Analytic Methods for PDEs

Examples 3: standard form of second order linear PDEs

1. Show that the equation

$$\frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic in $x > 0$ and reduce it to standard (canonical) form in this region.

2. Classify the equation

$$4x^2 \frac{\partial^2 u}{\partial x^2} + 4x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

By making a suitable transformation of independent variables, reduce the equation to standard (canonical) form.

3. Classify the equation

$$4 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} + u = 0.$$

By making a suitable transformation of independent variables, reduce the equation to standard (canonical) form.

4. Show that the equation

$$4y^2 \frac{\partial^2 u}{\partial x^2} + 2(1-y^2) \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} - \frac{2y}{1+y^2} \left(2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) = 0$$

is hyperbolic.

If

$$u = f(x) \quad \text{and} \quad \frac{\partial u}{\partial y} = g(x) \quad \text{at} \quad y = 0$$

show that the solution is

$$u(x,y) = f(x - 2y^3/3) + \frac{1}{2} \int_{x-2y^3/3}^{x+2y} g(s)\, \mathrm{d}s.$$