Analytic Methods for PDEs

Examples 5: parabolic equations

1. The function $u(x,t)$ satisfies the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ x \in \mathbb{R},\ t > 0,$$

subject to the conditions

$$u = f(x) \text{ at } t = 0,\ x \in \mathbb{R} \quad \text{with} \quad \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x < \infty.$$

Express the solution to this problem in terms of an integral involving the fundamental solution $K(x,t)$.

For the specific case $f(x) = \exp(-x^2)$, show that this integral reduces to

$$u(x,t) = \frac{1}{\sqrt{1+4t}}\ \exp(-x^2/(1+4t)).$$

Show how

$$u(0,t) \quad \text{and} \quad \int_{-\infty}^{+\infty} u(x,t)\,\mathrm{d}x$$

vary with t. What do these results say about the nature of the solution?

2. The function $u(x,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

on $t > 0$ and the semi-infinite strip $0 < x < +\infty$, subject to the conditions

$$u = 0 \text{ at } t = 0,\ x \in \mathbb{R}_+^*,$$
$$u = u_0\, t^{-1/2} \text{ on } x = 0,\ t > 0,$$
$$u \to 0 \text{ as } x \to +\infty,\ t > 0,$$

where u_0 is a positive constant.

Show that $\eta = x/2t^{1/2}$ is an appropriate 'diffusion variable' for the problem. By looking for a solution in the form $u(x,t) = u_0 t^{-1/2} v(\eta)$ determine the ordinary differential equation and boundary conditions satisfied by $v(\eta)$. Then, show that

$$v(\eta) = \mathrm{e}^{-\eta^2}$$

satisfies this problem.