Chapter 5

Parabolic Equations

Contents

5.1 Definitions and Properties

Unlike elliptic equations, which describes a steady state, parabolic (and hyperbolic) evolution equations describe processes that are evolving in time. For such an equation the initial state of the system is part of the auxiliary data for a well-posed problem.

The archetypal parabolic evolution equation is the "heat conduction" or "diffusion" equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{(1-dimensional)},$$

or more generally, for $\kappa > 0$,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla \cdot (\kappa \nabla u) \\ &= \kappa \nabla^2 u \quad (\kappa \text{ constant}), \\ &= \kappa \frac{\partial^2 u}{\partial x^2} \quad \text{(1-D)}.\end{aligned}$$

Problems which are well-posed for the heat equation will be well-posed for more general parabolic equation.

5.1.1 Well-Posed Cauchy Problem (Initial Value Problem)

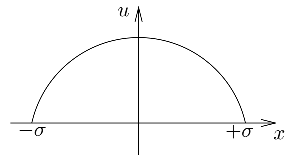
Consider $\kappa > 0$,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \kappa \nabla^2 u \quad \text{in } \mathbb{R}^n,\ t > 0, \\ \text{with} \quad u &= f(\mathbf{x}) \quad \text{in } \mathbb{R}^n \text{ at } t = 0, \\ \text{and} \quad |u| &< \infty \quad \text{in } \mathbb{R}^n,\ t > 0.\end{aligned}$$

Note that we require the solution $u(\mathbf{x}, t)$ bounded in $\mathbb{R}^n$ for all t. In particular we assume that the boundedness of the smooth function u at infinity gives $\nabla u|_\infty = 0$. We also impose conditions on f,

$$\int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} < \infty \Rightarrow f(\mathbf{x}) \to 0 \text{ as } |\mathbf{x}| \to \infty.$$

Sometimes $f(\mathbf{x})$ has compact support, i.e. $f(\mathbf{x}) = 0$ outside some finite region.(E.g., in 1-D, see graph hereafter.)



5.1.2 Well-Posed Initial-Boundary Value Problem

Consider an open bounded region Ω of $\mathbb{R}^n$ and $\kappa > 0$;

$$\begin{aligned}\frac{\partial u}{\partial t} &= \kappa \nabla^2 u \quad \text{in } \Omega,\ t > 0, \\ \text{with} \quad u &= f(\mathbf{x}) \quad \text{at } t = 0 \text{ in } \Omega, \\ \text{and} \quad \alpha\, u(\mathbf{x}, t) + \beta \frac{\partial u}{\partial n}(\mathbf{x}, t) &= g(\mathbf{x}, t) \quad \text{on the boundary } \partial\Omega.\end{aligned}$$

Then, $\beta = 0$ gives the Dirichlet problem, $\alpha = 0$ gives the Neumann problem ($\partial u/\partial n = 0$ on the boundary is the zero-flux condition) and $\alpha \neq 0$, $\beta \neq 0$ gives the Robin or radiation problem. (The problem can also have mixed boundary conditions.)
If Ω is not bounded (e.g. half-plane), then additional behavior-at-infinity condition may be needed.

5.1.3 Time Irreversibility of the Heat Equation

If the initial conditions in a well-posed initial value or initial-boundary value problem for an evolution equation are replaced by conditions on the solution at other than initial time, the resulting problem may not be well-posed (even when the total number of auxiliary conditions is unchanged). E.g. the backward heat equation in 1-D is ill-posed; this problem,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \kappa \frac{\partial^2 u}{\partial x^2} \quad \text{in } 0 < x < l,\ 0 < t < T, \\ \text{with} \quad u &= f(\mathbf{x}) \quad \text{at } t = T,\ x \in (0, l), \\ \text{and} \quad u(0, t) &= u(l, t) = 0 \quad \text{for } t \in (0, T),\end{aligned}$$

which is to find previous states $u(x, t)$, $(t < T)$ which will have evolved into the state $f(x)$, has no solution for arbitrary $f(x)$. Even when a the solution exists, it does not depend continuously on the data.

The heat equation is irreversible in the mathematical sense that forward time is distinguishable from backward time (i.e. it models physical processes irreversible in the sense of the Second Law of Thermodynamics).

5.1.4 Uniqueness of Solution for Cauchy Problem:

The 1-D initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ x \in \mathbb{R},\ t > 0,$$
$$\text{with} \quad u = f(x) \text{ at } t = 0\ (x \in \mathbb{R}),\ \text{such that} \int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x < \infty.$$

has a unique solution.

Proof:

We can prove the uniqueness of the solution of Cauchy problem using the energy method. Suppose that u_1 and u_2 are two bounded solutions. Consider $w = u_1 - u_2$; then w satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \quad (-\infty < x < \infty,\ t > 0),$$
$$\text{with} \quad w = 0 \quad \text{at } t = 0 \quad (-\infty < x < \infty) \quad \text{and} \quad \left.\frac{\partial w}{\partial x}\right|_{\infty} = 0,\ \forall t.$$

Consider the function of time

$$I(t) = \frac{1}{2}\int_{-\infty}^{\infty} w^2(x,t)\,\mathrm{d}x, \quad \text{such that} \quad I(0) = 0 \quad \text{and} \quad I(t) \geq 0\ \forall t \quad (\text{as } w^2 \geq 0),$$

which represents the energy of the function w. Then,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial w^2}{\partial t}\,\mathrm{d}x = \int_{-\infty}^{\infty} w\,\frac{\partial w}{\partial t}\,\mathrm{d}x = \int_{-\infty}^{\infty} w\,\frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x \quad (\text{from the heat equation}),$$
$$= \left[w\,\frac{\partial w}{\partial x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \quad (\text{integration by parts}),$$
$$= -\int_{-\infty}^{\infty} \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \leq 0 \quad \text{since} \quad \left.\frac{\partial w}{\partial x}\right|_{\infty} = 0.$$

Then,

$$0 \leq I(t) \leq I(0) = 0,\ \forall t > 0,$$

since $\mathrm{d}I/\mathrm{d}t < 0$. So, $I(t) = 0$ and $w \equiv 0$ i.e. $u_1 = u_2,\ \forall t > 0$.

5.1.5 Uniqueness of Solution for Initial-Boundary Value Problem:

Similarly we can make use of the energy method to prove the uniqueness of the solution of the 1-D Dirichlet or Neumann problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in } 0 < x < l,\ t > 0,$$
$$\text{with} \quad u = f(x) \quad \text{at } t = 0,\ x \in (0, l),$$
$$u(0,t) = g_0(t) \quad \text{and} \quad u(l,t) = g_l(t),\ \forall t > 0 \quad (\text{Dirichlet}),$$
$$\text{or} \quad \frac{\partial u}{\partial x}(0,t) = g_0(t) \quad \text{and} \quad \frac{\partial u}{\partial x}(l,t) = g_l(t),\ \forall t > 0 \quad (\text{Neumann}).$$

Suppose that u_1 and u_2 are two solutions and consider $w = u_1 - u_2$; then w satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \quad (0 < x < l,\ t > 0),$$
$$\text{with} \quad w = 0 \quad \text{at } t = 0 \quad (0 < x < l),$$
$$\text{and} \quad w(0,t) = w(l,t) = 0,\ \forall t > 0 \quad (\text{Dirichlet}),$$
$$\text{or} \quad \frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(l,t) = 0,\ \forall t > 0 \quad (\text{Neumann}).$$

Consider the function of time

$$I(t) = \frac{1}{2}\int_0^l w^2(x,t)\,\mathrm{d}x, \quad \text{such that} \quad I(0) = 0 \quad \text{and} \quad I(t) \geq 0\ \forall t \quad (\text{as } w^2 \geq 0),$$

which represents the energy of the function w. Then,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{2}\int_0^l \frac{\partial w^2}{\partial t}\,\mathrm{d}x = \int_0^l w\,\frac{\partial^2 w}{\partial x^2},$$
$$= \left[w\,\frac{\partial w}{\partial x}\right]_0^l - \int_0^l \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x = -\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \leq 0.$$

Then,

$$0 \leq I(t) \leq I(0) = 0,\ \forall t > 0,$$

since $\mathrm{d}I/\mathrm{d}t < 0$. So $I(t) = 0\ \forall t >$ and $w \equiv 0$ and $u_1 = u_2$.

5.2 Fundamental Solution of the Heat Equation

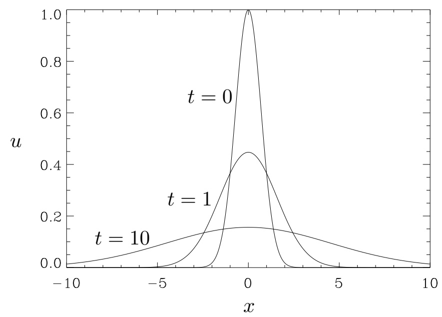
Consider the 1-D Cauchy problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{on } -\infty < x < \infty,\ t > 0,$$
$$\text{with} \quad u = f(x) \quad \text{at } t = 0 \quad (-\infty < x < \infty),$$
$$\text{such that} \quad \int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x < \infty.$$

Example: To illustrate the typical behaviour of the solution of this Cauchy problem, consider the specific case where $u(x,0) = f(x) = \exp(-x^2)$; the solution is

$$u(x,t) = \frac{1}{(1+4t)^{1/2}} \exp\left(-\frac{x^2}{1+4t}\right) \quad (\text{exercise: check this}).$$

Starting with $u(x,0) = \exp(-x^2)$ at $t = 0$, the solution becomes $u(x,t) \sim 1/2\sqrt{t}\,\exp(-x^2/4t)$, for t large, i.e. the amplitude of the solution scales as $1/\sqrt{t}$ and its width scales as $\sqrt{t}$.



Spreading of the Solution: The solution of the Cauchy problem for the heat equation spreads such that its integral remains constant:

$$Q(t) = \int_{-\infty}^{\infty} u \, \mathrm{d}x = \text{constant}.$$

Proof: Consider

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} \, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} \, \mathrm{d}x \quad \text{(from equation)},$$
$$= \left[\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} = 0 \quad \text{(from conditions on } u\text{)}.$$

So, $Q =$ constant.

5.2.1 Integral Form of the General Solution

To find the general solution of the Cauchy problem we define the Fourier transform of $u(x,t)$ and its inverse by

$$U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,t) \, \mathrm{e}^{-ikx} \, \mathrm{d}x,$$
$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U(k,t) \, \mathrm{e}^{ikx} \, \mathrm{d}k.$$

So, the heat equation gives,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{\partial U(k,t)}{\partial t} + k^2 U(k,t)\right] \mathrm{e}^{ikx} \, \mathrm{d}k = 0 \quad \forall x,$$

which implies that the Fourier transform $U(k,t)$ satisfies the equation

$$\frac{\partial U(k,t)}{\partial t} + k^2 U(k,t) = 0.$$

The solution of this linear equation is

$$U(k,t) = F(k) \, \mathrm{e}^{-k^2 t},$$

where $F(k)$ is the Fourier transform of the initial data, $u(x, t = 0)$,

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \, \mathrm{e}^{-ikx} \, \mathrm{d}x.$$

(This requires $\int_{-\infty}^{+\infty} |f(x)|^2 \, \mathrm{d}x < \infty$.) Then, we back substitute $U(k,t)$ in the integral form of $u(x,t)$ to find,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k) \, \mathrm{e}^{-k^2 t} \, \mathrm{e}^{ikx} \, \mathrm{d}k = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(\xi) \, \mathrm{e}^{-ik\xi} \, \mathrm{d}\xi \right) \mathrm{e}^{-k^2 t} \, \mathrm{e}^{ikx} \, \mathrm{d}k,$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) \int_{-\infty}^{+\infty} \mathrm{e}^{-k^2 t} \, \mathrm{e}^{ik(x-\xi)} \, \mathrm{d}k \, \mathrm{d}\xi.$$

Now consider

$$H(x,t,\xi) = \int_{-\infty}^{+\infty} \mathrm{e}^{-k^2 t} \, \mathrm{e}^{ik(x-\xi)} \, \mathrm{d}k = \int_{-\infty}^{+\infty} \exp\left[-t\left(k - i\frac{x-\xi}{2t}\right)^2 - \frac{(x-\xi)^2}{4t}\right] \mathrm{d}k,$$

since the exponent satisfies

$$-k^2 t + ik(x-\xi) = -t\left(k^2 - ik\frac{x-\xi}{t}\right) = -t\left[\left(k - i\frac{x-\xi}{2t}\right)^2 + \frac{(x-\xi)^2}{4t^2}\right],$$

and set $k - i(x-\xi)/2t = s/\sqrt{t}$, with $\mathrm{d}k = \mathrm{d}s$, such that

$$H(x,t,\xi) = \int_{-\infty}^{+\infty} \mathrm{e}^{-s^2} \exp\left[-\frac{(x-\xi)^2}{4t}\right] \frac{\mathrm{d}s}{\sqrt{t}} = \sqrt{\frac{\pi}{t}} \, \mathrm{e}^{-(x-\xi)^2/4t},$$

since $\int_{-\infty}^{+\infty} \mathrm{e}^{-s^2} \, \mathrm{d}s = \sqrt{\pi}$ (see appendix A).

So,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] \mathrm{d}\xi = \int_{-\infty}^{+\infty} K(x-\xi,t) \, f(\xi) \, \mathrm{d}\xi$$

Where the function

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{x^2}{4t}\right]$$

is called the fundamental solution — or source function, Green's function, propagator, diffusion kernel — of the heat equation.

5.2.2 Properties of the Fundamental Solution

The function $K(x,t)$ is solution (positive) of the heat equation for $t > 0$ (check this) and has a singularity only at $x = 0$, $t = 0$:

1. $K(x,t) \to 0$ as $t \to 0^+$ with $x \neq 0$ ($K \sim O(1/\sqrt{t} \, \exp[-1/t])$),
2. $K(x,t) \to +\infty$ as $t \to 0^+$ with $x = 0$ ($K \sim O(1/\sqrt{t})$),
3. $K(x,t) \to 0$ as $t \to +\infty$ ($K \sim O(1/\sqrt{t})$),
4. $\int_{-\infty}^{\infty} K(x-\xi,t) \, \mathrm{d}\xi = 1$

At any time $t > 0$ (no matter how small), the solution to the initial value problem for the heat equation at an arbitrary point x depends on all of the initial data, i.e. the data propagate with an infinite speed. (As a consequence, the problem is well posed only if behaviour-at-infinity conditions are imposed.) However, the influence of the initial state dies out very rapidly with the distance (as $\exp(-r^2)$).

5.2.3 Behaviour at large t

Suppose that the initial data have a compact support — or decays to zero sufficiently quickly as $|x| \to \infty$ and that we look at the solution of the heat equation on spatial scales, x, large compared to the spatial scale of the data ξ and at t large. Thus, we assume the ordering $x^2/t \sim O(1)$ and $\xi^2/t \sim O(\varepsilon)$ where $\varepsilon \ll 1$ (so that, $x\xi/t \sim O(\varepsilon^{1/2})$). Then, the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi\, t}} \int_{-\infty}^{+\infty} f(\xi)\, \mathrm{e}^{-(x-\xi)^2/4t}\, \mathrm{d}\xi = \frac{\mathrm{e}^{-x^2/4t}}{\sqrt{4\pi\, t}} \int_{-\infty}^{+\infty} f(\xi)\, \mathrm{e}^{-\xi^2/4t} \mathrm{e}^{-x\xi/2t}\, \mathrm{d}\xi,$$

$$\simeq \frac{\mathrm{e}^{-x^2/4t}}{\sqrt{4\pi\, t}} \int_{-\infty}^{+\infty} f(\xi)\, \mathrm{d}\xi \simeq \frac{F(0)}{\sqrt{2\, t}} \exp\left(-\frac{x^2}{4t}\right),$$

where $F(0)$ is the Fourier transform of f at $k = 0$, i.e.

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x \Rightarrow \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = \sqrt{2\pi}\, F(0).$$

So, at large t, on large spatial scales x the solution evolves as $u \simeq u_0/\sqrt{t}\, \exp(-\eta^2)$ where u_0 is a constant and $\eta = x/\sqrt{2t}$ is the diffusion variable. This solution spreads and decreases as t increases.

5.3 Similarity Solution

For some equations, like the heat equation, the solution depends on a certain grouping of the independent variables rather than depending on each of the independent variables independently. Consider the heat equation in 1-D

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0,$$

and introduce the dilatation transformation

$$\xi = \varepsilon^a\, x, \quad \tau = \varepsilon^b\, t \quad \text{and} \quad w(\xi,\tau) = \varepsilon^c\, u(\varepsilon^{-a}\xi, \varepsilon^{-b}\tau),\ \varepsilon \in \mathbb{R}.$$

This change of variables gives

$$\frac{\partial u}{\partial t} = \varepsilon^{-c}\, \frac{\partial w}{\partial \tau}\, \frac{\partial \tau}{\partial t} = \varepsilon^{b-c}\, \frac{\partial w}{\partial \tau}, \quad \frac{\partial u}{\partial x} = \varepsilon^{-c}\, \frac{\partial w}{\partial \xi}\, \frac{\partial \xi}{\partial x} = \varepsilon^{a-c}\, \frac{\partial w}{\partial \xi}$$

$$\text{and} \quad \frac{\partial^2 u}{\partial x^2} = \varepsilon^{a-c}\, \frac{\partial^2 w}{\partial \xi^2}\, \frac{\partial \xi}{\partial x} = \varepsilon^{2a-c}\, \frac{\partial^2 w}{\partial \xi^2}.$$

So, the heat equation transforms into

$$\varepsilon^{b-c}\, \frac{\partial w}{\partial \tau} - \varepsilon^{2a-c} D\, \frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{i.e.} \quad \varepsilon^{b-c}\left(\frac{\partial w}{\partial \tau} - \varepsilon^{2a-b} D\, \frac{\partial^2 w}{\partial \xi^2}\right) = 0,$$

and is invariant under the dilatation transformation (i.e. $\forall \varepsilon$) if $b = 2a$. Thus, if u solves the equation at x, t then $w = \varepsilon^{-c} u$ solve the equation at $x = \varepsilon^{-a}\, \xi$, $t = \varepsilon^{-b}\, \tau$.
Note also that we can build some groupings of independent variables which are invariant under this transformation, such as

$$\frac{\xi}{\tau^{a/b}} = \frac{\varepsilon^a\, x}{(\varepsilon^b\, t)^{a/b}} = \frac{x}{t^{a/b}}$$

which defines the dimensionless similarity variable $\eta(x,t) = x/\sqrt{2Dt}$, since $b = 2a$. ($\eta \to \infty$ if $x \to \infty$ or $t \to 0$ and $\eta = 0$ if $x = 0$.) Also,

$$\frac{w}{\tau^{c/b}} = \frac{\varepsilon^c\, u}{(\varepsilon^b\, t)^{c/b}} = \frac{u}{t^{c/b}} = v(\eta)$$

suggests that we look for a solution of the heat equation of the form $u = t^{c/2a}\, v(\eta)$. Indeed, since the heat equation is invariant under the dilatation transformation, then we also expect the solution to be invariant under that transformation. Hence, the partial derivatives become,

$$\frac{\partial u}{\partial t} = \frac{c}{2a}\, t^{c/2a-1}\, v(\eta) + t^{c/2a}\, v'(\eta)\, \frac{\partial \eta}{\partial t} = \frac{1}{2}\, t^{c/2a-1} \left(\frac{c}{a}\, v(\eta) - \eta\, v'(\eta)\right),$$

since $\partial\eta/\partial t = -x/(2t\sqrt{2Dt}) = -\eta/2t$, and

$$\frac{\partial u}{\partial x} = t^{c/2a}\, v'(\eta)\, \frac{\partial \eta}{\partial x} = \frac{t^{c/2a-1/2}}{\sqrt{2D}}\, v'(\eta), \quad \frac{\partial^2 u}{\partial x^2} = \frac{t^{c/2a-1}}{2D}\, v''(\eta).$$

Then, the heat equation reduces to an ODE

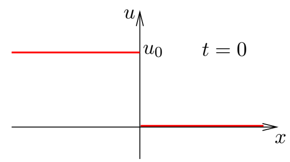
$$t^{\gamma/2-1}\left(v''(\eta) + \eta\, v'(\eta) - \gamma\, v(\eta)\right) = 0. \tag{5.1}$$

with $\gamma = c/a$, such that $u = t^{\gamma/2} v$ and $\eta = x/\sqrt{2Dt}$. So, we may be able to solve the heat equation through (5.1) if we can write the auxiliary conditions on u, x and t as conditions on v and η. Note that, in general, the integral transform method is able to deal with more general boundary conditions; on the other hand, looking for similarity solution permits to solve other types of problems (e.g. weak solutions).

5.3.1 Infinite Region

Consider the problem

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad \text{on } -\infty < x < \infty,\ t > 0,$$
$$\text{with} \quad u = u_0 \quad \text{at } t = 0,\ x \in \mathbb{R}_-^*, \quad u = 0 \quad \text{at } t = 0,\ x \in \mathbb{R}_+^*,$$
$$\text{and} \quad u \to u_0 \text{ as } x \to -\infty, \quad u \to 0 \text{ as } x \to \infty,\ \forall t > 0.$$



We look for a solution of the form $u = t^{\gamma/2}\, v(\eta)$, where $\eta(x,t) = x/\sqrt{2Dt}$, such that $v(\eta)$ is solution of equation (5.1). Moreover, since $u = t^{\gamma/2}\, v(\eta) \to u_0$ as $\eta \to -\infty$, where u_0 does not depend on t, γ must be zero. Hence, v is solution of the linear second order ODE

$$v''(\eta) + \eta\, v'(\eta) = 0 \quad \text{with} \quad v \to u_0 \text{ as } \eta \to -\infty \quad \text{and} \quad v \to 0 \text{ as } \eta \to +\infty.$$

Making use of the integrating factor method,

$$\mathrm{e}^{\eta^2/2}\, v''(\eta) + \eta \exp\left(\frac{\eta^2}{2}\right) v'(\eta) = \frac{\partial}{\partial \eta}\left(\mathrm{e}^{\eta^2/2}\, v'(\eta)\right) = 0 \Rightarrow \mathrm{e}^{\eta^2/2}\, v'(\eta) = \lambda_0,$$

$$v'(\eta) = \lambda_0\, \mathrm{e}^{-\eta^2/2} \Rightarrow v(\eta) = \lambda_0 \int_{-\infty}^{\eta} \mathrm{e}^{-h^2/2}\, \mathrm{d}h + \lambda_1 = \lambda_2 \int_{-\infty}^{\eta/\sqrt{2}} \mathrm{e}^{-s^2}\, \mathrm{d}s + \lambda_1.$$

Now, apply the initial conditions to determine the constants λ_2 and λ_1. As $\eta \to -\infty$, we have $v = \lambda_1 = u_0$ and as $\eta \to \infty$, $v = \lambda_2\sqrt{\pi} + u_0 = 0$, so $\lambda_2 = -u_0/\sqrt{\pi}$. Hence, the solution to this Cauchy problem in the infinite region is

$$v(\eta) = u_0 \left(1 - \int_{-\infty}^{\eta/\sqrt{2}} \mathrm{e}^{-s^2}\, \mathrm{d}s\right) \quad \text{i.e.} \quad u(x,t) = u_0 \left(1 - \int_{-\infty}^{x^2/\sqrt{4Dt}} \mathrm{e}^{-s^2}\, \mathrm{d}s\right).$$

5.3.2 Semi-Infinite Region

Consider the problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad \text{on } 0 < x < \infty,\ t > 0,\\
\text{with} \quad & u = 0 \quad \text{at } t = 0,\ x \in \mathbb{R}_+^*,\\
\text{and} \quad & \frac{\partial u}{\partial x} = -q \quad \text{at } x = 0,\ t > 0, \quad u \to 0 \text{ as } x \to \infty,\ \forall t > 0.
\end{aligned}$$

Again, we look for a solution of the form $u = t^{\gamma/2}\, v(\eta)$, where $\eta(x,t) = x/\sqrt{2Dt}$, such that $v(\eta)$ is solution of equation (5.1). However, the boundary conditions are now different

$$\frac{\partial u}{\partial x} = t^{\gamma/2}\, v'(\eta)\frac{\partial \eta}{\partial x} = \frac{t^{(\gamma-1)/2}}{\sqrt{2D}}\, v'(\eta) \Rightarrow \left.\frac{\partial \eta}{\partial x}\right|_{x=0} = \frac{t^{(\gamma-1)/2}}{\sqrt{2D}}\, v'(0) = -q,$$

since q does not depend on t, $\gamma - 1$ must be zero. Hence from equation (5.1), the function v, such that $u = v\sqrt{t}$, is solution of the linear second order ODE

$$v''(\eta) + \eta\, v'(\eta) - v(\eta) = 0 \quad \text{with} \quad v'(0) = -q\sqrt{2D} \quad \text{and} \quad v \to 0 \text{ as } \eta \to +\infty.$$

Since the function $v^* = \lambda\, \eta$ is solution of the above ODE, we seek for solutions of the form $v(\eta) = \eta\, \lambda(\eta)$ such that

$$v' = \lambda + \eta\, \lambda' \quad \text{and} \quad v'' = 2\lambda' + \eta\, \lambda''.$$

Then, back-substitute in the ODE

$$\eta\, \lambda'' + 2\lambda' + \eta^2\, \lambda' + \eta\lambda - \eta\lambda = 0 \quad \text{i.e.} \quad \frac{\lambda''}{\lambda'} = -\frac{2 + \eta^2}{\eta} = -\frac{2}{\eta} - \eta.$$

After integration (integrating factor method or another), we get

$$\ln|\lambda'| = -2\ln\eta - \frac{\eta^2}{2} + k = \ln\frac{1}{\eta^2} - \frac{\eta^2}{2} + k \Rightarrow \lambda' = \kappa_0 \frac{\mathrm{e}^{-\eta^2/2}}{\eta^2} \Rightarrow \lambda = \kappa_0 \int^{\eta} \frac{\mathrm{e}^{-s^2/2}}{s^2}\, \mathrm{d}s + \kappa_1.$$

An integration by part gives

$$\lambda(\eta) = \kappa_0 \left(\left[-\frac{\mathrm{e}^{-s^2/2}}{s}\right]^{\eta} - \int^{\eta} \mathrm{e}^{-s^2/2}\, \mathrm{d}s\right) + \kappa_1. = \kappa_2 \left(\frac{\mathrm{e}^{-\eta^2/2}}{\eta} + \int_0^{\eta} \mathrm{e}^{-s^2/2}\, \mathrm{d}s\right) + \kappa_3.$$

Hence, the solution becomes,

$$v(\eta) = \kappa_2 \left(\mathrm{e}^{-\eta^2/2} + \eta \int_0^{\eta} \mathrm{e}^{-s^2/2}\, \mathrm{d}s\right) + \kappa_3\eta,$$

where the constants of integration κ_2 and κ_3 are determined by the initial conditions:

$$v' = \kappa_2 \left(-\eta\mathrm{e}^{-\eta^2/2} + \int_0^{\eta} \mathrm{e}^{-s^2/2}\, \mathrm{d}s + \eta\mathrm{e}^{-\eta^2/2}\right) + \kappa_3 = \kappa_2 \int_0^{\eta} \mathrm{e}^{-s^2/2}\, \mathrm{d}s + \kappa_3,$$

so that $v'(0) = \kappa_3 = -q\sqrt{2D}$. Also

$$\text{as } \eta \to +\infty,\ v \sim \eta \left(\kappa_2 \int_0^{\infty} \mathrm{e}^{-s^2/2}\, \mathrm{d}s + \kappa_3\right) = 0 \Rightarrow \kappa_2 = -\kappa_3\sqrt{\frac{2}{\pi}},$$

$$\text{since} \quad \int_0^{\infty} \mathrm{e}^{-s^2/2}\, \mathrm{d}s = \sqrt{2} \int_0^{\infty} \mathrm{e}^{-h^2}\, \mathrm{d}h = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}.$$

The solution of the equation becomes

$$\begin{aligned}
v(\eta) &= \kappa_2 \left(\mathrm{e}^{-\eta^2/2} + \eta\sqrt{2} \int_0^{\eta/\sqrt{2}} \mathrm{e}^{-h^2}\, \mathrm{d}h\right) + \kappa_3\eta,\\
&= \kappa_2 \left(\mathrm{e}^{-\eta^2/2} - \eta\sqrt{2} \int_{\eta/\sqrt{2}}^{+\infty} \mathrm{e}^{-h^2}\, \mathrm{d}h\right) + \eta \left(\kappa_2\sqrt{\frac{\pi}{2}} + \kappa_3\right),\\
&= q\sqrt{\frac{4D}{\pi}} \left(\mathrm{e}^{-\eta^2/2} - \eta\sqrt{2} \int_{\eta/\sqrt{2}}^{+\infty} \mathrm{e}^{-h^2}\, \mathrm{d}h\right),\\
u(x,y) &= q\sqrt{\frac{4Dt}{\pi}} \left(\mathrm{e}^{-x^2/4Dt} - \frac{x}{\sqrt{Dt}} \int_{x/\sqrt{4Dt}}^{+\infty} \mathrm{e}^{-h^2}\, \mathrm{d}h\right).
\end{aligned}$$

5.4 Maximum Principles and Comparison Theorems

Like the elliptic PDEs, the heat equation or parabolic equations of most general form satisfy a maximum-minimum principle.
Consider the Cauchy problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad -\infty < x < \infty,\ 0 \le t \le T.$$

and define the two sets V and V_T as

$$\begin{aligned}
V &= \{(x,t) \in (-\infty, +\infty) \times (0, T)\},\\
\text{and} \quad V_T &= \{(x,t) \in (-\infty, +\infty) \times (0, T]\}.
\end{aligned}$$

Lemma: Suppose
$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}<0 \quad \text{in } V \quad \text{and} \quad u(x,0)\le M,$$
then $u(x,t)<M$ in V_T.

Proof: Suppose $u(x,t)$ achieves a maximum in V, at the point (x_0,t_0). Then, at this point,
$$\frac{\partial u}{\partial t}=0, \quad \frac{\partial u}{\partial x}=0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}\le 0.$$
But, $\partial^2 u/\partial x^2 \le 0$ is contradictory with the hypothesis $\partial^2 u/\partial x^2 > \partial u/\partial t = 0$ at (x_0,t_0). Moreover, if we now suppose that the maximum occurs in $t=T$ then, at this point
$$\frac{\partial u}{\partial t}\ge 0, \quad \frac{\partial u}{\partial x}=0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}\le 0,$$
which again leads to a contradiction.

5.4.1 First Maximum Principle

Suppose
$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\le 0 \quad \text{in } V \quad \text{and} \quad u(x,0)\le M,$$
then $u(x,t)\le M$ in V_T.

Proof: Suppose there is some point (x_0,t_0) in V_T $(0<t\le T)$ at which $u(x_0,t_0)=M_1>M$. Put $w(x,t)=u(x,t)-(t-t_0)\,\varepsilon$ where $\varepsilon=(M_1-M)/t_0<0$. Then,
$$\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}=\underbrace{\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}}_{\le 0}-\underbrace{\varepsilon}_{>0}<0 \quad \text{(in form of lemma)},$$
and by lemma,
$$\begin{aligned} w(x,t) &< \max\{w(x,0)\} \quad \text{in } V_T,\\ &< M+\varepsilon\, t_0,\\ &< M+\frac{M_1-M}{t_0}\,t_0,\\ \Rightarrow w(x,t) &< M_1 \quad \text{in } V_T. \end{aligned}$$
But, $w(x_0,t_0)=u(x_0,t_0)-(t_0-t_0)\,\varepsilon=u(x_0,t_0)=M_1$; since $(x_0,t_0)\in V_T$ we have a contradiction.

Appendix A

Integral of e^{-x^2} in $\mathbb{R}$

Consider the integrals

$$I(R) = \int_0^R e^{-s^2}\, ds \quad \text{and} \quad I = \int_0^{+\infty} e^{-s^2}\, ds$$

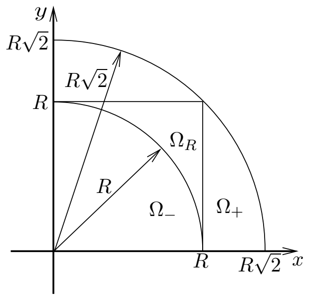
such that $I(R) \to I$ as $R \to +\infty$. Then,

$$I^2(R) = \int_0^R e^{-x^2}\, dx \int_0^R e^{-y^2}\, dy = \int_0^R\int_0^R e^{-(x^2+y^2)}\, dx\, dy = \int_{\Omega_R} e^{-(x^2+y^2)}\, dx\, dy.$$

Since its integrand is positive, $I^2(R)$ is bounded by the following integrals

$$\int_{\Omega_-} e^{-(x^2+y^2)}\, dx\, dy < \int_{\Omega_R} e^{-(x^2+y^2)}\, dx\, dy < \int_{\Omega_+} e^{-(x^2+y^2)}\, dx\, dy,$$

where $\Omega_- : \{x \in \mathbb{R}^+,\ y \in \mathbb{R}^+ | x^2 + y^2 = R^2\}$ and $\Omega_+ : \{x \in \mathbb{R}^+,\ y \in \mathbb{R}^+ | x^2 + y^2 = 2R^2\}$.



Hence, after polar coordinates transformation, ($x = \rho\cos\theta$, $y = \rho\sin\theta$), with $dx\, dy = \rho\, d\rho\, d\theta$ and $x^2 + y^2 = \rho^2$, this relation becomes

$$\int_0^{\pi/2}\int_0^R \rho\, e^{-\rho^2}\, d\rho\, d\theta < I^2(R) < \int_0^{\pi/2}\int_0^{R\sqrt{2}} \rho\, e^{-\rho^2}\, d\rho\, d\theta.$$

Put, $s = \rho^2$ so that $ds = 2\rho\, d\rho$, to get

$$\frac{\pi}{4}\int_0^{R^2} e^{-s}\, ds < I^2(R) < \frac{\pi}{4}\int_0^{2R^2} e^{-s}\, ds \quad \text{i.e.} \quad \frac{\pi}{4}\left(1 - e^{-R^2}\right) < I^2(R) < \frac{\pi}{4}\left(1 - e^{-2R^2}\right).$$

Take the limit $R \to +\infty$ to state that

$$\frac{\pi}{4} \le I^2 \le \frac{\pi}{4} \quad \text{i.e.} \quad I^2 = \frac{\pi}{4} \quad \text{and} \quad I = \frac{\sqrt{\pi}}{2}\ (I > 0).$$

So,

$$\int_0^{+\infty} e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2} \ \Rightarrow\ \int_{-\infty}^{+\infty} e^{-s^2}\, ds = \sqrt{\pi},$$

since $\exp(-x^2)$ is even on $\mathbb{R}$.