Chapter 3

Second Order Linear and Semilinear Equations in Two Variables

Contents

3.1 Classification and Standard Form Reduction

Consider a general second order linear equation in two independent variables

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y);$$

in the case of a semilinear equation, the coefficients d, e, f and g could be functions of $\partial_x u, \partial_y u$ and u as well.
Recall, for a first order linear and semilinear equation, $a\,\partial u/\partial x + b\,\partial u/\partial y = c$, we could define new independent variables, $\xi(x,y)$ and $\eta(x,y)$ with $J = \partial(\xi,\eta)/\partial(x,y) \neq \{0,\infty\}$, to reduce the equation to the simpler form, $\partial u/\partial \xi = \kappa(\xi,\eta)$.
For the second order equation, can we also transform the variables from (x,y) to (ξ,η) to put the equation into a simpler form?

So, consider the coordinate transform $(x,y) \to (\eta,\xi)$ where ξ and η are such that the Jacobian,

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq \{0,\infty\}.$$

Then by inverse theorem there is an open neighbourhood of (x,y) and another neighbourhood of (ξ,η) such that the transformation is invertible and one-to-one on these neighbourhoods. As before we compute chain rule derivations

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}.$$

The equation becomes

$$A\frac{\partial^2 u}{\partial \xi^2} + 2B\frac{\partial^2 u}{\partial \xi \partial \eta} + C\frac{\partial^2 u}{\partial \eta 2} + F(u_\xi, u_\eta, u, \xi, \eta) = 0, \tag{3.1}$$

where

$$A = a\left(\frac{\partial \xi}{\partial x}\right)^2 + 2b\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + c\left(\frac{\partial \xi}{\partial y}\right)^2,$$

$$B = a\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + b\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + c\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y},$$

$$C = a\left(\frac{\partial \eta}{\partial x}\right)^2 + 2b\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + c\left(\frac{\partial \eta}{\partial y}\right)^2.$$

We write explicitly only the principal part of the PDE, involving the highest-order derivatives of u (terms of second order).

It is easy to verify that

$$(B^2 - AC) = (b^2 - ac)\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)^2$$

where $(\partial_x \xi \partial_y \eta - \partial_y \xi \partial_x \eta)^2$ is just the Jacobian squared. So, provided $J \neq 0$ we see that the sign of the discriminant $b^2 - ac$ is invariant under coordinate transformations. We can use this invariance properties to classify the equation.

Equation (3.1) can be simplified if we can choose ξ and η so that some of the coefficients A, B or C are zero. Let us define,

$$D_\xi = \frac{\partial \xi/\partial x}{\partial \xi/\partial y} \quad \text{and} \quad D_\eta = \frac{\partial \eta/\partial x}{\partial \eta/\partial y};$$

then we can write

$$A = \left(aD_\xi^2 + 2bD_\xi + c\right)\left(\frac{\partial \xi}{\partial y}\right)^2,$$

$$B = \left(aD_\xi D_\eta + b\left(D_\xi + D_\eta\right) + c\right)\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y},$$

$$C = \left(aD_\eta^2 + 2bD_\eta + c\right)\left(\frac{\partial \eta}{\partial y}\right)^2.$$

Now consider the quadratic equation

$$aD^2 + 2bD + c = 0, \tag{3.2}$$

whose solution is given by

$$D = \frac{-b \pm \sqrt{b^2 - ac}}{a}.$$

If the discriminant $b^2 - ac \neq 0$, equation (3.2) has two distinct roots; so, we can make both coefficients A and C zero if we arbitrarily take the root with the negative sign for D_ξ and the one with the positive sign for D_η,

$$D_\xi = \frac{\partial\xi/\partial x}{\partial\xi/\partial y} = \frac{-b - \sqrt{b^2 - ac}}{a} \Rightarrow A = 0, \tag{3.3}$$

$$D_\eta = \frac{\partial\eta/\partial x}{\partial\eta/\partial y} = \frac{-b + \sqrt{b^2 - ac}}{a} \Rightarrow C = 0.$$

Then, using $D_\xi D_\eta = c/a$ and $D_\xi + D_\eta = -2b/a$ we have

$$B = \frac{2}{a}\left(ac - b^2\right)\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} \Rightarrow B \neq 0.$$

Furthermore, if the discriminant $b^2 - ac > 0$ then D_ξ and D_η as well as ξ and η are real. So, we can define two families of one-parameter characteristics of the PDE as the curves described by the equation $\xi(x, y) =$ constant and the equation $\eta(x, y) =$ constant. Differentiate ξ along the characteristic curves given by $\xi =$ constant,

$$\mathrm{d}\xi = \frac{\partial\xi}{\partial x}\mathrm{d}x + \frac{\partial\xi}{\partial y}\mathrm{d}y = 0,$$

and make use of (3.3) to find that this characteristics satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b + \sqrt{b^2 - ac}}{a}. \tag{3.4}$$

Similarly we find that the characteristic curves described by $\eta(x, y) =$ constant satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b - \sqrt{b^2 - ac}}{a}. \tag{3.5}$$

If the discriminant $b^2 - ac = 0$, equation (3.2) has one unique root and if we take this root for D_ξ say, we can make the coefficient A zero,

$$D_\xi = \frac{\partial\xi/\partial x}{\partial\xi/\partial y} = -\frac{b}{a} \Rightarrow A = 0.$$

To get η independent of ξ, D_η has to be different from D_ξ, so $C \neq 0$ in this case, but B is now given by

$$B = \left(-a\frac{b}{a}D_\eta + b\left(-\frac{b}{a} + D_\eta\right) + c\right)\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} = \left(-\frac{b^2}{a} + c\right)\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y},$$

so that $B = 0$. When $b^2 - ac = 0$ the PDE has only one family of characteristic curves, for $\xi(x, y) =$ constant, whose equation is now

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}. \tag{3.6}$$

Thus we have to consider three different cases.

1. If $b^2 > ac$ we can apply the change of variable $(x, y) \to (\eta, \xi)$ to transform the original PDE to
$$\frac{\partial^2 u}{\partial\xi\partial\eta} + (\text{lower order terms}) = 0.$$
In this case the equation is said to be hyperbolic and has two families of characteristics given by equation (3.4) and equation (3.5).

2. If $b^2 = ac$, a suitable choice for ξ still simplifies the PDE, but now we can choose η arbitrarily — provided η and ξ are independent — and the equation reduces to the form
$$\frac{\partial^2 u}{\partial\eta^2} + (\text{lower order terms}) = 0.$$
The equation is said to be parabolic and has only one family of characteristics given by equation (3.6).

3. If $b^2 < ac$ we can again apply the change of variables $(x, y) \to (\eta, \xi)$ to simplify the equation but now this functions will be complex conjugate. To keep the transformation real, we apply a further change of variables $(\xi, \eta) \to (\alpha, \beta)$ via
$$\alpha = \xi + \eta = 2\,\Re(\eta),$$
$$\beta = i(\xi - \eta) = 2\,\Im(\eta),$$
$$\text{i.e., } \frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} \qquad \text{(via the chain rule);}$$
so, the equation can be reduced to
$$\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} + (\text{lower order terms}) = 0.$$
In this case the equation is said to be elliptic and has no real characteristics.

The above forms are called the canonical (or standard) forms of the second order linear or semilinear equations (in two variables).

Summary:

$b^2 - ac$	> 0	$= 0$	< 0
Canonical form	$\frac{\partial^2 u}{\partial\xi\partial\eta} + \ldots = 0$	$\frac{\partial^2 u}{\partial\eta^2} + \ldots = 0$	$\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} + \ldots = 0$
Type	Hyperbolic	Parabolic	Elliptic

E.g.

- The wave equation,
$$\frac{\partial^2 u}{\partial t^2} - c_w^2\frac{\partial^2 u}{\partial x^2} = 0,$$
is hyperbolic ($b^2 - ac = c_w^2 > 0$) and the two families of characteristics are described by $\mathrm{d}x/\mathrm{d}t = \pm c_w$ i.e. $\xi = x - c_w t$ and $\eta = x + c_w t$. So, the equation transforms into its canonical form $\partial^2 u/\partial\xi\partial\eta = 0$ whose solutions are waves travelling in opposite direction at speed c_w.

- The diffusion (heat conduction) equation,
$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u}{\partial t} = 0,$$
is parabolic ($b^2 - ac = 0$). The characteristics are given by $\mathrm{d}t/\mathrm{d}x = 0$ i.e. $\xi = t =$ constant.

- Laplace's equation,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
is elliptic ($b^2 - ac = -1 < 0$).

The type of a PDE is a local property. So, an equation can change its form in different regions of the plane or as a parameter is changed. E.g. Tricomi's equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (b^2 - ac = 0 - y = -y)$$

is elliptic in $y > 0$, parabolic for $y = 0$ and hyperbolic in $y < 0$, and for small disturbances in incompressible (inviscid) flow

$$\frac{1}{1-m^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (b^2 - ac = -\frac{1}{1-m^2})$$

is elliptic if $m < 1$ and hyperbolic if $m > 1$.

Example 1: Reduce to the canonical form

$$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{xy}\left(y^3\frac{\partial u}{\partial x} + x^3\frac{\partial u}{\partial y}\right).$$

Here $\left.\begin{array}{l} a = y^2 \\ b = -xy \\ c = x^2 \end{array}\right\}$ so $b^2 - ac = (xy)^2 - x^2y^2 = 0 \Rightarrow$ parabolic equation.

On $\xi =$ constant,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b + \sqrt{b^2 - ac}}{a} = \frac{b}{a} = -\frac{x}{y} \Rightarrow \xi = x^2 + y^2.$$

We can choose η arbitrarily provided ξ and η are independent. We choose $\eta = y$. (Exercise, try it with $\eta = x$.) Then

$$\frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial \xi}, \qquad \frac{\partial u}{\partial y} = 2y\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}, \qquad \frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial \xi} + 4x^2\frac{\partial^2 u}{\partial \xi^2},$$
$$\frac{\partial^2 u}{\partial x \partial y} = 4xy\frac{\partial^2 u}{\partial \xi^2} + 2x\frac{\partial^2 u}{\partial \xi \partial \eta}, \qquad \frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial \xi} + 4y^2\frac{\partial^2 u}{\partial \xi^2} + 4y\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

and the equation becomes

$$2y^2\frac{\partial u}{\partial \xi} + 4x^2y^2\frac{\partial^2 u}{\partial \xi^2} - 8x^2y^2\frac{\partial^2 u}{\partial \xi^2} - 4x^2y\frac{\partial^2 u}{\partial \xi \partial \eta} + 2x^2\frac{\partial u}{\partial \xi} + 4x^2y^2\frac{\partial^2 u}{\partial \xi^2}$$
$$+ 4x^2y\frac{\partial^2 u}{\partial \xi \partial \eta} + x^2\frac{\partial^2 u}{\partial \eta^2} = \frac{1}{xy}\left(2xy^3\frac{\partial u}{\partial \xi} + 2x^3y\frac{\partial u}{\partial \xi} + x^3\frac{\partial u}{\partial \eta}\right),$$

$$\text{i.e.} \quad \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial u}{\partial \eta} = 0. \qquad \text{(canonical form)}$$

This has solution

$$u = f(\xi) + \eta^2\, g(\xi),$$

where f and g are arbitrary functions (via integrating factor method), i.e.

$$u = f(x^2 + y^2) + y^2\, g(x^2 + y^2).$$

We need to impose two conditions on u or its partial derivatives to determine the functions f and g i.e. to find a particular solution.

Example 2: Reduce to canonical form and then solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} + 1 = 0 \quad \text{in } 0 \le x \le 1, y > 0, \text{ with } u = \frac{\partial u}{\partial y} = x \text{ on } y = 0.$$

Here $\left.\begin{array}{l} a = 1 \\ b = 1/2 \\ c = -2 \end{array}\right\}$ so $b^2 - ac = 9/4\ (> 0) \Rightarrow$ equation is hyperbolic.

Characteristics:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \pm \frac{3}{2} = -1 \text{ or } 2 \qquad \left(= -\frac{\partial\xi/\partial x}{\partial\xi/\partial y} \text{ or } -\frac{\partial\eta/\partial x}{\partial\eta/\partial y}\right).$$

Two methods of solving:

1. directly:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \Rightarrow x - \frac{1}{2}y = \text{constant} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -1 \Rightarrow x + y = \text{constant}.$$

2. simultaneous equations:
$$\left.\begin{array}{l} -\dfrac{\partial\xi}{\partial x} = 2\dfrac{\partial\xi}{\partial y} \\ -\dfrac{\partial\eta}{\partial x} = -\dfrac{\partial\eta}{\partial y} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \xi = x - \dfrac{y}{2} \\ \eta = x + y \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} x = \dfrac{1}{3}(\eta + 2\xi) \\ y = \dfrac{2}{3}(\eta - \xi) \end{array}\right\}.$$

So,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}, \qquad \frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$
$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{2}\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

and the equation becomes

$$\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{2}\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} - 2\frac{\partial^2 u}{\partial \eta^2} + 1 = 0,$$

$$\Rightarrow \frac{9}{2}\frac{\partial^2 u}{\partial \xi \partial \eta} + 1 = 0, \quad \text{canonical form.}$$

So $\partial^2 u/\partial\xi\partial\eta = -2/9$ and general solution is given by

$$\Rightarrow u(\xi,\eta) = -\frac{2}{9}\,\xi\,\eta + f(\xi) + g(\eta),$$

where f and g are arbitrary functions; now, we need to apply two conditions to determine these functions.

When, $y=0$, $\xi=\eta=x$ so the condition $u=x$ at $y=0$ gives

$$u(\xi=x,\eta=x) = -\frac{2}{9}\,x^2 + f(x) + g(x) = x \Longleftrightarrow f(x)+g(x) = x + \frac{2}{9}\,x^2. \tag{3.7}$$

Also, using the relation

$$\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} = \frac{1}{9}\,\eta - \frac{1}{2}f'(\xi) - \frac{2}{9}\,\xi + g'(\eta),$$

the condition $\partial u/\partial y = x$ at $y=0$ gives

$$\frac{\partial u}{\partial y}(\xi=x,\eta=x) = \frac{1}{9}\,x - \frac{1}{2}f'(x) - \frac{2}{9}\,x + g'(x) = x \Longleftrightarrow g'(x) - \frac{1}{2}f'(x) = \frac{10}{9}\,x,$$

$$\text{and after integration,}\quad g(x) - \frac{1}{2}f(x) = \frac{5}{9}\,x^2 + k, \tag{3.8}$$

where k is a constant. Solving equation (3.7) and equation (3.8) simultaneously gives,

$$f(x) = \frac{2}{3}x - \frac{2}{9}x^2 - \frac{2}{3}k \quad\text{and}\quad g(x) = \frac{1}{3}x + \frac{4}{9}x^2 + \frac{2}{3}k,$$

$$\text{or, in terms of } \xi \text{ and } \eta \quad f(\xi) = \frac{2}{3}\xi - \frac{2}{9}\xi^2 - \frac{2}{3}k \quad\text{and}\quad g(\eta) = \frac{1}{3}\eta + \frac{4}{9}\eta^2 + \frac{2}{3}k.$$

So, full solution is

$$\begin{aligned} u(\xi,\eta) &= -\frac{2}{9}\xi\eta + \frac{2}{3}\xi - \frac{2}{9}\xi^2 + \frac{1}{3}\eta + \frac{4}{9}\eta^2, \\ &= \frac{1}{3}(2\xi+\eta) + \frac{2}{9}(\eta-\xi)(2\eta+\xi). \end{aligned}$$

$$\Rightarrow u(x,y) = x + xy + \frac{y^2}{2}. \qquad \text{(check this solution.)}$$

Example 3: Reduce to canonical form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\text{Here } \left.\begin{aligned} a &= 1 \\ b &= 1/2 \\ c &= 1 \end{aligned}\right\} \text{ so } b^2 - ac = -3/4\ (<0) \Rightarrow \text{equation is elliptic.}$$

Find ξ and η via

$$\left.\begin{aligned} \xi &= \text{constant on} \quad \mathrm{d}y/\mathrm{d}x = (1+i\sqrt{3})/2 \\ \eta &= \text{constant on} \quad \mathrm{d}y/\mathrm{d}x = (1-i\sqrt{3})/2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \xi &= y - \tfrac{1}{2}(1+i\sqrt{3})x \\ \eta &= y - \tfrac{1}{2}(1-i\sqrt{3})x \end{aligned}\right\}.$$

To obtain a real transformation, put

$$\alpha = \eta + \xi = 2y - x \quad\text{and}\quad \beta = i(\xi - \eta) = x\sqrt{3}.$$

So,

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \alpha} + \sqrt{3}\,\frac{\partial u}{\partial \beta}, \quad \frac{\partial u}{\partial y} = 2\,\frac{\partial u}{\partial \alpha}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \alpha^2} - 2\sqrt{3}\,\frac{\partial^2 u}{\partial\alpha\partial\beta} + 3\,\frac{\partial^2 u}{\partial \beta^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = -2\,\frac{\partial^2 u}{\partial \alpha^2} + 2\sqrt{3}\,\frac{\partial^2 u}{\partial\alpha\partial\beta}, \quad \frac{\partial^2 u}{\partial y^2} = 4\,\frac{\partial^2 u}{\partial \alpha^2},$$

and the equation transforms to

$$\frac{\partial^2 u}{\partial \alpha^2} - 2\sqrt{3}\,\frac{\partial^2 u}{\partial\alpha\partial\beta} + 3\,\frac{\partial^2 u}{\partial \beta^2} - 2\,\frac{\partial^2 u}{\partial \alpha^2} + 2\sqrt{3}\,\frac{\partial^2 u}{\partial\alpha\partial\beta} + 4\,\frac{\partial^2 u}{\partial \alpha^2} = 0.$$

$$\Rightarrow \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = 0, \quad \text{canonical form.}$$

3.2 Extensions of the Theory

3.2.1 Linear second order equations in n variables

There are two obvious ways in which we might wish to extend the theory.

To consider quasilinear second order equations (still in two independent variables.) Such equations can be classified in an invariant way according to rule analogous to those developed above for linear equations. However, since a, b and c are now functions of $\partial_x u, \partial_y u$ and u its type turns out to depend in general on the particular solution searched and not just on the values of the independent variables.

To consider linear second order equations in more than two independent variables. In such cases it is not usually possible to reduce the equation to a simple canonical form. However, for the case of an equation with constant coefficients such a reduction is possible. Although this seems a rather restrictive class of equations, we can regard the classification obtained as a local one, at a particular point.
Consider the linear PDE

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = d.$$

Without loss of generality we can take the matrix $A = (a_{ij}),\ i,j = 1\cdots n$, to be symmetric (assuming derivatives commute). For any real symmetric matrix A, there is an associate orthogonal matrix P such that $P^T A P = \Lambda$ where Λ is a diagonal matrix whose element are the eigenvalues, λ_i, of A and the columns of P the linearly independent eigenvectors of A, $\mathbf{e}_i = (e_{1i}, e_{2i}, \cdots, e_{ni})$. So

$$P = (e_{ij}) \quad\text{and}\quad \Lambda = (\lambda_i \delta_{ij}), \ \ i,j = 1, \cdots, n.$$

Now consider the transformation $\mathbf{x} = P\boldsymbol{\xi}$, i.e. $\boldsymbol{\xi} = P^{-1}\mathbf{x} = P^T\mathbf{x}$ (P orthogonal) where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$; this can be written as

$$x_i = \sum_{j=1}^{n} e_{ij}\,\xi_j \quad\text{and}\quad \xi_j = \sum_{i=1}^{n} e_{ij}\,x_i.$$

So,

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k} e_{ik} \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,r=1}^{n} \frac{\partial u}{\partial \xi_k \partial \xi_r} e_{ik}\, e_{jr}.$$

The original equation becomes,

$$\sum_{i,j=1}^{n} \sum_{k,r=1}^{n} \frac{\partial u}{\partial \xi_k \partial \xi_r} a_{ij}\, e_{ik}\, e_{jr} + (\text{lower order terms}) = 0.$$

But by definition of the eigenvectors of A,

$$\mathbf{e}_k^T A\, \mathbf{e}_r = \sum_{i,j=1}^{n} e_{ik}\, a_{ij}\, e_{jr} \equiv \lambda_r\, \delta_{rk}.$$

Then equation simplifies to

$$\sum_{k=1}^{n} \lambda_k \frac{\partial u}{\partial \xi_k^2} + (\text{lower order terms}) = 0.$$

We are now in a position to classify the equation.

- Equation is elliptic if and only if all λ_k are non-zero and have the same sign. E.g. Laplace's equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$
- When all the λ_k are non-zero and have the same sign except for precisely one of them, the equation is hyperbolic. E.g. the wave equation
$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0.$$
- When all the λ_k are non-zero and there are at least two of each sign, the equation ultra-hyperbolic. E.g.
$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2};$$
such equation do not often arise in mathematical physics.
- If any of the λ_k vanish the equation is parabolic. E.g. heat equation
$$\frac{\partial u}{\partial t} - \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0.$$

3.2.2 The Cauchy Problem

Consider the problem of finding the solution of the equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + F(\partial_x u, \partial_y u, u, x, y) = 0$$

which takes prescribed values on a given curve Γ which we assume is represented parametrically in the form

$$x = \phi(\sigma), \quad y = \theta(\sigma),$$

for $\sigma \in I$, where I is an interval, $\sigma_0 \le \sigma \le \sigma_1$, say. (Usually consider piecewise smooth curves.)
We specify Cauchy data on Γ: u, $\partial u/\partial x$ and $\partial u/\partial y$ are given $\forall \sigma \in I$, but note that we cannot specify all these quantities arbitrarily. To show this, suppose u is given on Γ by $u = f(\sigma)$; then the derivative tangent to Γ, $\mathrm{d}u/\mathrm{d}\sigma$, can be calculated from $\mathrm{d}u/\mathrm{d}\sigma = f'(\sigma)$ but also

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\sigma} &= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\sigma} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\sigma}, \\ &= \phi'(\sigma)\frac{\partial u}{\partial x} + \theta'(\sigma)\frac{\partial u}{\partial y} = f'(\sigma),\end{aligned}$$

so, on Γ, the partial derivatives $\partial u/\partial x, \partial u/\partial y$ and u are connected by the above relation. Only derivatives normal to Γ and u can be prescribed independently.

So, the Cauchy problem consists in finding the solution $u(x, y)$ which satisfies the following conditions

$$\left.\begin{aligned} u(\phi(\sigma), \theta(\sigma)) &= f(\sigma) \\ \text{and} \quad \frac{\partial u}{\partial n}(\phi(\sigma), \theta(\sigma)) &= g(\sigma) \end{aligned}\right\},$$

where $\sigma \in I$ and $\partial/\partial n = \mathbf{n} \cdot \nabla$ denotes a normal derivative to Γ (e.g. $\mathbf{n} = [\theta', -\phi']^T$); the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are uniquely determined on Γ by these conditions.

Set, $p = \partial u/\partial x$ and $q = \partial u/\partial y$ so that on Γ, p and q are known; then

$$\frac{\mathrm{d}p}{\mathrm{d}s} = \frac{\partial^2 u}{\partial x^2}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial^2 u}{\partial x \partial y}\frac{\mathrm{d}y}{\mathrm{d}s} \quad \text{and} \quad \frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\partial^2 u}{\partial x \partial y}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial^2 u}{\partial y^2}\frac{\mathrm{d}y}{\mathrm{d}s}.$$

Combining these two equations with the original PDE gives the following system of equations for $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$ and $\partial^2 u/\partial y^2$ on Γ (in matrix form),

$$\mathcal{M} \begin{pmatrix} \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial^2 u}{\partial x \partial y} \\ \dfrac{\partial^2 u}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -F \\ \dfrac{\mathrm{d}p}{\mathrm{d}s} \\ \dfrac{\mathrm{d}q}{\mathrm{d}s} \end{pmatrix} \quad \text{where} \quad \mathcal{M} = \begin{pmatrix} a & 2b & c \\ \dfrac{\mathrm{d}x}{\mathrm{d}s} & \dfrac{\mathrm{d}y}{\mathrm{d}s} & 0 \\ 0 & \dfrac{\mathrm{d}x}{\mathrm{d}s} & \dfrac{\mathrm{d}y}{\mathrm{d}s} \end{pmatrix}.$$

So, if $\det(\mathcal{M}) \neq 0$ we can solve the equations uniquely and find $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$ and $\partial^2 u/\partial y^2$ on Γ. By successive differentiations of these equations it can be shown that the derivatives of u of all orders are uniquely determined at each point on Γ for which $\det(\mathcal{M}) \neq 0$. The values of u at neighbouring points can be obtained using Taylor's theorem.

So, we conclude that the equation can be solved uniquely in the vicinity of Γ provided $\det(\mathcal{M}) \neq 0$ (Cauchy-Kowaleski theorem provides a majorant series ensuring convergence of Taylor's expansion).

Consider what happens when $\det(\mathcal{M}) = 0$, so that $\mathcal{M}$ is singular and we cannot solve uniquely for the second order derivatives on Γ. In this case the determinant $\det(\mathcal{M}) = 0$ gives,

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 - 2b\,\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}y}{\mathrm{d}s} + c\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 = 0.$$

But,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}s}{\mathrm{d}x/\mathrm{d}s}$$

and so (dividing through by $\mathrm{d}x/\mathrm{d}s$), $\mathrm{d}y/\mathrm{d}x$ satisfies the equation,

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2b\,\frac{\mathrm{d}y}{\mathrm{d}x} + c = 0, \qquad \text{i.e.} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}.$$

The exceptional curves Γ, on which, if u and its normal derivative are prescribed, no unique solution can be found satisfying these conditions, are the characteristics curves.