Analytic Solutions of Partial Differential Equations

MATH3414

School of Mathematics, University of Leeds

- 15 credits
- Taught Semester 1,
- Year running 2003/04
- Pre-requisites MATH2360 or MATH2420 or equivalent.
- Co-requisites None.
- Objectives: To provide an understanding of, and methods of solution for, the most important types of partial differential equations that arise in Mathematical Physics. On completion of this module, students should be able to: a) use the method of characteristics to solve first-order hyperbolic equations; b) classify a second order PDE as elliptic, parabolic or hyperbolic; c) use Green's functions to solve elliptic equations; d) have a basic understanding of diffusion; e) obtain a priori bounds for reaction-diffusion equations.
- Syllabus: The majority of physical phenomena can be described by partial differential equations (e.g. the Navier-Stokes equation of fluid dynamics, Maxwell's equations of electromagnetism). This module considers the properties of, and analytical methods of solution for some of the most common first and second order PDEs of Mathematical Physics. In particular, we shall look in detail at elliptic equations (Laplace?s equation), describing steady-state phenomena and the diffusion / heat conduction equation describing the slow spread of concentration or heat. The topics covered are: First order PDEs. Semilinear and quasilinear PDEs; method of characteristics. Characteristics crossing. Second order PDEs. Classification and standard forms. Elliptic equations: weak and strong minimum and maximum principles; Green's functions. Parabolic equations: exemplified by solutions of the diffusion equation. Bounds on solutions of reaction-diffusion equations.
- Form of teaching
  Lectures: 26 hours. 7 examples classes.
- Form of assessment
  One 3 hour examination at end of semester (100%).

Details:

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Schedule: three lectures every week, for eleven weeks (from 27/09 to 10/12).

Tuesday	13:00–14:00	RSLT 03
Wednesday	10:00–11:00	RSLT 04
Friday	11:00–12:00	RSLT 06

Pre-requisite: elementary differential calculus and several variables calculus (e.g. partial differentiation with change of variables, parametric curves, integration), elementary algebra (e.g. partial fractions, linear eigenvalue problems), ordinary differential equations (e.g. change of variable, integrating factor), and vector calculus (e.g. vector identities, Green's theorem).

Outline of course:

Introduction:
- definitions
- examples

First order PDEs:
- linear & semilinear
- characteristics
- quasilinear
- nonlinear
- system of equations

Second order linear PDEs:
- classification
- elliptic
- parabolic

Book list:

P. Prasad & R. Ravindran, "Partial Differential Equations", Wiley Eastern, 1985.

W. E. Williams, "Partial Differential Equations", Oxford University Press, 1980.

P. R. Garabedian, "Partial Differential Equations", Wiley, 1964.

Thanks to Prof. D. W. Hughes, Prof. J. H. Merkin and Dr. R. Sturman for their lecture notes.

Course Summary

- Definitions of different type of PDE (linear, quasilinear, semilinear, nonlinear)
- Existence and uniqueness of solutions
- Solving PDEs analytically is generally based on finding a change of variable to transform the equation into something soluble or on finding an integral form of the solution.

First order PDEs

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c.$$

Linear equations: change coordinate using $\eta(x,y)$, defined by the characteristic equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a},$$

and $\xi(x,y)$ independent (usually $\xi = x$) to transform the PDE into an ODE.

Quasilinear equations: change coordinate using the solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = b \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s} = c$$

to get an implicit form of the solution $\phi(x,y,u) = F(\psi(x,y,u))$.

Nonlinear waves: region of solution.

System of linear equations: linear algebra to decouple equations.

Second order PDEs

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g.$$

Classification	Type	Canonical form	Characteristics
$b^2 - ac > 0$	Hyperbolic	$\frac{\partial^2 u}{\partial \xi \partial \eta} + \ldots = 0$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}$
$b^2 - ac = 0$	Parabolic	$\frac{\partial^2 u}{\partial \eta^2} + \ldots = 0$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}, \eta = x$ (say)
$b^2 - ac < 0$	Elliptic	$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + \ldots = 0$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}, \begin{cases} \alpha = \xi + \eta \\ \beta = i(\xi - \eta) \end{cases}$

Elliptic equations: (Laplace equation.) Maximum Principle. Solutions using Green's functions (uses new variables and the Dirac δ-function to pick out the solution). Method of images.

Parabolic equations: (heat conduction, diffusion equation.) Derive a fundamental solution in integral form or make use of the similarity properties of the equation to find the solution in terms of the diffusion variable

$$\eta = \frac{x}{2\sqrt{t}}.$$

First and Second Maximum Principles and Comparison Theorem give bounds on the solution, and can then construct invariant sets.