Computing with stable corank 1 liftable vector fields from n-space to (n+1)-space

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Outline

Introduction

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- Generic geometry
- μ versus τ results in local complex analytic geometry (Mond conjecture)

The vector fields are given explicitly for the minimal cross cap of multiplicity k. The fields can contain lots of monomials (none worse than a quadratic!) and so we use computer packages to work with them.

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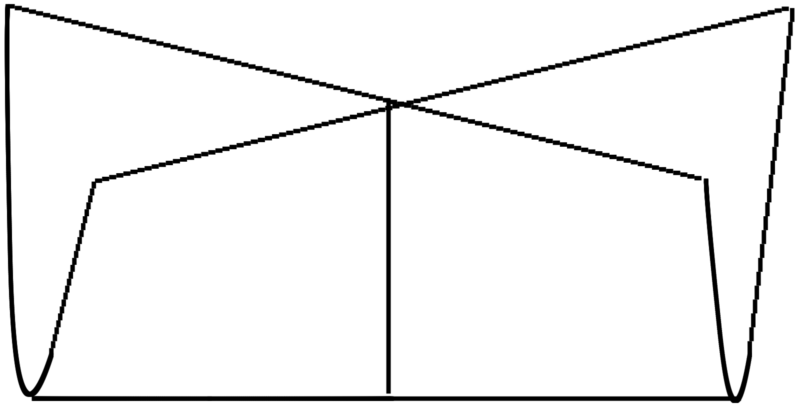

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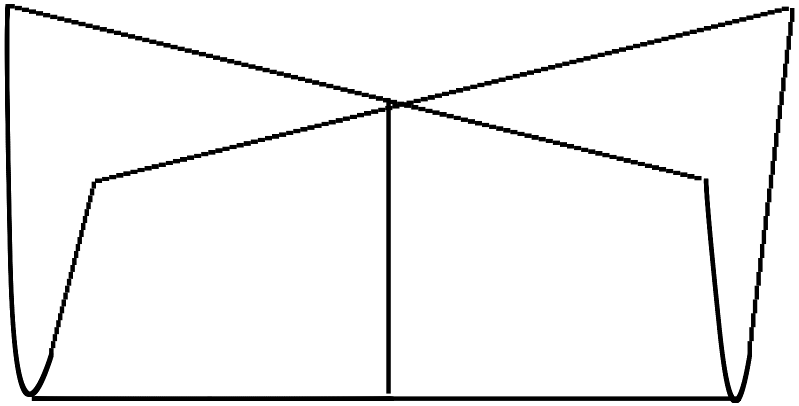

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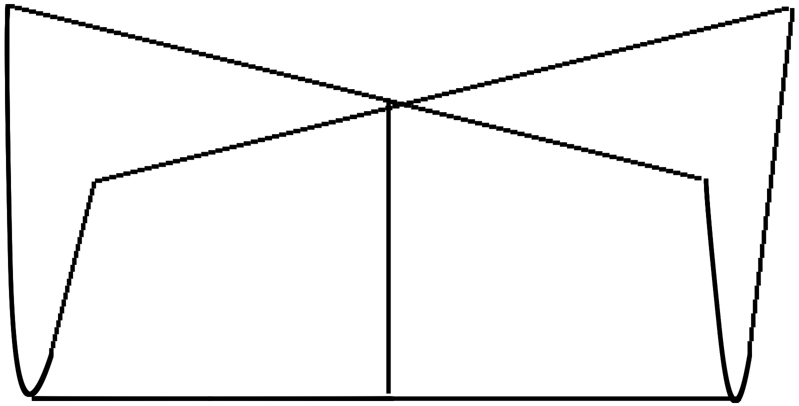

- corank 1, i.e., the rank of the Jacobian drops by at most 1 at the singular point,
- stable, i.e., small perturbations give (up to diffeomorphism) the same map.

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In the complex case these are equivalent to the liftable vector fields.

Liftable Vector Fields for Cross Caps

Definition

Let f be a smooth mapping $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. A vector field ξ on $(\mathbb{K}^p, 0)$ is **liftable over** f if there is a vector field η on $(\mathbb{K}^n, 0)$ such that $df \circ \eta = \xi \circ f$. That is, the following diagram commutes

$$\begin{tikzcd}
T(\mathbb{K}^n, 0) \arrow[r, "df"] & T(\mathbb{K}^p, 0) \\
(\mathbb{K}^n, 0) \arrow[u, "\eta"] \arrow[r, "f"] & (\mathbb{K}^p, 0). \arrow[u, "\xi"']
\end{tikzcd}$$

In these circumstances η is called **lowerable**.

Example

For the cross-cap $\varphi(v_1, y) = (v_1, y^2, v_1 y) = (V_1, W_1, W_2)$ the following are liftable vector fields:

$$\begin{pmatrix} W_2 \\ 0 \\ V_1 W_1 \end{pmatrix}, \begin{pmatrix} -V_1 \\ 2W_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2W_2 \\ V_1^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} V_1 \\ 2W_1 \\ 2W_2 \end{pmatrix}.$$

The corresponding lowerable vector fields are (respectively)

$$\begin{pmatrix} v_1 y \\ 0 \end{pmatrix}, \begin{pmatrix} -v_1 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ y \end{pmatrix}.$$

These vector fields generate the module of liftable vector fields.

Example

Taking the second vector field in the list we have

$$\begin{aligned} d\varphi \circ \begin{pmatrix} -v_1 \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & v_1 \end{pmatrix} \begin{pmatrix} -v_1 \\ y \end{pmatrix} \\ &= \begin{pmatrix} -v_1 \\ 2y^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -V_1 \\ 2W_1 \\ 0 \end{pmatrix} \circ \varphi. \end{aligned}$$

Minimal cross-cap

Definition

For $k \geq 2$ the **minimal cross-cap mapping of multiplicity k** is the map $\varphi_k : (\mathbb{K}^{2k-2}, 0) \to (\mathbb{K}^{2k-1}, 0)$ given by

$$\begin{aligned} &\varphi_k(u_1, \dots, u_{k-2}, v_1, \dots, v_{k-1}, y) \\ &= \left(u_1, \dots, u_{k-2}, v_1, \dots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i \right) \end{aligned}$$

We shall label the coordinates of the target $U_1, \dots, U_{k-2}, V_1, \dots, V_{k-1}, W_1$ and W_2, respectively.

Example

The Whitney cross-cap is $\varphi_2(v_1, y) = (v_1, y^2, v_1 y)$.

- Stable

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- Corank 1

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- Any stable corank 1 mono-germ from n-space to $(n+1)$-space is $\mathcal{A}$-equivalent to a trivial unfolding of such a map.

Definition

Two smooth multi-germs $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and $\widetilde{f} : (\mathbb{K}^n, \widetilde{S}) \to (\mathbb{K}^p, 0)$ are $\mathcal{A}$**-equivalent** if there exist diffeomorphisms φ and ψ for which the following diagram commutes.

$$\begin{tikzcd}
(\mathbb{K}^n, S) \arrow[r, "f"] \arrow[d, "\varphi"'] & (\mathbb{K}^p, 0) \arrow[d, "\psi"] \\
(\mathbb{K}^n, \widetilde{S}) \arrow[r, "\widetilde{f}"] & (\mathbb{K}^p, 0).
\end{tikzcd}$$

This is also known as **Right-Left-equivalence**.

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- Thus image of φ_k is not a free divisor. Has $k - 1$ ‘extra’ generators. Compare with $n \geq p$ case.
- Used `Singular` for small k. Guessed form of lowerables. Then generalized form for liftables and lowerables. Used $d\varphi_k \circ \eta = \xi \circ \varphi_k$.

Euler vector field

One vector field is easy. The Euler vector field:

$$\xi_e = \begin{pmatrix} (k-1)U_1 \\ (k-2)U_2 \\ \vdots \\ 2U_{k-2} \\ (k-1)V_1 \\ (k-2)V_2 \\ \vdots \\ V_{k-1} \\ kW_1 \\ kW_2 \end{pmatrix}$$

The other $3k-3$ vector fields come in three families. We need some way of describing these liftables in a compact form.

$$\xi_j^f = \left(\begin{array}{c} A_{1,j}^f \\ \vdots \\ A_{k-2,j}^f \\ B_{1,j}^f \\ \vdots \\ B_{k-1,j}^f \\ C_{1,j}^f \\ C_{2,j}^f \end{array}\right) \quad \begin{array}{c} U_1 \\ \vdots \\ U_{k-2} \\ V_1 \\ \vdots \\ V_{k-1} \\ W_1 \\ W_2 \end{array}$$

$f = 1, 2, 3$ denotes the family.
$j = 1, 2, 3, \ldots, k-1$ denotes which member of the family it is.

First Family

Theorem (Littlestone)

For $1 \leq j \leq k-1$ the vector field given by the following components is liftable over φ_k:

$$\begin{aligned} A^1_{i,j} &= (k-i)(k-j)U_iU_j, \qquad 1 \leq i \leq k-2, \\ B^1_{i,j} &= k\sum_{r=1}^{i-1} U_{i+j-r}V_r - k\sum_{r=1}^{i} U_rV_{i+j-r} - (i-1)(k-j)U_jV_i \\ &\quad + kV_{i+j}W_1 - kU_{i+j}W_2, \qquad 1 \leq i \leq k-1, \\ C^1_{1,j} &= k(k-j)U_jW_1, \\ C^1_{2,j} &= -kV_jW_1 + (k-j)U_jW_2. \end{aligned}$$

We set $U_{k-1} = V_k = 0$, $U_k = 1$ and $U_i = V_i = 0$ for $i < 0$ and $i > k$.

Second family

Theorem (Littlestone)

For $1 \leq j \leq k-1$ the vector field given by the following components is liftable over φ_k:

$$\begin{aligned}
A^2_{i,j} &= -k(k+i-j+1)U_{k+i-j+1}W_1 + k\sum_{r=1}^{i}(k+i-j-2r+1)U_rU_{k+i-j-r+1} \\
&\quad -j(i+1)U_{i+1}U_{k-j}, \qquad 1 \leq i \leq k-2, \\
B^2_{i,j} &= -k(k+i-j+1)V_{k+i-j+1}W_1 + k\sum_{r=1}^{i}(k+i-j-r+1)U_rV_{k+i-j-r+1} \\
&\quad -k\sum_{r=1}^{i} rU_{k+i-j-r+1}V_r - j(i+1)U_{k-j}V_{i+1}, \qquad 1 \leq i \leq k-1, \\
C^2_{1,j} &= k(k-j+1)U_{k-j+1}W_1 + jU_1U_{k-j}, \\
C^2_{2,j} &= k(k-j+1)V_{k-j+1}W_1 + jV_1U_{k-j}.
\end{aligned}$$

Third family

Theorem (Littlestone)

For $1 \leq j \leq k-1$ the vector field given by the following components is liftable over φ_k:

$$\begin{aligned}
A^3_{i,j} &= -k(k+i-j+1)U_{k+i-j+1}W_2 + k\sum_{r=1}^{i}(k+i-j-r+1)U_{k+i-j-r+1}V_r \\
&\quad -k\sum_{r=1}^{i} rU_rV_{k+i-j-r+1} - k(i+1)U_{i+1}V_{k-j}, \qquad 1 \leq i \leq k-2, \\
B^3_{i,j} &= -k(k+i-j+1)V_{k+i-j+1}W_2 + k\sum_{r=1}^{i}(k+i-j-2r+1)V_rV_{k+i-j-r+1} \\
&\quad -k(i+1)V_{i+1}V_{k-j}, \qquad 1 \leq i \leq k-1, \\
C^3_{1,j} &= k(k-j+1)U_{k-j+1}W_2 + kU_1V_{k-j} \\
C^3_{2,j} &= k(k-j+1)V_{k-j+1}W_2 + kV_1V_{k-j}.
\end{aligned}$$

The proofs are just long calculations involving writing down a lowerable and liftable and showing that $d\varphi_k \circ \eta = \xi \circ \varphi_k$.

An example

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Liftable vector fields for $k = 3$

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$$\xi_1^2 = \begin{pmatrix} 6U_1 \\ -3V_1 \\ -6V_2 \\ 9W_1 \\ 0 \end{pmatrix} \qquad \xi_2^2 = \begin{pmatrix} -9W_1 \\ 2U_1V_2 \\ -3V_1 \\ 2U_1^2 \\ 6V_2W_1 + 2U_1V_1 \end{pmatrix}$$

$$\xi_1^3 = \begin{pmatrix} 9V_1 \\ -6V_2^2 \\ 0 \\ 9W_2 + 3U_1V_2 \\ 3V_1V_2 \end{pmatrix} \qquad \xi_2^3 = \begin{pmatrix} -9W_2 - 3U_1V_2 \\ -3V_1V_2 \\ 0 \\ 3U_1V_1 \\ 6V_2W_2 + 3V_1^2 \end{pmatrix}$$

$$\xi_e = \begin{pmatrix} 2U_1 \\ 2V_1 \\ V_2 \\ 3W_1 \\ 3W_2 \end{pmatrix}.$$

They generate Derlog(V)

Theorem (H—)

The module of vector fields liftable over $\varphi_k : (\mathbb{C}^{2k-2}, 0) \to (\mathbb{C}^{2k-1}, 0)$ is generated by

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- For any liftable vector field ξ divide by our vector fields so that have remainder r. This is liftable.
- Division algorithm says that remainder has no W_1s and W_2s in most positions.
- Show no lowerable exists for the remainder unless the remainder was 0.

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To answer the sceptics I programmed the liftables and lowerables into Maple. We can check quite quickly for large values of k.

Application: ${}_V\mathcal{K}$-equivalence

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We shall investigate the latter.

Recall the following:

Definition

Let $h, \widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be two smooth map-germs. We say that h and $\widetilde{h}$ are **$\mathcal{K}$-equivalent** if there exists a diffeomorphism $\Psi : (\mathbb{K}^p \times \mathbb{K}^q, 0 \times 0) \to (\mathbb{K}^p \times \mathbb{K}^q, 0 \times 0)$ such that

1. $\Psi(\mathbb{K}^p \times \{0\}, 0 \times 0) = (\mathbb{K}^p \times \{0\}, 0 \times 0)$,
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The map Ψ determines a diffeomorphism $\psi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$.

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Example

$(x, y) \mapsto (x, y^2, y^3 \pm x^2 y)$.

We now use a definition of Damon:

Definition

Let $(V, 0)$ be subset of $(\mathbb{K}^p, 0)$ (we do not assume that it is analytic). Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ and $\widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be smooth maps. We say that h and $\widetilde{h}$ are **${}_V\mathcal{K}$-equivalent** if

1. h and $\widetilde{h}$ are $\mathcal{K}$-equivalent, and
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The action of $\mathcal{K}$ is often described as the semi-direct product of $\mathcal{R}$ and $\mathcal{C}$. If we let ${}_V\mathcal{R}$ denote the diffeomorphisms that preserve V, then the action of ${}_V\mathcal{K}$ is given by the semi-direct product of ${}_V\mathcal{R}$ and $\mathcal{C}$.

This equivalence is intimately connected to $\mathcal{A}$-equivalence. Thus we can perform $\mathcal{A}$-classifications using a $\mathcal{K}$-type equivalence.

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And we have these for stable corank 1 maps from $(\mathbb{K}^n, 0)$ to $(\mathbb{K}^{n+1}, 0)$!

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The **Jacobian of** h **with respect to** V, denoted $J_V(h)$ is the module

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The **extended ${}_V\mathcal{K}$-tangent space of h with respect to V** is

$$T_V\mathcal{K}_e(h) = J_V(h) + h^*(\mathfrak{m}_q)\mathcal{E}_p^q.$$

($\mathcal{E}_r$ is module of germs of smooth functions on $(\mathbb{K}^r, 0)$ and $\mathfrak{m}_r$ is its maximal ideal.)

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The **${}_V\mathcal{K}_e$-codimension of h with respect to V** is

$${}_V\mathcal{K}_e - \operatorname{cod}(h) = \dim_{\mathbb{K}} \frac{\mathcal{E}_p^q}{T_V\mathcal{K}(h)}.$$

Example

Let $F : (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ be the trivial extension of the Whitney umbrella given by $F(x, v_1, y) = (x, v_1, y^2, v_1 y)$. Use coordinates (X, V_1, W_1, W_2).

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&= \dim_{\mathbb{C}} \frac{\mathcal{E}_2}{\langle W_1 \frac{\partial p}{\partial W_1}, \frac{\partial p}{\partial X}, p\rangle}
\end{aligned}$$

A complete transversal theorem

Theorem (Complete transversal theorem for ${}_V\mathcal{K}$-equivalence)

Suppose that $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ *is a smooth map and* V *is an analytic subgerm of* $(\mathbb{K}^p, 0)$.

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If $g_1, \dots, g_s$ *are homogeneous polynomial maps of degree* $k+1$ *such that*

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then every g *with* $j^k(h) = j^k(g)$ *is* ${}_V\mathcal{K}$*-equivalent to some* f *where* $j^{k+1}(f)$ *is of the form* $j^k(h) + \sum_{i=1}^s \alpha_i g_i$*,* $\alpha_i \in \mathbb{K}$*.*

Follows from Bruce, du Plessis and Kirk, *Nonlinearity*, 1997.

Application to $\mathcal{A}$-equivalence classifications

Let's study the following:

$$(\mathbb{K}^n, S) \xrightarrow{F} (\mathbb{K}^p, 0) \xrightarrow{h} (\mathbb{K}^q, 0)$$

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We study h under ${}_V\mathcal{K}$-equivalence.

Definition

Suppose that $F : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is stable map and that $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ is a smooth map. We define $h^\#(F)$ to be the multi-germ given by
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We have the following commutative diagram:

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(\mathbb{K}^n, S) \arrow[r, "F"] & (\mathbb{K}^p, 0) \arrow[r, "h"] & (\mathbb{K}^q, 0) \\
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where f is the pullback of F by g.

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Theorem

We have $f \sim_{\mathcal{A}} h^{\#}(F)$.

Example

Consider the earlier example where $F : (\mathbb{K}^3, 0) \to (\mathbb{K}^4, 0)$ is the trivial extension of the Whitney umbrella given by $F(x, v_1, y) = (x, v_1, y^2, v_1 y)$ and $h(X, V_1, W_1, W_2) = V_1 - p(X, W_1)$ for a function p.

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Then the image of g is equal to $h^{-1}(0)$ and g is transverse to F.

The pull-back of F by g is a map of the form

$$f(x, y) = (x, y^2, yp(x, y^2)).$$

$\mathcal{A}$ versus ${}_V\mathcal{K}$

We now connect $\mathcal{A}$-equivalence and ${}_V\mathcal{K}$-equivalence.

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Theorem

Let $F : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, $n < p$, be a stable map with discriminant V.
Suppose that $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ and $\widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ are submersions with F transverse to $h^{-1}(0)$ and $\widetilde{h}^{-1}(0)$.
Then

$$h^{\#}(F) \sim_{\mathcal{A}} \widetilde{h}^{\#}(F) \iff h \sim_{{}_V\mathcal{K}} \widetilde{h}.$$

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$$h^{\#}(F) \sim_{\mathcal{A}} \widetilde{h}^{\#}(F) \iff h \sim_{{}_V\mathcal{K}} \widetilde{h}.$$

Can generalize to $\widetilde{h}$ on codomain of $\widetilde{F}$ where $F \sim_{\mathcal{A}} \widetilde{F}$.

Proof

[$\Longrightarrow$] To lighten notation denote $h^{\#}(F)$ and $\widetilde{h}^{\#}(F)$ by f and $\widetilde{f}$ respectively.

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\quad \text{and} \quad
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(\mathbb{K}^n, \widetilde{S}) \arrow[r, "F"] & (\mathbb{K}^p, 0) \\
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where $n' = n - q$, $p' = p - q$, the map j parametrizes $h^{-1}(0)$, f is the pull-back of F by j, i is the natural map arising from that pull-back,

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Proof

As f and $\widetilde{f}$ are $\mathcal{A}$-equivalent there exist diffeomorphism germs $\rho : (\mathbb{K}^{n'}, S) \to (\mathbb{K}^{n'}, \widetilde{S})$ and $\lambda : (\mathbb{K}^{p'}, 0) \to (\mathbb{K}^{p'}, 0)$ so that the following commutes

$$\begin{tikzcd}
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(\mathbb{K}^{n'}, S) \arrow[u, "\rho"] \arrow[r, "f"] & (\mathbb{K}^{p'}, 0). \arrow[u, "\lambda"']
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As both (F, i, j) and $(F, \widetilde{i} \circ \rho, \widetilde{j} \circ \lambda)$ unfold f (and F is stable), they are isomorphic as unfoldings.

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$$\begin{tikzcd}
(\mathbb{K}^n, S) \arrow[rrr, "F"] \arrow[dd, "\varphi"'] & & & (\mathbb{K}^p, 0) \arrow[dd, "\psi"] \\
& (\mathbb{K}^{n'}, S) \arrow[ul, "i"'] \arrow[dl, "\widetilde{i}\circ\rho"] \arrow[r, "f"] & (\mathbb{K}^{p'}, 0) \arrow[ur, "j"] \arrow[dr, "\widetilde{j}\circ\lambda"'] & \\
(\mathbb{K}^n, S) \arrow[rrr, "F"] & & & (\mathbb{K}^p, 0)
\end{tikzcd}$$

Proof

As $j(\mathbb{K}^{p'}) = h^{-1}(0)$ and $\widetilde{j} \circ \psi(\mathbb{K}^{p'}) = \widetilde{h}^{-1}(0)$ we have that h and $\widetilde{h}$ are $\mathcal{K}$-equivalent. Furthermore as the diagram above commutes ψ has the property that it preserves the discriminant of F. That is, h and $\widetilde{h}$ are ${}_V\mathcal{K}$-equivalent as required.

Proof

[$\Longleftarrow$] First note that $h \sim_{{}_V\mathcal{K}} \widetilde{h}$ implies that h and $\widetilde{h}$ are $\mathcal{K}$-equivalent and the induced diffeomorphism of $(\mathbb{K}^p, 0)$, denoted ψ, preserves the discriminant of F, i.e., $\psi(V) = V$.

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This $\mathcal{K}$-equivalence implies that $h^{-1}(0)$ and $(\widetilde{h} \circ \psi)^{-1}(0)$ are diffeomorphic, so $h^{-1}(0)$ and $\widetilde{h}^{-1}(0)$ are diffeomorphic by ψ.

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As ψ preserves the discriminant of F we have that f and $\widetilde{f}$ have diffeomorphic discriminants. As $n < p$ we see that the domains of these maps are normalizations of the discriminants. This implies that f and $\widetilde{f}$ are $\mathcal{A}$-equivalent.

$\square$

Main point

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$$h^{\#}(F) \sim_{\mathcal{A}} \widetilde{h}^{\#}(F) \iff h \sim_{{}_V\mathcal{K}} \widetilde{h}.$$

result allows us to classify maps under $\mathcal{A}$-equivalence using ${}_V\mathcal{K}$-equivalence. The latter behaves a bit like $\mathcal{K}$-equivalence, e.g., has a good 'tangent space'.

Coda

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- Classify corank 1 maps from $(\mathbb{K}^n, 0)$ to $(\mathbb{K}^{n+1}, 0)$ under $\mathcal{A}$-equivalence.

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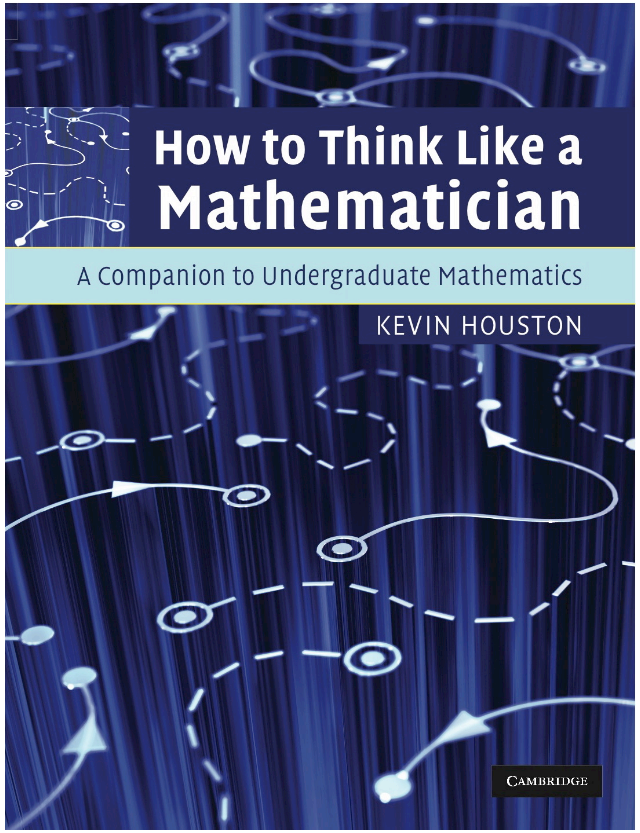
Shameless advert for book

My book for undergraduates ‘How to Think Like a Mathematician’ is now available:



Videos available on my YouTube channel

`http://www.youtube.com/user/DrKevinHouston`