of exponents*:

(1) $a^u a^v = a^{u+v}$;
(2) $a^u/a^v = a^{u-v}$;
(3) $(a^u)^v = a^{uv}$;
(4) $(ab)^u = a^u b^u$;
(5) $(a/b)^u = a^u/b^u$.

DEFINITION F.0.3. If a is a positive real number other than 1, then the *logarithm* of x with base a is defined by $y = \log_a x$ if and only if $x = a^y$ for every $x > 0$ and every real number y.

Thus logarithm is inverse to exponential function. As consequences logarithm one-to-one function, for $a > 1$ it is an increasing function, for $a < 1$ it is decreasing.

EXERCISE F.0.4. Find solution

(1) $\log_2(x^2 - x) = \log_2 2$;
(2) $\log_{0.5} |2x - 5| > \log_{0.5} 4$.

There are corresponding *laws of logarithms*

(1) $\log_a(uv) = \log_a u + \log_a v$;
(2) $\log_a(u/v) = \log_a u - \log_a v$;
(3) $\log_a(u^c) = c \log_a u$ for any real number c.

The *change-of-base formula for logarithms*: if $x > 0$ and if a and b are positive real numbers other than 1, then

$$\log_b x = \frac{\log_a x}{\log_a b}. \tag{87}$$

EXERCISE F.0.5. Find solution: $\log_x(3x - 1) = 2$.

Bibliography

[1] Earl Swokowski, Michael Olinick, and Dennis Pence, *Calculus*, 6-th, PWS Publishing, Boston, 1994. ↑61

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