A BOSE TYPE FORMULA FOR THE INTERNAL MEDIAL AXIS OF AN EMBEDDED MANIFOLD.

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Abstract. We establish a three dimensional variant of the Bose formula for the internal medial axis of a closed plane curve. We generalize the result to dimensions less than or equal to 6, and we apply the result to cut-loci, and spines of 3-manifolds. Also, we describe the difference between the main theorems and recent work of Sedykh.

Keywords: Medial axis, Bose formula, cut-locus
MSC(2000) Subject classification: 53A07, 58K70

1. Introduction

The medial axis is an invariant, somewhere between geometry and topology, of the embedding of a manifold. In this paper we produce a relationship between the Euler characteristic of a closed surface and the Euler characteristic of the part of the medial axis that lies in the interior of the surface in terms of features (strata) of the medial axis.

Our formula is similar to the relationship $e-y=2$ expressed in the Bose Theorem in the one-dimensional case, where e is number of endpoints and y is number of triple points on the medial axis, see the original paper [4] and the extensive treatment in [19]. The proof is similar to the proof of the main results in [9, 10, 11] and is based on a counting argument involving triangulations of the manifold and the medial axis.

Definition 1.1. *Let M be a smooth orientable compact surface without boundary in $\mathbb{R}^3$. The* internal medial axis $\mathrm{IntMed}(M)$ *is the closure of the locus of centers of spheres which are tangent to M at two (or more) points and are contained inside M. The* external medial axis $\mathrm{ExtMed}(M)$ *is the closure of the locus of centers of spheres which are tangent to M at two (or more) points and contain M in their interior.*

Using this definition the classification of the local structure of a medial axis for a generic surface is given in [15] and [21]. The local pictures are shown in Figure 1. We will assume that M is a generic manifold so that the medial axis has only the local singularities described above and hence can be triangulated. We partition the set of points where $\mathrm{IntMed}(M)$ is not a manifold according to the following local classification:

- Edge is the set of points of $\mathrm{IntMed}(M)$ at which it is locally an edge: the edge locus.

Date: March 6, 2008.

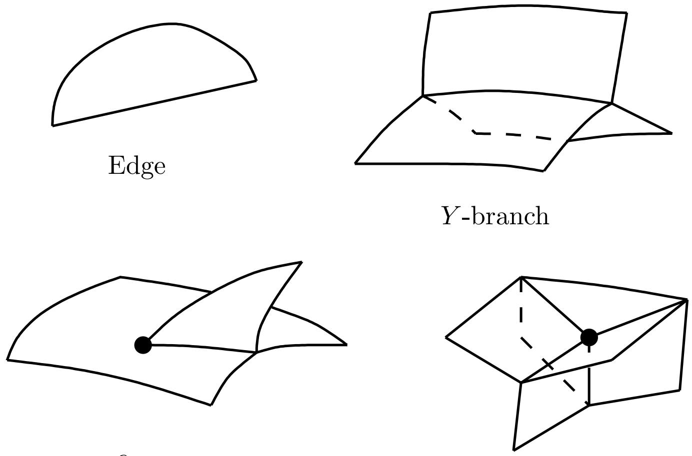


FIGURE 1. Local structure for medial axis.

- YB denotes the Y-branch locus: the set of points where locally IntMed(M) has a Y-branch.
- F is the number of fin point vertices.
- J is the number of 6-junction vertices.

Theorem 1.2. *For a generic smooth compact surface without boundary embedded in $\mathbb{R}^3$ we have*

$$\chi(\overline{\mathrm{Edge}}) = J + \chi(\overline{\mathrm{YB}}),$$

where $\overline{Z}$ denotes the closure of the set Z and χ as usual denotes the Euler characteristic.

In the second section we give a completely elementary proof of Theorem 1.2. The third section contains a review of some fundamentals on the medial axis and related phenomena. In the fourth section we formulate some higher dimensional analogues of Theorem 1.2. This we achieve as up to dimension 6 we know the normal forms of the medial axis, see [15]. In Section 5 we describe the relationship with the work of Sedykh (see for example, [16, 17, 18]). The main point is that we calculate the Euler characteristic of the *closure* of strata while Sedykh calculates the Euler characteristic (sometimes with compact supports) of the *open* strata. A few applications are presented in the last section. In particular we adapt Theorem 1.2 and its generalizations to give a result on the topology of cut-loci of 3-dimensional and 4-dimensional manifolds.

Remark 1.3. *A different (and very interesting) approach to the topology of the internal medial axis is contained in the papers of J. Damon, see* [7].

2. An elementary proof of Theorem 1.2

In order to prove Theorem 1.2 we remark that there is a continuous map $p\colon M \to \mathrm{IntMed}(M)$. Indeed M can be viewed as a tubular neighborhood of IntMed(M), every point of M comes from a unique point on IntMed(M). We can use this map to pull back any triangulation of IntMed(M) to one of M. The precise technical statement is given in Lemma 3.4.

We use the following symbols to count the number of 0-, 1-, and 2-simplices in a given triangulation K of IntMed(M).

- Vertices (0-simplices):

   - F is the number of fin point vertices,
   - J is the number of 6-junction vertices,
   - v_E is the number of vertices in the edges of the medial axis – but not counting fin vertices,
   - v_Y is the number of vertices in the Y-branch locus – but not counting fin vertices and 6-junction vertices,
   - v_{FC} is the number of vertices lying in the faces (that is, the non-singular parts of the medial axis).
- Edges (1-simplices):
   - E is the number of 1-simplices that lie in medial axis edges,
   - Y is the number in the Y-branch locus,
   - e_{FC} is the number that lie in the faces.
- Faces (2-simplices):
   - FC is the number of 2-simplices in the triangulation.

We count the number of 0-, 1-, and 2-simplices in the triangulation $\tilde{K}$ of M induced by the map p from the triangulation K of $\mathrm{IntMed}(M)$.

Generically p has exactly two preimages. Fin points also have exactly two preimages. Junctions have four preimages. Vertices in Edge have one preimage. Vertices on the Y-branch locus YB have three preimages.

Using this information we can write down the Euler characteristic of $\mathrm{IntMed}(M)$,

$$\chi(\mathrm{IntMed}(M)) = F + J + v_E + v_Y + v_{FC} - (E + Y + e_{FC}) + FC$$

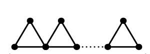
and the Euler characteristic of the manifold M,

$$\chi(M) = 2F + 4J + v_E + 3v_Y + 2v_{FC} - (E + 3Y + 2e_{FC}) + 2FC.$$

From these we can deduce

$$\begin{aligned}\chi(M) - 2\chi(\mathrm{IntMed}(M)) &= 2J - v_E + v_Y + E - Y \\ &= 2J - (v_E - E) + (v_Y - Y) + F - F \\ &= J - (v_E + F - E) + (v_Y + J + F - Y) \\ &= J - \chi(\overline{\mathrm{Edge}}) + \chi(\overline{\mathrm{YB}}).\end{aligned}$$

It remains to show that $\chi(M)-2\chi(\mathrm{IntMed}(M)) = 0$. This follows from Lemma 3.5 below but there is another completely elementary way of seeing this for surfaces. Any compact two dimensional oriented manifold without boundary is a Riemann surface of genus g. It is a 'donut' with g holes and so has Euler characteristic $2-2g$. The interior of the region bounded by M is homotopically equivalent to the medial axis. This interior is also homotopically equivalent to g aligned triangles, each one sharing a vertex with the successive one:



g triangles

Thus the interior has Euler characteristic $1 - g$. A Riemann surface of genus g has Euler characteristic $2 - 2g$. It follows that $\chi(M) - 2\chi(\mathrm{IntMed}) = 0$.

The proof of Theorem 1.2 is complete.

3. Some preliminaries

Before we state and prove the higher dimensional analogues of Theorem 1.2 we recall some preliminaries on medial axes.

In this and the next section let M be a compact orientable manifold of dimension $n-1$ without boundary embedded as a submanifold in $\mathbb{R}^n$ by a map $\phi\colon M \to \mathbb{R}^n$. The image of M divides $\mathbb{R}^n$ into a bounded and an unbounded component. We will call the bounded component $\mathcal{B}$. If the embedding is tame, then $\mathcal{B}$ is a compact manifold with boundary M. We now repeat Definition 1.1 for this more general case:

Definition 3.1. *The* internal medial axis $\mathrm{IntMed}(M)$ *is the closure of the locus of centers of hyperspheres which are tangent to M at two (or more) points and are contained inside M. The* external medial axis $\mathrm{ExtMed}(M)$ *is the closure of the locus of centers of hyperspheres which are tangent to M at two (or more) points and contain M in their interior.*

For every $x \in \mathbb{R}^n$ define the function $d_x \colon M \to \mathbb{R}$ by

$$d_x(s) = \|x - \phi(s)\|^2.$$

Definition 3.2. *The closure of the set of those x for which d_x has a non-unique global minimum is called the* medial axis*, denoted* $\mathrm{Med}(M)$*. The closure of the set of those x for which d_x has a non-unique global maximum is called the* maximal medial axis*, denoted* $\mathrm{MaxMed}(M)$*.*

From Definitions 3.1 and 3.2 we see that we have the inclusion $\mathrm{IntMed}(M) \subseteq \mathrm{Med}(M)$ and the equality $\mathrm{ExtMed}(M) = \mathrm{MaxMed}(M)$. The inclusion $\mathrm{IntMed}(M) \subseteq \mathrm{Med}(M)$ is strict in general. The union $\mathrm{MaxMed}(M) \cup \mathrm{Med}(M)$ is contained in a larger structure called the symmetry set, and this inclusion is also strict in general, see [5]. We encourage the reader unfamiliar with these spaces to draw a couple of closed curves in the plane that exemplify these assertions.

Lemma 3.3. *For a generic embedding of a compact manifold of codimension one without boundary in $\mathbb{R}^n$ the possible singularities of the maximal medial axis are the same as those of the medial axis.*

Proof. See Figure 2. Take a sphere S strictly containing the sphere that defines the singular point x_0 on $\mathrm{ExtMed}(M)$ and not centered at x_0. Denote the inversion through S by I_S. Locally $\mathrm{ExtMed}(M) = I_S(\mathrm{IntMed}(I_S(M)))$. The map I_S is a diffeomorphism and thus the local normal form of $\mathrm{ExtMed}(M)$ at x_0 is the local normal form of $\mathrm{IntMed}(I_S(M))$ at $I_S(x_0)$. □

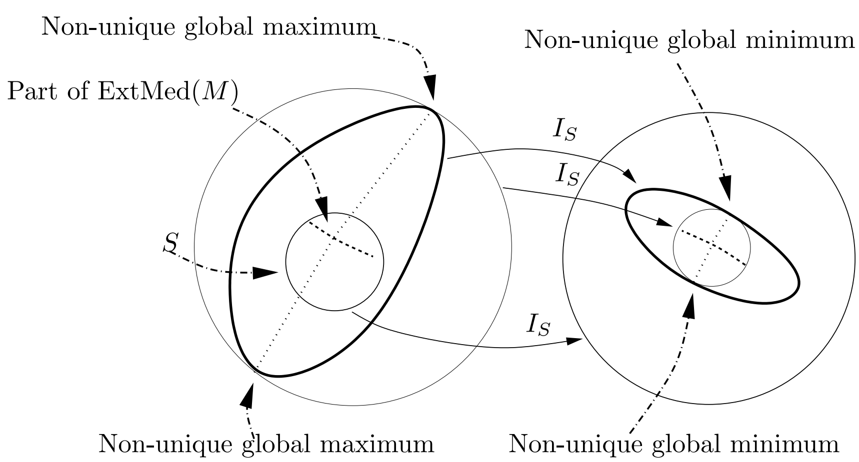


FIGURE 2. The inversion I_S locally maps $\mathrm{ExtMed}(M)$ to $\mathrm{IntMed}(I_S(M))$.

We will assume from now on that the embedding ϕ is generic in the sense of Looijenga, [13]. This implies in particular that the embedding ϕ is tame, and that the medial axis is triangulable. The following is well-known but we include a proof for completeness.

Lemma 3.4. *There is a finite-to-one map*

$$p\colon M \to \mathrm{IntMed}(M).$$

Let $\mu\colon K \to \mathrm{IntMed}(M)$ be a triangulation of the internal medial axis. Then there is a triangulation $\tilde{\mu}\colon \tilde{K} \to M$ and a simplicial map $\tilde{p}$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{K} & \xrightarrow{\tilde{\mu}} & M \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ K & \xrightarrow{\mu} & \mathrm{IntMed}(M). \end{array}$$

Proof. A normal $\nu(q)$ to $\phi(M)$ at $q \in M$ has a positive and a negative side because M is orientable. We take the positive side to be the part of the normal that points inside $\mathcal{B}$. For $s \in M$ let $S_r(q)$ be a hypersphere in $\mathbb{R}^n$, tangent to $\phi(M)$ at $\phi(s) = q$, and whose center lies at distance r from q on the positive side of the normal to M at q. There is some $r \neq 0$ such that $S_r(q)$ is contained in $\mathcal{B}$, because M has a tubular neighborhood. Take $r = r_0$ the supremum for which $S_r(q)$ is contained in $\mathcal{B}$. We will have $q \in S_{r_0}(q) \cap \phi(M)$. We let $p(s)$ be the center of the hypersphere $S_{r_0}(q)$. We now need to show that the map is well defined. Let there be another point $q' = \phi(s')$ in $S_{r_0}(q) \cap \phi(M)$. We need to show that $p(s) = p(s')$. This is clearly the case because there are points outside $\mathcal{B}$ on $S_r(q')$ if $r > r_0$.

The second statement is true for each stratum, because when restricted to the stratum the map p is a covering. Patching the strata together we obtain the required statement. □

Singularities of the medial axis up to local diffeomorphism are classified using R^+-equivalence of unfoldings induced by the distance function

$$F(x, s) = \frac{1}{2}\|x - \phi(s)\|^2$$

which unfolds $f_{x_0}(s) = F(x_0, s)$, see Chapter 1, Section 3 of [2]. For definitions and proofs we refer to [8]. For a generic embedding this unfolding will be versal.

Obviously there can only be multiple minima with equal critical value near x_0 if f_{x_0} has a local minimum. We call such germs *minimum germs*. For example, s_1^4 is a minimum germ, but, for every $m \geq 0$, $s_1^2 s_2 + s_2^m$ is not a minimum germ. Therefore, looking at the list of equivalence classes of unfoldings we should pick out those that unfold a minimum germ. When $n \leq 6$ the minimum germs are

$$\mathcal{A}_\mu\colon s^{\mu+1}, \text{ with } \mu \text{ an odd number}$$

with miniversal unfoldings

$$s^{\mu+1} + \sum_{i=1}^{\mu-1} x_i s^i.$$

When $n = 7$ a minimum germ with a modulus appears: $s_1^2 + s_2^2 + s_3^4$, and thus we limit our generalizations to $n \leq 6$.

We have to determine the adjacencies of the singularities $\mathcal{A}_1$, $\mathcal{A}_3$, $\mathcal{A}_5$, $\mathcal{A}_7$ and the multi-singularities that can be composed from them.

Let

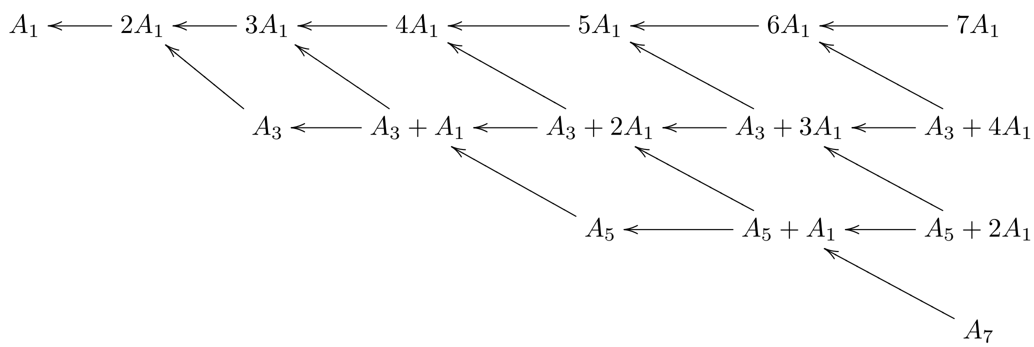
$$A_{\mu_1} + A_{\mu_2} + \cdots + A_{\mu_k} = \{x \in \mathbb{R}^n \mid f_x \text{ has singularities of type } \mathcal{A}_{\mu_1},\ \mathcal{A}_{\mu_2},\ \ldots,\ \mathcal{A}_{\mu_k}\}.$$

We will abbreviate the sum of k copies of A_μ to kA_μ. Thus $A_5 + 3A_1$ will denote $A_5 + A_1 + A_1 + A_1$.

We will abuse notation and also use $A_{\mu_1} + A_{\mu_2} + \cdots + A_{\mu_k}$ to denote the singularity type and hence can talk about adjacencies between them.

The sum $\mu_1 + \mu_2 + \cdots + \mu_k$ is called the *degree* of the multi-singularity. For a generic manifold with $n \leq 6$ only multi-singularities of degree less than or equal to

7 can occur. For higher n moduli can appear. Hence, we shall restrict ourselves to the adjacencies for degree less than or equal to 7. These are as follows:



Additionally we have the adjacencies $A_3+2A_1 \leftarrow 2A_3 \leftarrow A_7$, $A_3+3A_1 \leftarrow 2A_3+A_1$, and $2A_3 \leftarrow 2A_3 + A_1$. These adjacencies are explained in [3] Section 2.3, but note that $2A_3 \leftarrow A_7$ is missing there. To explain the adjacencies, denote the germs schematically as:

- A_1 : , the germ s^2,
- A_3 : , $s(s-\lambda)(s-2\lambda)(s-3\lambda)$ in the miniversal unfolding of s^4,
- A_5 : , $s(s-\lambda)(s-2\lambda)(s-3\lambda)(s-4\lambda)(s-5\lambda)$ in the miniversal unfolding of s^6,
- A_7 : , $s(s-\lambda)(s-2\lambda)(s-3\lambda)(s-4\lambda)(s-5\lambda)(s-6\lambda)(s-7\lambda)$ in the miniversal unfolding of s^8.

Generically the natural map $p : M \to \mathrm{IntMed}(M)$ is finite-to-one. For $\mu_i \geq 1$, at a point of $A_{\mu_1} + A_{\mu_2} + \cdots + A_{\mu_k}$ the number of preimages under p is k. Thus at A_1, A_3, A_5, and A_7 points the number of preimages is one. For each of the singularities in the above diagram we have one of these pictures for each preimage. So $2A_3$ can be denoted by (,). We have an adjacency $T \leftarrow S$ if S can be constructed from T by one of the two following operations.

- Two preimages of p become a more degenerate one. For instance + becomes .
- The number of preimages is augmented with one simple minimum. For instance (,) becomes (, ,).

From these rules it is easy to determine the adjacencies. As a further example of the first rule, consider the adjacencies $A_3 + 2A_1 \leftarrow 2A_3 \leftarrow A_7$. The multi-germ $A_3 + 2A_1$ is represented by (, ,). Connect the two minima . We get (,), which is a $2A_3$. If you connect the two A_3 germs in $2A_3$ you get , which is an A_7. As a further example for the second rule consider the adjacency $2A_3 \leftarrow 2A_3 + A_1$. Namely $2A_3 + A_1$ is represented by (, ,), which is nothing but $2A_3$ with an added .

For the convenience of the reader we point out the correspondence between this new notation, which we will use in the sequel, and the notation used in the previous section. The notation A_3 refers to ends e if $n = 2$, and to edges in Edge if $n = 3$. The notation $3A_1$ refers to trivalent vertices y if $n = 2$, and to Y-branches if $n = 3$. The notation $4A_1$ refers to 6-junctions if $n = 3$. The notation $A_3 + A_1$ refers to fins if $n = 3$. With this notation the Bose formula is

$$\chi(\overline{A_3}) - \chi(\overline{3A_1}) = 2 \qquad (1)$$

and Theorem 1.2 becomes

$$\chi(\overline{A_3}) - \chi(\overline{3A_1}) - \chi(\overline{4A_1}) = 0. \qquad (2)$$

In equations (1) and (2) we denoted by for instance $\chi(\overline{3A_1})$ the Euler characteristic of the closure of the $3A_1$ stratum.

We end with another well-known result which we will require later.

Lemma 3.5. *If M is a compact smooth orientable manifold with boundary embedded as a hypersurface in $\mathbb{R}^n$, then*

$$\chi(M) - 2\chi(\mathrm{IntMed}(M)) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ -2\chi(\mathrm{IntMed}(M)), & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is even, then $\dim(M)$ is odd. It is a straightforward consequence of Poincaré duality that $\chi(M) = 0$ for compact orientable manifolds without boundary.

The manifold M bounds a region $\mathcal{B}$, which is in its turn a compact manifold with boundary. The manifold $\mathcal{B}$ has the same homotopy type as $\mathrm{IntMed}(M)$ because $\mathrm{IntMed}(M)$ is a deformation retract of $\mathcal{B}$. If n is odd, then $\dim(\mathcal{B})$ is odd as well. For odd-dimensional compact manifolds with boundary we have $0 = \chi(\partial\mathcal{B}) - 2\chi(\mathcal{B}) = \chi(M) - 2\chi(\mathrm{IntMed}(M))$. (See Chapter IX in [14] for example.) □

4. A relationship between the Euler characteristic of a generic manifold and its internal medial axis

With the knowledge of the previous section we can produce some interesting relations between the Euler characteristic of a generic manifold and its internal medial axis. Denote by S_k an $n-k$ dimensional stratum of the internal medial axis. (A stratum is just the set of points with the same singularity type.) We write $f_i(S_k)$ for the number of interior faces of dimension i on the stratum. Moreover write formally

$$\check{\chi}(S_k) = \sum_{i=0}^{n-k} (-1)^i f_i(S_k).$$

Since this is just a formal sum it does not necessarily calculate the standard Euler characteristic of the set S_k and hence we use the $\check{\chi}$ notation. Using Lemma 3.4 we deduce the formulae:

$$\chi(M) = \sum_{\text{strata } T \text{ of } \mathrm{IntMed}(M)} (\#\text{preimages of } T)\check{\chi}(T), \text{ and}$$

$$\chi(\mathrm{IntMed}(M)) = \sum_{\text{strata } T \text{ of } \mathrm{IntMed}(M)} \check{\chi}(T). \tag{3}$$

Let S be a stratum. Then, adjacencies can be used to partition the closure $\overline{S}$ of S:

$$\chi(\overline{S}) = \check{\chi}(S) + \sum_{\text{strata } T \text{ adjacent to } S} \check{\chi}(T). \tag{4}$$

We now come to the main propositions in the paper. These concern the generalization of Theorem 1.2 to higher dimensions.

Proposition 4.1. *If M is a compact smooth orientable manifold without boundary generically embedded as a hypersurface in $\mathbb{R}^4$, then*

$$\chi(\mathrm{IntMed}(M)) = \frac{1}{2}\left(\chi(\overline{A_3}) + \chi(\overline{A_5}) - \chi(\overline{3A_1}) - \chi(\overline{4A_1}) - \chi(\overline{5A_1})\right).$$

Proof. In this four-dimensional case we get the following formulae from equation (4):

$$\begin{aligned}
\chi(\overline{3A_1}) &= \check{\chi}(3A_1) + \check{\chi}(4A_1) + \check{\chi}(5A_1) + \check{\chi}(A_3 + A_1) + \check{\chi}(A_3 + 2A_1) + \check{\chi}(A_5), \\
\chi(\overline{4A_1}) &= \check{\chi}(4A_1) + \check{\chi}(5A_1) + \check{\chi}(A_3 + 2A_1), \\
\chi(\overline{A_3}) &= \check{\chi}(A_3) + \check{\chi}(A_3 + A_1) + \check{\chi}(A_3 + 2A_1) + \check{\chi}(A_5).
\end{aligned}$$

From equation (3) we get

$$\begin{aligned}\chi(M)-2\chi(\mathrm{IntMed}(M)) =&\check{\chi}(3A_1)+2\check{\chi}(4A_1)+3\check{\chi}(5A_1)-\check{\chi}(A_3)+\check{\chi}(A_3+2A_1)-\check{\chi}(A_5)\\ =&\chi(\overline{3A_1})+\chi(\overline{4A_1})+\chi(\overline{5A_1})-\chi(\overline{A_3})-\chi(\overline{A_5}).\end{aligned}$$

After applying Lemma 3.5 we are done. □

In the five and six dimensional cases one can do exactly the same calculations.

Proposition 4.2. *If M is a compact smooth manifold without boundary generically embedded as a hypersurface in $\mathbb{R}^5$, then*

$$\sum_{i=3}^{6}\chi(\overline{iA_1})=\chi(\overline{A_3})+\chi(\overline{A_5})+\chi(2A_3).$$

Proof. Again we use equation (4):

$$\begin{aligned}\chi(\overline{3A_1}) &= \check{\chi}(3A_1)+\check{\chi}(4A_1)+\check{\chi}(5A_1)+\check{\chi}(6A_1)+\check{\chi}(A_3+A_1)+\check{\chi}(A_3+2A_1)\\ &\quad+\check{\chi}(A_3+3A_1)+\check{\chi}(A_5)+\check{\chi}(A_5+A_1)+\check{\chi}(2A_3),\\ \chi(\overline{4A_1}) &= \check{\chi}(4A_1)+\check{\chi}(5A_1)+\check{\chi}(6A_1)+\check{\chi}(A_3+2A_1)+\check{\chi}(A_3+3A_1)\\ &\quad+\check{\chi}(A_5+A_1)+\check{\chi}(2A_3),\\ \chi(\overline{5A_1}) &= \check{\chi}(5A_1)+\check{\chi}(6A_1)+\check{\chi}(A_3+3A_1),\\ \chi(\overline{A_3}) &= \check{\chi}(A_3)+\check{\chi}(A_3+A_1)+\check{\chi}(A_3+2A_1)+\check{\chi}(A_5)+\check{\chi}(A_5+A_1)+\check{\chi}(2A_3),\\ \chi(\overline{A_3+A_1}) &= \check{\chi}(A_3+A_1)+\check{\chi}(A_3+2A_1)+\check{\chi}(A_5)+\check{\chi}(A_5+A_1)+\check{\chi}(2A_3),\\ \chi(\overline{A_3+2A_1}) &= \check{\chi}(A_3+2A_1)+\check{\chi}(A_5)+\check{\chi}(2A_3),\\ \chi(\overline{A_5}) &= \check{\chi}(A_5)+\check{\chi}(A_5+A_1).\end{aligned}$$

Then,

$$\begin{aligned}\sum_{i=3}^{6}\chi(\overline{iA_1})-\chi(\overline{A_3})-\chi(\overline{A_5}) &= \check{\chi}(3A_1)+2\check{\chi}(4A_1)+3\check{\chi}(5A_1)+4\check{\chi}(6A_1)-\check{\chi}(A_3)\\ &\quad+\check{\chi}(A_3+2A_1)+2\check{\chi}(A_3+3A_1)-\check{\chi}(A_5)+\check{\chi}(2A_3).\end{aligned}$$

Also, we have,

$$\begin{aligned}\chi(M)-2\chi(\mathrm{IntMed}(M)) &= \check{\chi}(3A_1)+2\check{\chi}(4A_1)+3\check{\chi}(5A_1)+4\check{\chi}(6A_1)-\check{\chi}(A_3)\\ &\quad+\check{\chi}(A_3+2A_1)+2\check{\chi}(A_3+3A_1)-\check{\chi}(A_5)\\ &= \sum_{i=3}^{6}\chi(\overline{iA_1})-\chi(\overline{A_3})-\chi(\overline{A_5})-\chi(2A_3).\end{aligned}$$

□

Finally, we leave the proof of the six dimensional case to the reader:

Proposition 4.3. *If M is a compact smooth manifold without boundary generically embedded as a hypersurface in $\mathbb{R}^6$, then*

$$2\chi(\mathrm{IntMed}(M))=\chi(\overline{A_3})+\chi(\overline{A_5})+\chi(\overline{2A_3})-\sum_{i=3}^{7}\chi(\overline{iA_1}).$$

5. Relation with work of Sedykh

In a series of papers Sedykh has produced formulae for relations between the Euler characteristic of strata in the image of a map. In particular, [16] investigates the case of medial axis (though the paper is framed in the language of Maxwell sets) and [18] is on symmetry sets, of which medial axes are subsets. See also [17].

The main difference between his work and ours is that his formulae give relations between *strata*, i.e., sets which are usually open, whereas ours give relations for the *closure of strata*. Furthermore, in [18] he uses the Euler characteristic of manifolds with compact supports, whereas we use the ordinary Euler characteristic. In the curve in the plane case the relevant strata are zero-dimensional and hence closed. Thus both approaches coincide and produce the Bose formula.

This difference between looking at open and closed sets means that in more general cases the approaches diverge. However, for example, one can use Sedykh's formulae (in conjunction with some elementary topological ideas such as the Mayer-Vietoris sequence and knowledge of the local structure of the singularities) to deduce Theorem 1.2. It should be noted that our proof in this case has the advantage that it is very direct, the proof in [16] rests upon a statement in an earlier paper which was determined via computer. Hence, deducing the formulae from his would be an unnecessarily complicated method.

Let us see how Sedykh's relations can be used to prove Theorem 1.2. In Table 2 of [16] we find (our notation is identical to his):

$$\begin{aligned} 8\chi(3A_1) &= 16\chi(4A_1) + 4\chi(A_3 + A_1) \\ \text{and } 8\chi(A_3) &= 4\chi(A_3 + A_1), \end{aligned}$$

where χ denotes the usual Euler characteristic of a set. Therefore, we can deduce that

$$\chi(3A_1) = 2\chi(4A_1) + \chi(A_3).$$

For clarity, we can put this in the notation of Theorem 1.2:

$$\chi(\mathrm{YB}) = 2J + \chi(\mathrm{Edge}). \tag{5}$$

We can show that $\chi(\mathrm{Edge}) = \chi(\overline{\mathrm{Edge}})$ and, using the Mayer-Vietoris Theorem in an elementary way, that $\chi(\mathrm{YB}) = \chi(\overline{\mathrm{YB}}) + 3J$. Thus, from equation 5, we deduce Theorem 1.2:

$$\chi(\overline{\mathrm{YB}}) + J = \chi(\overline{\mathrm{Edge}}).$$

Therefore, one can see that the two approaches are essentially different. Furthermore, in the next section, we shall give different applications.

6. Some applications

Let N be a connected compact Riemannian manifold of dimension m, with a distinguished point q on it. The *cut-locus* $C(N, q)$ of q in N is the closure of the set of those $p \in N$ such that there exist two different globally minimizing geodesics from p to q. The unit tangent bundle SN of N is the subbundle of TN of tangent vectors of length 1. We have a map P, similar to the map p of Lemma 3.4, from S_qN to $C(N, q)$. For $v \in S_qN$, $P(v)$ is the first point of $C(N, q)$ that a geodesic starting from (q, v) meets. If the metric on N is a generic metric in the sense of [6] then we can use it to pull back a triangulation K of $C(N, q)$ to a triangulation $\widetilde{K}$ of S_qN. Moreover, for a generic metric the cut-locus locally has one of the normal forms $A_{\mu_1} + \cdots + A_{\mu_k}$, with degree less than or equal to 7, see [6]. We thus find ourselves in exactly the same situation as in Section 4.

Theorem 6.1. *Let N be a connected compact manifold of dimension m. Equip N with a generic metric. Let $C(N, q)$ be the cut-locus of a point $q \in N$. If $m = 2$ and*

N is orientable, then

$$\chi(\overline{3A_1}) - \chi(\overline{A_3}) = 4g - 2, \tag{6}$$

where $g \geq 0$ is the genus of the surface. If $m = 2$ and N is not orientable, then

$$2k - 4 = \chi(\overline{3A_1}) - \chi(\overline{A_3}), \tag{7}$$

where $k \geq 1$ is the number of projective planes of which N is a connected sum. If $m = 3$ and N is orientable, then

$$\chi(\overline{3A_1}) + \chi(\overline{4A_1}) - \chi(\overline{A_3}) = 0. \tag{8}$$

If $m = 3$ and N is not orientable, then

$$2 - 2\chi(N) = \chi(\overline{3A_1}) + \chi(\overline{4A_1}) - \chi(\overline{A_3}).$$

If $m = 4$ and N is orientable, then

$$2(\chi(N) - 1) = \chi(\overline{A_3}) + \chi(\overline{A_5}) - \chi(\overline{3A_1}) - \chi(\overline{4A_1}) - \chi(\overline{5A_1}).$$

If $m = 4$ and N is not orientable, then

$$2\chi(N) = \chi(\overline{A_3}) + \chi(\overline{A_5}) - \chi(\overline{3A_1}) - \chi(\overline{4A_1}) - \chi(\overline{5A_1}). \tag{9}$$

Proof. Where in the previous section we had to calculate $\chi(M) - 2\chi(\text{IntMed}(M))$, now we have to calculate $\chi(S_qN) - 2\chi(C(N, q))$. We will express this quantity using $\chi(N)$ only.

The inclusion $C(N, q) \to N$ gives an isomorphism in homology for $i = 0, \dots, m-1$, see [12], Section 2.1. Therefore

$$\chi(N) - \chi(C(N, q)) = (-1)^m \operatorname{rank} H_m(N, \mathbb{Q}) \tag{10}$$

Because N is assumed to be connected we have $\operatorname{rank} H_m(N, \mathbb{Q}) = 1$ or $= 0$, depending on whether N is orientable or not. If m is odd and N orientable, $\chi(N) = 0$ by Poincaré duality, and, because of the orientability, also $\operatorname{rank} H_n(N, \mathbb{Q}) = 1$. So by (10) $\chi(C(N, q)) = 1$. It follows that $\chi(S^{m-1}) - 2\chi(C(N, q)) = 0$.

If m is odd and N is not orientable, $\chi(N) = \chi(C(N, q))$ by (10). It follows that $\chi(S^{m-1}) - 2\chi(C(N, q)) = 2 - 2\chi(N)$.

If m is even and N is orientable, $\chi(N) = \chi(C(N, q)) + 1$ also by (10). It follows that $\chi(S^{m-1}) - 2\chi(C(N, q)) = 2 - 2\chi(N)$.

If m is even and N is not orientable, $\chi(N) = \chi(C(N, q))$ by (10). It follows that $\chi(S^{m-1}) - 2\chi(C(N, q)) = -2\chi(N)$.

A two dimensional compact connected manifold is either a connected sum of k projective planes (non-orientable), or it is the connected sum of g tori (orientable). In the first case we have $\chi(N) = 2 - k$, in the other case we have $\chi(N) = 2 - 2g$. Applying the results of Section 4 the proof is complete. □

Note that formulae (6) and (7) are just Bose formulae for the cut-locus. Formulae for higher dimensional cases can be created by the interested reader.

If the cut-locus of a compact manifold has only transversal self-intersections, i.e., its only singularities are kA_1, $3 \leq k \leq \dim N + 2$, then it is called *a simple spine*. A result of Weinstein, see [20], says that any manifold of dimension greater than 2 admits a metric such that some cut-locus is a simple spine. In this case the above formulae become a lot simpler. These simpler formulae are stated in [1], Section 5.2. For the special cases in equations (8) to (9) just drop the terms not involving kA_1. In general:

Proposition 6.2. *Let N be a connected compact manifold of dimension $m > 2$. Equip N with a metric so that there is a point q whose cut-locus $C(N, q)$ only has transversal self-intersections. If N is orientable, then*

$$\sum_{i=3}^{m+1} \chi(\overline{iA_1}) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 2 - 2\chi(N) & \text{if } m \text{ is even.} \end{cases}$$

If N is not orientable, then

$$\sum_{i=3}^{m+1} \chi(\overline{iA_1}) = 1 - (-1)^m - 2\chi(N).$$

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