Chapter 2

First Order Equations

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2.1 Linear and Semilinear Equations

2.1.1 Method of Characteristic

We consider linear first order partial differential equation in two independent variables:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y), \tag{2.1}$$

where a, b, c and f are continuous in some region of the plane and we assume that $a(x,y)$ and $b(x,y)$ are not zero for the same (x,y).
In fact, we could consider semilinear first order equation (where the nonlinearity is present only in the right-hand side) such as

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = \kappa(x,y,u), \tag{2.2}$$

instead of a linear equation as the theory of the former does not require any special treatment as compared to that of the latter.

The key to the solution of the equation (2.1) is to find a change of variables (or a change of coordinates)

$$\xi \equiv \xi(x,y), \eta \equiv \eta(x,y)$$

which transforms (2.1) into the simpler equation

$$\frac{\partial w}{\partial \xi} + h(\xi,\eta)w = F(\xi,\eta) \tag{2.3}$$

where $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$.

We shall define this transformation so that it is one-to-one, at least for all (x,y) in some set D of points in the (x-y) plane. Then, on D we can (in theory) solve for x and y as functions of ξ, η. To ensure that we can do this, we require that the Jacobian of the transformation does not vanish in D:

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} \neq \{0, \infty\}$$

for (x,y) in D. We begin looking for a suitable transformation by computing derivatives via the chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y}.$$

We substitute these into equation (2.1) to obtain

$$a\left(\frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x}\right) + b\left(\frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y}\right) + cw = f.$$

We can rearrange this as

$$\left(a\frac{\partial \xi}{\partial x} + b\frac{\partial \xi}{\partial y}\right)\frac{\partial w}{\partial \xi} + \left(a\frac{\partial \eta}{\partial x} + b\frac{\partial \eta}{\partial y}\right)\frac{\partial w}{\partial \eta} + cw = f. \tag{2.4}$$

This is close to the form of equation (2.1) if we can choose $\eta \equiv \eta(x,y)$ so that

$$a\frac{\partial \eta}{\partial x} + b\frac{\partial \eta}{\partial y} = 0 \quad \text{for} \quad (x,y) \quad \text{in} \quad D.$$

Provided that $\partial\eta/\partial y \neq 0$ we can express this required property of η as

$$\frac{\partial_x \eta}{\partial_y \eta} = -\frac{b}{a}.$$

Suppose we can define a new variable (or coordinate) η which satisfies this constraint. What is the equation describing the curves of constant η? Putting $\eta \equiv \eta(x,y) = k$ (k an arbitrary constant), then

$$\mathrm{d}\eta = \frac{\partial \eta}{\partial x}\mathrm{d}x + \frac{\partial \eta}{\partial y}\mathrm{d}y = 0$$

implies that $\mathrm{d}y/\mathrm{d}x = -\partial_x\eta/\partial_y\eta = b/a$. So, the equation $\eta(x,y) = k$ defines solutions of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y)}{a(x,y)}. \tag{2.5}$$

Equation (2.5) is called the characteristic equation of the linear equation (2.1). Its solution can be written in the form $F(x,y,\eta) = 0$ (where η is the constant of integration) and defines a family of curves in the plane called characteristics or characteristic curves of (2.1). (More on characteristics later.) Characteristics represent curves along which the independent variable η of the new coordinate system (ξ,η) is constant.
So, we have made the coefficient of $\partial w/\partial\eta$ vanish in the transformed equation (2.4), by choosing $\eta \equiv \eta(x,y)$, with $\eta(x,y) = k$ an equation defining the solution of the characteristic equation (2.5). We can now choose ξ arbitrarily (or at least to suit our convenience), providing we still have $J \neq 0$. An obvious choice is

$$\xi \equiv \xi(x,y) = x.$$

Then

$$J = \begin{vmatrix} 1 & 0 \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \frac{\partial \eta}{\partial y},$$

and we have already assumed this on-zero.
Now we see from equation (2.4) that this change of variables,

$$\xi = x, \eta \equiv \eta(x, y),$$

transforms equation (2.1) to

$$\alpha(x, y)\frac{\partial w}{\partial \xi} + c(x, y)w = f(x, y).,$$

where $\alpha = a\partial\xi/\partial x + b\partial\xi/\partial y$. To complete the transformation to the form of equation (2.3), we first write $\alpha(x, y), c(x, y)$ and $f(x, y)$ in terms of ξ and η to obtain

$$A(\xi, \eta)\frac{\partial w}{\partial \xi} + C(\xi, \eta)w = \rho(\xi, \eta).$$

Finally, restricting the variables to a set in which $A(\xi, \eta) \neq 0$ we have

$$\frac{\partial w}{\partial \xi} + \frac{C}{A}w = \frac{\rho}{A},$$

which is in the form of (2.3) with

$$h(\xi, \eta) = \frac{C(\xi, \eta)}{A(\xi, \eta)} \quad \text{and} \quad F(\xi, \eta) = \frac{\rho(\xi, \eta)}{A(\xi, \eta)}.$$

The characteristic method applies to first order semilinear equation (2.2) as well as linear equation (2.1); similar change of variables and basic algebra transform equation (2.2) to

$$\frac{\partial w}{\partial \xi} = \frac{\mathcal{K}}{A},$$

where the nonlinear term $\mathcal{K}(\xi, \eta, w) = \kappa(x, y, u)$ and restricting again the variables to a set in which $A(\xi, \eta) = \alpha(x, y) \neq 0$.

Notation: It is very convenient to use the function u in places where rigorously the function w should be used. E.g., the equation here above can identically be written as $\partial u/\partial \xi = \mathcal{K}/A$.

Example: Consider the linear first order equation

$$x^2\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + xyu = 1.$$

This is equation (2.1) with $a(x, y) = x^2$, $b(x, y) = y$, $c(x, y) = xy$ and $f(x, y) = 1$. The characteristic equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a} = \frac{y}{x^2}.$$

Solve this by separation of variables

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x^2}\,\mathrm{d}x \Rightarrow \ln y + \frac{1}{x} = k, \quad \text{for} \quad y > 0, \quad \text{and} \quad x \neq 0.$$

This is an integral of the characteristic equation describing curves of constant η and so we choose

$$\eta \equiv \eta(x, y) = \ln y + \frac{1}{x}.$$

Graphs of $\ln y + 1/x$ are the characteristics of this PDE. Choosing $\xi = x$ we have the Jacobian

$$J = \frac{\partial \eta}{\partial y} = \frac{1}{y} \neq 0 \quad \text{as required.}$$

Since $\xi = x$,

$$\eta = \ln y + \frac{1}{\xi} \Rightarrow y = e^{\eta - 1/\xi}.$$

Now we apply the transformation $\xi = x, \eta = \ln y + 1/x$ with $w(\xi, \eta) = u(x, y)$ and we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta}\cdot\left(-\frac{1}{x^2}\right) = \frac{\partial w}{\partial \xi} - \frac{1}{\xi^2}\frac{\partial w}{\partial \eta}, \\
\frac{\partial u}{\partial y} &= \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y} = 0 + \frac{\partial w}{\partial \eta}\cdot\frac{1}{y} = \frac{1}{e^{\eta - 1/\xi}}\frac{\partial w}{\partial \eta}.
\end{aligned}$$

Then the PDE becomes

$$\xi^2\left(\frac{\partial w}{\partial \xi} - \frac{1}{\xi^2}\frac{\partial w}{\partial \eta}\right) + e^{\eta - 1/\xi}\frac{1}{e^{\eta - 1/\xi}}\frac{\partial w}{\partial \eta} + \xi e^{\eta - 1/\xi}w = 1,$$

which simplifies to

$$\xi^2\frac{\partial w}{\partial \xi} + \xi e^{\eta - 1/\xi}w = 1 \quad \text{then to} \quad \frac{\partial w}{\partial \xi} + \frac{1}{\xi}e^{\eta - 1/\xi}w = \frac{1}{\xi^2}.$$

We have transformed the equation into the form of equation (2.3), for any region of (ξ, η)-space with $\xi \neq 0$.

2.1.2 Equivalent set of ODEs

The point of this transformation is that we can solve equation (2.3). Think of

$$\frac{\partial w}{\partial \xi} + h(\xi, \eta)w = F(\xi, \eta)$$

as a linear first order ordinary differential equation in ξ, with η carried along as a parameter. Thus we use an integrating factor method

$$e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\frac{\partial w}{\partial \xi} + h(\xi, \eta)\,e^{\int h(\xi,\eta)\,\mathrm{d}\xi}w = F(\xi, \eta)\,e^{\int h(\xi,\eta)\,\mathrm{d}\xi},$$

$$\frac{\partial}{\partial \xi}\left(e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\,w\right) = F(\xi, \eta)\,e^{\int h(\xi,\eta)\,\mathrm{d}\xi}.$$

Now we integrate with respect to ξ. Since η is being carried as a parameter, the constant of integration may depend on η

$$e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\, w = \int F(\xi,\eta)\, e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\,\mathrm{d}\xi + g(\eta)$$

in which g is an arbitrary differentiable function of one variable. Now the general solution of the transformed equation is

$$w(\xi,\eta) = e^{-\int h(\xi,\eta)\,\mathrm{d}\xi} \int F(\xi,\eta)\, e^{\int h(\xi,\eta)\,\mathrm{d}\xi}\,\mathrm{d}\xi + g(\eta)\, e^{-\int h(\xi,\eta)\,\mathrm{d}\xi}.$$

We obtain the general form of the original equation by substituting back $\xi(x,y)$ and $\eta(x,y)$ to get

$$u(x,y) = e^{\alpha(x,y)}\left[\beta(x,y) + g(\eta(x,y))\right]. \tag{2.6}$$

A certain class of first order PDEs (linear and semilinear PDEs) can then be reduced to a set of ODEs. This makes use of the general philosophy that ODEs are easier to solve than PDEs.

Example: Consider the constant coefficient equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0$$

where $a, b, c \in \mathbb{R}$. Assume $a \neq 0$, the characteristic equation is

$$\mathrm{d}y/\mathrm{d}x = b/a$$

with general solution defined by the equation

$$bx - ay = k, k \quad \text{constant.}$$

So the characteristics of the PDE are the straight line graphs of $bx - ay = k$ and we make the transformation with

$$\xi = x, \eta = bx - ay.$$

Using the substitution we find the equation transforms to

$$\frac{\partial w}{\partial \xi} + \frac{c}{a} w = 0.$$

The integrating factor method gives

$$\frac{\partial}{\partial \xi}\left(e^{c\xi/a} w\right) = 0$$

and integrating with respect to ξ gives

$$e^{c\xi/a} w = g(\eta),$$

where g is any differentiable function of one variable. Then

$$w = g(\eta)\, e^{-c\xi/a}$$

and in terms of x and y we back transform

$$u(x,y) = g(bx - ay)\, e^{-cx/a}.$$

Exercise: Verify the solution by substituting back into the PDE.

Note: Consider the difference between general solution for linear ODEs and general solution for linear PDEs. For ODEs, the general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = p(x)$$

contains an arbitrary constant of integration. For different constants you get different curves of solution in (x-y)-plane. To pick out a unique solution you use some initial condition (say $y(x_0) = y_0$) to specify the constant.
For PDEs, if u is the general solution to equation (2.1), then $z = u(x,y)$ defines a family of integral surfaces in 3D-space, each surface corresponding to a choice of arbitrary function g in (2.6). We need some kind of information to pick out a unique solution; i.e., to chose the arbitrary function g.

2.1.3 Characteristic Curves

We investigate the significance of characteristics which, defined by the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y)}{a(x,y)},$$

represent a one parameter family of curves whose tangent at each point is in the direction of the vector $\mathbf{e} = (a,b)$. (Note that the left-hand side of equation (2.2) is the derivation of u in the direction of the vector $\mathbf{e}$, $\mathbf{e}\cdot\nabla u$.) Their parametric representation is $(x = x(s), y = y(s))$ where $x(s)$ and $y(s)$ satisfy the pair of ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x,y), \quad \frac{\mathrm{d}y}{\mathrm{d}s} = b(x,y). \tag{2.7}$$

The variation of u with respect $x = \xi$ along these characteristic curves is given by

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\partial u}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + \frac{b}{a}\frac{\partial u}{\partial y},\\ &= \frac{\kappa(x,y,u)}{a(x,y)} \quad \text{from equation (2.2),}\end{aligned}$$

such that, in term of the curvilinear coordinate s, the variation of u along the curves becomes

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \kappa(x,y,u).$$

The one parameter family of characteristic curves is parameterised by η (each value of η represents one unique characteristic). The solution of equation (2.2) reduces to the solution of the family of ODEs

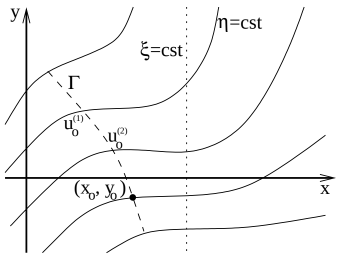
$$\frac{\mathrm{d}u}{\mathrm{d}s} = \kappa(x,y,u) \quad \left(\text{or similarly} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{\kappa(x,y,u)}{a(x,y)}\right) \tag{2.8}$$

along each characteristics (i.e. for each value of η).
Characteristic equations (2.7) have to be solved together with equation (2.8), called the compatibility equation, to find a solution to semilinear equation (2.2).

Cauchy Problem: Consider a curve Γ in (x, y)-plane whose parametric form is $(x = x_0(\sigma), y = y_0(\sigma))$. The Cauchy problem is to determine a solution of the equation

$$F(x, y, u, \partial_x u, \partial_y u) = 0$$

in a neighbourhood of Γ such that u takes prescribed values $u_0(\sigma)$ called Cauchy data on Γ.



Notes:

1. u can only be found in the region between the characteristics drawn through the endpoint of Γ.
2. Characteristics are curves on which the values of u combined with the equation are not sufficient to determine the normal derivative of u.
3. A discontinuity in the initial data propagates onto the solution along the characteristics. These are curves across which the derivatives of u can jump while u itself remains continuous.

Existence & Uniqueness: Why do some choices of Γ in (x, y)-space give a solution and other give no solution or an infinite number of solutions? It is due to the fact that the Cauchy data (initial conditions) may be prescribed on a curve Γ which is a characteristic of the PDE. To understand the definition of characteristics in the context of existence and uniqueness of solution, return to the general solution (2.6) of the linear PDE:

$$u(x, y) = e^{\alpha(x,y)} \left[\beta(x, y) + g(\eta(x, y))\right].$$

Consider the Cauchy data, u_0, prescribed along the curve Γ whose parametric form is $(x = x_0(\sigma), y = y_0(\sigma))$ and suppose $u_0(x_0(\sigma), y_0(\sigma)) = q(\sigma)$. If Γ is not a characteristic, the problem is well-posed and there is a unique function g which satisfies the condition

$$q(\sigma) = e^{\alpha(x_0(\sigma),y_0(\sigma))} \left[\beta(x_0(\sigma), y_0(\sigma)) + g(x_0(\sigma), y_0(\sigma))\right].$$

If on the other hand $(x = x_0(\sigma), y = y_0(\sigma))$ is the parametrisation of a characteristic $(\eta(x, y) = k$, say), the relation between the initial conditions q and g becomes

$$q(\sigma) = e^{\alpha(x_0(\sigma),y_0(\sigma))} \left[\beta(x_0(\sigma), y_0(\sigma)) + G\right], \tag{2.9}$$

where $G = g(k)$ is a constant; the problem is ill-posed. The functions $\alpha(x, y)$ and $\beta(x, y)$ are determined by the PDE, so equation (2.9) places a constraint on the given data function $q(x)$. If $q(\sigma)$ is not of this form for any constant G, then there is no solution taking on these prescribed values on Γ. On the other hand, if $q(\sigma)$ is of this form for some G, then there are infinitely many such solutions, because we can choose for g any differentiable function so that $g(k) = G$.

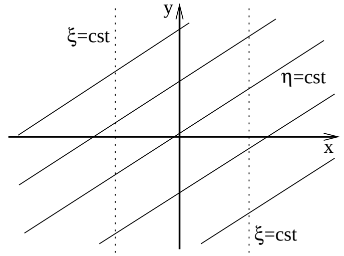
Example 1: Consider

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} + 8u = 0.$$

The characteristic equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}$$

and the characteristics are the straight line graphs $3x - 2y = c$. Hence we take $\eta = 3x - 2y$ and $\xi = x$.



(We can see that an η and ξ cross only once they are independent, i.e. $J \neq 0$; η and ξ have been properly chosen.)

This gives the solution

$$u(x, y) = e^{-4x}\, g(3x - 2y)$$

where g is a differentiable function defined over the real line. Simply specifying the solution at a given point (as in ODEs) does not uniquely determine g; we need to take a curve of initial conditions.
Suppose we specify values of $u(x, y)$ along a curve Γ in the plane. For example, let's choose Γ as the x-axis and gives values of $u(x, y)$ at points on Γ, say

$$u(x, 0) = \sin(x).$$

Then we need

$$u(x, 0) = e^{-4x}\, g(3x) = \sin(x) \quad \text{i.e.} \quad g(3x) = \sin(x) e^{4x},$$

and putting $t = 3x$,

$$g(t) = \sin(t/3)\, \mathrm{e}^{4t/3}.$$

This determines g and the solution satisfying the condition $u(x, 0) = \sin(x)$ on Γ is

$$u(x, y) = \sin(x - 2y/3)\, e^{-8y/3}.$$

We have determined the unique solution of the PDE with u specified along the x-axis.
We do not have to choose an axis — say, along $x = y$, $u(x, y) = u(x, x) = x^4$. From the general solution this requires,

$$u(x, x) = e^{-4x}\, g(x) = x^4, \quad \text{so} \quad g(x) = x^4\, e^{4x}$$

to give the unique solution

$$u(x, y) = (3x - 2y)^4 e^{8(x-y)}$$

satisfying $u(x, x) = x^4$.

However, not every curve in the plane can be used to determine g. Suppose we choose Γ to be the line $3x - 2y = 1$ and prescribe values of u along this line, say

$$u(x, y) = u(x, (3x - 1)/2) = x^2.$$

Now we must choose g so that

$$e^{-4x}\, g(3x - (3x - 1)) = x^2.$$

This requires $g(1) = x^2\, e^{4x}$ (for all x). This is impossible and hence there is no solution taking the value x^2 at points (x, y) on this line.
Last, we consider again Γ to be the line $3x - 2y = 1$ but choose values of u along this line to be

$$u(x, y) = u(x, (3x - 1)/2) = e^{-4x}.$$

Now we must choose g so that

$$e^{-4x}\, g(3x - (3x - 1)) = e^{-4x}.$$

This requires $g(1) = 1$, condition satisfied by an infinite number of functions and hence there is an infinite number of solutions taking the values e^{-4x} on the line $3x - 2y = 1$.
Depending on the initial conditions, the PDE has one unique solution, no solution at all or an infinite number or solutions. The difference is that the x-axis and the line $y = x$ are not the characteristics of the PDE while the line $3x - 2y = 1$ is a characteristic.

Example 2:

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u \quad \text{with} \quad u = x^2 \quad \text{on} \quad y = x, 1 \le y \le 2$$

Characteristics:

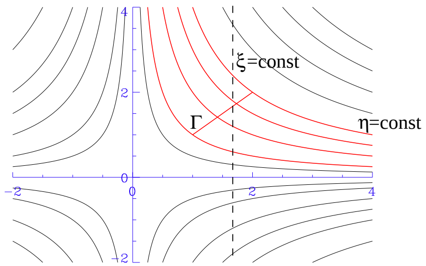
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} \Rightarrow \mathrm{d}(xy) = 0 \Rightarrow xy = c, \quad \text{constant.}$$

So, take $\eta = xy$ and $\xi = x$. Then the equation becomes

$$xy\frac{\partial w}{\partial \eta} + x\frac{\partial w}{\partial \xi} - xy\frac{\partial w}{\partial \eta} = w \quad \Rightarrow \quad \xi\frac{\partial w}{\partial \xi} - w = 0 \quad \Rightarrow \quad \frac{\partial}{\partial \xi}\frac{w}{\xi} = 0.$$

Finally the general solution is, $w = \xi\, g(\eta)$ or equivalently $u(x, y) = x\, g(xy)$. When $y = x$ with $1 \le y \le 2$, $u = x^2$; so $x^2 = x\, g(x^2) \Rightarrow g(x) = \sqrt{x}$ and the solution is

$$u(x, y) = x\,\sqrt{xy}.$$



This figure presents the characteristic curves given by $xy =$ constant. The red characteristics show the domain where the initial conditions permit us to determine the solution.

Alternative approach to solving example 2:

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u \quad \text{with} \quad u = x^2 \quad \text{on} \quad y = x, 1 \le y \le 2$$

This method is not suitable for finding general solutions but it works for Cauchy problems. The idea is to integrate directly the characteristic and compatibility equations in curvilinear coordinates. (See also "alternative method for solving the characteristic equations" for quasilinear equations hereafter.)
The solution of the characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = -y$$

gives the parametric form of the characteristic curves, while the integration of the compatibility equation

$$\frac{\mathrm{d}u}{\mathrm{d}s} = u$$

gives the solution $u(s)$ along these characteristics curves.
The solution of the characteristic equations is

$$x = c_1\,\mathrm{e}^s \quad \text{and} \quad y = c_2\,\mathrm{e}^{-s},$$

where the parametric form of the data curve Γ permits us to find the two constants of integration c_1 & c_2 in terms of the curvilinear coordinate along Γ.
The curve Γ is described by

$$x_0(\theta) = \theta \quad \text{and} \quad y_0(\theta) = \theta \quad \text{with} \quad \theta \in [2, 1]$$

and we consider the points on Γ to be the origin of the coordinate s along the characteristics (i.e. $s = 0$ on Γ). So,

$$\text{on } \Gamma\ (s = 0) \quad \left| \begin{array}{l} x_0 = \theta = c_1 \\ y_0 = \theta = c_2 \end{array} \right\} \Rightarrow \left| \begin{array}{l} x(s, \theta) = \theta\,\mathrm{e}^s \\ y(s, \theta) = \theta\,\mathrm{e}^{-s} \end{array} \right\}, \quad \forall \theta \in [0, 1].$$

For linear or semilinear problems we can solve the compatibility equation independently of the characteristic equations. (This property is not true for quasilinear equations.) Along the characteristics u is determined by

$$\frac{\mathrm{d}u}{\mathrm{d}s} = u \Rightarrow u = c_3\,\mathrm{e}^s.$$

Now we can make use of the Cauchy data to determine the constant of integration c_3,

$$\text{on } \Gamma, \text{ at } s = 0, \quad u_0(x_0(\theta), y_0(\theta)) \equiv u_0(\theta) = \theta^2 = c_3.$$

Then, we have the parametric forms of the characteristic curves and the solution

$$x(s, \theta) = \theta\,\mathrm{e}^s, \quad y(s, \theta) = \theta\,\mathrm{e}^{-s} \quad \text{and} \quad u(s, \theta) = \theta^2\,\mathrm{e}^s,$$

in terms of two parameters, s the curvilinear coordinate along the characteristic curves and θ the curvilinear coordinate along the data curve Γ. From the two first ones we get s and θ in terms of x and y.

$$\frac{x}{y} = \mathrm{e}^{2s} \Rightarrow s = \ln\sqrt{\frac{x}{y}} \quad \text{and} \quad xy = \theta^2 \Rightarrow \theta = \sqrt{xy} \quad (\theta \ge 0).$$

Then, we substitute s and θ in $u(s, \theta)$ to find

$$u(x, y) = xy\,\exp\left(\ln\sqrt{\frac{x}{y}}\right) = xy\sqrt{\frac{x}{y}} = x\sqrt{xy}.$$

2.2 Quasilinear Equations

Consider the first order quasilinear PDE

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \tag{2.10}$$

where the functions a, b and c can involve u but not its derivatives.

2.2.1 Interpretation of Quasilinear Equation

We can represent the solutions $u(x,y)$ by the integral surfaces of the PDE, $z = u(x,y)$, in (x,y,z)-space. Define the Monge direction by the vector (a,b,c) and recall that the normal to the integral surface is $(\partial_x u, \partial_y u, -1)$. Thus quasilinear equation (2.10) says that the normal to the integral surface is perpendicular to the Monge direction; i.e. integral surfaces are surfaces that at each point are tangent to the Monge direction,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} \partial_x u \\ \partial_y u \\ -1 \end{pmatrix} = a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} - c(x,y,u) = 0.$$

With the field of Monge direction, with direction numbers (a,b,c), we can associate the family of Monge curves which at each point are tangent to that direction field. These are defined by

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c\,\mathrm{d}y - b\,\mathrm{d}z \\ a\,\mathrm{d}z - c\,\mathrm{d}x \\ b\,\mathrm{d}x - a\,\mathrm{d}y \end{pmatrix} = 0 \Leftrightarrow \frac{\mathrm{d}x}{a(x,y,u)} = \frac{\mathrm{d}y}{b(x,y,u)} = \frac{\mathrm{d}z}{c(x,y,u)} (= \mathrm{d}s),$$

where $\mathrm{d}\mathbf{l} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ is an arbitrary infinitesimal vector parallel to the Monge direction. In the linear case, characteristics were curves in the (x,y)-plane (see § 2.1.3). For the quasilinear equation, we consider Monge curves in (x,y,u)-space defined by

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x,y,u),$$
$$\frac{\mathrm{d}y}{\mathrm{d}s} = b(x,y,u),$$
$$\frac{\mathrm{d}u}{\mathrm{d}s} = c(x,y,u).$$

Characteristic equations $(\mathrm{d}\{x,y\}/\mathrm{d}s)$ and compatibility equation $(\mathrm{d}u/\mathrm{d}s)$ are simultaneous first order ODEs in terms of a dummy variable s (curvilinear coordinate along the characteristics); we cannot solve the characteristic equations and compatibility equation independently as it is for a semilinear equation. Note that, in cases where $c \equiv 0$, the solution remains constant on the characteristics.
The rough idea in solving the PDE is thus to build up the integral surface from the Monge curves, obtained by solution of the ODEs.

Note that we make the difference between Monge curve or direction in (x,y,z)-space and characteristic curve or direction, their projections in (x,y)-space.

2.2.2 General solution:

Suppose that the characteristic and compatibility equations that we have defined have two independent first integrals (function, $f(x,y,u)$, constant along the Monge curves)

$$\phi(x,y,u) = c_1 \quad \text{and} \quad \psi(x,y,u) = c_2.$$

Then the solution of equation (2.10) satisfies $F(\phi,\psi) = 0$ for some arbitrary function F (equivalently, $\phi = G(\psi)$ for some arbitrary G), where the form of F (or G) depends on the initial conditions.

Proof: Since ϕ and ψ are first integrals of the equation,

$$\begin{aligned}\phi(x,y,u) &= \phi(x(s), y(s), u(s)), \\ &= \phi(s) = c_1.\end{aligned}$$

We have the chain rule

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{\partial\phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial\phi}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} = 0,$$

and then from the characteristic equations

$$a\frac{\partial\phi}{\partial x} + b\frac{\partial\phi}{\partial y} + c\frac{\partial\phi}{\partial u} = 0.$$

And similarly for ψ

$$a\frac{\partial\psi}{\partial x} + b\frac{\partial\psi}{\partial y} + c\frac{\partial\psi}{\partial u} = 0.$$

Solving for c gives

$$a\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial u} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial u}\right) + b\left(\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial u} - \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial u}\right) = 0$$

$$\text{or} \quad a\,J[u,x] = b\,J[y,u] \quad \text{where} \quad J[x_1,x_2] = \begin{vmatrix} \frac{\partial\phi}{\partial x_1} & \frac{\partial\phi}{\partial x_2} \\ \frac{\partial\psi}{\partial x_1} & \frac{\partial\psi}{\partial x_2} \end{vmatrix}.$$

And similarly, solving for a,

$$b\,J[x,y] = c\,J[u,x].$$

Thus, we have $J[u,x] = J[x,y]\,b/c$ and $J[y,u] = J[u,x]\,a/b = J[x,y]\,a/c$.
Now consider $F(\phi,\psi) = 0$ — remember $F(\phi(x,y,u(x,y)), \psi(x,y,u(x,y)))$ — and differentiate

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y = 0$$

Then, the derivative with respect to x is zero,

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial\phi}\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}\right) + \frac{\partial F}{\partial\psi}\left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x}\right) = 0,$$

as well as the derivative with respect to y

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial\phi}\left(\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}\right) + \frac{\partial F}{\partial\psi}\left(\frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial u}\frac{\partial u}{\partial y}\right) = 0.$$

For a non-trivial solution of this we must have

$$\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}\right)\left(\frac{\partial\psi}{\partial y}+\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial y}\right)-\left(\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}\right)=0,$$
$$\Rightarrow\left(\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial u}-\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial u}\right)\frac{\partial u}{\partial x}+\left(\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial u}\frac{\partial\phi}{\partial x}\right)\frac{\partial u}{\partial y}=\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y},$$
$$\Rightarrow J[y,u]\,\frac{\partial u}{\partial x}+J[u,x]\,\frac{\partial u}{\partial y}=J[x,y].$$

Then from the previous expressions for a, b, and c

$$a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=c,$$

i.e., $F(\phi,\psi)=0$ defines a solution of the original equation.

Example 1:

$$(y+u)\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x-y \quad \text{in} \quad y>0, -\infty<x<\infty,$$
$$\text{with} \quad u=1+x \quad \text{on} \quad y=1.$$

We first look for the general solution of the PDE before applying the initial conditions. Combining the characteristic and compatibility equations,

$$\frac{\mathrm{d}x}{\mathrm{d}s}=y+u, \tag{2.11}$$
$$\frac{\mathrm{d}y}{\mathrm{d}s}=y, \tag{2.12}$$
$$\frac{\mathrm{d}u}{\mathrm{d}s}=x-y \tag{2.13}$$

we seek two independent first integrals. Equations (2.11) and (2.13) give

$$\frac{\mathrm{d}}{\mathrm{d}s}(x+u)=x+u,$$

and equation (2.12)

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}s}=1.$$

Now, consider

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x+u}{y}\right)=\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}s}(x+u)-\frac{x+u}{y^2}\frac{\mathrm{d}y}{\mathrm{d}s},$$
$$=\frac{x+u}{y}-\frac{x+u}{y}=0.$$

So, $(x+u)/y=c_1$ is constant. This defines a family of solutions of the PDE; so, we can choose

$$\phi(x,y,u)=\frac{x+u}{y},$$

such that $\phi=c_1$ determines one particular family of solutions. Also, equations (2.11) and (2.12) give

$$\frac{\mathrm{d}}{\mathrm{d}s}(x-y)=u,$$

and equation (2.13)

$$(x-y)\,\frac{\mathrm{d}}{\mathrm{d}s}(x-y)=u\frac{\mathrm{d}u}{\mathrm{d}s}.$$

Now, consider

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[(x-y)^2-u^2\right]=\frac{\mathrm{d}}{\mathrm{d}s}\left[(x-y)^2\right]-\frac{\mathrm{d}}{\mathrm{d}s}\left(u^2\right),$$
$$=2(x-y)\,\frac{\mathrm{d}}{\mathrm{d}s}(x-y)-2u\,\frac{\mathrm{d}u}{\mathrm{d}s}=0.$$

Then, $(x-y)^2-u^2=c_2$ is constant and defines another family of solutions of the PDE. So, we can take

$$\psi(x,y,u)=(x-y)^2-u^2.$$

The general solution is

$$F\left(\frac{x+u}{y},(x-y)^2-u^2\right)=0 \quad \text{or} \quad (x-y)^2-u^2=G\left(\frac{x+u}{y}\right),$$

for some arbitrary functions F or G.
Now to get a particular solution, apply initial conditions ($u=1+x$ when $y=1$)

$$(x-1)^2-(x+1)^2=G(2x+1)\Rightarrow G(2x+1)=-4x.$$

Substitute $\theta=2x+1$, i.e. $x=(\theta-1)/2$, so $G(\theta)=2(1-\theta)$. Hence,

$$(x-y)^2-u^2=2\left(1-\frac{x+u}{y}\right)=\frac{2}{y}(y-x-u).$$

We can regard this as a quadratic equation for u:

$$u^2-\frac{2}{y}u-\left[2\frac{x-y}{y}+(x-y)^2\right]=0,$$
$$u^2-\frac{2}{y}u-\left(x-y+\frac{1}{y}\right)^2+\frac{1}{y^2}=0.$$

Then,

$$u_\pm=\frac{1}{y}\pm\sqrt{\frac{1}{y^2}+\left(x-y+\frac{1}{y}\right)^2-\frac{1}{y^2}}=\frac{1}{y}\pm\left(x-y+\frac{1}{y}\right).$$

Consider again the initial condition $u=1+x$ on $y=1$

$$u_\pm(x,y=1)=1\pm(x-1+1)=1\pm x\Rightarrow \text{take the positive root.}$$

Hence,

$$u(x,y)=x-y+\frac{2}{y}.$$

Example 2: using the same procedure solve

$$x(y-u)\frac{\partial u}{\partial x}+y(x+u)\frac{\partial u}{\partial y}=(x+y)u \quad \text{with} \quad u=x^2+1 \quad \text{on} \quad y=x.$$

Characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}s}=x(y-u), \tag{2.14}$$
$$\frac{\mathrm{d}y}{\mathrm{d}s}=y(x+u), \tag{2.15}$$
$$\frac{\mathrm{d}u}{\mathrm{d}s}=(x+y)u. \tag{2.16}$$

Again, we seek to independent first integrals. On the one hand, equations (2.14) and (2.15) give

$$y\frac{\mathrm{d}x}{\mathrm{d}s}+x\frac{\mathrm{d}y}{\mathrm{d}s}=xy^2-xyu+yx^2+xyu=xy\,(x+y),$$
$$=xy\,\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}s} \quad \text{from equation (2.16).}$$

Now, consider

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}s}=\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}s} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}s}\ln\left(\frac{xy}{u}\right)=0.$$

Hence, $xy/u=c_1$ is constant and

$$\phi(x,y,u)=\frac{xy}{u}$$

is a first integral of the PDE. On the other hand,

$$\frac{\mathrm{d}x}{\mathrm{d}s}-\frac{\mathrm{d}y}{\mathrm{d}s}=xy-xu-xy-yu=-u(x+y)=-\frac{\mathrm{d}u}{\mathrm{d}s},$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}s}(x+u-y)=0.$$

Hence, $x+u-y=c_2$ is also a constant on the Monge curves and another first integral is given by

$$\psi(x,y,u)=x+u-y,$$

so the general solution is

$$\frac{xy}{u}=G(x+u-y).$$

Now, we make use of the initial conditions, $u=x^2+1$ on $y=x$, to determine G:

$$\frac{x^2}{1+x^2}=G(x^2+1);$$

set $\theta=x^2+1$, i.e. $x^2=\theta-1$, then

$$G(\theta)=\frac{\theta-1}{\theta},$$

and finally the solution is

$$\frac{xy}{u}=\frac{x+u-y-1}{x+u-y}. \quad \text{Rearrange to finish!}$$

Alternative approach: Solving the characteristic equations. Illustration by an example,

$$x^2\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=1, \quad \text{with} \quad u=0 \quad \text{on} \quad x+y=1.$$

The characteristic equations are

$$\frac{\mathrm{d}x}{\mathrm{d}s}=x^2, \quad \frac{\mathrm{d}y}{\mathrm{d}s}=u \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s}=1,$$

which we can solve to get

$$x=\frac{1}{c_1-s}, \tag{2.17}$$
$$y=\frac{s^2}{2}+c_2\,s+c_3, \tag{2.18}$$
$$u=c_2+s, \quad \text{for constants} \quad c_1,c_2,c_3. \tag{2.19}$$

We now parameterise the initial line in terms of θ:

$$\theta=x, y=1-\theta,$$

and apply the initial data at $s=0$. Hence,

$$\begin{aligned} &(2.17) \quad \text{gives} \quad \theta=\frac{1}{c_1}\Rightarrow c_1=\frac{1}{\theta},\\ &(2.18) \quad \text{gives} \quad 1-\theta=c_3\Rightarrow c_3=1-\theta,\\ &(2.19) \quad \text{gives} \quad 0=c_2\Rightarrow c_2=0. \end{aligned}$$

Hence, we found the parametric form of the surface integral,

$$x=\frac{\theta}{1-s\theta}, \quad y=\frac{s^2}{2}+1-\theta \quad \text{and} \quad u=s.$$

Eliminate s and θ,

$$x=\frac{\theta}{1-s\theta}\Rightarrow\theta=\frac{x}{1+sx},$$

then

$$y=\frac{u^2}{2}+1-\frac{x}{1+ux}.$$

Invariants, or first integrals, are (from solution (2.17), (2.18) and (2.19), keeping arbitrary $c_2=0$) $\phi=u^2/2-y$ and $\psi=x/(1+ux)$.

Alternative approach to example 1:

$$(y+u)\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x-y \quad \text{in} \quad y>0, -\infty<x<\infty,$$
$$\text{with} \quad u=1+x \quad \text{on} \quad y=1.$$

Characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}s}=y+u, \tag{2.20}$$
$$\frac{\mathrm{d}y}{\mathrm{d}s}=y, \tag{2.21}$$
$$\frac{\mathrm{d}u}{\mathrm{d}s}=x-y. \tag{2.22}$$

Solve with respect to the dummy variable s; (2.21) gives,

$$y = c_1\, e^s,$$

(2.20) and (2.22) give

$$\frac{\mathrm{d}}{\mathrm{d}s}(x+u) = x+u \Rightarrow x+u = c_2\, e^s,$$

and (2.20) give

$$\frac{\mathrm{d}x}{\mathrm{d}s} = c_1\, e^s + c_2\, e^s - x,$$

$$\text{so,}\quad x = c_3\, e^{-s} + \frac{1}{2}(c_1+c_2)\, e^s \quad\text{and}\quad u = -c_3\, e^{-s} + \frac{1}{2}(c_2-c_1)\, e^s.$$

Now, at $s=0$, $y=1$ and $x=\theta$, $u=1+\theta$ (parameterising initial line Γ),

$$c_1 = 1,\quad c_2 = 1+2\theta \quad\text{and}\quad c_3 = -1.$$

Hence, the parametric form of the surface integral is,

$$x = -\,e^{-s} + (1+\theta)\, e^s,\quad y = e^s \quad\text{and}\quad u = e^{-s} + \theta\, e^s.$$

Then eliminate θ and s:

$$x = -\frac{1}{y} + (1+\theta)\, y \Rightarrow \theta = \frac{1}{y}\left(x - y + \frac{1}{y}\right),$$

so

$$u = \frac{1}{y} + \frac{1}{y}\left(x - y + \frac{1}{y}\right) y.$$

Finally,

$$u = x - y + \frac{2}{y},\quad \text{as before.}$$

To find invariants, return to solved characteristics equations and solve for constants in terms of x, y and u. We only need two, so put for instance $c_1 = 1$ and so $y = e^s$. Then,

$$x = \frac{c_3}{y} + \frac{1}{2}(1+c_2)\, y \quad\text{and}\quad u = -\frac{c_3}{y} + \frac{1}{2}(c_2-1)\, y.$$

Solve for c_2

$$c_2 = \frac{x+u}{y},\quad\text{so}\quad \phi = \frac{x+u}{y},$$

and solve for c_3

$$c_3 = \frac{1}{2}(x-u-y)y,\quad\text{so}\quad \psi = (x-u-y)y.$$

Observe ψ is different from last time, but this does not as we only require two independent choices for ϕ and ψ. In fact we can show that our previous ψ is also constant,

$$\begin{aligned}(x-y)^2 - u^2 &= (x-y+u)(x-y-u),\\ &= (\phi y - y)\frac{\psi}{y},\\ &= (\phi-1)\psi \quad\text{which is also constant.}\end{aligned}$$

Summary: Solving the characteristic equations — two approaches.

1. Manipulate the equations to get them in a 'directly integrable' form, e.g.

$$\frac{1}{x+u}\frac{\mathrm{d}}{\mathrm{d}s}(x+u) = 1$$

and find some combination of the variables which differentiates to zero (first integral), e.g.

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x+u}{y}\right) = 0.$$

2. Solve the equations with respect to the dummy variable s, and apply the initial data (parameterised by θ) at $s=0$. Eliminate θ and s; find invariants by solving for constants.

2.3 Wave Equation

We consider the equation

$$\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} = 0 \quad\text{with}\quad u(0,x) = f(x),$$

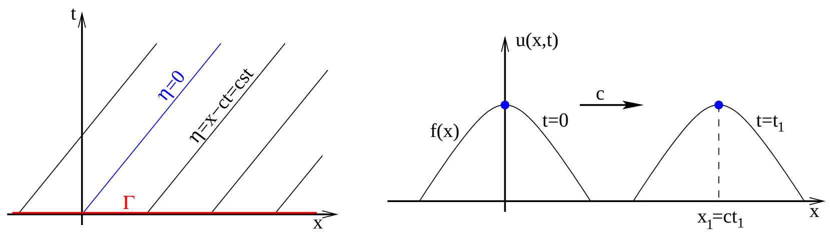
where c is some positive constant.

2.3.1 Linear Waves

If u is small (i.e. $u^2 \ll u$), then the equation approximate to the linear wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad\text{with}\quad u(x,0) = f(x).$$

The solution of the equation of characteristics, $\mathrm{d}x/\mathrm{d}t = c$, gives the first integral of the PDE, $\eta(x,t) = x - ct$, and then general solution $u(x,t) = g(x-ct)$, where the function g is determined by the initial conditions. Applying $u(x,0) = f(x)$ we find that the linear wave equation has the solution $u(x,t) = f(x-ct)$, which represents a wave (unchanging shape) propagating with constant wave speed c.



Note that u is constant where $x - ct$ =constant, i.e. on the characteristics.

2.3.2 Nonlinear Waves

For the nonlinear equation,

$$\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} = 0,$$

the characteristics are defined by

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = c + u \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s} = 0,$$

which we can solve to give two independent first integrals $\phi = u$ and $\psi = x - (u+c)t$. So,

$$u = f[x - (u+c)t],$$

according to initial conditions $u(x,0) = f(x)$. This is similar to the previous result, but now the "wave speed" involves u.

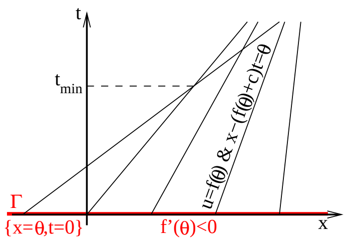
However, this form of the solution is not very helpful; it is more instructive to consider the characteristic curves. (The PDE is homogeneous, so the solution u is constant along the Monge curves — this is not the case in general — which can then be reduced to their projections in the (x,t)-plane.) By definition, $\psi = x-(c+u)t$ is constant on the characteristics (as well as u); differentiate ψ to find that the characteristics are described by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + c.$$

These are straight lines,

$$x = (f(\theta) + c)t + \theta,$$

expressed in terms of a parameter θ. (If we make use of the parametric form of the data curve Γ: $\{x = \theta, t = 0, \theta \in \mathbb{R}\}$ and solve directly the Cauchy problem in terms of the coordinate $s = t$, we similarly find, $u = f(\theta)$ and $x = (u+c)t + \theta$.) The slope of the characteristics, $1/(c+u)$, varies from one line to another, and so, two curves can intersect.



Consider two crossing characteristics expressed in terms of θ_1 and θ_2,

$$\text{i.e.} \quad x = (f(\theta_1) + c)t + \theta_1,$$
$$x = (f(\theta_2) + c)t + \theta_2.$$

(These correspond to initial values given at $x = \theta_1$ and $x = \theta_2$.) These characteristics intersect at the time

$$t = -\frac{\theta_1 - \theta_2}{f(\theta_1) - f(\theta_2)},$$

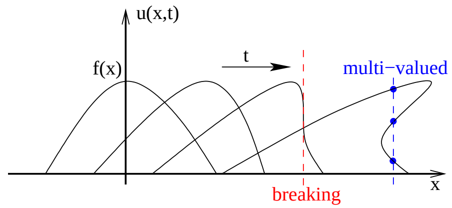
and if this is positive it will be in the region of solution. At this point u will not be single-valued and the solution breaks down. By letting $\theta_2 \to \theta_1$ we can see that the characteristics intersect at

$$t = -\frac{1}{f'(\theta)},$$

and the minimum time for which the solution becomes multi-valued is

$$t_{\min} = \frac{1}{\max[-f'(\theta)]},$$

i.e. the solution is single valued (i.e. is physical) only for $0 \le t < t_{\min}$. Hence, when $f'(\theta) < 0$ we can expect the solution to exist only for a finite time. In physical terms, the equation considered is purely advective; in real waves, such as shock waves in gases, when very large gradients are formed then diffusive terms (e.g. $\partial_{xx}u$) become vitally important.



To illustrate finite time solutions of the nonlinear wave equation, consider

$$f(\theta) = \theta(1-\theta), \quad (0 \le \theta \le 1),$$
$$f'(\theta) = 1 - 2\theta.$$

So, $f'(\theta) < 0$ for $1/2 < \theta < 1$ and we can expect the solution not to remain single-valued for all values of t. ($\max[-f'(\theta)] = 1$ so $t_{\min} = 1$. Now,

$$u = f(x - (u+c)t),$$
$$\text{so} \quad u = [x - (u+c)t] \times [1 - x + (u+c)t], \quad (ct \le x \le 1 + ct),$$

which we can express as

$$t^2u^2 + (1 + t - 2xt + 2ct^2)u + (x^2 - x - 2ctx + ct + c^2t^2) = 0,$$

and solving for u (we take the positive root from initial data)

$$u = \frac{1}{2t^2}\left(2t(x-ct) - (1+t) + \sqrt{(1+t)^2 - 4t(x-ct)}\right).$$

Now, at $t = 1$,

$$u = x - (c+1) + \sqrt{1 + c - x},$$

so the solution becomes singular as $t \to 1$ and $x \to 1 + c$.

2.3.3 Weak Solution

When wave breaking occurs (multi-valued solutions) we must re-think the assumptions in our model. Consider again the nonlinear wave equation,

$$\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} = 0,$$

and put $w(x,t) = u(x,t) + c$; hence the PDE becomes the inviscid Burger's equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0,$$

or equivalently in a conservative form

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{w^2}{2}\right) = 0,$$

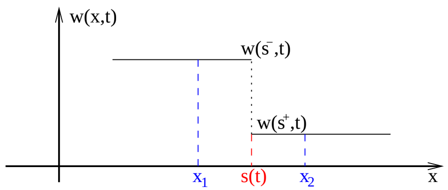
where $w^2/2$ is the flux function. We now consider its integral form,

$$\int_{x_1}^{x_2}\left[\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{w^2}{2}\right)\right]\mathrm{d}x = 0 \quad\Leftrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} w(x,t)\,\mathrm{d}x = -\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\frac{w^2}{2}\right)\mathrm{d}x$$

where $x_2 > x1$ are real. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} w(x,t)\,\mathrm{d}x = \frac{w^2(x_1,t)}{2} - \frac{w^2(x_2,t)}{2}.$$

Let us now relax the assumption regarding the differentiability of the our solution; suppose that w has a discontinuity in $x = s(t)$ with $x_1 < s(t) < x_2$.



Thus, splitting the interval $[x_1, x_2]$ in two parts, we have

$$\begin{aligned}\frac{w^2(x_1,t)}{2} - \frac{w^2(x_2,t)}{2} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{s(t)} w(x,t)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t)}^{x_2} w(x,t)\,\mathrm{d}x,\\ &= w(s^-,t)\,\dot{s}(t) + \int_{x_1}^{s(t)}\frac{\partial w}{\partial t}\,\mathrm{d}x - w(s^+,t)\,\dot{s}(t) + \int_{s(t)}^{x_2}\frac{\partial w}{\partial t}\,\mathrm{d}x,\end{aligned}$$

where $w(s^-(t),t)$ and $w(s^+(t),t)$ are the values of w as $x \to s$ from below and above respectively; $\dot{s} = \mathrm{d}s/\mathrm{d}t$.

Now, take the limit $x_1 \to s^-(t)$ and $x_2 \to s^+(t)$. Since $\partial w/\partial t$ is bounded, the two integrals tend to zero. We then have

$$\frac{w^2(s^-,t)}{2} - \frac{w^2(s^+,t)}{2} = \dot{s}\left(w(s^-,t) - w(s^+,t)\right).$$

The velocity of the discontinuity of the shock velocity $U = \dot{s}$. If [] indicates the jump across the shock then this condition may be written in the form

$$-U\,[w] = \left[\frac{w^2}{2}\right].$$

The shock velocity for Burger's equation is

$$U = \frac{1}{2}\frac{w^2(s^+) - w^2(s^-)}{w(s^+) - w(s^-)} = \frac{w(s^+) + w(s^-)}{2}.$$

The problem then reduces to fitting shock discontinuities into the solution in such a way that the jump condition is satisfied and multi-valued solution are avoided. A solution that satisfies the original equation in regions and which satisfies the integral form of the equation is called a weak solution or generalised solution.

Example: Consider the inviscid Burger's equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0,$$

with initial conditions

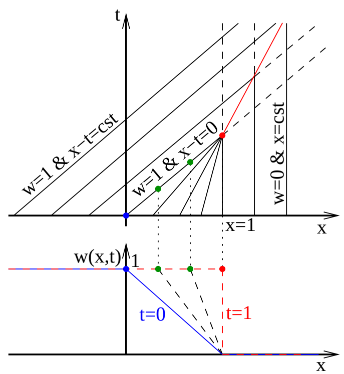
$$w(x=\theta, t=0) = f(\theta) = \begin{cases} 1 & \text{for} \quad \theta \le 0,\\ 1-\theta & \text{for} \quad 0 \le \theta \le 1,\\ 0 & \text{for} \quad \theta \ge 1.\end{cases}$$

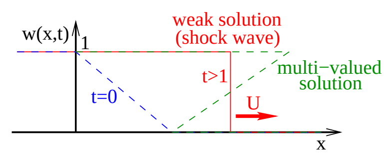
As seen before, the characteristics are curves on which $w = f(\theta)$ as well as $x - f(\theta)\,t = \theta$ are constant, where θ is the parameter of the parametric form of the curve of initial data, Γ. For all $\theta \in (0,1)$, $f'(\theta) = -1$ is negative ($f' = 0$ elsewhere), so we can expect that all the characteristics corresponding to these values of θ intersect at the same point; the solution of the inviscid Burger's equation becomes multi-valued at the time

$$t_{\min} = 1/\max[-f'(\theta)] = 1, \forall\theta \in (0,1).$$

Then, the position where the singularity develops at $t = 1$ is

$$x = -f(\theta)\,t + \theta = 1 - \theta + \theta = 1.$$





As time increases, the slope of the solution,

$$w(x,t) = \begin{cases} 1 & \text{for} \quad x \le t, \\ \dfrac{1-x}{1-t} & \text{for} \quad t \le x \le 1, \\ 0 & \text{for} \quad x \ge 1, \end{cases} \qquad \text{with} \quad 0 \le t < 1,$$

becomes steeper and steeper until it becomes vertical at $t = 1$; then the solution is multi-valued. Nevertheless, we can define a generalised solution, valid for all positive time, by introducting a shock wave.
Suppose shock at $s(t) = U\,t + \alpha$, with $w(s^-,t) = 1$ and $w(s^+,t) = 0$. The jump condition gives the shock velocity,

$$U = \frac{w(s^+) + w(s^-)}{2} = \frac{1}{2};$$

furthermore, the shock starts at $x = 1, t = 1$, so $\alpha = 1 - 1/2 = 1/2$. Hence, the weak solution of the problem is, for $t \ge 1$,

$$w(x,t) = \begin{cases} 0 & \text{for} \quad x < s(t), \\ 1 & \text{for} \quad x > s(t), \end{cases} \qquad \text{where} \quad s(t) = \frac{1}{2}(t+1).$$

2.4 Systems of Equations

2.4.1 Linear and Semilinear Equations

These are equations of the form

$$\sum_{j=1}^{n} \left(a_{ij}\, u_x^{(j)} + b_{ij}\, u_y^{(j)} \right) = c_i, \quad i = 1, 2, \dots, n \quad \left(\frac{\partial u}{\partial x} = u_x \right),$$

for the unknowns $u^{(1)}, u^{(j)}, \dots, u^{(n)}$ and when the coefficients a_{ij} and b_{ij} are functions only of x and y. (Though the c_i could also involve $u^{(k)}$.)
In matrix notation

$$A\mathbf{u}_x + B\mathbf{u}_y = \mathbf{c},$$

where

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \qquad B = (b_{ij}) = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(n)} \end{bmatrix}.$$

E.g.,

$$u_x^{(1)} - 2u_x^{(2)} + 3u_y^{(1)} - u_y^{(2)} = x + y,$$
$$u_x^{(1)} + u_x^{(2)} - 5u_y^{(1)} + 2u_y^{(2)} = x^2 + y^2,$$

can be expressed as

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_x^{(1)} \\ u_x^{(2)} \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} u_y^{(1)} \\ u_y^{(2)} \end{bmatrix} = \begin{bmatrix} x+y \\ x^2+y^2 \end{bmatrix},$$

or $A\mathbf{u}_x + B\mathbf{u}_y = \mathbf{c}$ where

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} x+y \\ x^2+y^2 \end{bmatrix}.$$

If we multiply by $A^{-1} \left(= \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} \right)$

$$A^{-1}A\,\mathbf{u}_x + A^{-1}B\,\mathbf{u}_y = A^{-1}\mathbf{c},$$

$$\text{we obtain} \quad \mathbf{u}_x + D\mathbf{u}_y = d,$$

where $D = A^{-1}B \left(= \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 7/3 & 1 \\ -8/3 & 1 \end{bmatrix} \right)$ and $d = A^{-1}c$.
We now assume that the matrix A is non-singular (i.e., the inverse A^{-1} exists) — at least there is some region of the (x, y)-plane where it is non-singular. Hence, we need only to consider systems of the form

$$\mathbf{u}_x + D\mathbf{u}_y = \mathbf{d}.$$

We also limit our attention to totally hyperbolic systems, i.e. systems where the matrix D has n distinct real eigenvalues (or at least there is some region of the plane where this holds).
D has the n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ where $\det(\lambda_i I - D) = 0$ $(i = 1, \dots, n)$, with $\lambda_i \ne \lambda_j$ $(i \ne j)$ and the n corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ so that

$$D\mathbf{e}_i = \lambda_i \mathbf{e}_i.$$

The matrix $P = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n]$ diagonalises D via $P^{-1}DP = \Lambda$,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & 0 & \lambda_{n-1} & 0 \\ 0 & \dots & \dots & 0 & \lambda_n \end{bmatrix}.$$

We now put $\mathbf{u} = P\mathbf{v}$,

$$\text{then} \quad P\mathbf{v}_x + P_x\mathbf{v} + DP\mathbf{v}_y + DP_y\mathbf{v} = \mathbf{d},$$
$$\text{and} \quad P^{-1}P\mathbf{v}_x + P^{-1}P_x\mathbf{v} + P^{-1}DP\mathbf{v}_y + P^{-1}DP_y\mathbf{v} = P^{-1}\mathbf{d},$$

which is of the form

$$\mathbf{v}_x + \Lambda\mathbf{v}_y = \mathbf{q},$$

where

$$\mathbf{q} = P^{-1}\mathbf{d} - P^{-1}P_x\mathbf{v} - P^{-1}DP_y\mathbf{v}.$$

The system is now of the form

$$v_x^{(i)} + \lambda_i\, v_y^{(i)} = q_i \quad (i = 1, \dots, n),$$

where q_i can involve $\{v^{(1)}, v^{(2)}, \dots, v^{(n)}\}$ and with n characteristics given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_i.$$

This is the canonical form of the equations.

Example 1: Consider the linear system

$$\left.\begin{aligned} u_x^{(1)} + 4\,u_y^{(2)} &= 0, \\ u_x^{(2)} + 9\,u_y^{(1)} &= 0, \end{aligned}\right\} \quad \text{with initial conditions } \mathbf{u} = [2x, 3x]^{\mathrm{T}} \text{ on } y = 0.$$

Here, $\mathbf{u}_x + D\mathbf{u}_y = 0$ with $D = \begin{bmatrix} 0 & 4 \\ 9 & 0 \end{bmatrix}$.

Eigenvalues:

$$\det(D - \lambda I) = 0 \Rightarrow \lambda^2 - 36 = 0 \Rightarrow \lambda = \pm 6.$$

Eigenvectors:

$$\begin{bmatrix} -6 & 4 \\ 9 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{for } \lambda = 6,$$

$$\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad \text{for } \lambda = -6.$$

Then,

$$P = \begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix}, \quad P^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \quad \text{and} \quad P^{-1}DP = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}.$$

So we put $\mathbf{u} = \begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix}\mathbf{v}$ and $\mathbf{v}_x + \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{v}_y = 0$, which has general solution

$$v^{(1)} = f(6x - y) \quad \text{and} \quad v^{(2)} = g(6x + y),$$

i.e.

$$u^{(1)} = 2v^{(1)} + 2v^{(2)} \quad \text{and} \quad u^{(1)} = 3v^{(1)} - 3v^{(2)}.$$

Initial conditions give

$$\begin{aligned} 2x &= 2f(6x) + 2g(6x), \\ 3x &= 3f(6x) - 3g(6x), \end{aligned}$$

so, $f(x) = x/6$ and $g(x) = 0$; then

$$\begin{aligned} u^{(1)} &= \frac{1}{3}(6x - y), \\ u^{(2)} &= \frac{1}{2}(6x - y). \end{aligned}$$

Example 2: Reduce the linear system

$$\mathbf{u}_x + \begin{bmatrix} 4y - x & 2x - 2y \\ 2y - 2x & 4x - y \end{bmatrix}\mathbf{u}_x = 0$$

to canonical form in the region of the (x, y)-space where it is totally hyperbolic.
Eigenvalues:

$$\det\begin{bmatrix} 4y - x - \lambda & 2x - 2y \\ 2y - 2x & 4x - y - \lambda \end{bmatrix} = 0 \Rightarrow \lambda \in \{3x, 3y\}.$$

The system is totally hyperbolic everywhere expect where $x = y$.

Eigenvalues:

$$\begin{aligned} \lambda_1 &= 3x \Rightarrow \mathbf{e}_1 = [1, 2]^{\mathrm{T}}, \\ \lambda_2 &= 3y \Rightarrow \mathbf{e}_2 = [2, 1]^{\mathrm{T}}. \end{aligned}$$

So,

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad P^{-1}DP = \begin{bmatrix} 3x & 0 \\ 0 & 3y \end{bmatrix}.$$

Then, with $\mathbf{u} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{v}$ we obtain

$$\mathbf{v}_x + \begin{bmatrix} 3x & 0 \\ 0 & 3y \end{bmatrix}\mathbf{v}_y = 0.$$

2.4.2 Quasilinear Equations

We consider systems of n equations, involving n functions $u^{(i)}(x, y)$ $(i = 1, \dots, n)$, of the form

$$\mathbf{u}_x + D\mathbf{u}_y = \mathbf{d},$$

where D as well as $\mathbf{d}$ may now depend on $\mathbf{u}$. (We have already shown how to reduce a more general system $A\mathbf{u}_x + B\mathbf{u}_y = \mathbf{c}$ to that simpler form.) Again, we limit our attention to totally hyperbolic systems; then

$$\Lambda = P^{-1}DP \iff D = P\Lambda P^{-1},$$

using the same definition of P, P^{-1} and the diagonal matrix Λ, as for the linear and semilinear cases. So, we can transform the system in its normal form as,

$$P^{-1}\mathbf{u}_x + \Lambda P^{-1}\mathbf{u}_y = P^{-1}\mathbf{d},$$

such that it can be written in component form as

$$\sum_{j=1}^{n} P_{ij}^{-1}\left(\frac{\partial}{\partial x}u^{(j)} + \lambda_i \frac{\partial}{\partial y}u^{(j)}\right) = \sum_{j=1}^{n} P_{ij}^{-1} d_j, \quad (i = 1, \dots, n)$$

where λ_i is the ith eigenvalue of the matrix D and wherein the ith equation involves differentiation only in a single direction — the direction $\mathrm{d}y/\mathrm{d}x = \lambda_i$. We define the ith characteristic, with curvilinear coordinate s_i, as the curve in the (x, y)-plane along which

$$\frac{\mathrm{d}x}{\mathrm{d}s_i} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}s_i} = \lambda_i \quad \text{or equivalently} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_i.$$

Hence, the directional derivative parallel to the characteristic is

$$\frac{\mathrm{d}}{\mathrm{d}s_i}u^{(j)} = \frac{\partial}{\partial x}u^{(j)} + \lambda_i \frac{\partial}{\partial y}u^{(j)},$$

and the system in normal form reduces to n ODEs involving different components of $\mathbf{u}$

$$\sum_{j=1}^{n} P_{ij}^{-1}\frac{\mathrm{d}}{\mathrm{d}s_i}u^{(j)} = \sum_{j=1}^{n} P_{ij}^{-1} d_j \quad (i = 1, \dots, n).$$

Example: Unsteady, one-dimensional motion of an inviscid compressible adiabatic gas. Consider the equation of motion (Euler equation)

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = -\frac{1}{\rho}\,\frac{\partial P}{\partial x},$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \rho\,\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0.$$

If the entropy is the same everywhere in the motion then $P\rho^{-\gamma} =$ constant, and the motion equation becomes

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\,\frac{\partial \rho}{\partial x} = 0,$$

where $c^2 = \mathrm{d}P/\mathrm{d}\rho = \gamma P/\rho$ is the sound speed. We have then a system of two first order quasilinear PDEs; we can write these as

$$\frac{\partial \mathbf{w}}{\partial t} + D\frac{\partial \mathbf{w}}{\partial x} = 0,$$

with

$$\mathbf{w} = \begin{bmatrix} u \\ \rho \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} u & c^2/\rho \\ \rho & u \end{bmatrix}.$$

The two characteristics of this hyperbolic system are given by $\mathrm{d}x/\mathrm{d}t = \lambda$ where λ are the eigenvalues of D;

$$\det(D - \lambda I) = \begin{vmatrix} u-\lambda & c^2/\rho \\ \rho & u-\lambda \end{vmatrix} = 0 \Rightarrow (u-\lambda)^2 = c^2 \quad \text{and} \quad \lambda_\pm = u \pm c.$$

The eigenvectors are $[c, -\rho]^T$ for λ_- and $[c, \rho]^T$ for λ_+, such that the usual matrices are

$$T = \begin{bmatrix} c & c \\ -\rho & \rho \end{bmatrix}, \quad T^{-1} = \frac{1}{2\rho c}\begin{bmatrix} \rho & -c \\ \rho & c \end{bmatrix}, \quad \text{such that} \quad \Lambda = T^{-1}DT = \begin{bmatrix} u-c & 0 \\ 0 & u+c \end{bmatrix}.$$

Put α and β the curvilinear coordinates along the characteristics $\mathrm{d}x/\mathrm{d}t = u - c$ and $\mathrm{d}x/\mathrm{d}t = u + c$ respectively; then the system transforms to the canonical form

$$\frac{\mathrm{d}t}{\mathrm{d}\alpha} = \frac{\mathrm{d}t}{\mathrm{d}\beta} = 1, \quad \frac{\mathrm{d}x}{\mathrm{d}\alpha} = u - c, \quad \frac{\mathrm{d}x}{\mathrm{d}\beta} = u + c, \quad \rho\frac{\mathrm{d}u}{\mathrm{d}\alpha} - c\frac{\mathrm{d}\rho}{\mathrm{d}\alpha} = 0 \quad \text{and} \quad \rho\frac{\mathrm{d}u}{\mathrm{d}\beta} + c\frac{\mathrm{d}\rho}{\mathrm{d}\beta} = 0.$$