Analytic Methods for PDEs

Examples 4: elliptic equations

1. Consider the Cauchy problem for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with Cauchy data

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{1}{n}\sin(ny) \text{ on } \Gamma : \{x = 0,\ -\infty < y < +\infty\}.$$

Using the method of separation of variables, show that the solution to this Cauchy problem is

$$u(x, y) = \frac{1}{n^2}\sinh(nx)\ \sin(ny).$$

What can you conclude about the stability of the solution if you consider this Cauchy problem in the limit $n \to \infty$?

2. Consider the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1 \text{ on the square } |x| < 1, |y| < 1,$$

with $u = 0$ on the boundaries. By considering the function

$$\phi = u - \frac{1}{4}(x^2 + y^2)$$

find an upper and lower bound for $u(0, 0)$.

3. Consider the boundary-value problem

$$\nabla^2 u + ku = F \text{ in } V,$$
$$\text{with} \quad u = f \text{ on } S,$$

for given functions F and f, where k is a constant and where S is the boundary of V.

Show that, if $k < 0$, then the solution to the problem is unique. Would you expect the solution to be unique if $k > 0$?

4. Solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } x > 0,\, y > 0$$

with

$$\frac{\partial u}{\partial y} = f(x) \text{ on } y = 0 \ (x \geq 0) \quad \text{and} \quad \frac{\partial u}{\partial x} = g(y) \text{ on } x = 0 \ (y \geq 0),$$

$$\text{where} \quad \int_0^{+\infty} [f(\lambda) + g(\lambda)] \, \mathrm{d}\lambda = 0,$$

by first obtaining the appropriate Green's function.