Expanding Analysis
Real, Complex, Quaternionic, Clifford and Beyond

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Exercise
Find the interval of convergence of the Taylor series:

\[
\frac{1}{1 + x^2} = 1 - x^2 + x^4 - \ldots
\]

Answer: \((-1, 1)\).

Exercise
Explain the significance of the endpoints \(-1\) and \(1\) of the convergence interval for the function \(\frac{1}{1 + x^2}\).

Answer (Real Analysis viewpoint): Hm-mm…

Answer (Complex Analysis perspective): The function \(\frac{1}{1 + z^2}\) on \(\mathbb{C}\) has singular points \(-i\) and \(i\), thus the radius of convergence is 1, so…
Benefits from Complex Analysis

- \( \mathbb{C} \) is algebraically closed field, the maximal one! (Frobenius theorem)
- The possibility to factorise in linear terms: \( x^2 + y^2 = (x + iy)(x - iy) \)
- Trigonometry is simplified by identities like \( \cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \)
- Mysterious identities like \( e^{i\pi} + 1 = 0 \) or \( i^i = e^{-\frac{\pi}{2}} \)
- Analytic number theory, e.g. Waring’s problem
- Elliptic differential equations, elliptic boundary value problems
- Theory of harmonic functions (null solutions of the Laplacian)
- Mathematical physics, quantum mechanics
- Operator theory, functional models
- Analytic functional calculus, spectral theory:

\[
f(A) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{A - z} \, dz
\]
Cauchy–Riemann equations \( \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f = 0 \).

Factorisation of Laplacian \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \).

Complex differentiability

Cauchy Theorem \( \int_{\Gamma} f(z) \, dz = 0 \)

Cauchy Integral \( f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{z-t} \, dt \)

Taylor and Laurent expansions

Calculus of residues, integration

Hardy and Bergman spaces of square integrable analytic functions

Conformal maps, Möbius transformations

Riemann theorem—the group of automorphisms of a connected simply connected domain with a boundary is \( SL_2(\mathbb{R}) \)
A generalisation: Several Complex Variables

For a function \( f : \mathbb{C}^n \rightarrow \mathbb{C} \):

- ✔ Cauchy–Riemann equations \( \left( \frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right) f = 0, \ k = 1, \ldots, n \) but is overdetermined
- ❌ No factorisation of Laplacian \( \Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \)
do not work for harmonic functions in \( \mathbb{R}^n \)
- ✔ Taylor and Laurent expansions
- ✔ Cauchy theorem \( \int_{\Gamma} f \, dz = 0 \)
- ❖ Cauchy integral formula \( f(z) = \int_{\Gamma} f(t) K(t, z) \, dz \), domain dependent
- ❌ Riemann theorem, not every domain is domain of analyticity
- ❖ Conformal maps, Möbius transformations

Summary (One and Several Complex Variables)

They look like different subjects done by different mathematicians and taught in different textbooks.

Vladimir V. Kisil (http://maths.leeds.ac.uk/~kisilv) Expanding Analysis
Analysis in higher dimension requires more imaginary units.

**Example (Quaternions—Hamilton, 1843)**

A four dimensional non-commutative division ring (the maximal one!) with the basis $1, i, j, k$ and multiplication table:

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$ 

Very suitable for geometry and analysis in $\mathbb{R}^3$ and $\mathbb{R}^4$, e.g.:

$$a \cdot b = \frac{1}{2}(ab + ba), \quad a \times b = \frac{1}{2}(ab - ba).$$

for

$$a = a_1i + a_2j + a_3k, \quad b = b_1i + b_2j + b_3k.$$
A move to dimensionality higher than 4 requires higher losses—the division property this time.

**Example (Clifford algebras—Clifford, 187?)**

A non-commutative algebra generated by $1, e_1, e_2, \ldots, e_n$ bound by the identities:

$$e_i^2 = -1, \text{ (or 1 and even 0)}, \quad e_i e_j = -e_j e_i.$$

The total dimensionality of the Clifford algebra is $2^n$. Linear combinations of imaginary units are called *vectors* and identified with points of $\mathbb{R}^n$. Although there are divisors of zero all vectors are invertible due to:

$$(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)^2 = -x_1^2 - x_2^2 - \cdots - x_n^2.$$
Quaternionic and Clifford Analysis: Check-list

- Cauchy–Riemann–Dirac equations
  \[
  \left( e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + \cdots + e_n \frac{\partial}{\partial x_n} \right) f = 0.
  \]

- Factorisation of Laplacian
  \[
  \Delta = \sum_1^n \frac{\partial^2}{\partial x_k^2} = -\left( \sum_1^n e_k \frac{\partial}{\partial x_k} \right)^2
  \]

- Quaternionic/Clifford differentiability (smartly revised!)

- Cauchy Theorem
  \[
  \int_{\Omega} f(x) \, dn(x) = 0
  \]

- Cauchy Integral
  \[
  f(x) = \int_{\Omega} f(t) \, dn(t) \, E(x - t) \quad \text{(domain-independent!)}
  \]

- Taylor and Laurent expansions (again revised!)

- Calculus of residues, integration

- Hardy and Bergman spaces of square integrable analytic functions

- Conformal maps, Möbius transformations

- Riemann theorem (not yet?)
Hyperbolic and parabolic analytic functions in non-division rings.

Example (Dual and double numbers vs complex one)
In $\mathbb{R}^2$ there are imaginary units $\varepsilon^2 = 0$ and $\varepsilon^2 = 1$ along with $i^2 = -1$.

Alternative entry points, e.g. Möbius transformations.

Example ($SL_2(\mathbb{R})$ action on $\mathbb{R}^2$)
Möbius action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$ for complex, double and dual numbers.

Application to non-elliptic differential equations and BVP.

Example (Wave and heat equation)
Treat $u''_t - u''_x$ and $u'_t - u''_x$ along with the elliptic $u''_t + u''_x$.

Application to functional calculus.

Example (Covariant functional calculus)
Action of $SL_2(\mathbb{R})$ in Banach algebras gives a rich notion of spectrum