

# Analytic Methods for PDEs

## Examples 5: parabolic equations

1. The function  $u(x, t)$  satisfies the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0,$$

subject to the conditions

$$u = f(x) \text{ at } t = 0, \quad x \in \mathbb{R} \quad \text{with} \quad \int_{-\infty}^{+\infty} f(x) \, dx < \infty.$$

Express the solution to this problem in terms of an integral involving the fundamental solution  $K(x, t)$ .

For the specific case  $f(x) = \exp(-x^2)$ , show that this integral reduces to

$$u(x, t) = \frac{1}{\sqrt{1+4t}} \exp(-x^2/(1+4t)).$$

Show how

$$u(0, t) \quad \text{and} \quad \int_{-\infty}^{+\infty} u(x, t) \, dx$$

vary with  $t$ . What do these results say about the nature of the solution?

2. The function  $u(x, t)$  satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

on  $t > 0$  and the semi-infinite strip  $0 < x < +\infty$ , subject to the conditions

$$\begin{aligned} u &= 0 \text{ at } t = 0, \quad x \in \mathbb{R}_+^*, \\ u &= u_0 t^{-1/2} \text{ on } x = 0, \quad t > 0, \\ u &\rightarrow 0 \text{ as } x \rightarrow +\infty, \quad t > 0, \end{aligned}$$

where  $u_0$  is a positive constant.

Show that  $\eta = x/2t^{1/2}$  is an appropriate 'diffusion variable' for the problem. By looking for a solution in the form  $u(x, t) = u_0 t^{-1/2} v(\eta)$  determine the ordinary differential equation and boundary conditions satisfied by  $v(\eta)$ . Then, show that

$$v(\eta) = e^{-\eta^2}$$

satisfies this problem.