Chapter 4

Elliptic Equations

Contents

4.1 Definitions

Elliptic equations are typically associated with steady-state behavior. The archetypal elliptic equation is Laplace's equation

$$\nabla^2 u = 0, \quad \text{e.g.} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in 2-D},$$

and describes

- steady, irrotational flows,
- electrostatic potential in the absence of charge,
- equilibrium temperature distribution in a medium.

Because of their physical origin, elliptic equations typically arise as boundary value problems (BVPs). Solving a BVP for the general elliptic equation

$$\mathcal{L}[u] = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = F$$

(recall: all the eigenvalues of the matrix $A = (a_{ij}), i, j = 1 \cdots n$, are non-zero and have the same sign) is to find a solution u in some open region Ω of space, with conditions imposed on $\partial\Omega$ (the boundary of Ω) or at infinity. E.g. inviscid flow past a sphere is determined by boundary conditions on the sphere ($\mathbf{u} \cdot \mathbf{n} = 0$) and at infinity ($\mathbf{u} = \text{Const}$).
There are three types of boundary conditions for well-posed BVPs,

1. Dirichlet condition — u takes prescribed values on the boundary $\partial\Omega$ (first BVP).

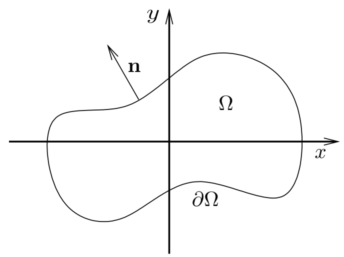
2. Neumann conditions — the normal derivative, $\partial u/\partial n = \mathbf{n} \cdot \nabla u$ is prescribed on the boundary $\partial\Omega$ (second BVP).

   In this case we have compatibility conditions (i.e. global constraints):
   E.g., suppose u satisfies $\nabla^2 u = F$ on Ω and $\mathbf{n} \cdot \nabla u = \partial_n u = f$ on $\partial\Omega$. Then,

$$\int_\Omega \nabla^2 u \,\mathrm{d}V = \int_\Omega \nabla \cdot \nabla u \,\mathrm{d}V = \int_{\partial\Omega} \nabla u \cdot \mathbf{n} \,\mathrm{d}S = \int_{\partial\Omega} \frac{\partial u}{\partial n} \,\mathrm{d}S \qquad \text{(divergence theorem)},$$

$$\Rightarrow \int_\Omega F \,\mathrm{d}V = \int_{\partial\Omega} f \,\mathrm{d}S \qquad \text{for the problem to be well-defined.}$$

3. Robin conditions — a combination of u and its normal derivative such as $\partial u/\partial n + \alpha u$ is prescribed on the boundary $\partial\Omega$ (third BVP).



Sometimes we may have a mixed problem, in which u is given on part of $\partial\Omega$ and $\partial u/\partial n$ given on the rest of $\partial\Omega$.
If Ω encloses a finite region, we have an interior problem; if, however, Ω is unbounded, we have an exterior problem, and we must impose conditions 'at infinity'.
Note that initial conditions are irrelevant for these BVPs and the Cauchy problem for elliptic equations is not always well-posed (even if Cauchy-Kowaleski theorem states that the solution exist and is unique).

As a general rule, it is hard to deal with elliptic equations since the solution is global, affected by all parts of the domain. (Hyperbolic equations, posed as initial value or Cauchy problem, are more localised.)

From now, we shall deal mainly with the Helmholtz equation $\nabla^2 u + P\,u = F$, where P and F are functions of $\mathbf{x}$, and particularly with the special one if $P = 0$, Poisson's equation, or Laplace's equation, if $F = 0$ too. This is not too severe restriction; recall that any linear elliptic equation can be put into the canonical form

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} + \cdots = 0$$

and that the lower order derivatives do not alter the overall properties of the solution.

4.2 Properties of Laplace's and Poisson's Equations

Definition: A continuous function satisfying Laplace's equation in an open region Ω, with continuous first and second order derivatives, is called an harmonic function. Functions u

in $\mathcal{C}^2(\Omega)$ with $\nabla^2 u \geq 0$ (respectively $\nabla^2 u \leq 0$) are call subharmonic (respectively superharmonic).

4.2.1 Mean Value Property

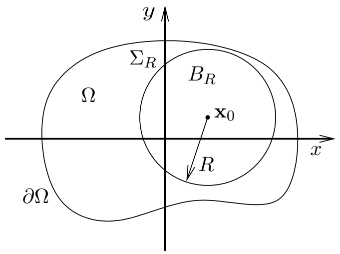
Definition: Let $\mathbf{x}_0$ be a point in Ω and let $B_R(\mathbf{x}_0)$ denote the open ball having centre $\mathbf{x}_0$ and radius R. Let $\Sigma_R(\mathbf{x}_0)$ denote the boundary of $B_R(\mathbf{x}_0)$ and let $A(R)$ be the surface area of $\Sigma_R(\mathbf{x}_0)$. Then a function u has the mean value property at a point $\mathbf{x}_0 \in \Omega$ if

$$u(\mathbf{x}_0) = \frac{1}{A(R)} \int_{\Sigma_R} u(\mathbf{x})\, \mathrm{d}S$$

for every $R > 0$ such that $B_R(\mathbf{x}_0)$ is contained in Ω. If instead $u(\mathbf{x}_0)$ satisfies

$$u(\mathbf{x}_0) = \frac{1}{V(R)} \int_{B_R} u(\mathbf{x})\, \mathrm{d}V,$$

where $V(R)$ is the volume of the open ball $B_R(\mathbf{x}_0)$, we say that $u(\mathbf{x}_0)$ has the second mean value property at a point $\mathbf{x}_0 \in \Omega$. The two mean value properties are equivalent.



Theorem: If u is harmonic in Ω an open region of $\mathbb{R}^n$, then u has the mean value property on Ω.

Proof: We need to make use of Green's theorem which says,

$$\int_S \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) \mathrm{d}S = \int_V \left(v\,\nabla^2 u - u\,\nabla^2 v \right) \mathrm{d}V. \tag{4.1}$$

(Recall: Apply divergence theorem to the function $v\nabla u - u\nabla v$ to state Green's theorem.)

Since u is harmonic, it follows from equation (4.1), with $v = 1$, that

$$\int_S \frac{\partial u}{\partial n}\, \mathrm{d}S = 0.$$

Now, take $v = 1/r$, where $r = |\mathbf{x} - \mathbf{x}_0|$, and the domain V to be $B_r(\mathbf{x}_0) - B_\varepsilon(\mathbf{x}_0)$, $0 < \varepsilon < R$. Then, in $\mathbb{R}^n - \mathbf{x}_0$,

$$\nabla^2 v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) = 0$$

so v is harmonic too and equation (4.1) becomes

$$\int_{\Sigma_r} u\frac{\partial v}{\partial n}\, \mathrm{d}S + \int_{\Sigma_\varepsilon} u\frac{\partial v}{\partial n}\, \mathrm{d}S = \int_{\Sigma_r} u\frac{\partial v}{\partial r}\, \mathrm{d}S - \int_{\Sigma_\varepsilon} u\frac{\partial v}{\partial r}\, \mathrm{d}S = 0$$

$$\Rightarrow \int_{\Sigma_\varepsilon} u\frac{\partial v}{\partial r}\, \mathrm{d}S = \int_{\Sigma_r} u\frac{\partial v}{\partial r}\, \mathrm{d}S \quad \text{i.e} \quad \frac{1}{\varepsilon^2}\int_{\Sigma_\varepsilon} u\, \mathrm{d}S = \frac{1}{r^2}\int_{\Sigma_r} u\, \mathrm{d}S.$$

Since u is continuous, then as $\varepsilon \to 0$ the LHS converges to $4\pi\, u(x_0, y_0, z_0)$ (with $n = 3$, say), so

$$u(\mathbf{x}_0) = \frac{1}{A(r)} \int_{\Sigma_r} u\, \mathrm{d}S.$$

Recovering the second mean value property (with $n = 3$, say) is straightforward

$$\int_0^r \rho^2\, u(\mathbf{x}_0)\, \mathrm{d}\rho = \frac{r^3}{3}\, u(\mathbf{x}_0) = \frac{1}{4\pi}\int_0^r \int_{\Sigma_\rho} u\, \mathrm{d}S\, \mathrm{d}\rho = \frac{1}{4\pi}\int_{B_r} u\, \mathrm{d}V.$$

The inverse of this theorem holds too, but is harder to prove. If u has the mean value property then u is harmonic.

4.2.2 Maximum-Minimum Principle

One of the most important features of elliptic equations is that it is possible to prove theorems concerning the boundedness of the solutions.

Theorem: Suppose that the subharmonic function u satisfies

$$\nabla^2 u = F \quad \text{in } \Omega, \text{ with } F > 0 \text{ in } \Omega.$$

Then $u(x, y)$ attains his maximum on $\partial\Omega$.

Proof: (Theorem stated in 2-D but holds in higher dimensions.) Suppose for a contradiction that u attains its maximum at an interior point (x_0, y_0) of Ω. Then at (x_0, y_0),

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial x^2} \leq 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} \leq 0,$$

since it is a maximum. So,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \leq 0, \quad \text{which contradicts } F > 0 \text{ in } \Omega.$$

Hence u must attain its maximum on $\partial\Omega$, i.e. if $u \leq M$ on $\partial\Omega$, $u < M$ in Ω.

Theorem: The weak Maximum-Minimum Principle for Laplace's equation. Suppose that u satisfies

$$\nabla^2 u = 0 \quad \text{in a bounded region } \Omega;$$

if $m \leq u \leq M$ on $\partial\Omega$, then $m \leq u \leq M$ in Ω.

Proof: (Theorem stated in 2-D but holds in higher dimensions.) Consider the function $v = u + \varepsilon(x^2 + y^2)$, for any $\varepsilon > 0$. Then $\nabla^2 v = 4\varepsilon > 0$ in Ω (since $\nabla^2(x^2 + y^2) = 4$), and using the previous theorem,

$$v \le M + \varepsilon R^2 \quad \text{in } \Omega,$$

where $u \le M$ on $\partial\Omega$ and R is the radius of the circle containing Ω. As this holds for any ε, let $\varepsilon \to 0$ to obtain

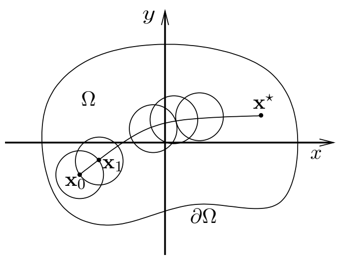
$$u \le M \quad \text{in } \Omega,$$

i.e., if u satisfies $\nabla^2 u = 0$ in Ω, then u cannot exceed M, the maximum value of u on $\partial\Omega$.
Also, if u is a solution of $\nabla^2 u = 0$, so is $-u$. Thus, we can apply all of the above to $-u$ to get a minimum principle: if $u \ge m$ on $\partial\Omega$, then $u \ge m$ in Ω.

This theorem does not say that harmonic function cannot also attain m and M inside Ω though. We shall now progress into the strong Maximum-Minimum Principle.

Theorem: Suppose that u has the mean value property in a bounded region Ω and that u is continuous in $\bar{\Omega} = \Omega \cup \partial\Omega$. If u is not constant in Ω then u attains its maximum value on the boundary $\partial\Omega$ of Ω, not in the interior of Ω.

Proof: Since u is continuous in the closed, bounded domain $\bar{\Omega}$ then it attains its maximum M somewhere in $\bar{\Omega}$.
Our aim is to show that, if u attains its max. at an interior point of Ω, then u is constant in $\bar{\Omega}$.
Suppose $u(\mathbf{x}_0) = M$ and let $\mathbf{x}^\star$ be some other point of Ω. Join these points with a path covered by a sequence of overlapping balls, B_r.



Consider the ball with $\mathbf{x}_0$ at its center. Since u has the mean value property then

$$M = u(\mathbf{x}_0) = \frac{1}{A(r)} \int_{\Sigma_r} u \, \mathrm{d}S \le M.$$

This equality must hold throughout this statement and $u = M$ throughout the sphere surrounding $\mathbf{x}_0$. Since the balls overlap, there is $\mathbf{x}_1$, centre of the next ball such that $u(\mathbf{x}_1) = M$; the mean value property implies that $u = M$ in this sphere also. Continuing like this gives $u(\mathbf{x}^\star) = M$. Since $\mathbf{x}^\star$ is arbitrary, we conclude that $u = M$ throughout Ω, and by continuity throughout $\bar{\Omega}$. Thus if u is not a constant in Ω it can attain its maximum value only on the boundary $\partial\Omega$.

Corollary: Applying the above theorem to $-u$ establishes that if u is non constant it can attain its minimum only on $\partial\Omega$.
Also as a simple corollary, we can state the following theorem. (The proof follows immediately the previous theorem and the weak Maximum-Minimum Principle.)

Theorem: The strong Maximum-Minimum Principle for Laplace's equation.
Let u be harmonic in $\bar{\Omega}$, i.e. solution of $\nabla^2 u = 0$ in Ω and continuous in $\bar{\Omega}$, with M and m the maximum and minimum values respectively of u on the boundary $\partial\Omega$. Then, either $m < u < M$ in Ω or else $m = u = M$ in $\bar{\Omega}$.

Note that it is important that Ω be bounded for the theorem to hold. E.g., consider $u(x, y) = \mathrm{e}^x \sin y$ with $\Omega = \{(x, y) | -\infty < x < +\infty,\ 0 < y < 2\pi\}$. Then $\nabla^2 u = 0$ and on the boundary of Ω we have $u = 0$, so that $m = M = 0$. But of course u is not identically zero in Ω.

Corollary: If $u = C$ is constant on $\partial\Omega$, then $u = C$ is constant in Ω. Armed with the above theorems we are in position to prove the uniqueness and the stability of the solution of Dirichlet problem for Poisson's equation.
Consider the Dirichlet BVP

$$\nabla^2 u = F \text{ in } \Omega \quad \text{with} \quad u = f \text{ on } \partial\Omega$$

and suppose u_1, u_2 two solutions to the problem. Then $v = u_1 - u_2$ satisfies

$$\nabla^2 v = \nabla^2 (u_1 - u_2) = 0 \quad \text{in } \Omega, \text{ with } v = 0 \text{ on } \partial\Omega.$$

Thus, $v \equiv 0$ in Ω, i.e. $u_1 = u_2$; the solution is unique.
To establish the continuous dependence of the solution on the prescribed data (i.e. the stability of the solution) let u_1 and u_2 satisfy

$$\nabla^2 u_{\{1,2\}} = F \text{ in } \Omega \quad \text{with} \quad u_{\{1,2\}} = f_{\{1,2\}} \text{ on } \partial\Omega,$$

with $\max |f_1 - f_2| = \varepsilon$. Then $v = u_1 - u_2$ is harmonic with $v = f_1 - f_2$ on $\partial\Omega$. As before, v must have its maximum and minimum values on $\partial\Omega$; hence $|u_1 - u_2| \le \varepsilon$ in $\bar{\Omega}$. So, the solution is stable — small changes in the boundary data lead to small changes in the solution.

We may use the Maximum-Minimum Principle to put bounds on the solution of an equation without solving it.

The strong Maximum-Minimum Principle may be extended to more general linear elliptic equations

$$\mathcal{L}[u] = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = F,$$

and, as for Poisson's equation it is possible then to prove that the solution to the Dirichlet BVP is unique and stable.

4.3 Solving Poisson Equation Using Green's Functions

We shall develop a formal representation for solutions to boundary value problems for Poisson's equation.

4.3.1 Definition of Green's Functions

Consider a general linear PDE in the form

$$\mathcal{L}(\mathbf{x})u(\mathbf{x}) = F(\mathbf{x}) \text{ in } \Omega,$$

where $\mathcal{L}(\mathbf{x})$ is a linear (self-adjoint) differential operator, $u(\mathbf{x})$ is the unknown and $F(\mathbf{x})$ is the known homogeneous term.
(Recall: $\mathcal{L}$ is self-adjoint if $\mathcal{L} = \mathcal{L}^\star$, where $\mathcal{L}^\star$ is defined by $\langle v|\mathcal{L}u\rangle = \langle \mathcal{L}^\star v|u\rangle$ and where $\langle v|u\rangle = \int v(\mathbf{x})w(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x}$ (($w(\mathbf{x})$ is the weight function).)

The solution to the equation can be written formally

$$u(\mathbf{x}) = \mathcal{L}^{-1}F(\mathbf{x}),$$

where $\mathcal{L}^{-1}$, the inverse of $\mathcal{L}$, is some integral operator. (We can expect to have $\mathcal{L}\mathcal{L}^{-1} = \mathcal{L}\mathcal{L}^{-1} = \mathcal{I}$, identity.) We define the inverse $\mathcal{L}^{-1}$ using a Green's function: let

$$u(\mathbf{x}) = \mathcal{L}^{-1}F(\mathbf{x}) = -\int_\Omega G(\mathbf{x},\boldsymbol{\xi})F(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{4.2}$$

where $G(\mathbf{x},\boldsymbol{\xi})$ is the Green's function associated with $\mathcal{L}$ (G is the kernel). Note that G depends on both the independent variables $\mathbf{x}$ and the new independent variables $\boldsymbol{\xi}$, over which we integrate.

Recall the Dirac δ-function (more precisely distribution or generalised function) $\delta(\mathbf{x})$ which has the properties,

$$\int_{\mathbb{R}^n} \delta(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^n} \delta(\mathbf{x}-\boldsymbol{\xi})\,h(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = h(\mathbf{x}).$$

Now, applying $\mathcal{L}$ to equation (4.2) we get

$$\mathcal{L}u(\mathbf{x}) = F(\mathbf{x}) = -\int_\Omega \mathcal{L}G(\mathbf{x},\boldsymbol{\xi})F(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi};$$

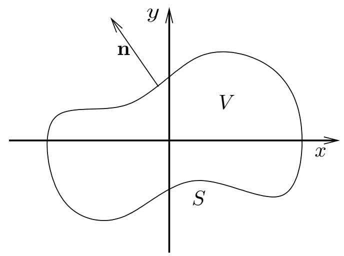
hence, the Green's function $G(\mathbf{x},\boldsymbol{\xi})$ satisfies

$$u(\mathbf{x}) = -\int_\Omega G(\mathbf{x},\boldsymbol{\xi})\,F(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \quad \text{with} \quad \mathcal{L}\,G(\mathbf{x},\boldsymbol{\xi}) = -\delta(\mathbf{x}-\boldsymbol{\xi}) \text{ and } \mathbf{x},\boldsymbol{\xi} \in \Omega.$$

4.3.2 Green's function for Laplace Operator

Consider Poisson's equation in the open bounded region V with boundary S,

$$\nabla^2 u = F \text{ in } V. \tag{4.3}$$



Then, Green's theorem ($\mathbf{n}$ is normal to S outward from V), which states

$$\int_V \left(u\nabla^2 v - v\nabla^2 u\right)\mathrm{d}V = \int_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S,$$

for any functions u and v, with $\partial h/\partial n = \mathbf{n}\cdot\nabla h$, becomes

$$\int_V u\nabla^2 v\,\mathrm{d}V = \int_V vF\,\mathrm{d}V + \int_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S;$$

so, if we choose $v \equiv v(\mathbf{x},\boldsymbol{\xi})$, singular at $\mathbf{x} = \boldsymbol{\xi}$, such that $\nabla^2 v = -\delta(\mathbf{x}-\boldsymbol{\xi})$, then u is solution of the equation

$$u(\boldsymbol{\xi}) = -\int_V vF\,\mathrm{d}V - \int_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S \tag{4.4}$$

which is an integral equation since u appears in the integrand. To address this we consider another function, $w \equiv w(\mathbf{x},\boldsymbol{\xi})$, regular at $\mathbf{x} = \boldsymbol{\xi}$, such that $\nabla^2 w = 0$ in V. Hence, apply Green's theorem to the function u and w

$$\int_S \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right)\mathrm{d}S = \int_V \left(u\nabla^2 w - w\nabla^2 u\right)\mathrm{d}V = -\int_V wF\,\mathrm{d}V.$$

Combining this equation with equation (4.4) we find

$$u(\boldsymbol{\xi}) = -\int_V (v+w)F\,\mathrm{d}V - \int_S \left(u\frac{\partial}{\partial n}(v+w) - (v+w)\frac{\partial u}{\partial n}\right)\mathrm{d}S,$$

so, if we consider the fundamental solution of Laplace's equation, $G = v + w$, such that $\nabla^2 G = -\delta(\mathbf{x}-\boldsymbol{\xi})$ in V,

$$u(\boldsymbol{\xi}) = -\int_V GF\,\mathrm{d}V - \int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)\mathrm{d}S. \tag{4.5}$$

Note that if, F, f and the solution u are sufficiently well-behaved at infinity this integral equation is also valid for unbounded regions (i.e. for exterior BVP for Poisson's equation).

The way to remove u or $\partial u/\partial n$ from the RHS of the above equation depends on the choice of boundary conditions.

Dirichlet Boundary Conditions

Here, the solution to equation (4.3) satisfies the condition $u = f$ on S. So, we choose w such that $w = -v$ on S, i.e. $G = 0$ on S, in order to eliminate $\partial u/\partial n$ form the RHS of equation (4.5).
Then, the solution of the Dirichlet BVP for Poisson's equation

$$\nabla^2 u = F \text{ in } V \text{ with } u = f \text{ on } S$$

is

$$u(\boldsymbol{\xi}) = -\int_V GF \,\mathrm{d}V - \int_S f\frac{\partial G}{\partial n}\,\mathrm{d}S,$$

where $G = v+w$ (w regular at $\mathbf{x} = \boldsymbol{\xi}$) with $\nabla^2 v = -\delta(\mathbf{x}-\boldsymbol{\xi})$ and $\nabla^2 w = 0$ in V and $v+w = 0$ on S. So, the Green's function G is solution of the Dirichlet BVP

$$\nabla^2 G = -\delta(\mathbf{x}-\boldsymbol{\xi}) \text{ in } V,$$
$$\text{with} \quad G = 0 \text{ on } S.$$

Neumann Boundary Conditions

Here, the solution to equation (4.3) satisfies the condition $\partial u/\partial n = f$ on S. So, we choose w such that $\partial w/\partial n = -\partial v/\partial n$ on S, i.e. $\partial G/\partial n = 0$ on S, in order to eliminate u from the RHS of equation (4.5).
However, the Neumann BVP

$$\nabla^2 G = -\delta(\mathbf{x}-\boldsymbol{\xi}) \text{ in } V,$$
$$\text{with} \quad \frac{\partial G}{\partial n} = 0 \text{ on } S,$$

which does not satisfy a compatibility equation, has no solution. Recall that the Neumann BVP $\nabla^2 u = F$ in V, with $\partial u/\partial n = f$ on S, is ill-posed if

$$\int_V F\mathrm{d}V \neq \int_S f\mathrm{d}S.$$

We need to alter the Green's function a little to satisfy the compatibility equation; put $\nabla^2 G = -\delta + C$, where C is a constant, then the compatibility equation for the Neumann BVP for G is

$$\int_V (-\delta + C)\,\mathrm{d}V = \int_S 0\,\mathrm{d}S = 0 \Rightarrow C = \frac{1}{\mathcal{V}},$$

where $\mathcal{V}$ is the volume of V. Now, applying Green's theorem to G and u:

$$\int_V \left(G\nabla^2 u - u\nabla^2 G\right)\mathrm{d}V = \int_S \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)\mathrm{d}S$$

we get

$$u(\boldsymbol{\xi}) = -\int_V GF\,\mathrm{d}V + \int_S Gf\,\mathrm{d}S + \underbrace{\frac{1}{\mathcal{V}}\int_V u\,\mathrm{d}V}_{\bar{u}}.$$

This shows that, whereas the solution of Poisson's equation with Dirichlet boundary conditions is unique, the solution of the Neumann problem is unique up to an additive constant $\bar{u}$ which is the mean value of u over Ω.

Thus, the solution of the Neumann BVP for Poisson's equation

$$\nabla^2 u = F \text{ in } V \text{ with } \frac{\partial u}{\partial n} = f \text{ on } S$$

is

$$u(\boldsymbol{\xi}) = \bar{u} - \int_V GF\,\mathrm{d}V + \int_S Gf\,\mathrm{d}S,$$

where $G = v + w$ (w regular at $\mathbf{x} = \boldsymbol{\xi}$) with $\nabla^2 v = -\delta(\mathbf{x}-\boldsymbol{\xi}), \nabla^2 w = 1/\mathcal{V}$ in V and $\partial w/\partial n = -\partial v/\partial n$ on S. So, the Green's function G is solution of the Neumann BVP

$$\nabla^2 G = -\delta(\mathbf{x}-\boldsymbol{\xi}) + \frac{1}{\mathcal{V}} \text{ in } V,$$
$$\text{with} \quad \frac{\partial G}{\partial n} = 0 \text{ on } S.$$

Robin Boundary Conditions

Here, the solution to equation (4.3) satisfies the condition $\partial u/\partial n + \alpha u = f$ on S. So, we choose w such that $\partial w/\partial n + \alpha w = -\partial v/\partial n - \alpha v$ on S, i.e. $\partial G/\partial n + \alpha G = 0$ on S. Then,

$$\int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)\mathrm{d}S = \int_S \left(u\frac{\partial G}{\partial n} + G(\alpha u - f)\right)\mathrm{d}S = -\int_S Gf\,\mathrm{d}S.$$

Hence, the solution of the Robin BVP for Poisson's equation

$$\nabla^2 u = F \text{ in } V \text{ with } \frac{\partial u}{\partial n} + \alpha u = f \text{ on } S$$

is

$$u(\boldsymbol{\xi}) = -\int_V GF\,\mathrm{d}V + \int_S Gf\,\mathrm{d}S,$$

where $G = v + w$ (w regular at $\mathbf{x} = \boldsymbol{\xi}$) with $\nabla^2 v = -\delta(\mathbf{x}-\boldsymbol{\xi})$ and $\nabla^2 w = 0$ in V and $\partial w/\partial n + \alpha w = -\partial v/\partial n - \alpha v$ on S. So, the Green's function G is solution of the Robin BVP

$$\nabla^2 G = -\delta(\mathbf{x}-\boldsymbol{\xi}) \text{ in } V,$$
$$\text{with} \quad \frac{\partial G}{\partial n} + \alpha G = 0 \text{ on } S.$$

Symmetry of Green's Functions

The Green's function is symmetric (i.e., $G(\mathbf{x},\boldsymbol{\xi}) = G(\boldsymbol{\xi},\mathbf{x})$). To show this, consider two Green's functions, $G_1(\mathbf{x}) \equiv G(\mathbf{x},\boldsymbol{\xi}_1)$ and $G_2(\mathbf{x}) \equiv G(\mathbf{x},\boldsymbol{\xi}_2)$, and apply Green's theorem to these,

$$\int_V \left(G_1\nabla^2 G_2 - G_2\nabla^2 G_1\right)\mathrm{d}V = \int_S \left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right)\mathrm{d}S.$$

Now, since, G_1 and G_2 are by definition Green's functions, $G_1 = G_2 = 0$ on S for Dirichlet boundary conditions, $\partial G_1/\partial n = \partial G_2/\partial n = 0$ on S for Neumann boundary conditions or $G_2\,\partial G_1/\partial n = G_1\,\partial G_2/\partial n$ on S for Robin boundary conditions, so in any case the right-hand side is equal to zero. Also, $\nabla^2 G_1 = -\delta(\mathbf{x}-\boldsymbol{\xi}_1)$, $\nabla^2 G_2 = -\delta(\mathbf{x}-\boldsymbol{\xi}_2)$ and the equation becomes

$$\int_V G(\mathbf{x},\boldsymbol{\xi}_1)\,\delta(\mathbf{x}-\boldsymbol{\xi}_2)\,\mathrm{d}V = \int_V G(\mathbf{x},\boldsymbol{\xi}_2)\,\delta(\mathbf{x}-\boldsymbol{\xi}_1)\,\mathrm{d}V,$$
$$G(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) = G(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2).$$

Nevertheless, note that for Neumann BVPs, the term $1/\mathcal{V}$ which provides the additive constant to the solution to Poisson's equation breaks the symmetry of G.

Example:

Consider the 2-dimensional Dirichlet problem for Laplace's equation,

$$\nabla^2 u = 0 \text{ in } V, \text{ with } u = f \text{ on } S \text{ (boundary of } V).$$

Since u is harmonic in V (i.e. $\nabla^2 u = 0$) and $u = f$ on S, then Green's theorem gives

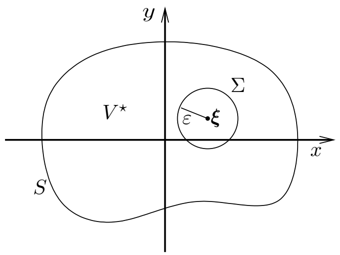
$$\int_V u\nabla^2 v \, \mathrm{d}V = \int_S \left(f\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}S.$$

Note that we have no information about $\partial u/\partial n$ on S or u in V. Suppose we choose,

$$v = -\frac{1}{4\pi} \ln\left((x-\xi)^2 + (y-\eta)^2 \right),$$

then $\nabla^2 v = 0$ on V for all points except $P \equiv (x = \xi, y = \eta)$, where it is undefined.
To eliminate this singularity, we 'cut this point P out' — i.e, surround P by a small circle of radius $\varepsilon = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ and denote the circle by Σ, whose parametric form in polar coordinates is

$$\Sigma : \{x - \xi = \varepsilon\cos\theta, y - \eta = \varepsilon\sin\theta \text{ with } \varepsilon > 0 \text{ and } \theta \in (0, 2\pi)\}.$$



Hence, $v = -1/2\pi \ \ln\varepsilon$ and $\mathrm{d}v/\mathrm{d}\varepsilon = -1/2\pi\varepsilon$ and applying Green's theorem to u and v in this new region $V^\star$ (with boundaries S and Σ), we get

$$\int_S \left(f\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}S + \int_\Sigma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}S = 0. \tag{4.6}$$

since $\nabla^2 u = \nabla^2 v = 0$ for all point in $V^\star$. By transforming to polar coordinates, $\mathrm{d}S = \varepsilon\mathrm{d}\theta$ and $\partial u/\partial n = -\partial u/\partial\varepsilon$ (unit normal is in the direction ε) onto Σ; then

$$\int_\Sigma v\frac{\partial u}{\partial n} \, \mathrm{d}S = \frac{\varepsilon\ln\varepsilon}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial\varepsilon} \, \mathrm{d}\theta \to 0 \text{ as } \varepsilon \to 0,$$

and also

$$\int_\Sigma u\frac{\partial v}{\partial n}\mathrm{d}S = -\int_0^{2\pi} u\,\frac{\partial v}{\partial\varepsilon}\,\varepsilon\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} u\,\varepsilon\frac{1}{\varepsilon}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} u\,\mathrm{d}\theta \to u(\xi,\eta) \text{ as } \varepsilon \to 0,$$

and so, in the limit $\varepsilon \to 0$, equation (4.6) gives

$$u(\xi,\eta) = \int_S \left(v\frac{\partial u}{\partial n} - f\frac{\partial v}{\partial n} \right) \mathrm{d}S, \quad \text{where} \quad v = -\frac{1}{4\pi}\ln\left((x-\xi)^2 + (y-\eta)^2 \right).$$

now, consider w, such that $\nabla^2 w = 0$ in V but with w regular at $(x = \xi, y = \eta)$, and with $w = -v$ on S. Then Green's theorem gives

$$\int_V \left(u\nabla^2 w - w\nabla^2 u \right) \mathrm{d}V = \int_S \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n} \right) \mathrm{d}S \Leftrightarrow \int_S \left(f\frac{\partial w}{\partial n} + v\frac{\partial u}{\partial n} \right) \mathrm{d}S = 0$$

since $\nabla^2 u = \nabla^2 w = 0$ in V and $w = -v$ on S. Then, subtract this equation from equation above to get

$$u(\xi,\eta) = \int_S \left(v\frac{\partial u}{\partial n} - f\frac{\partial v}{\partial n} \right) \mathrm{d}S - \int_S \left(f\frac{\partial w}{\partial n} + v\frac{\partial u}{\partial n} \right) \mathrm{d}S = -\int_S f\,\frac{\partial}{\partial n}\,(v+w)\,\mathrm{d}S.$$

Setting $G(x,y;\xi,\eta) = v + w$, then

$$u(\xi,\eta) = -\int_S f\,\frac{\partial G}{\partial n}\,\mathrm{d}S.$$

Such a function G then has the properties,

$$\nabla^2 G = -\delta(\mathbf{x} - \boldsymbol{\xi}) \text{ in } V, \quad \text{with} \quad G = 0 \text{ on } S.$$

4.3.3 Free Space Green's Function

We seek a Green's function G such that,

$$G(\mathbf{x},\boldsymbol{\xi}) = v(\mathbf{x},\boldsymbol{\xi}) + w(\mathbf{x},\boldsymbol{\xi}) \quad \text{where} \quad \nabla^2 v = -\delta(\mathbf{x} - \boldsymbol{\xi}) \text{ in } V.$$

How do we find the free space Green's function v defined such that $\nabla^2 v = -\delta(\mathbf{x} - \boldsymbol{\xi})$ in V? Note that it does not depend on the form of the boundary. (The function v is a 'source term' and for Laplace's equation is the potential due to a point source at the point $\mathbf{x} = \boldsymbol{\xi}$.)

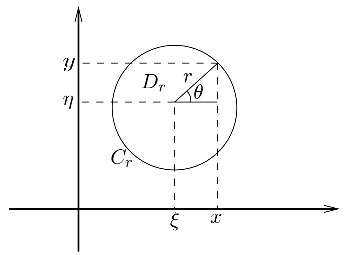
As an illustration of the method, we can derive that, in two dimensions,

$$v = -\frac{1}{4\pi}\ \ln\left((x-\xi)^2 + (y-\eta)^2 \right),$$

as we have already seen. We move to polar coordinate around (ξ,η),

$$x - \xi = r\cos\theta \quad \& \quad y - \eta = r\sin\theta,$$

and look for a solution of Laplace's equation which is independent of θ and which is singular as $r \to 0$.



Laplace's equation in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = 0$$

which has solution $v = B\ln r + A$ with A and B constant. Put $A = 0$ and, to determine the constant B, apply Green's theorem to v and 1 in a small disc D_r (with boundary C_r), of radius r around the origin (η, ξ),

$$\int_{C_r} \frac{\partial v}{\partial n}\,\mathrm{d}S = \int_{D_r} \nabla^2 v\,\mathrm{d}V = -\int_{D_r} \delta(\mathbf{x}-\boldsymbol{\xi})\,\mathrm{d}V = -1,$$

so we choose B to make

$$\int_{C_r} \frac{\partial v}{\partial n}\,\mathrm{d}S = -1.$$

Now, in polar coordinates, $\partial v/\partial n = \partial v/\partial r = B/r$ and $\mathrm{d}S = r\mathrm{d}\theta$ (going around circle C_r). So,

$$\int_0^{2\pi} \frac{B}{r}\,r\mathrm{d}\theta = B\int_0^{2\pi} \mathrm{d}\theta = -1 \Rightarrow B = -\frac{1}{2\pi}.$$

Hence,

$$v = -\frac{1}{2\pi}\ln r = -\frac{1}{4\pi}\ln r^2 = -\frac{1}{4\pi}\ln\left((x-\xi)^2 + (y-\eta)^2\right).$$

(We do not use the boundary condition in finding v.)

Similar (but more complicated) methods lead to the free-space Green's function v for the Laplace equation in n dimensions. In particular,

$$v(\mathbf{x}, \boldsymbol{\xi}) = \begin{cases} -\dfrac{1}{2}|x-\xi|, & n = 1, \\ -\dfrac{1}{4\pi}\ln\left(|\mathbf{x}-\boldsymbol{\xi}|^2\right), & n = 2, \\ -\dfrac{1}{(2-n)A_n(1)}\,|\mathbf{x}-\boldsymbol{\xi}|^{2-n}, & n \geq 3, \end{cases}$$

where $\mathbf{x}$ and $\boldsymbol{\xi}$ are distinct points and $A_n(1)$ denotes the area of the unit n-sphere. We shall restrict ourselves to two dimensions for this course.

Note that Poisson's equation, $\nabla^2 u = F$, is solved in unbounded $\mathbb{R}^n$ by

$$u(\mathbf{x}) = -\int_{\mathbb{R}^n} v(\mathbf{x}, \boldsymbol{\xi})\,F(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$

where from equation (4.2) the free space Green's function v, defined above, serves as Green's function for the differential operator ∇^2 when no boundaries are present.

4.3.4 Method of Images

In order to solve BVPs for Poisson's equation, such as $\nabla^2 u = F$ in an open region V with some conditions on the boundary S, we seek a Green's function G such that, in V

$$G(\mathbf{x}, \boldsymbol{\xi}) = v(\mathbf{x}, \boldsymbol{\xi}) + w(\mathbf{x}, \boldsymbol{\xi}) \quad \text{where} \quad \nabla^2 v = -\delta(\mathbf{x}-\boldsymbol{\xi}) \quad \text{and} \quad \nabla^2 w = 0 \text{ or } 1/\mathcal{V}(V).$$

Having found the free space Green's function v — which does not depend on the boundary conditions, and so is the same for all problems — we still need to find the function w, solution of Laplace's equation and regular in $\mathbf{x} = \boldsymbol{\xi}$, which fixes the boundary conditions (v does not satisfies the boundary conditions required for G by itself). So, we look for the function which satisfies

$$\begin{aligned} &\nabla^2 w = 0 \text{ or } 1/\mathcal{V}(V) \text{ in } V, \quad \text{(ensuring } w \text{ is regular at } (\xi,\eta)), \\ &\text{with} \quad w = -v \quad \text{(i.e. } G = 0\text{) on } S \text{ for Dirichlet boundary conditions,} \\ &\text{or} \quad \frac{\partial w}{\partial n} = -\frac{\partial v}{\partial n} \text{ (i.e. } \frac{\partial G}{\partial n} = 0\text{) on } S \text{ for Neumann boundary conditions.} \end{aligned}$$

To obtain such a function we superpose functions with singularities at the image points of (ξ, η)). (This may be regarded as adding appropriate point sources and sinks to satisfy the boundary conditions.) Note also that, since G and v are symmetric then w must be symmetric too (i.e. $w(\mathbf{x}, \boldsymbol{\xi}) = w(\boldsymbol{\xi}, \mathbf{x})$).

Example 1

Suppose we wish to solve the Dirichlet BVP for Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } y > 0 \quad \text{with} \quad u = f(x) \text{ on } y = 0.$$

We know that in 2-D the free space function is

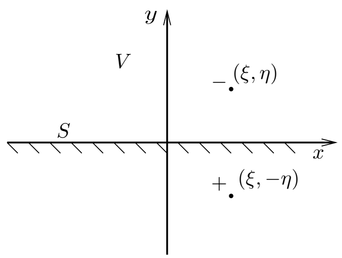
$$v = -\frac{1}{4\pi}\ln\left((x-\xi)^2 + (y-\eta)^2\right).$$

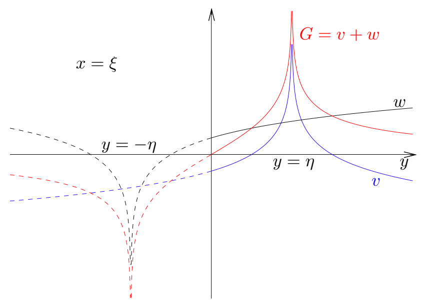
If we superpose to v the function

$$w = +\frac{1}{4\pi}\ln\left((x-\xi)^2 + (y+\eta)^2\right),$$

solution of $\nabla^2 w = 0$ in V and regular at $(x = \xi, y = \eta)$, then

$$G(x, y, \xi, \eta) = v + w = -\frac{1}{4\pi}\ln\left(\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}\right).$$





Note that, setting $y = 0$ in this gives,

$$G(x, 0, \xi, \eta) = -\frac{1}{4\pi}\ln\left(\frac{(x-\xi)^2 + \eta^2}{(x-\xi)^2 + \eta^2}\right) = 0, \text{ as required.}$$

The solution is then given by

$$u(\xi,\eta) = -\int_S f\,\frac{\partial G}{\partial n}\,\mathrm{d}S.$$

Now, we want $\partial G/\partial n$ for the boundary $y=0$, which is

$$\left.\frac{\partial G}{\partial n}\right|_S = -\left.\frac{\partial G}{\partial y}\right|_{y=0} = -\frac{1}{\pi}\frac{\eta}{(x-\xi)^2+\eta^2} \quad \text{(exercise, check this).}$$

Thus,

$$u(\xi,\eta) = \frac{\eta}{\pi}\int_{-\infty}^{+\infty}\frac{f(x)}{(x-\xi)^2+\eta^2}\,\mathrm{d}x,$$

and we can relabel to get in the original variables

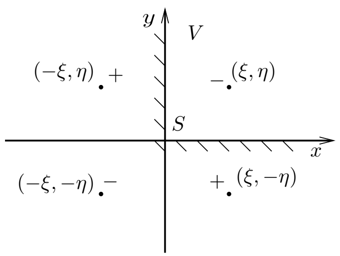
$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(\lambda)}{(\lambda-x)^2+y^2}\,\mathrm{d}\lambda.$$

Example 2

Find Green's function for the Dirichlet BVP

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F \text{ in the quadrant } x>0,\ y>0.$$

We use the same technique but now we have three images.



Then, the Green's function G is

$$G(x,y,\xi,\eta) = -\frac{1}{4\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right) + \frac{1}{4\pi}\ln\left((x-\xi)^2+(y+\eta)^2\right)$$
$$-\frac{1}{4\pi}\ln\left((x+\xi)^2+(y+\eta)^2\right) + \frac{1}{4\pi}\ln\left((x+\xi)^2+(y-\eta)^2\right).$$

So,

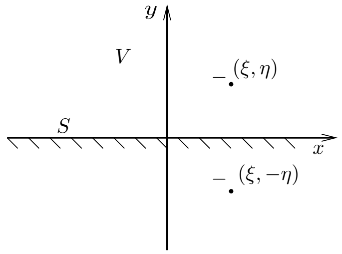
$$G(x,y,\xi,\eta) = -\frac{1}{4\pi}\ln\left[\frac{\left((x-\xi)^2+(y-\eta)^2\right)\left((x+\xi)^2+(y+\eta)^2\right)}{\left((x-\xi)^2+(y+\eta)^2\right)\left((x+\xi)^2+(y-\eta)^2\right)}\right],$$

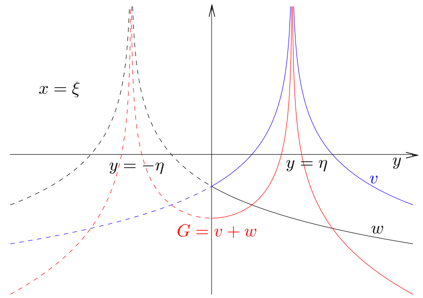
and again we can check that $G(0,y,\xi,\eta) = G(x,0,\xi,\eta) = 0$ as required for Dirichlet BVP.

Example 3

Consider the Neumann BVP for Laplace's equation in the upper half-plane,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } y>0 \quad \text{with} \quad \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y} = f(x) \text{ on } y=0.$$





Add an image to make $\partial G/\partial y = 0$ on the boundary:

$$G(x,y,\xi,\eta) = -\frac{1}{4\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right) - \frac{1}{4\pi}\ln\left((x-\xi)^2+(y+\eta)^2\right).$$

Note that,

$$\frac{\partial G}{\partial y} = -\frac{1}{4\pi}\left(\frac{2(y-\eta)}{(x-\xi)^2+(y-\eta)^2} + \frac{2(y+\eta)}{(x-\xi)^2+(y+\eta)^2}\right),$$

and as required for Neumann BVP,

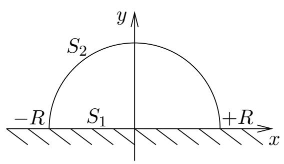
$$\left.\frac{\partial G}{\partial n}\right|_S = -\left.\frac{\partial G}{\partial y}\right|_{y=0} = \frac{1}{4\pi}\left(\frac{-2\eta}{(x-\xi)^2+\eta^2} + \frac{2\eta}{(x-\xi)^2+\eta^2}\right) = 0.$$

Then, since $G(x,0,\xi,\eta) = -1/2\pi\ \ln\left((x-\xi)^2+\eta^2\right)$,

$$u(\xi,\eta) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\ln\left((x-\xi)^2+\eta^2\right)\,\mathrm{d}x,$$

$$\text{i.e. } u(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\lambda)\ln\left((x-\lambda)^2+y^2\right)\,\mathrm{d}\lambda,$$

Remind that all the theory on Green's function has been developed in the case when the equation is given in a bounded open domain. In an infinite domain (i.e. for external problems) we have to be a bit careful since we have not given conditions on G and $\partial G/\partial n$ at infinity. For instance, we can think of the boundary of the upper half-plane as a semi-circle with $R \to +\infty$.



Green's theorem in the half-disc, for u and G, is

$$\int_V \left(G\nabla^2 u - u\nabla^2 G\right)\,\mathrm{d}V = \int_S \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)\,\mathrm{d}S.$$

Split S into S_1, the portion along the x-axis and S_2, the semi-circular arc. Then, in the above equation we have to consider the behaviour of the integrals

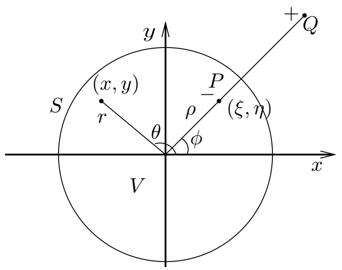
$$(1)\ \int_{S_2} G\frac{\partial u}{\partial n}\,\mathrm{d}S \sim \int_0^\pi G\frac{\partial u}{\partial R}\,R\,\mathrm{d}\theta \quad \text{and} \quad (2)\ \int_{S_2} u\frac{\partial G}{\partial n}\,\mathrm{d}S \sim \int_0^\pi u\frac{\partial G}{\partial R}\,R\,\mathrm{d}\theta$$

as $R \to +\infty$. Green's function G is $O(\ln R)$ on S_2, so from integral (1) we need $\partial u/\partial R$ to fall off sufficiently rapidly with the distance: faster than $1/(R\ln R)$ i.e. u must fall off faster than $\ln(\ln(R))$. In integral (2), $\partial G/\partial R = O(1/R)$ on S_2 provides a more stringent constraint since u must fall off more rapidly that $O(1)$ at large R. If both integrals over S_2 vanish as $R \to +\infty$ then we recover the previously stated results on Green's function.

Example 4

Solve the Dirichlet problem for Laplace's equation in a disc of radius a,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \text{ in } r < a \quad \text{with } u = f(\theta) \text{ on } r = a.$$



Consider image of point P at inverse point Q

$$P = (\rho\cos\phi, \rho\sin\phi),$$
$$Q = (q\cos\phi, q\sin\phi),$$

with $\rho\, q = a^2$ (i.e. $OP \cdot OQ = a^2$).

$$G(x,y,\xi,\eta) = -\frac{1}{4\pi}\ \ln\left((x-\xi)^2 + (y-\eta)^2\right)$$
$$+\frac{1}{4\pi}\ \ln\left((x - \frac{a^2}{\rho}\cos\phi)^2 + (y - \frac{a^2}{\rho}\sin\phi)^2\right) + h(x,y,\xi,\eta) \quad (\text{with } \xi^2+\eta^2 = \rho^2).$$

We need to consider the function $h(x,y,\xi,\eta)$ to make G symmetric and zero on the boundary. We can express this in polar coordinates, $x = r\cos\theta, y = r\sin\theta$,

$$G(r,\theta,\rho,\phi) = \frac{1}{4\pi}\ln\left(\frac{(r\cos\theta - a^2/\rho\,\cos\phi)^2 + (r\sin\theta - a^2/\rho\,\sin\phi)^2}{(r\cos\theta - \rho\cos\phi)^2 + (r\sin\theta - \rho\sin\phi)^2}\right) + h,$$
$$= \frac{1}{4\pi}\ln\left(\frac{r^2 + a^4/\rho^2 - 2a^2 r/\rho\,\cos(\theta-\phi)}{r^2+\rho^2 - 2r\rho\,\cos(\theta-\phi)}\right) + h.$$

Choose h such that $G = 0$ on $r = a$,

$$G|_{r=a} = \frac{1}{4\pi}\ln\left(\frac{a^2 + a^4/\rho^2 - 2a^3/\rho\,\cos(\theta-\phi)}{a^2+\rho^2-2a\rho\,\cos(\theta-\phi)}\right) + h,$$
$$= \frac{1}{4\pi}\ln\left(\frac{a^2}{\rho^2}\,\frac{\rho^2+a^2-2a\rho\,\cos(\theta-\phi)}{\rho^2+a^2-2a\rho\,\cos(\theta-\phi)}\right) + h = 0 \quad \Rightarrow h = \frac{1}{4\pi}\ln\left(\frac{\rho^2}{a^2}\right).$$

Note that,

$$w(r,\theta,\rho,\phi) = \frac{1}{4\pi}\ln\left(r^2 + \frac{a^4}{\rho^2} - 2\frac{a^2 r}{\rho}\,\cos(\theta-\phi)\right) + \frac{1}{4\pi}\ln\left(\frac{\rho^2}{a^2}\right)$$
$$= \frac{1}{4\pi}\ln\left(a^2 + \frac{r^2\rho^2}{a^2} - 2r\rho\,\cos(\theta-\phi)\right)$$

is symmetric, regular and solution of $\nabla^2 w = 0$ in V. So,

$$G(r,\theta,\rho,\phi) = v + w = \frac{1}{4\pi}\ln\left(\frac{a^2 + r^2\rho^2/a^2 - 2r\rho\,\cos(\theta-\phi)}{r^2+\rho^2-2r\rho\,\cos(\theta-\phi)}\right),$$

G is symmetric and zero on the boundary. This enable us to get the result for Dirichlet problem for a circle,

$$u(\rho,\phi) = -\int_0^{2\pi} f(\theta)\,\left.\frac{\partial G}{\partial r}\right|_{r=a} a\,\mathrm{d}\theta,$$

where

$$\frac{\partial G}{\partial r} = \frac{1}{4\pi}\left(\frac{2r\rho^2/a^2 - 2\rho\,\cos(\theta-\phi)}{a^2 + r^2\rho^2/a^2 - 2r\rho\,\cos(\theta-\phi)} - \frac{2r - 2\rho\,\cos(\theta-\phi)}{r^2+\rho^2-2r\rho\,\cos(\theta-\phi)}\right),$$

so

$$\left.\frac{\partial G}{\partial n}\right|_S = \left.\frac{\partial G}{\partial r}\right|_{r=a} = \frac{1}{2\pi}\left(\frac{\rho^2/a - \rho\,\cos(\theta-\phi)}{a^2+\rho^2-2a\rho\,\cos(\theta-\phi)} - \frac{a - \rho\,\cos(\theta-\phi)}{a^2+\rho^2-2a\rho\,\cos(\theta-\phi)}\right),$$
$$= \frac{1}{2\pi\,a}\,\frac{\rho^2 - a^2}{a^2+\rho^2-2a\rho\,\cos(\theta-\phi)}.$$

Then

$$u(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2-\rho^2}{a^2+\rho^2-2a\rho\,\cos(\theta-\phi)}\,f(\theta)\,\mathrm{d}\theta,$$

and relabelling,

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\phi)}{a^2+r^2-2a\,r\,\cos(\theta-\phi)}\,\mathrm{d}\phi.$$

Note that, from the integral form of $u(r,\theta)$ above, we can recover the Mean Value Theorem. If we put $r = 0$ (centre of the circle) then,

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,\mathrm{d}\phi,$$

i.e. the average of an harmonic function of two variables over a circle is equal to its value at the centre.

Furthermore we may introduce more subtle inequalities within the class of positive harmonic functions $u \geq 0$. Since $-1 \leq \cos(\theta-\phi) \leq 1$ then $(a-r)^2 \leq a^2-2ar\cos(\theta-\phi)+r^2 \leq (a+r)^2$. Thus, the kernel of the integrand in the integral form of the solution $u(r,\theta)$ can be bounded

$$\frac{1}{2\pi}\frac{a-r}{a+r} \leq \frac{1}{2\pi}\frac{a^2-r^2}{(a-r)^2 \leq a^2-2ar\cos(\theta-\phi)+r^2} \leq \frac{1}{2\pi}\frac{a+r}{a-r}.$$

For positive harmonic functions u, we may use these inequalities to bound the solution of Dirichlet problem for Laplace's equation in a disc

$$\frac{1}{2\pi}\frac{a-r}{a+r}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta \leq u(r,\theta) \leq \frac{1}{2\pi}\frac{a+r}{a-r}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta,$$

i.e. using the Mean Value Theorem we obtain Harnack's inequalities

$$\frac{a-r}{a+r}u(0) \leq u(r,\theta) \leq \frac{a+r}{a-r}u(0).$$

Example 5

Interior Neumann problem for Laplace's equation in a disc,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0 \text{ in } r < a,$$
$$\frac{\partial u}{\partial n} = f(\theta) \text{ on } r = a.$$

Here, we need

$$\nabla^2 G = -\delta(x-\xi)\delta(y-\eta) + \frac{1}{\mathcal{V}} \quad \text{with} \quad \left.\frac{\partial G}{\partial r}\right|_{r=a} = 0,$$

where $\mathcal{V} = \pi a^2$ is the surface area of the disc. In order to deal with this term we solve the equation

$$\nabla^2\kappa(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\kappa}{\partial r}\right) = \frac{1}{\pi a^2} \Rightarrow \kappa(r) = \frac{r^2}{4\pi a^2} + c_1\ln r + c_2,$$

and take the particular solution with $c_1 = c_2 = 0$. Then, add in source at inverse point and an arbitrary function h to fix the symmetry and boundary condition of G

$$G(r,\theta,\rho,\phi) = -\frac{1}{4\pi}\ln\left(r^2+\rho^2-2r\rho\cos(\theta-\phi)\right) - \frac{1}{4\pi}\ln\left[\frac{a^2}{\rho^2}\left(a^2+\frac{\rho^2 r^2}{a^2}-2r\rho\cos(\theta-\phi)\right)\right] + \frac{r^2}{4\pi a^2} + h.$$

So,

$$\frac{\partial G}{\partial r} = -\frac{1}{4\pi}\frac{2r-2\rho\cos(\theta-\phi)}{r^2+\rho^2-2r\rho\cos(\theta-\phi)} - \frac{1}{4\pi}\frac{2r-2a^2/\rho\cos(\theta-\phi)}{r^2+a^4/\rho^2-2a^2r/\rho\cos(\theta-\phi)} + \frac{r}{2\pi a^2} + \frac{\partial h}{\partial r},$$
$$\left.\frac{\partial G}{\partial r}\right|_{r=a} = -\frac{1}{2\pi}\left(\frac{a-\rho\cos(\theta-\phi)}{a^2+\rho^2-2a\rho\cos(\theta-\phi)} + \frac{a-a^2/\rho\cos(\theta-\phi)}{a^2+a^4/\rho^2-2a^3/\rho\cos(\theta-\phi)}\right) + \frac{1}{2\pi a} + \left.\frac{\partial h}{\partial r}\right|_{r=a},$$
$$= -\frac{1}{2\pi}\frac{a-\rho\cos(\theta-\phi)+\rho^2/a-\rho\cos(\theta-\phi)}{\rho^2+a^2-2a\rho\cos(\theta-\phi)} + \frac{1}{2\pi a} + \left.\frac{\partial h}{\partial r}\right|_{r=a},$$

$$\left.\frac{\partial G}{\partial r}\right|_{r=a} = -\frac{1}{2\pi\,a} + \frac{1}{2\pi a} + \left.\frac{\partial h}{\partial r}\right|_{r=a} \quad \text{and} \quad \left.\frac{\partial h}{\partial r}\right|_{r=a} = 0 \quad \text{implies} \quad \frac{\partial G}{\partial r} = 0 \text{ on the boundary.}$$

Then, put $h \equiv 1/2\pi\,\ln(a/\rho)$; so,

$$G(r,\theta,\rho,\phi) = -\frac{1}{4\pi}\ln\left[\left(r^2+\rho^2-2r\rho\,\cos(\theta-\phi)\right)\left(a^2+\frac{\rho^2r^2}{a^2}-2r\rho\,\cos(\theta-\phi)\right)\right] + \frac{r^2}{4\pi\,a^2}.$$

On $r = a$,

$$G|_{r=a} = -\frac{1}{4\pi}\ln\left[\left(a^2+\rho^2-2a\rho\,\cos(\theta-\phi)\right)^2\right] + \frac{1}{4\pi},$$
$$= -\frac{1}{2\pi}\left[\ln\left(a^2+\rho^2-2a\rho\,\cos(\theta-\phi)\right) - \frac{1}{2}\right].$$

Then,

$$u(\rho,\phi) = \bar{u} + \int_0^{2\pi} f(\theta)\,G|_{r=a}\,a\,\mathrm{d}\theta,$$
$$= \bar{u} - \frac{a}{2\pi}\int_0^{2\pi}\left[\ln\left(a^2+\rho^2-2a\rho\,\cos(\theta-\phi)\right) - \frac{1}{2}\right] f(\theta)\,\mathrm{d}\theta.$$

Now, recall the Neumann problem compatibility condition,

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = 0.$$

Indeed, $\int_V \nabla^2 u\,\mathrm{d}V = \int_S \frac{\partial u}{\partial n}\,\mathrm{d}S$ from divergence theorem $\Rightarrow \int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = 0.$

So the term involving $\int_0^{2\pi} f(\theta)\mathrm{d}\theta$ in the solution $u(\rho,\phi)$ vanishes; hence

$$u(\rho,\phi) = \bar{u} - \frac{a}{2\pi}\int_0^{2\pi}\ln\left(a^2+\rho^2-2a\rho\,\cos(\theta-\phi)\right) f(\theta)\,\mathrm{d}\theta,$$
$$\text{or} \quad u(r,\theta) = \bar{u} - \frac{a}{2\pi}\int_0^{2\pi}\ln\left(a^2+r^2-2ar\,\cos(\theta-\phi)\right) f(\phi)\,\mathrm{d}\phi.$$

Exercise: Exterior Neumann problem for Laplace's equation in a disc,

$$u(r,\theta) = \frac{a}{2\pi}\int_0^{2\pi}\ln\left(a^2+r^2-2ar\,\cos(\theta-\phi)\right) f(\phi)\,\mathrm{d}\phi.$$

4.4 Extensions of Theory:

- Alternative to the method of images to determine the Green's function G: (a) eigenfunction method when G is expended on the basis of the eigenfunction of the Laplacian operator; conformal mapping of the complex plane for solving 2-D problems.
- Green's function for more general operators.