Chapter 1

Introduction

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1.1 Motivation

Why do we study partial differential equations (PDEs) and in particular analytic solutions?
We are interested in PDEs because most of mathematical physics is described by such equations. For example, fluids dynamics (and more generally continuous media dynamics), electromagnetic theory, quantum mechanics, traffic flow. Typically, a given PDE will only be accessible to numerical solution (with one obvious exception — exam questions!) and analytic solutions in a practical or research scenario are often impossible. However, it is vital to understand the general theory in order to conduct a sensible investigation. For example, we may need to understand what type of PDE we have to ensure the numerical solution is valid. Indeed, certain types of equations need appropriate boundary conditions; without a knowledge of the general theory it is possible that the problem may be ill-posed of that the method is solution is erroneous.

1.2 Reminder

Partial derivatives: The differential (or differential form) of a function f of n independent variables, $(x_1, x_2, \dots, x_n)$, is a linear combination of the basis form $(\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_n)$

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \ldots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n,$$

where the partial derivatives are defined by

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, x_2, \dots, x_i + h, \dots, x_n) - f(x_1, x_2, \dots, x_i, \dots, x_n)}{h}.$$

The usual differentiation identities apply to the partial differentiations (sum, product, quotient, chain rules, etc.)

Notations: I shall use interchangeably the notations

$$\frac{\partial f}{\partial x_i} \equiv \partial_{x_i} f \equiv f_{x_i}, \qquad \frac{\partial^2 f}{\partial x_i \partial x_j} \equiv \partial^2_{x_i x_j} f \equiv f_{x_i x_j},$$

for the first order and second order partial derivatives respectively. We shall also use interchangeably the notations

$$\vec{u} \equiv \underline{u} \equiv \mathbf{u},$$

for vectors.

Vector differential operators: in three dimensional Cartesian coordinate system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ we consider $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ and $[u_x(x, y, z), u_y(x, y, z), u_z(x, y, z)] : \mathbb{R}^3 \to \mathbb{R}^3$.

Gradient: $\nabla f = \partial_x f\,\mathbf{i} + \partial_y f\,\mathbf{j} + \partial_z f\,\mathbf{k}$.

Divergence: $\operatorname{div}\mathbf{u} \equiv \nabla \cdot \mathbf{u} = \partial_x u_x + \partial_y u_y + \partial_z uz$.

Curl: $\nabla \times \mathbf{u} = (\partial_z u_y - \partial_y u_z)\,\mathbf{i} + (\partial_z u_x - \partial_x u_z)\,\mathbf{j} + (\partial_x u_y - \partial_y u_x)\,\mathbf{k}$.

Laplacian: $\Delta f \equiv \nabla^2 f = \partial^2_x f + \partial^2_y f + \partial^2_z f$.

Laplacian of a vector: $\Delta \mathbf{u} \equiv \nabla^2 \mathbf{u} = \nabla^2 u_x\,\mathbf{i} + \nabla^2 u_y\,\mathbf{j} + \nabla^2 u_z\,\mathbf{k}$.

Note that these operators are different in other systems of coordinate (cylindrical or spherical, say)

1.3 Definitions

A partial differential equation (PDE) is an equation for some quantity u (dependent variable) which depends on the independent variables $x_1, x_2, x_3, \dots, x_n,\ n \geq 2$, and involves derivatives of u with respect to at least some of the independent variables.

$$F(x_1, \dots, x_n, \partial_{x_1} u, \dots, \partial_{x_n} u, \partial^2_{x_1} u, \partial^2_{x_1 x_2} u, \dots, \partial^n_{x_1 \dots x_n} u) = 0.$$

Note:

1. In applications x_i are often space variables (e.g. x, y, z) and a solution may be required in some region Ω of space. In this case there will be some conditions to be satisfied on the boundary $\partial\Omega$; these are called boundary conditions (BCs).

2. Also in applications, one of the independent variables can be time (t say), then there will be some initial conditions (ICs) to be satisfied (i.e., u is given at $t = 0$ everywhere in Ω)

3. Again in applications, systems of PDEs can arise involving the dependent variables $u_1, u_2, u_3, \dots, u_m,\ m \geq 1$ with some (at least) of the equations involving more than one u_i.

The order of the PDE is the order of the highest (partial) differential coefficient in the equation.
As with ordinary differential equations (ODEs) it is important to be able to distinguish between linear and nonlinear equations.
A linear equation is one in which the equation and any boundary or initial conditions do not include any product of the dependent variables or their derivatives; an equation that is not linear is a nonlinear equation.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \text{first order linear PDE (simplest wave equation)},$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Phi(x, y), \quad \text{second order linear PDE (Poisson)}.$$

A nonlinear equation is semilinear if the coefficients of the highest derivative are functions of the independent variables only.

$$(x+3)\frac{\partial u}{\partial x} + xy^2\frac{\partial u}{\partial y} = u^3,$$

$$x\frac{\partial^2 u}{\partial x^2} + (xy + y^2)\frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial y} = u^4.$$

A nonlinear PDE of order m is quasilinear if it is linear in the derivatives of order m with coefficients depending only on $x, y, \ldots$ and derivatives of order $< m$.

$$\left[1 + \left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial x^2} - 2\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y}\,\frac{\partial^2 u}{\partial x \partial y} + \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]\frac{\partial^2 u}{\partial y^2} = 0.$$

Principle of superposition: A linear equation has the useful property that if u_1 and u_2 both satisfy the equation then so does $\alpha u_1 + \beta u_2$ for any $\alpha, \beta \in \mathbb{R}$. This is often used in constructing solutions to linear equations (for example, so as to satisfy boundary or initial conditions; c.f. Fourier series methods). This is not true for nonlinear equations, which helps to make this sort of equations more interesting, but much more difficult to deal with.

1.4 Examples

1.4.1 Wave Equations

Waves on a string, sound waves, waves on stretch membranes, electromagnetic waves, etc.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

or more generally

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

where c is a constant (wave speed).

1.4.2 Diffusion or Heat Conduction Equations

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2},$$

or more generally

$$\frac{\partial u}{\partial t} = \kappa\nabla^2 u,$$

or even

$$\frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u)$$

where κ is a constant (diffusion coefficient or thermometric conductivity).
Both those equations (wave and diffusion) are linear equations and involve time (t). They require some initial conditions (and possibly some boundary conditions) for their solution.

1.4.3 Laplace's Equation

Another example of a second order linear equation is the following.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

or more generally

$$\nabla^2 u = 0.$$

This equation usually describes steady processes and is solved subject to some boundary conditions.

One aspect that we shall consider is: why do the similar looking equations describes essentially different physical processes? What is there about the equations that make this the cases?

1.4.4 Other Common Second Order Linear PDEs

Poisson's equation is just the Lapace's equation (homogeneous) with a known source term (e.g. electric potential in the presence of a density of charge):

$$\nabla^2 u = \Phi.$$

The Helmholtz equation may be regarded as a stationary wave equation:

$$\nabla^2 u + k^2\, u = 0.$$

The Schrödinger equation is the fundamental equation of physics for describing quantum mechanical behavior; Schrödinger wave equation is a PDE that describes how the wavefunction of a physical system evolves over time:

$$-\nabla^2 u + V\, u = i\frac{\partial u}{\partial t}.$$

1.4.5 Nonlinear PDEs

An example of a nonlinear equation is the equation for the propagation of reaction-diffusion waves:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) \quad (2^{\text{nd}} \text{ order}),$$

or for nonlinear wave propagation:

$$\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} = 0; \quad (1^{\text{st}} \text{ order}).$$

The equation

$$x^2 u \frac{\partial u}{\partial x} + (y+u)\frac{\partial u}{\partial y} = u^3$$

is an example of quasilinear equation, and

$$y\frac{\partial u}{\partial x} + (x^3+y)\frac{\partial u}{\partial y} = u^3$$

is an example of semilinear equation.

1.4.6 System of PDEs

Maxwell equations constitute a system of linear PDEs:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}, \qquad \nabla \times \mathbf{B} = \mu \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

In empty space (free of charges and currents) this system can be rearranged to give the equations of propagation of the electromagnetic field,

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}, \qquad \frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B}.$$

Incompressible magnetohydrodynamic (MHD) equations combine Navier-Stokes equation (including the Lorentz force), the induction equation as well as the solenoidal constraints,

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} = -\nabla\Pi + \mathbf{B}\cdot\nabla\mathbf{B} + \nu\nabla^2\mathbf{U} + \mathbf{F},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U}\times\mathbf{B}) + \eta\nabla^2\mathbf{B},$$

$$\nabla\cdot\mathbf{U} = 0, \qquad \nabla\cdot\mathbf{B} = 0.$$

Both systems involve space and time; they require some initial and boundary conditions for their solution.

1.5 Existence and Uniqueness

Before attempting to solve a problem involving a PDE we would like to know if a solution exists, and, if it exists, if the solution is unique. Also, in problem involving time, whether a solution exists $\forall t > 0$ (global existence) or only up to a given value of t — i.e. only for $0 < t < t_0$ (finite time blow-up, shock formation). As well as the equation there could be certain boundary and initial conditions. We would also like to know whether the solution of the problem depends continuously of the prescribed data — i.e. small changes in boundary or initial conditions produce only small changes in the solution.

Illustration from ODEs:

1. $$\frac{\mathrm{d}u}{\mathrm{d}t} = u, \quad u(0) = 1.$$

   Solution: $u = e^t$ exists for $0 \le t < \infty$

2. $$\frac{\mathrm{d}u}{\mathrm{d}t} = u^2, \quad u(0) = 1.$$

   Solution: $u = 1/(1-t)$ exists for $0 \le t < 1$

3. $$\frac{\mathrm{d}u}{\mathrm{d}t} = \sqrt{u}, \quad u(0) = 0,$$

   has two solutions: $u \equiv 0$ and $u = t^2/4$ (non uniqueness).

We say that the PDE with boundary or initial condition is well-formed (or well-posed) if its solution exists (globally), is unique and depends continuously on the assigned data. If any of these three properties (existence, uniqueness and stability) is not satisfied, the problem (PDE, BCs and ICs) is said to be ill-posed. Usually problems involving linear systems are well-formed but this may not be always the case for nonlinear systems (bifurcation of solutions, etc.)

Example: A simple example of showing uniqueness is provided by:

$$\nabla^2 u = F \ \text{ in } \Omega \quad \text{(Poisson's equation)}.$$

with $u = 0$ on $\partial\Omega$, the boundary of Ω, and F is some given function of $\mathbf{x}$.

Suppose u_1 and u_2 two solutions satisfying the equation and the boundary conditions. Then consider $w = u_1 - u_2$; $\nabla^2 w = 0$ in Ω and $w = 0$ on $\partial\Omega$. Now the divergence theorem gives

$$\begin{aligned}\int_{\partial\Omega} w\nabla w\cdot\mathbf{n}\,\mathrm{d}S &= \int_\Omega \nabla\cdot(w\nabla w)\,\mathrm{d}V,\\ &= \int_\Omega \left(w\nabla^2 w + (\nabla w)^2\right)\,\mathrm{d}V\end{aligned}$$

where $\mathbf{n}$ is a unit normal outwards from Ω.

$$\int_{\Omega} (\nabla w)^2 \, \mathrm{d}V = \int_{\partial\Omega} w \frac{\partial w}{\partial n} \, \mathrm{d}S = 0.$$

Now the integrand $(\nabla w)^2$ is non-negative in Ω and hence for the equality to hold we must have $\nabla w \equiv 0$; i.e. $w =$ constant in Ω. Since $w = 0$ on $\partial\Omega$ and the solution is smooth, we must have $w \equiv 0$ in Ω; i.e. $u_1 = u_2$. The same proof works if $\partial u/\partial n$ is given on $\partial\Omega$ or for mixed conditions.