Review of 'Singular Points of Plane Curves by C.T.C. Wall'

Plane curves are a valuable source of mathematics. In many situations one can really see what is going on and test one's intuition against real pictures rather some abstract schematic. Many excellent books have been written on the subject. For example, the recent *Plane Algebraic Curves* by Gerd Fischer, and the older but identically named *Plane Algebraic Curves* by Egbert Brieskorn and Horst Knörrer. Is there a need for another? What sets *Singular Points of Plane Curves* by C.T.C. Wall apart from the others? I'll give three good reasons, concerning content, style and pedagogy.

First, the emphasis, as the title suggests, is on the theory arising when one considers non-manifold points. Mathematicians love manifolds, much of 20th Century mathematics was about manifolds. But most spaces are not manifolds. Most spaces have sharp, nasty, problematic singular points. And it is here that the real interest lies since in applications, and in theory, one quickly encounters singular points. One need only look into a coffee cup at the bright curve formed by reflection on the surface of the coffee to see a singular cusp. Small perturbations will not rid the curve of the cusp - the feature is robust and so cannot be ignored. Many other applications of singularity theory occur. For example, in Wall's book, it is shown the real link of a singular curve produces a knot. Hence we open a door from complex curves to knot theory.

The second reason for reading the book is one of style. Wall's love and enjoyment of computation and complex calculation is reflected throughout. The hands-on, explicit calculations provide examples and motivation for later theory. Thus the book avoids being merely a dull list of dry facts with no interesting or useful examples.

Third, Wall gets to some deep results. According to the introduction, the book grew out of an MSc course and so takes the reader to the edge of current research. The books mentioned earlier by Brieskorn and Knörrer, and Fischer focus more on the classical theory of curves, though did touch on recent results of 30 years ago. Here we get an insight into many different areas of current research.

The book is not a light read. The text is dense in places and is challenging as it approaches the subject from many different angles: algebra, analysis, topology and geometry. At least some of the material will be new to all but the most well-read of students or researchers.

There are eleven chapters. The first gives some necessary preliminaries. Chapters 2 to 5 were the basis of an MSc course and focus on Puiseux characteristics and resolution of singularities via blowing up. Both are classic subjects which one would expect students to know by the end of a course on plane curves. More modern developments such as the Eggers tree and Lê's carousel method are covered. The MSc course mentioned was also taught by the knot theorist Hugh Morton and so there are applications to the study of knots: A complex curve in the plane is a real two-dimensional body in four-dimensional space. The real link of a singularity on the curve is the intersection of a sufficiently small real 3-sphere centred at the singularity. If the singular point is isolated, then the real link is a one-dimensional manifold in a 3-dimensional space. Thus the link is of interest to knot theorists. The fifth chapter is concerned with the geometry of the link of a curve and gives a very good introduction to the carousel method. This method is quite complicated and so it is good to see it in such a low dimensional direct way.

Moving beyond the MSc part of the book Chapter 6 gives a rigorous explanation of the Milnor fibre of a function on $\mathbb{C}^2$. Much work has been done in higher dimensions for Milnor fibres so the restriction to 2 variables is in some sense limiting but is also liberating as it allows the reader to develop a feel for the Milnor number. Subjects, such as Morsification of functions, are covered with minimal pre-requisites and so are mostly self-contained. Even a proof of $\chi(E) = \chi(F)\chi(B)$ for the Euler characteristics of a fibration $f : E \to B$ with fibre F is included rather than have the reader rely on another text.

Chapter 7 is about curves and their duals so to begin with has a more classical flavour but does mention more recent results concerning integration with respect to Euler characteristic. Chapter 8 *Combinatorics on a resolution tree* contains the toughest material. One of the sections features the topological zeta function and as such can be considered a concrete introduction to a very modern and vibrant area of research, that of motivitic integration.

Chapter 9 concerns the decomposition of the Milnor fibre and so is essentially topological in nature preparing the reader for Chapter 10 on the monodromy of Milnor fibration. This chapter goes into significant detail on monodromy.

The final chapter looks at a different aspect: the theory of ideals. This relies less on what has gone before but does allow a student or researcher an introduction to a useful theory in the study of curves.

I do have some minor grumbles about the book; given the broad coverage of topics it is to be expected that there are some slips here and there. For example, the use of notation not previously introduced, see page 164 where F_{PL} is used but I can find no earlier reference to it. Another gripe is that the final chapter is quite dense, expanding it over more chapters may have been better. Alternatively since this chapter is not so dependent on earlier chapters it could have formed the start of a separate book. (And such a book needs to be written!)

Exercises, often involving an emphasis on calculation noted earlier, are given at the end of each chapter along with notes on sources and further reading.

On the whole a rewarding book to read.

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