Expanding Analysis

Real, Complex, Quaternionic, Clifford and Beyond

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Starting Point: Real Analysis

Exercise

Find the interval of convergence of the Taylor series:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

Answer: $(-1, 1)$.

Exercise

Explain the significance of the endpoints -1 and 1 of the convergence interval for the function $\frac{1}{1+x^2}$.

Answer (Real Analysis viewpoint): Hm-mm. . .

Answer (Complex Analysis perspective): The function $\frac{1}{1+z^2}$ on $\mathbb{C}$ has singular points $-i$ and i, thus the radius of convergence is 1, so. . .

Benefits from Complex Analysis

- $\mathbb{C}$ is algebraically closed field, the maximal one! (Frobenius theorem)
- The possibility to factorise in linear terms: $x^2 + y^2 = (x + iy)(x - iy)$
- Trigonometry is simplified by identities like $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$
- Mysterious identities like $e^{i\pi} + 1 = 0$ or $i^i = e^{-\frac{\pi}{2}}$
- Analytic number theory, e.g. Waring's problem
- *Elliptic* differential equations, *elliptic* boundary value problems
- Theory of harmonic functions (null solutions of the Laplacian)
- Mathematical physics, quantum mechanics
- Operator theory, functional models
- Analytic functional calculus, spectral theory:

$$f(A) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{A - z}\, dz$$

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Main components of Complex Analysis

- Cauchy–Riemann equations $\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) f = 0$.
- Factorisation of Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)$
- Complex differentiability
- Cauchy Theorem $\int_\Gamma f(z)\, dz = 0$
- Cauchy Integral $f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(t)}{z-t}\, dt$
- Taylor and Laurent expansions
- Calculus of residues, integration
- Hardy and Bergman spaces of square integrable analytic functions
- Conformal maps, Möbius transformations
- Riemann theorem—the group of automorphisms of a connected simply connected domain with a boundary is $SL_2(\mathbb{R})$

A generalisation: Several Complex Variables

For a function $f : \mathbb{C}^n \to \mathbb{C}$:

- ✓ Cauchy–Riemann equations $\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) f = 0$, $k = 1, \ldots, n$ but is overdetermined
- ✗ No factorisation of Laplacian $\Delta = \sum_1^n \frac{\partial^2}{\partial x_k^2}$ do not work for harmonic functions in $\mathbb{R}^n$
- ✓ Taylor and Laurent expansions
- ✓ Cauchy theorem $\int_\Gamma f\, dz = 0$
- ? Cauchy integral formula $f(z) = \int_\Gamma f(t)\, K(t, z)\, dz$, domain dependent
- ✗ Riemann theorem, not every domain is domain of analyticity
- ? Conformal maps, Möbius transformations

Summary (One and Several Complex Variables)

They look like different subjects done by different mathematicians and taught in different textbooks.

More Imaginary Units: Nice and Various

Analysis in higher dimension requires more imaginary units.

Example (Quaternions—Hamilton, 1843)

A four dimensional non-commutative division ring (the maximal one!) with the basis 1, i, j, k and multiplication table:

$$ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik.$$

Very suitable for geometry and analysis in $\mathbb{R}^3$ and $\mathbb{R}^4$, e.g.:

$$a \cdot b = \frac{1}{2}(ab + ba), \qquad a \times b = \frac{1}{2}(ab - ba).$$

for

$$a = a_1 i + a_2 j + a_3 k, \qquad b = b_1 i + b_2 j + b_3 k.$$

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Even More Imaginary Units (with a cost however)

A move to dimensionality higher than 4 requires higher losses—the division property this time.

Example (Clifford algebras—Clifford, 187?)

A non-commutative algebra generated by 1, e_1, e_2, ..., e_n bound by the identities:

$$e_i^2 = -1, \text{ (or 1 and even 0)}, \qquad e_i e_j = -e_j e_i.$$

The total dimensionality of the Clifford algebra is 2^n. Linear combinations of imaginary units are called *vectors* and identified with points of $\mathbb{R}^n$. Although there are divisors of zero all vectors are invertible due to:

$$(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)^2 = -x_1^2 - x_2^2 - \cdots - x_n^2.$$

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Quaternionic and Clifford Analysis: Check-list

- Cauchy–Riemann–Dirac equations
$$\left(e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+\cdots+e_n\frac{\partial}{\partial x_n}\right)f=0.$$
- ✓ Factorisation of Laplacian $\Delta=\sum_1^n\frac{\partial^2}{\partial x_k^2}=-\left(\sum_1^n e_k\frac{\partial}{\partial x_k}\right)^2$
- ✓ Quaternionic/Clifford differentiability (smartly revised!)
- ✓ Cauchy Theorem $\int_\Omega f(x)\,dn(x)=0$
- ✓ Cauchy Integral $f(x)=\int_\Omega f(t)\,dn(t)\,E(x-t)$ (domain-independent!)
- ✓ Taylor and Laurent expansions (again revised!)
- ✓ Calculus of residues, integration
- ✓ Hardy and Bergman spaces of square integrable analytic functions
- ✓ Conformal maps, Möbius transformations
- ✗ Riemann theorem (not yet?)

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Extending Analysis: a “To-Do” List

- Hyperbolic and parabolic analytic functions in non-division rings.

Example (Dual and double numbers vs complex one)

In $\mathbb{R}^2$ there are imaginary units $\varepsilon^2 = 0$ and $\epsilon^2 = 1$ along with $i^2 = -1$.

- Alternative entry points, e.g. Möbius transformations.

Example ($SL_2(\mathbb{R})$ action on $\mathbb{R}^2$)

Möbius action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$ for complex, double and dual numbers.

- Application to non-elliptic differential equations and BVP.

Example (Wave and heat equation)

Treat $u''_{tt} - u''_{xx}$ and $u'_t - u''_{xx}$ along with the elliptic $u''_{tt} + u''_{xx}$.

- Application to functional calculus.

Example (Covariant functional calculus)

Action of $SL_2(\mathbb{R})$ in Banach algebras gives a rich notion of spectrum