Global Properties of MagnetoRotational Instabilities in Accretion Discs

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MagnetoRotational Instability & Dynamo in Accretion Discs

MagnetoRotational Instability:

- Weakly magnetized ($\beta \gg 1$) and differential rotating flows unstable if $\mathrm{d}\Omega/\mathrm{d}r < 0$
- Free energy $\equiv$ differential rotation $\Rightarrow$ MRI extremely powerful $\gamma \sim r|\mathrm{d}\Omega/\mathrm{d}r|$

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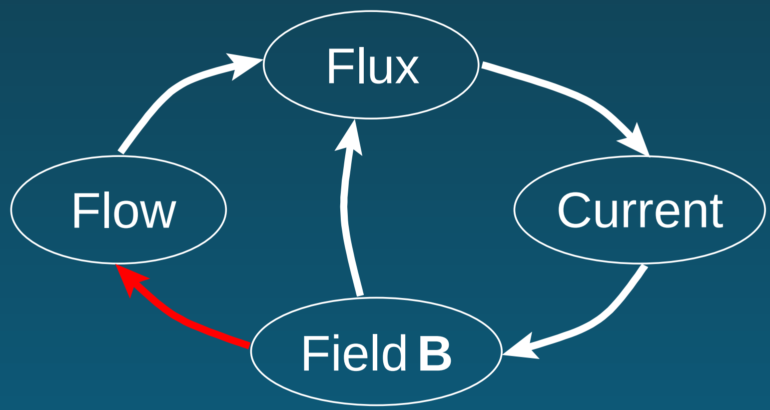

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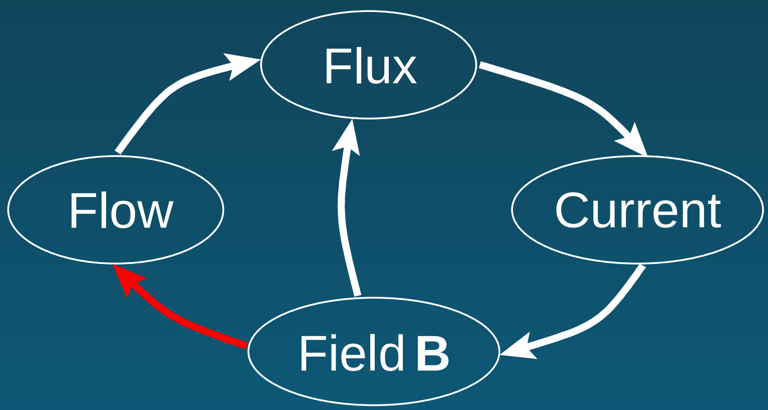
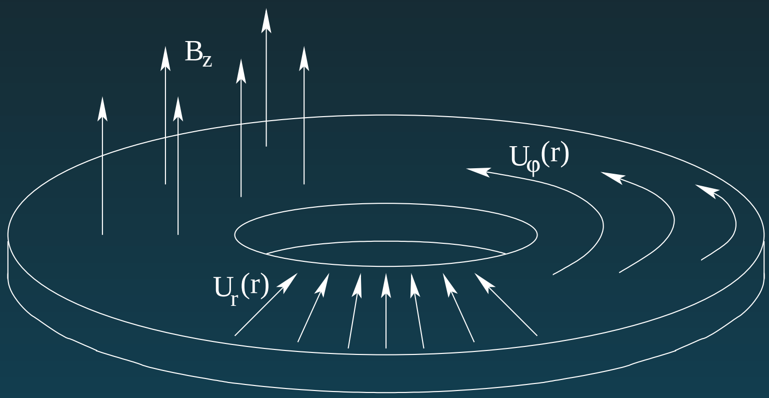


Main studies:

- Local radially (shearing sheet): vorticity gradient not taken into account
- Global: no accretion of material in the basic state
- Rely on numerical dissipation
- No strong conclusion about $\langle \mathbf{B} \rangle$

Global Dissipative Study



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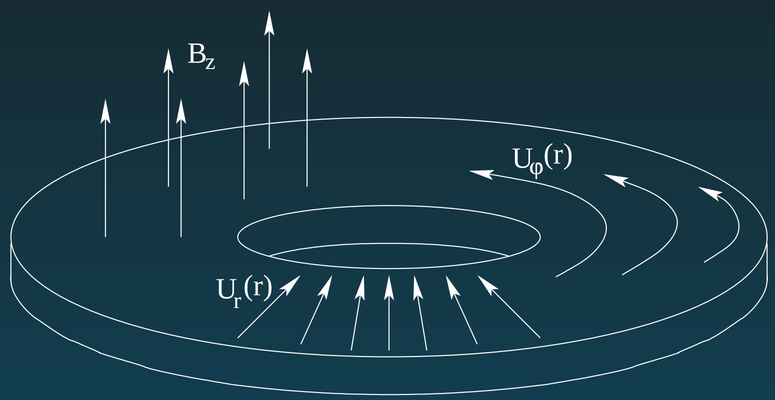

NL evolution equations:

$$(\partial_t + \mathbf{U}\cdot\nabla)\,\mathbf{U} = -\nabla\Phi - \nabla\Pi + \mathbf{B}\cdot\nabla\mathbf{B} + \nu\Delta\mathbf{U}$$

$$(\partial_t + \mathbf{U}\cdot\nabla)\,\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{U} + \eta\Delta\mathbf{B}$$

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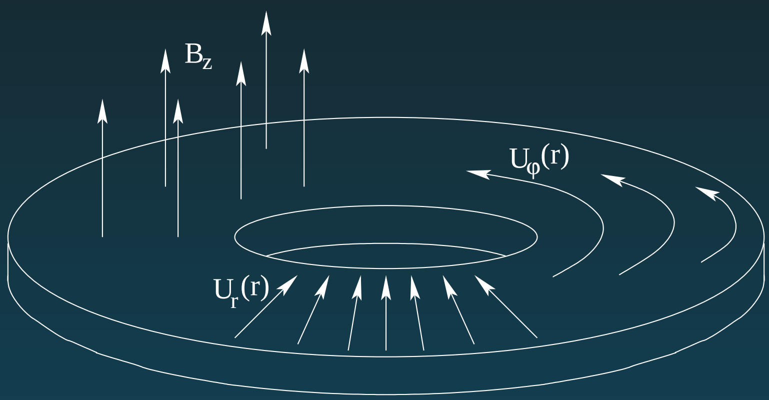

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Axisymmetric basic state:

$$\Phi = -GM_\star/r \Rightarrow U_\varphi = \sqrt{GM_\star/r}$$

$$U_r = -3\nu/2r \Rightarrow P = C - 9\nu/8r^2$$

$$U_z = 0$$

$\mathbf{B} = B_o\,\mathbf{e}_z$ or $\mathbf{B} = B_o/r\;\mathbf{e}_\varphi$

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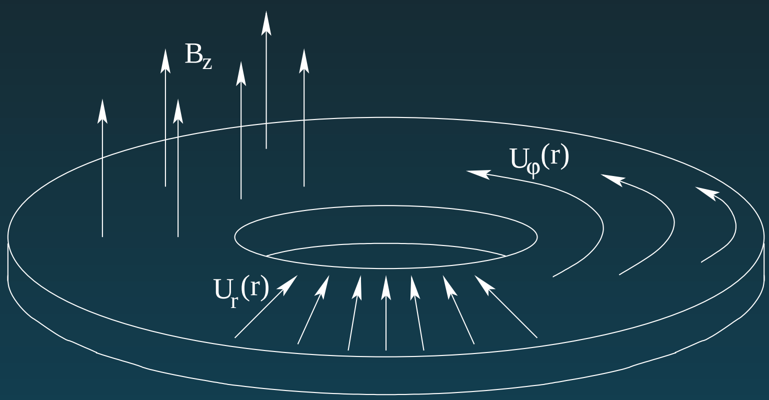

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BCs: $U_\varphi = \sqrt{GM_\star/r_{\{1,2\}}}, \ldots$

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The boundaries drive the shearing flow

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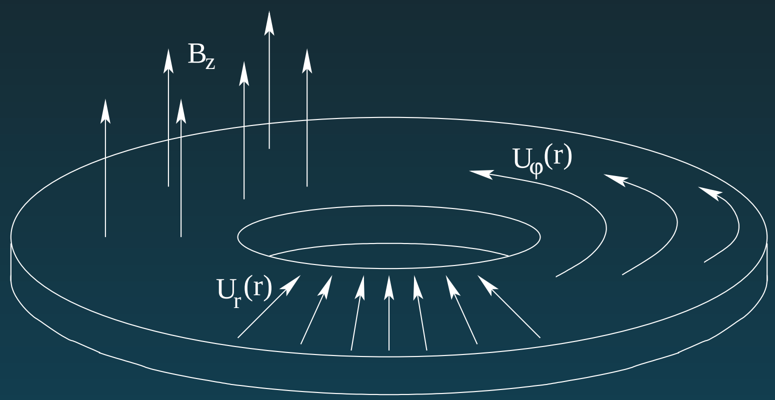

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The boundaries drive the shearing flow

Linear evolution equations:

- Normal modes:

  $$\underline{\mathcal{K}}(\mathbf{r},t) = \underline{\kappa}(r)\ \exp(\sigma t + im\,\varphi + ik\,z)$$

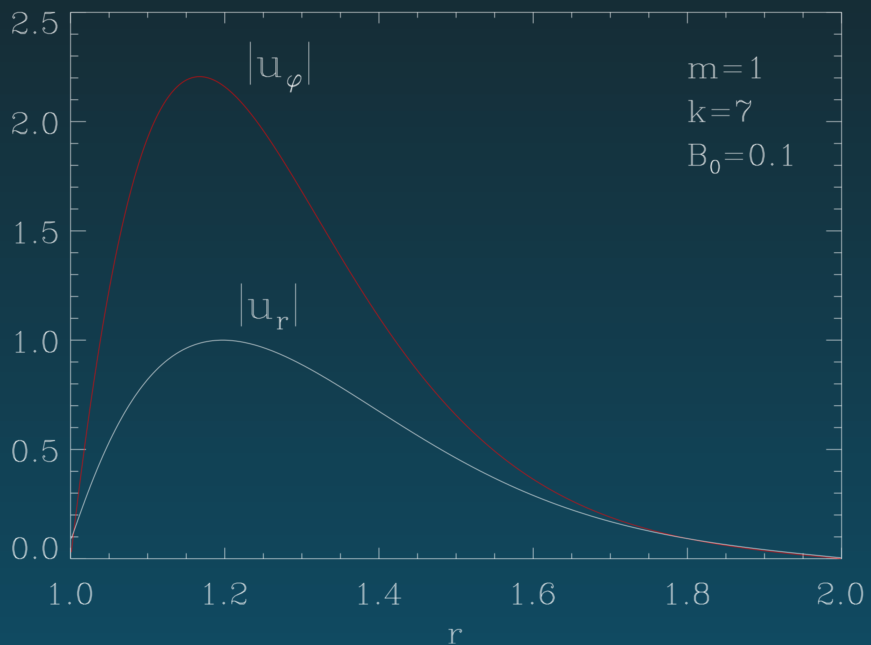
- 10th order linear system:

  $$\sigma\ \underline{\underline{\mathcal{I}}}(r)\ \underline{\kappa}(r) = \underline{\underline{\mathcal{L}}}(r)\ \underline{\kappa}(r)$$

- BCs: $u_\varphi = 0, \ldots$

Ideal MRI Modes

No inflow in the basic state



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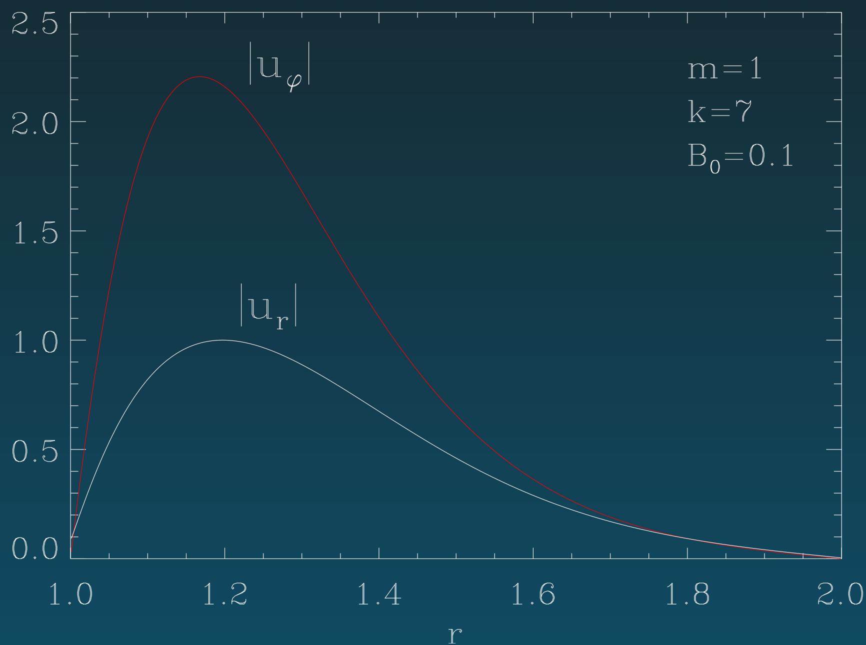
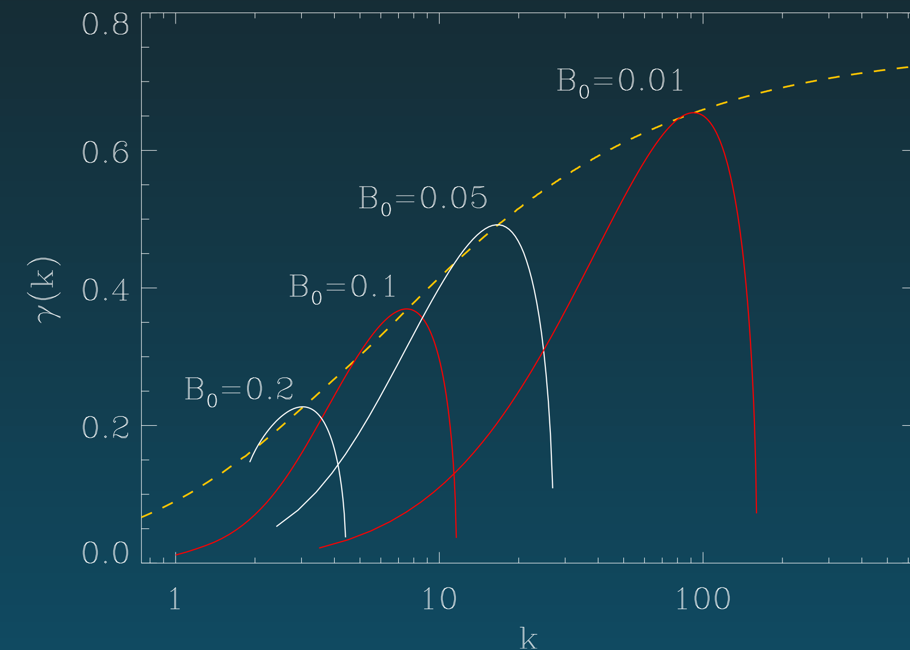
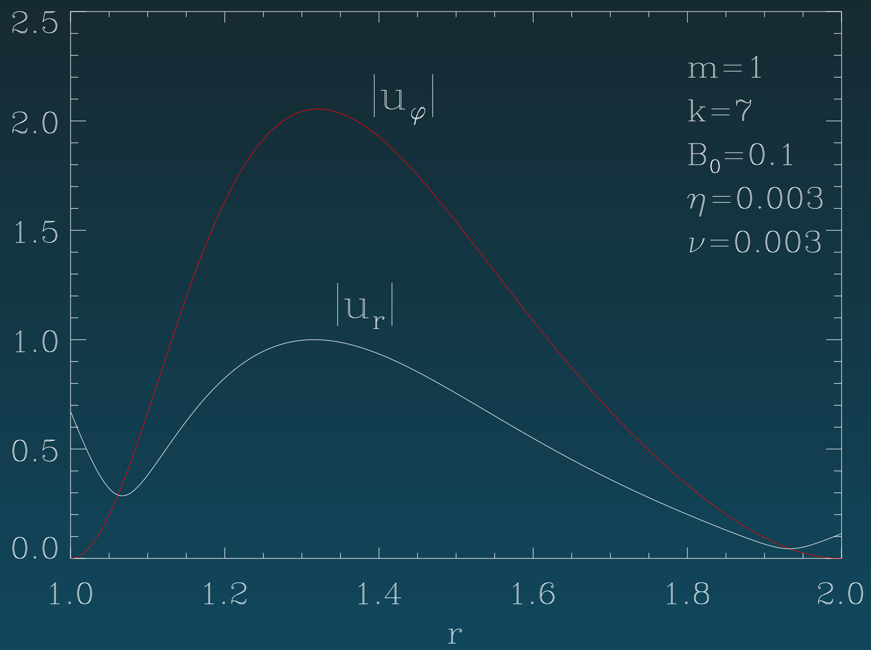




- Non axisymmetric $m = 1$ modes globally unstable
- Quenching by the magnetic tension
- Saturation: $\gamma_{\text{max}} \to r_1/2\ |\mathrm{d}\Omega/\mathrm{d}r|_{r_1}$

Dissipative MRI Modes



- Shape of the modes slightly modified by the inflow and the dissipation

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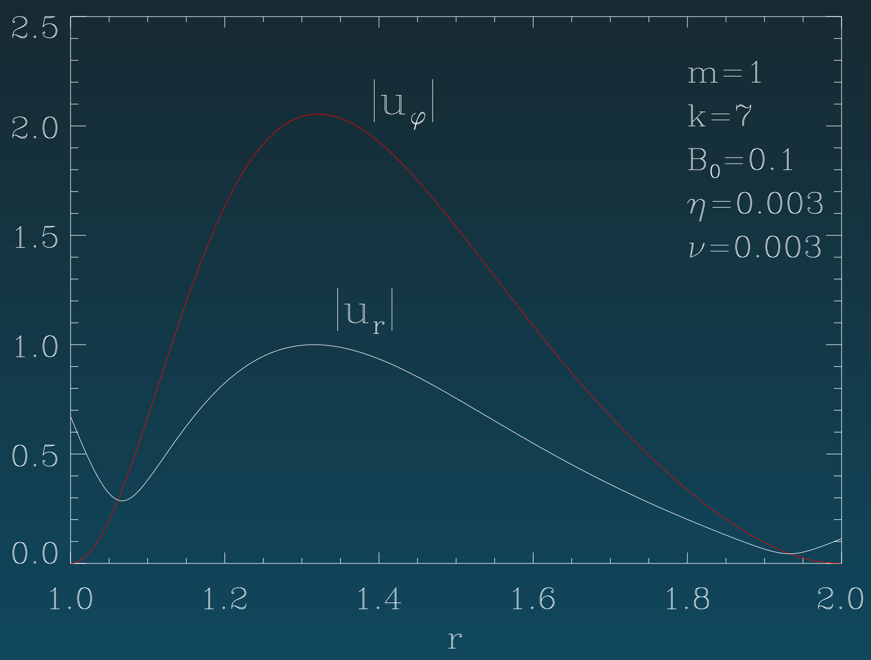
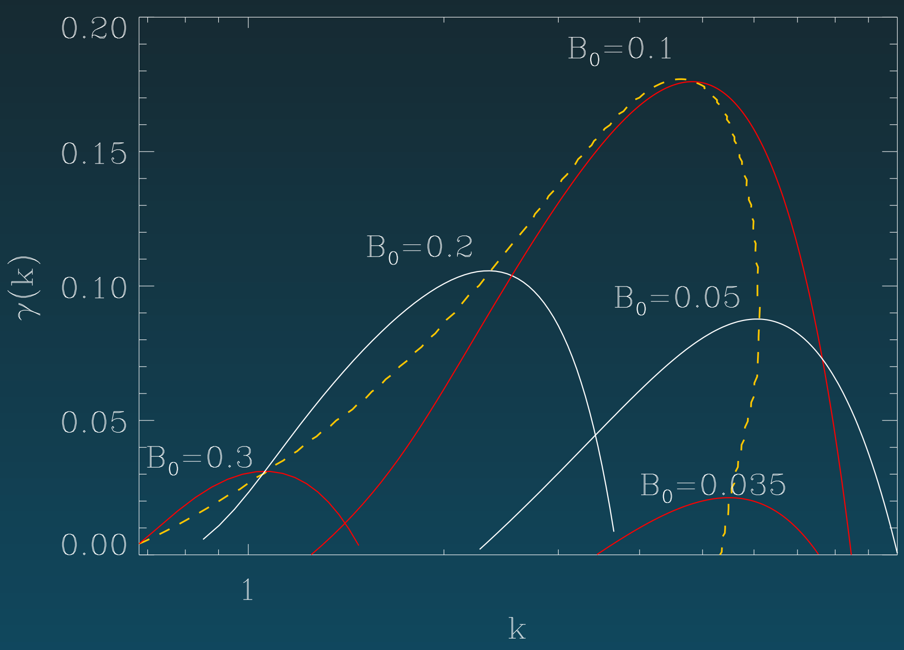
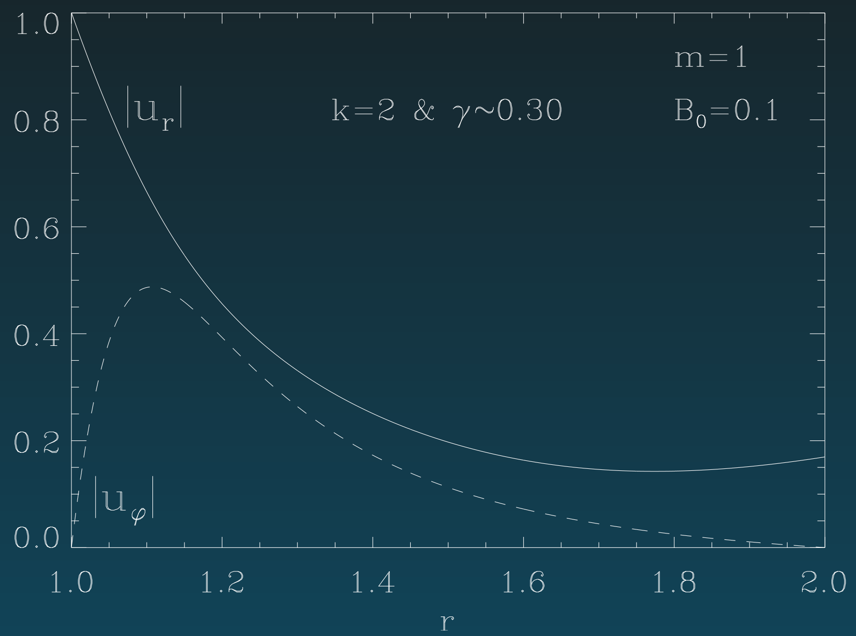




- Shape of the modes slightly modified by the inflow and the dissipation
- Growth rates globally reduced
- Damping of the small-scale modes

Wall Modes



- Wall modes solutions of the linear system too

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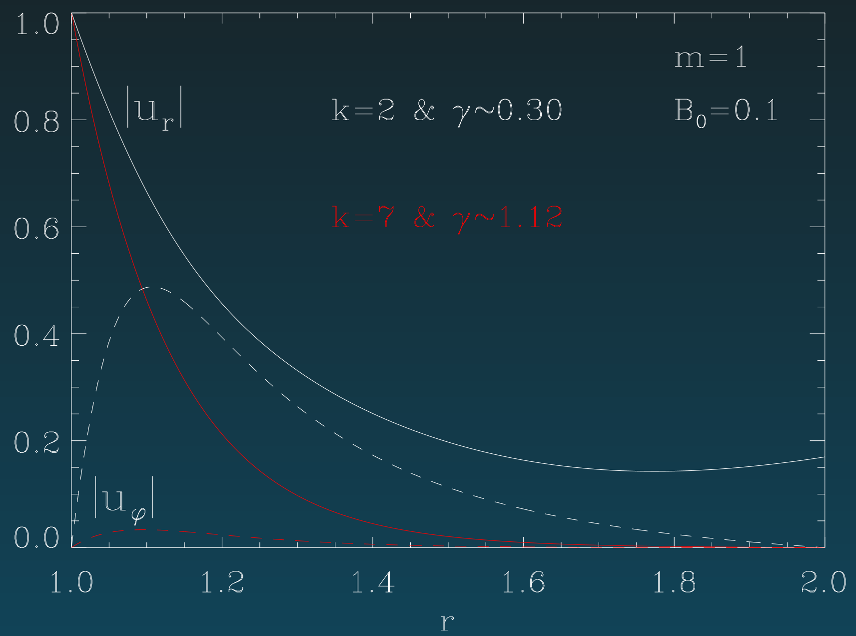

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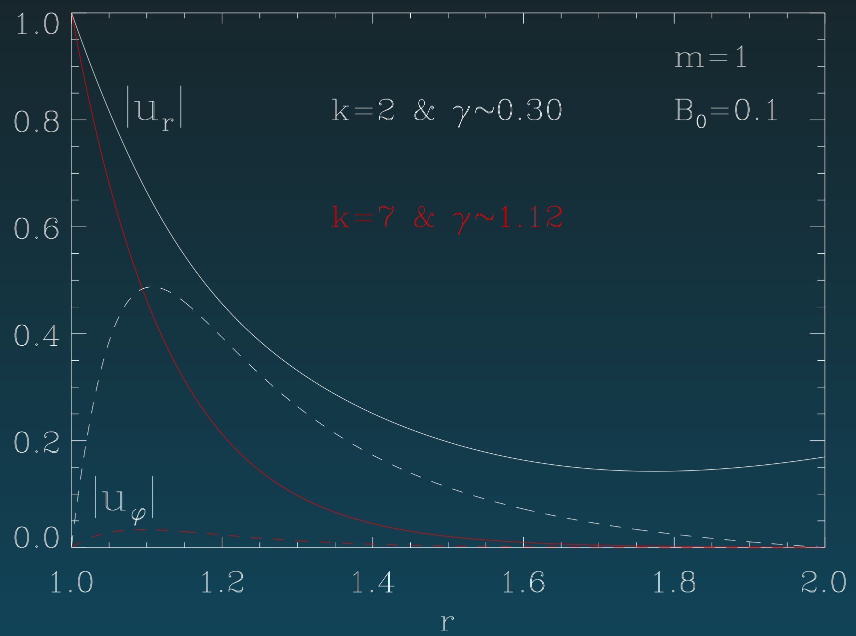

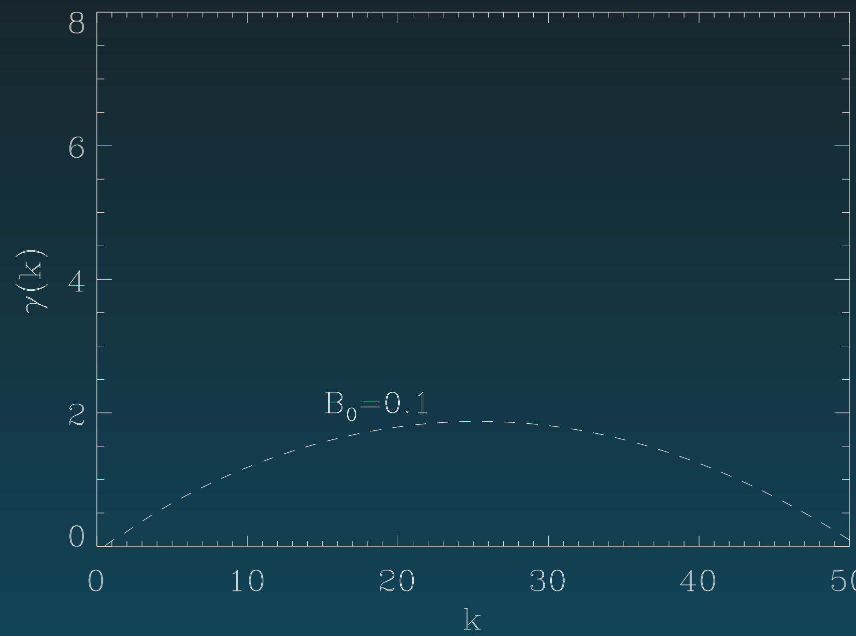

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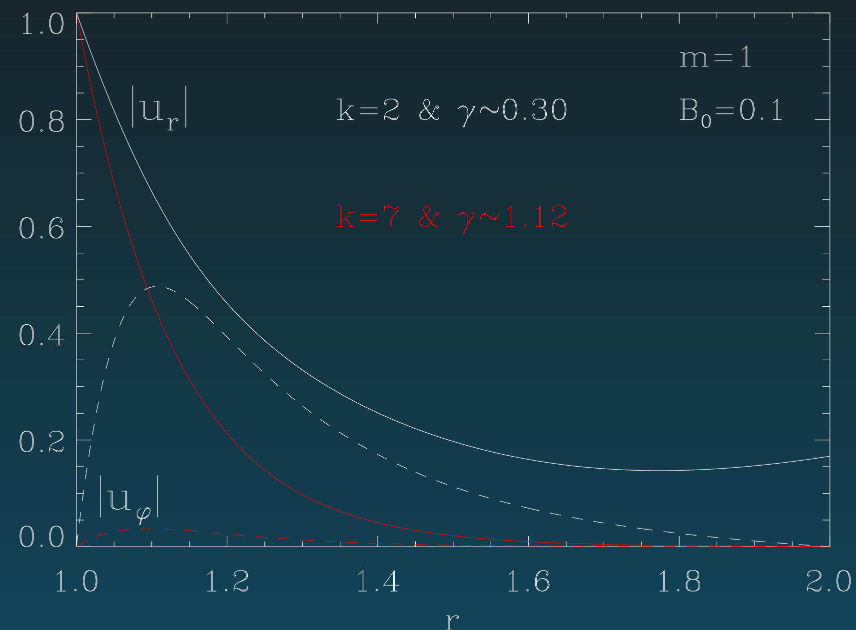

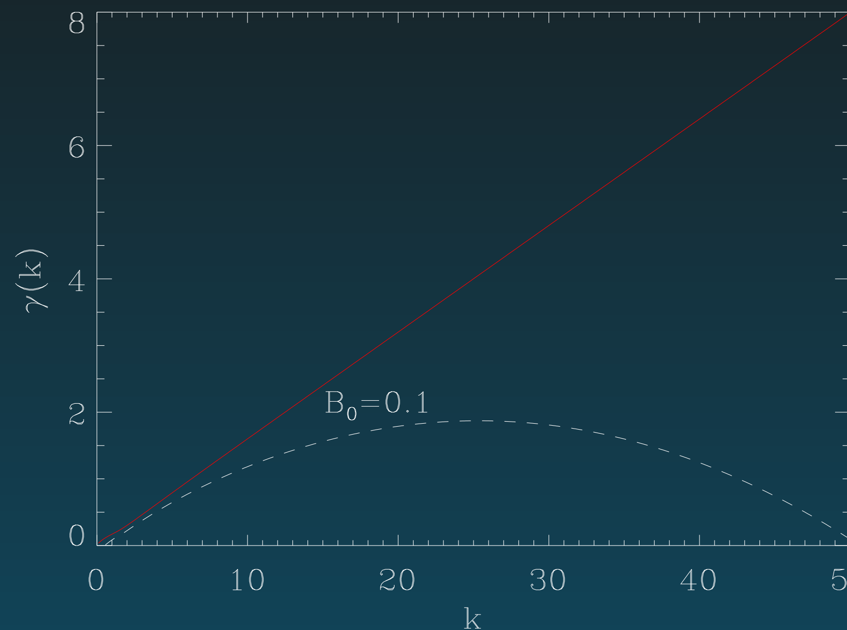

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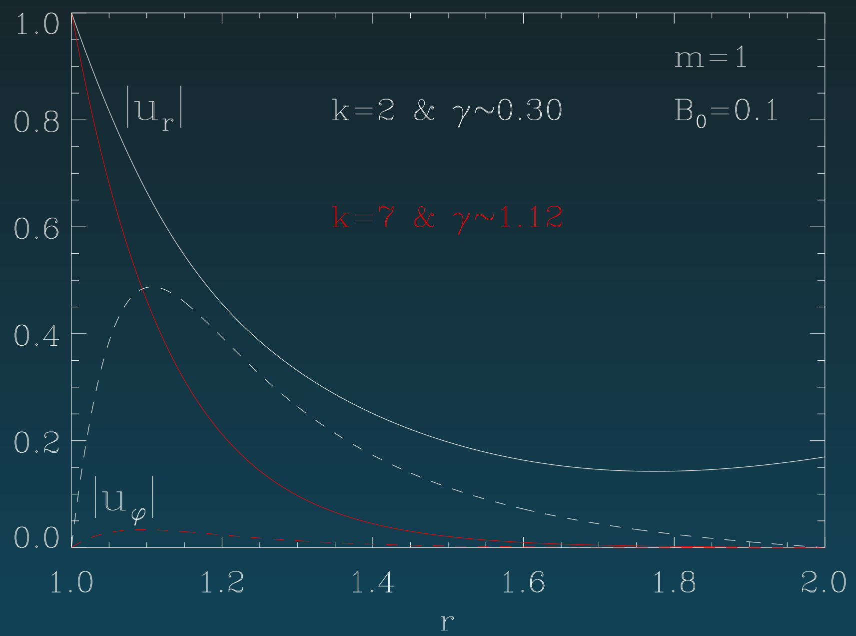

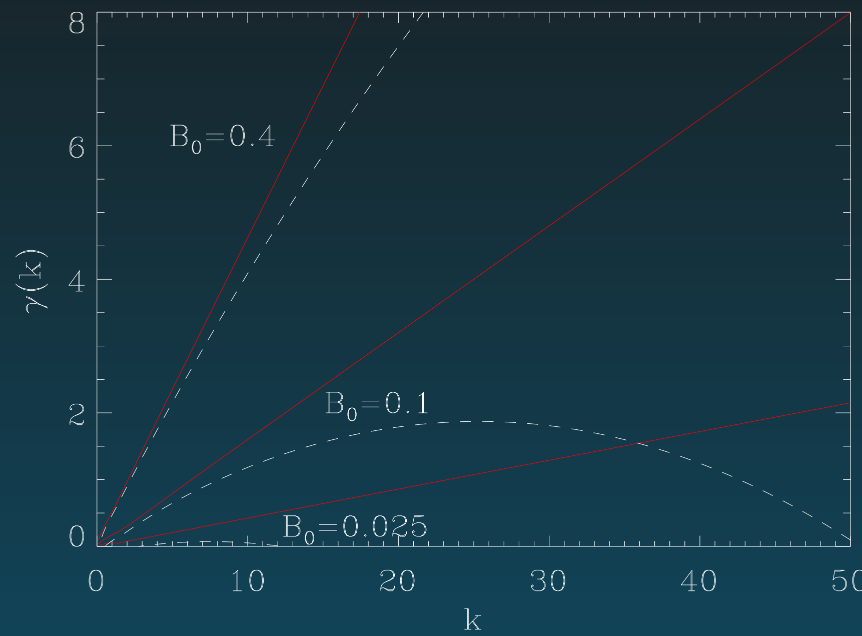

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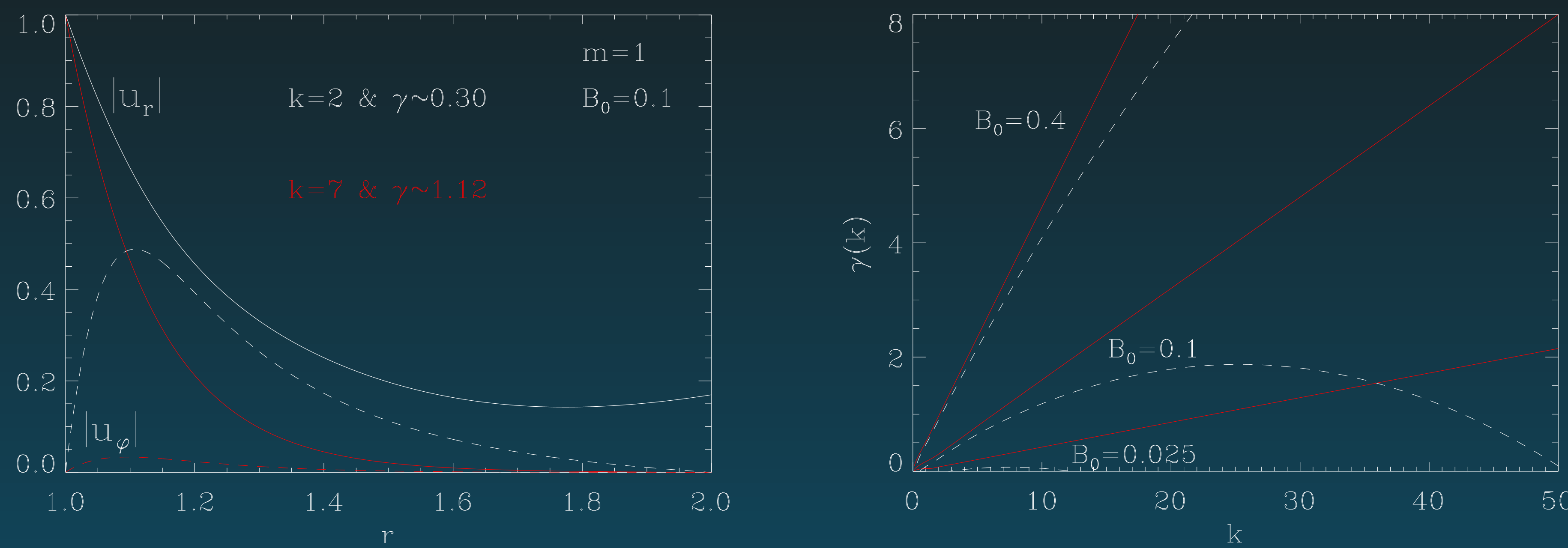
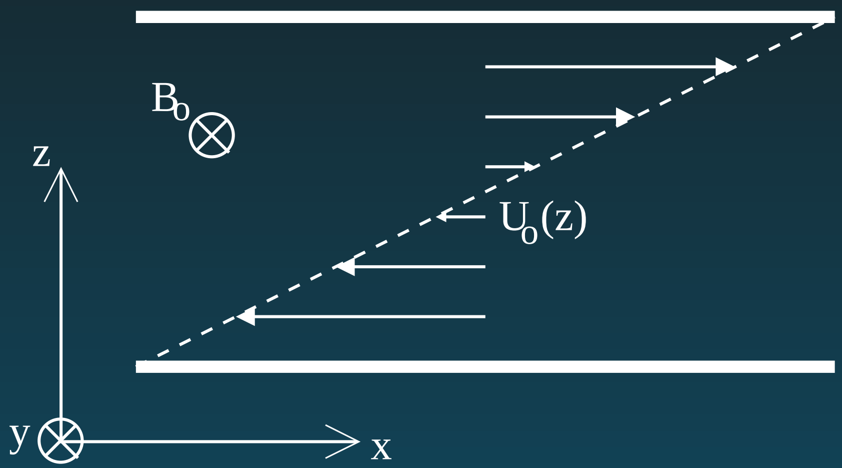


- Wall modes solutions of the linear system too
- γ too large for the free energy available and large range of k unstable
- Ideal case: γ scales linearly with k and increases rapidly with B_0
- Significant flux of energy through the boundaries to feed these modes
- Inflow, curvature and Coriolis force non crucial

Cartesian Linear Shearing Flow



Incompressible, non dissipative basic state:

$$\begin{aligned} \rho_0 &= 1 \\ \mathbf{U}_0 &= z\, \mathbf{e}_x, \quad z \in [-z_0, +z_0] \\ \mathbf{B}_0 &= B_0\, \mathbf{e}_y \end{aligned}$$

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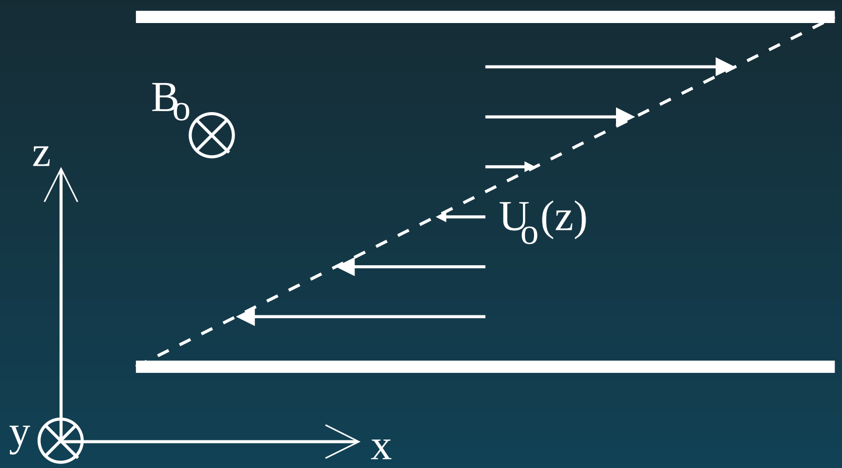

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2nd order system of linear ODEs:

$$\chi\mathcal{H}\,u_x = -U_0'\mathcal{H}\,u_z - ik_x\,\pi$$
$$\chi\mathcal{H}\,u_y = -ik_y\,\pi$$
$$\chi\mathcal{H}\,u_z = -\pi'$$
$$0 = ik_x\,u_x + ik_y\,u_y + u_z'$$

where,

$$\underline{\mathcal{K}}(\mathbf{x},t) = \underline{\kappa}(z)\ \exp(\sigma t + ik_x\,x + ik_y\,y)$$
$$\omega_a = kB_0$$
$$\chi = \sigma + ik_xU_0$$
$$\mathcal{H} = \left(1 + \omega_a^2/\chi^2\right)$$

Cartesian Wall Modes: HD Limit

Hydrodynamic limit: $\omega_a = 0$ and $\mathcal{H} = 1$

HD modes are solutions of $$\chi \left[u_z'' - \left(k^2 + \frac{\chi''}{\chi} \right) u_z \right] = 0$$

Linear shear $\Rightarrow \chi'' = 0$ and $u_z = c_- \exp(-kz) + c_+ \exp(kz)$

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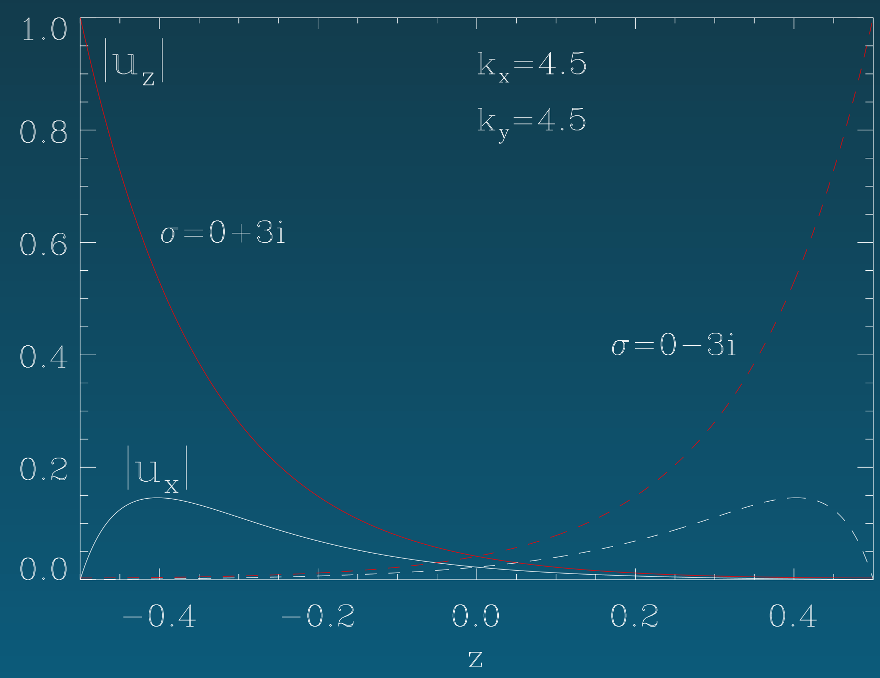
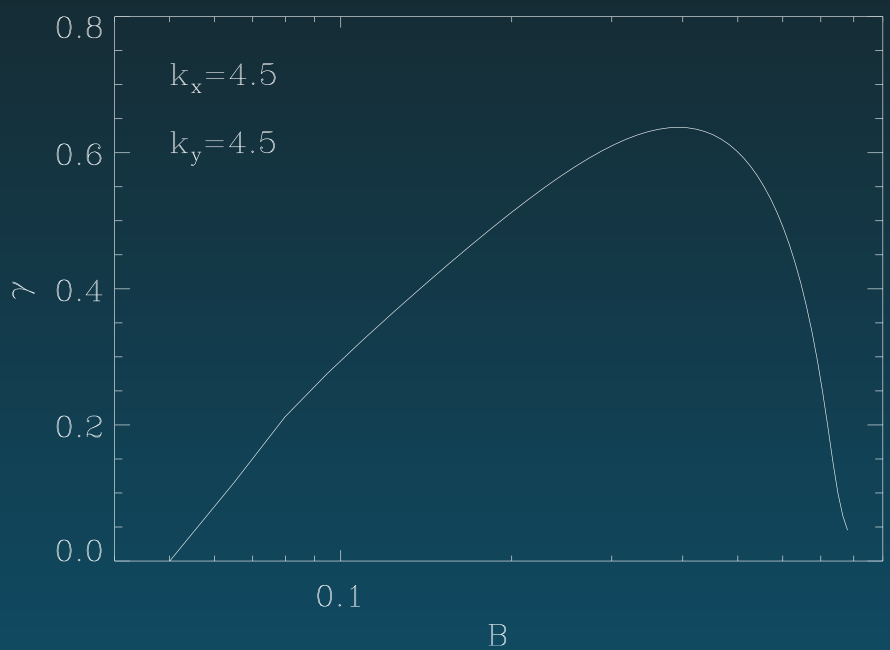
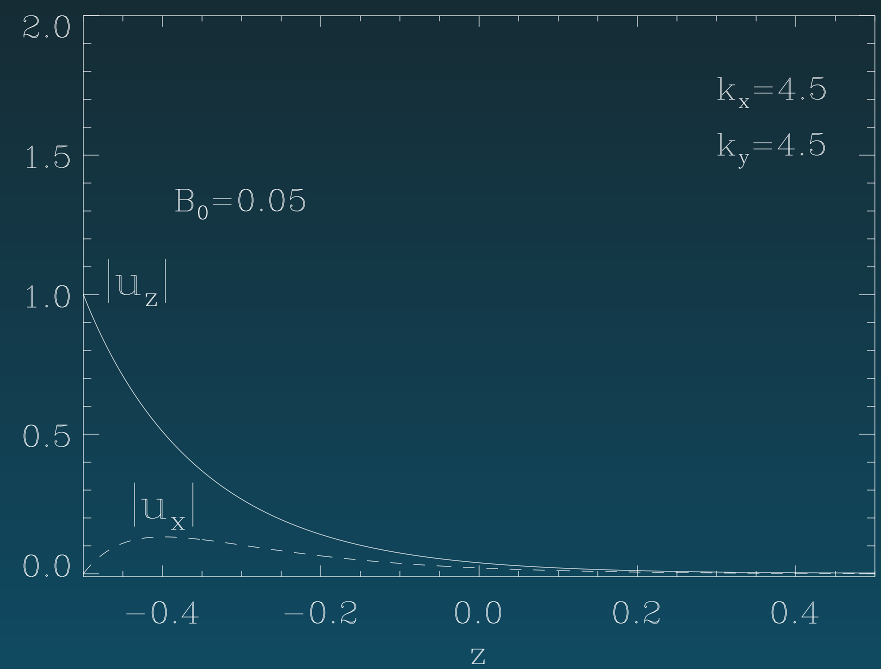


Cartesian Wall Modes in MHD

Magnetic field destabilizes the wall modes





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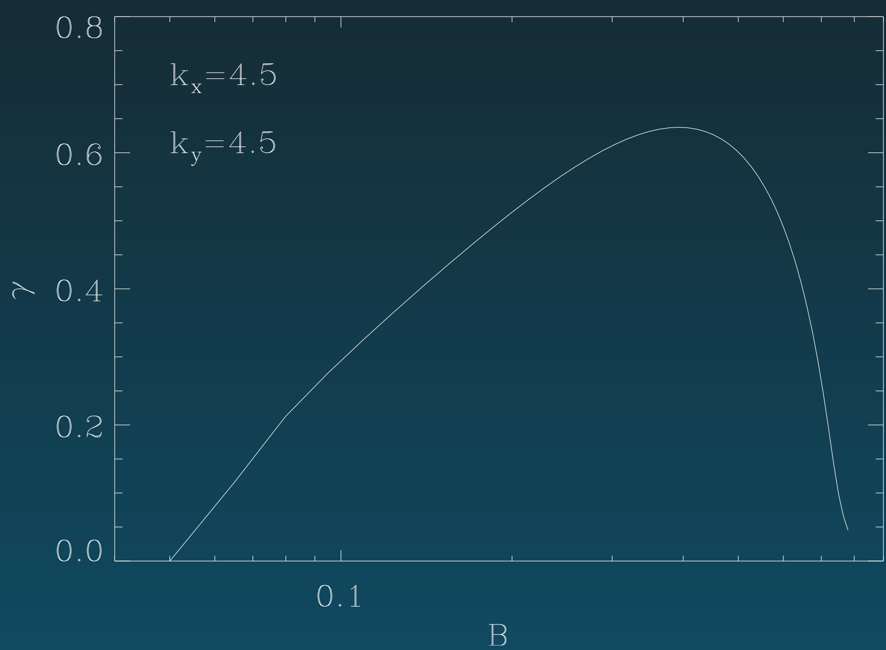

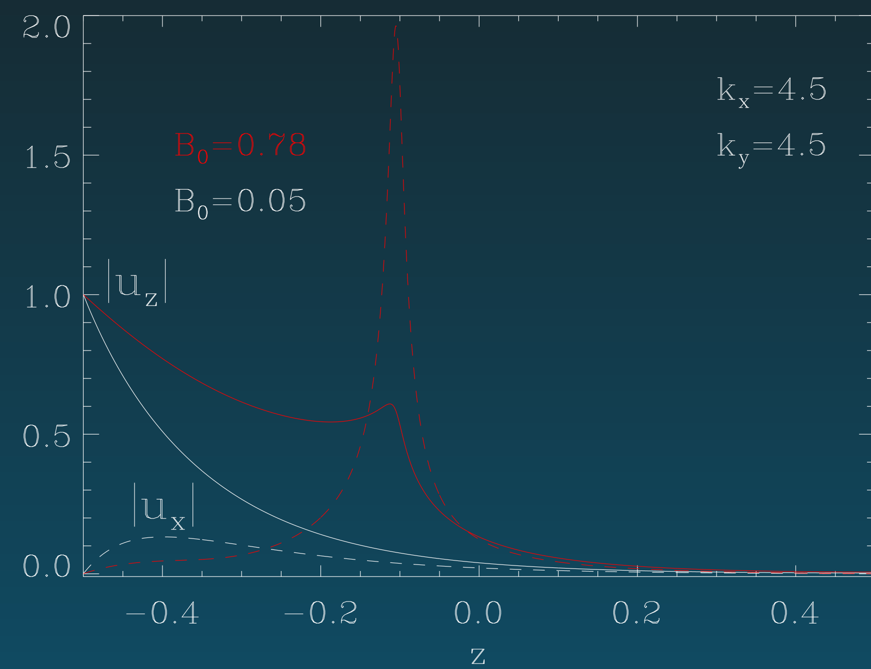

$$u_z'' - 2\frac{\omega_a^2}{\chi^2+\omega_a^2}\frac{\chi'}{\chi}u_z' - \left[k^2 + \frac{\chi''}{\chi} - 2\frac{\omega_a^2}{\chi^2+\omega_a^2}\left(\frac{\chi'}{\chi}\right)^2\right]u_z = 0$$

Singularity when $\gamma = 0$ and $\omega = -k_x U_0 \pm \omega_a$

Origin of the Instability

Analyse on the boundaries:

$$\gamma^2 = \frac{(\mathcal{S}^2\, U_0')^2}{(\mathcal{S}^2\, U_0')^2 + \omega_a^2\, k^2\, k_x^2} \left[(U_0')^2 \frac{\mathcal{K}^2}{k_x^2} + \omega_a^2\, k^2\, \mathcal{L}^2 - \omega_a^2 \right]$$

where $\mathcal{S}^2 = |\pi''|/|\pi|$, $\mathcal{K} = |\pi|'/|\pi|$, $\mathcal{L}^2 = |\pi|\,|\pi|''/|\pi''|^2$ and $k_x = k_y$

B_0 and U_0' both non zero at either of the boundaries $\Rightarrow$ wall modes unstable

Mechanism:

$$u_z \xrightarrow{k_y\, B_0} b_z \xrightarrow{U_0'} b_x \xrightarrow{k_y\, B_0} T_x = k_y\, B_0\, b_x$$

In the vicinity of the boundary $u_x \sim 0 \Rightarrow$ energy from the outside required to balance T_x

Conclusion about Accretion Discs

- Wall modes are solutions of incompressible shearing flows when rigid BCs are relaxed

- B_0 makes them linearly unstable if $u_\varphi = 0$ on the boundaries

Forcing the differential rotation of the boundaries impossible unless Ω_0' or B_0 locally zero

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or

BCs on the pressure to keep it low in agreement with quasi-keplerian accretion discs...