

Global Properties of MagnetoRotational Instabilities in Accretion Discs

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N. Weiss & G. Ogilvie (DAMTP, Cambridge)

MagnetoRotational Instability & Dynamo in Accretion Discs

MagnetoRotational Instability:

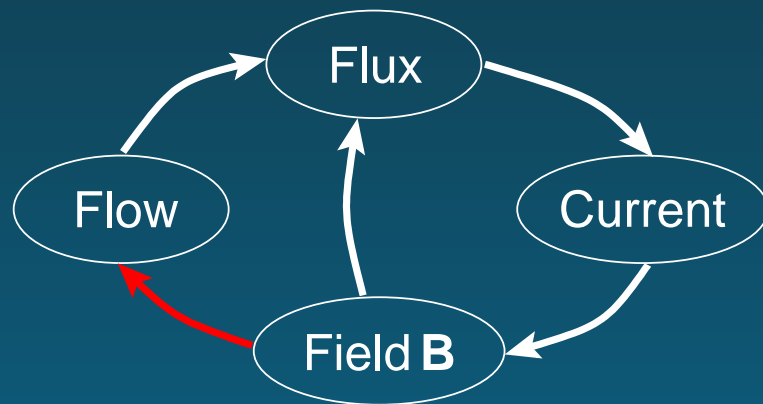
- **Weakly magnetized** ($\beta \gg 1$) and differential rotating flows unstable if $d\Omega/dr < 0$
- Free energy \equiv differential rotation \Rightarrow MRI **extremely powerful** $\gamma \sim r|d\Omega/dr|$

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Nonlinear dynamo:



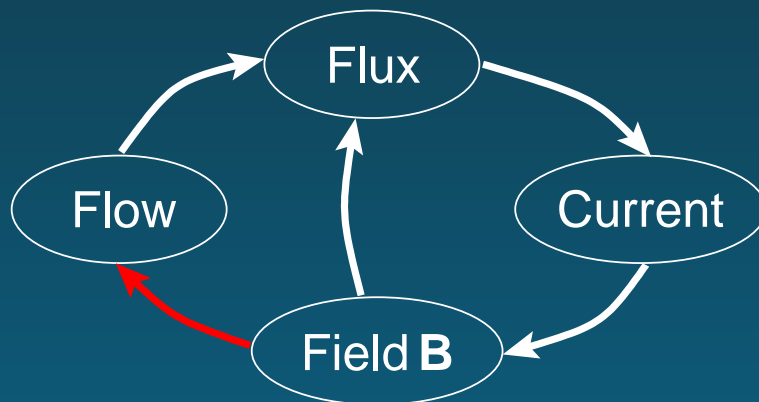
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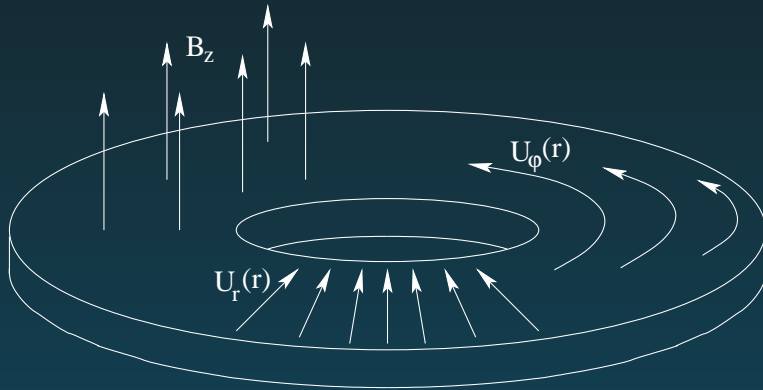
Main studies:

Nonlinear dynamo:

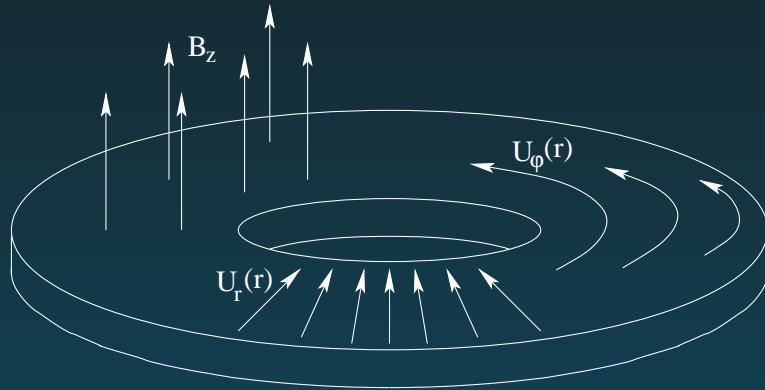


- Local radially (shearing sheet): **vorticity gradient not taken into account**
- Global: no accretion of material in the basic state
- Rely on **numerical dissipation**
- No strong conclusion about $\langle \mathbf{B} \rangle$

Global Dissipative Study



Global Dissipative Study



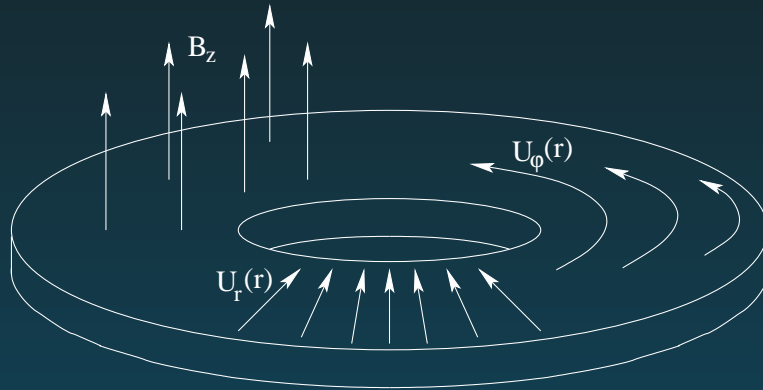
NL evolution equations:

$$(\partial_t + \mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla\Phi - \nabla\Pi + \mathbf{B} \cdot \nabla\mathbf{B} + \nu\Delta\mathbf{U}$$

$$(\partial_t + \mathbf{U} \cdot \nabla) \mathbf{B} = \mathbf{B} \cdot \nabla\mathbf{U} + \eta\Delta\mathbf{B}$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{U} = 0$$

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Axisymmetric basic state:

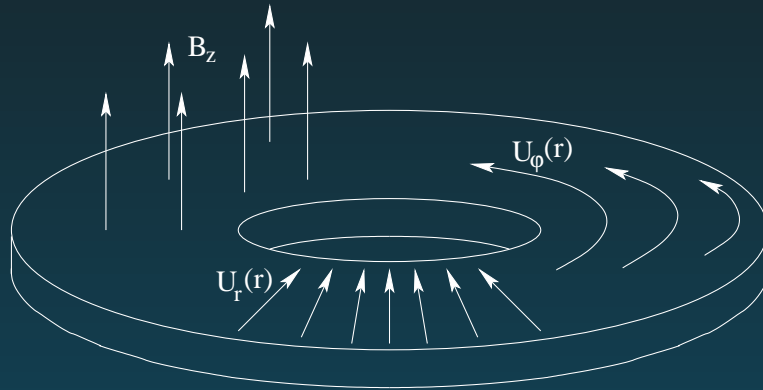
$$\Phi = -GM_\star/r \Rightarrow U_\phi = \sqrt{GM_\star/r}$$

$$U_r = -3\nu/2r \Rightarrow P = C - 9\nu/8r^2$$

$$U_z = 0$$

$$\mathbf{B} = B_o \mathbf{e}_z \quad \text{or} \quad \mathbf{B} = B_o/r \mathbf{e}_\phi$$

Global Dissipative Study



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$$\text{BCs: } U_\phi = \sqrt{GM_\star/r_{\{1,2\}}}, \dots$$

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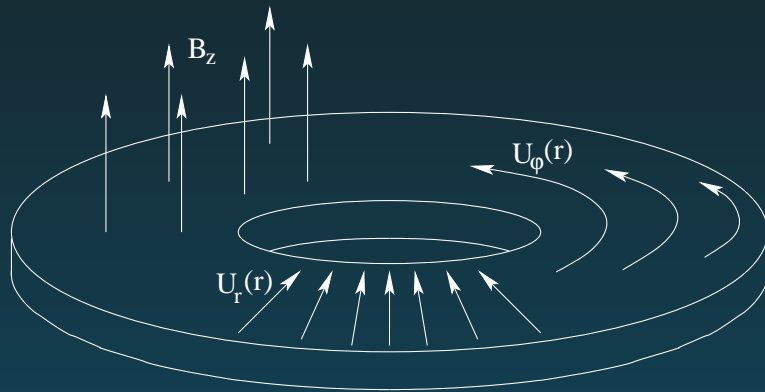
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The boundaries drive the shearing flow

Global Dissipative Study



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Linear evolution equations:

- Normal modes:

$$\underline{\mathcal{K}}(\mathbf{r}, t) = \underline{\kappa}(\mathbf{r}) \exp(\sigma t + im \varphi + ik z)$$

- 10th order linear system:

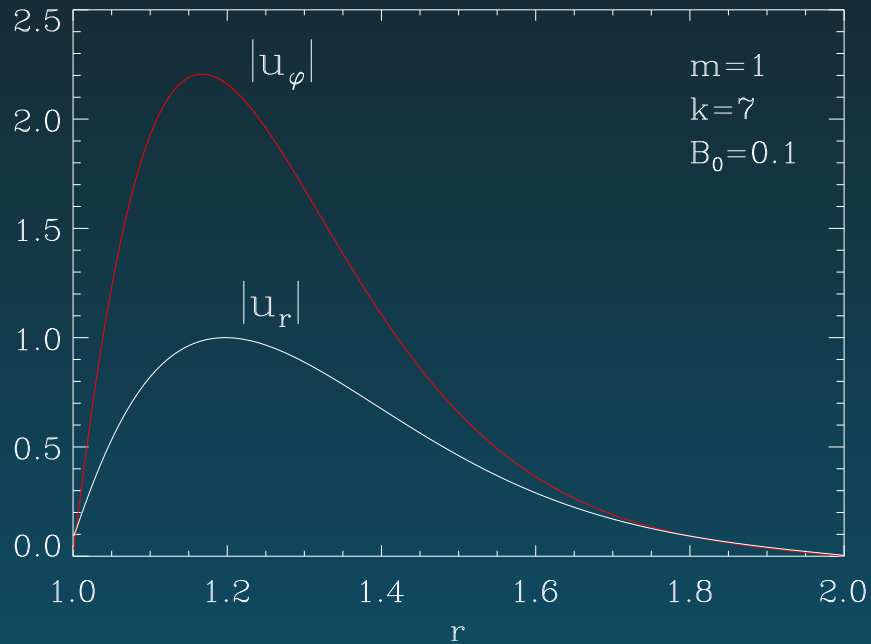
$$\sigma \underline{\mathcal{I}}(\mathbf{r}) \underline{\kappa}(\mathbf{r}) = \underline{\mathcal{L}}(\mathbf{r}) \underline{\kappa}(\mathbf{r})$$

- BCs: $u_\phi = 0, \dots$

The boundaries drive the shearing flow

Ideal MRI Modes

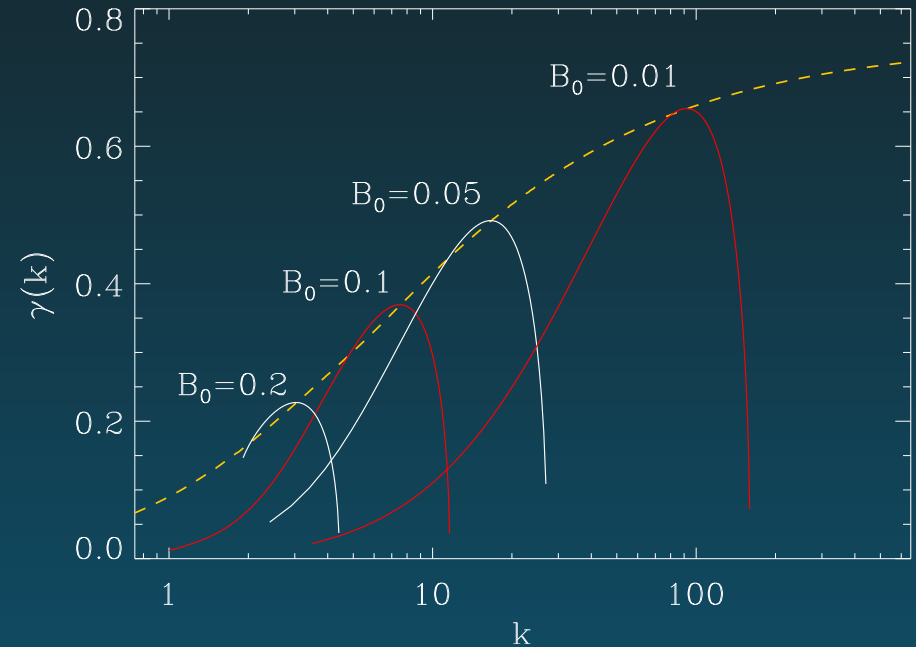
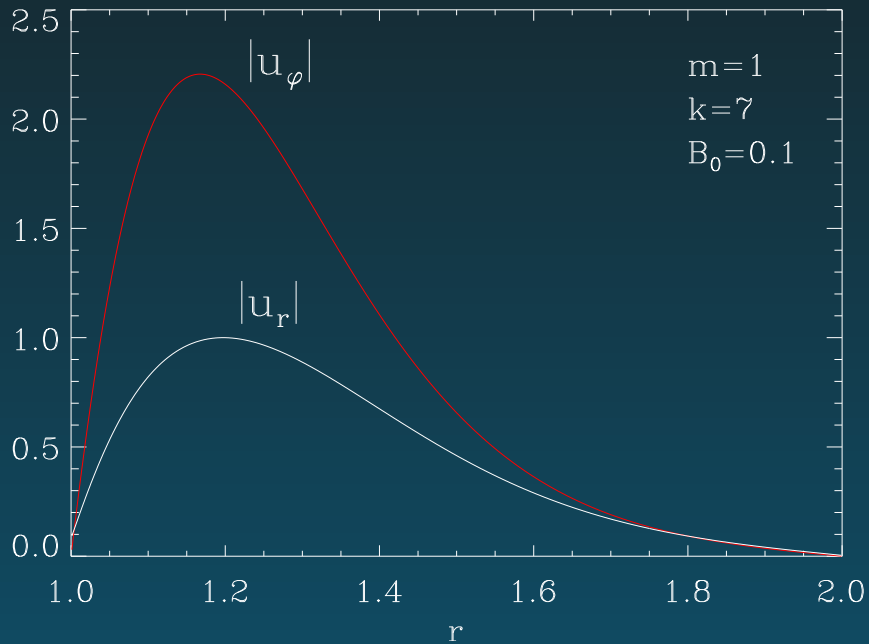
No inflow in the basic state



- Non axisymmetric $m = 1$ modes globally unstable

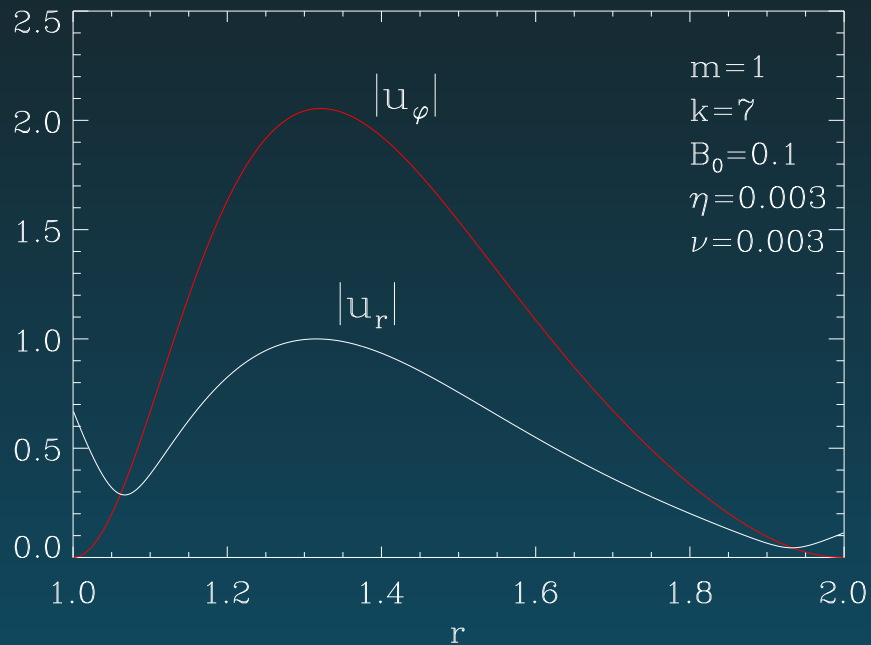
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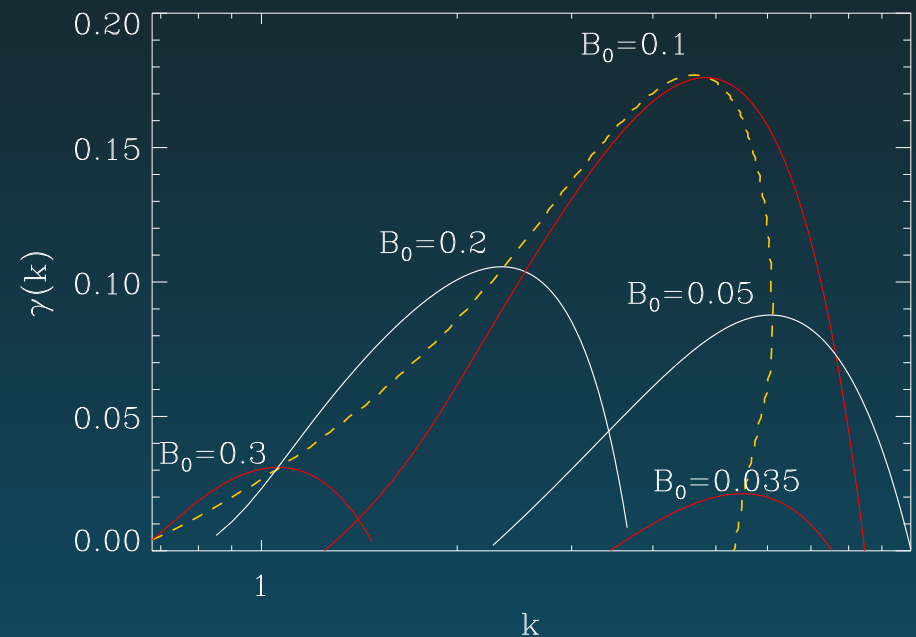
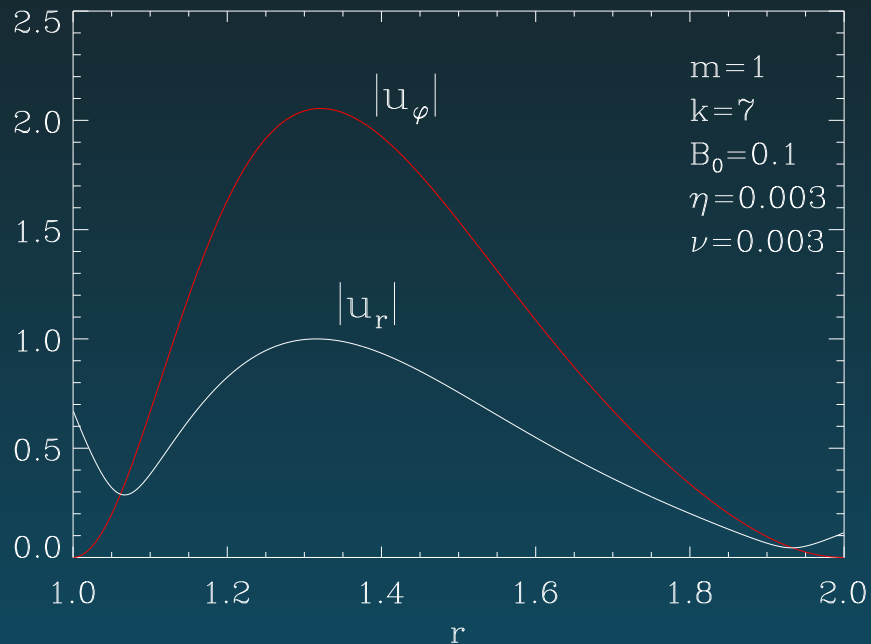
- Non axisymmetric $m = 1$ modes globally unstable
- Quenching by the magnetic tension
- **Saturation:** $\gamma_{\max} \rightarrow r_1/2 |d\Omega/dr|_{r_1}$

Dissipative MRI Modes



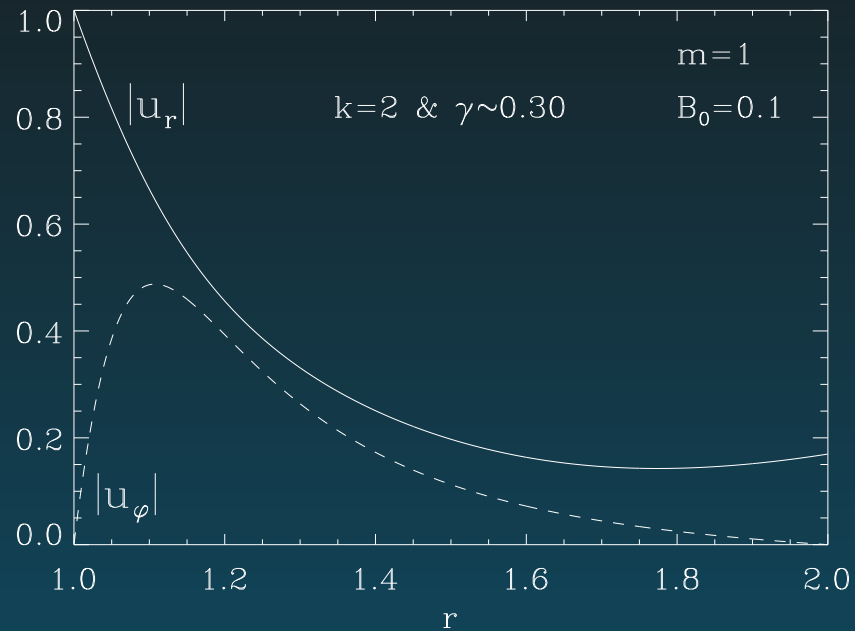
- Shape of the modes slightly modified by the inflow and the dissipation

Dissipative MRI Modes



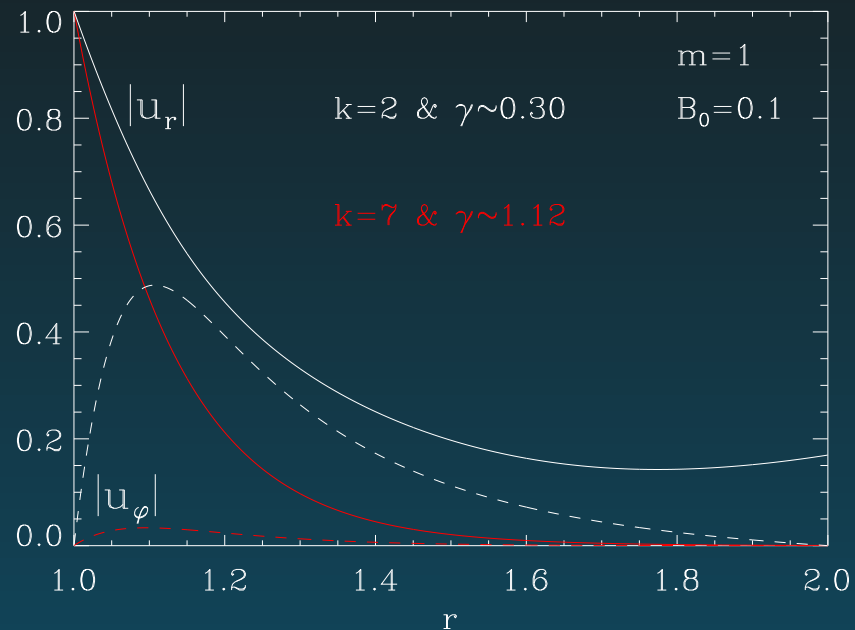
- Shape of the modes slightly modified by the inflow and the dissipation
- Growth rates globally reduced
- **Damping of the small-scale modes**

Wall Modes



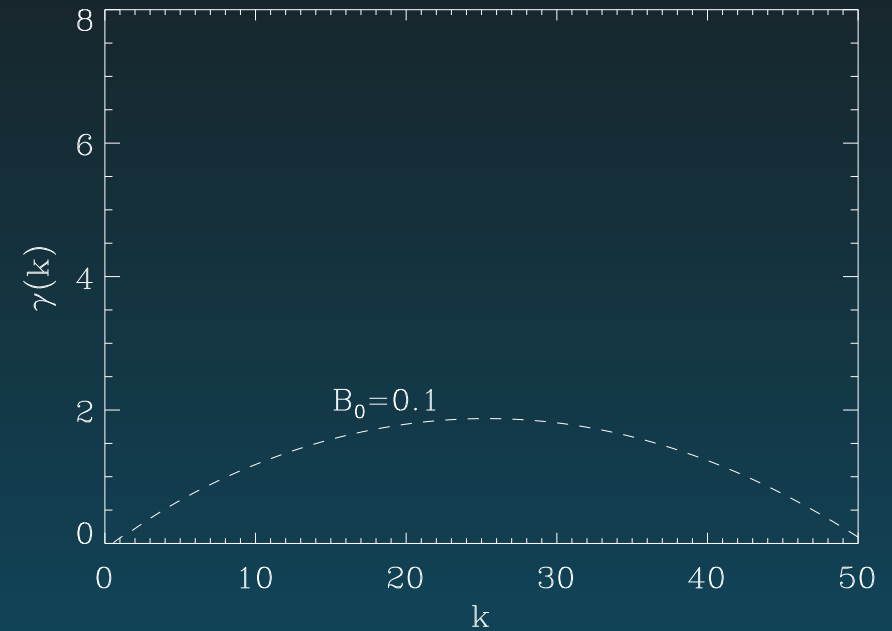
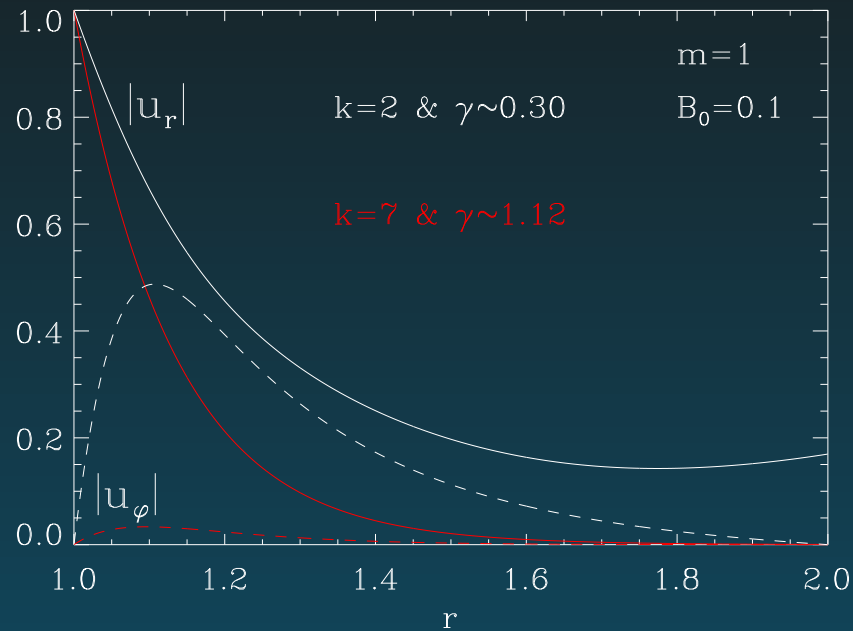
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Wall Modes



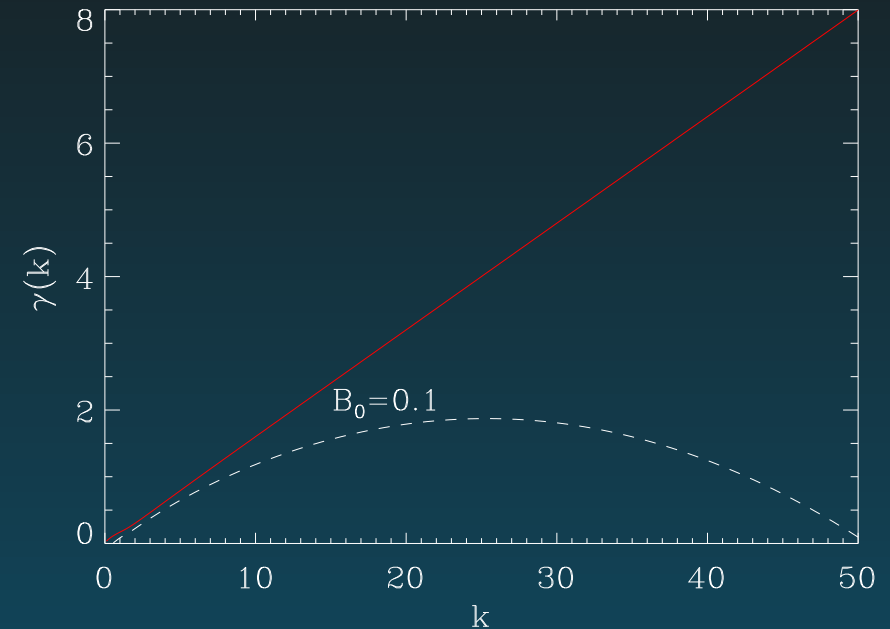
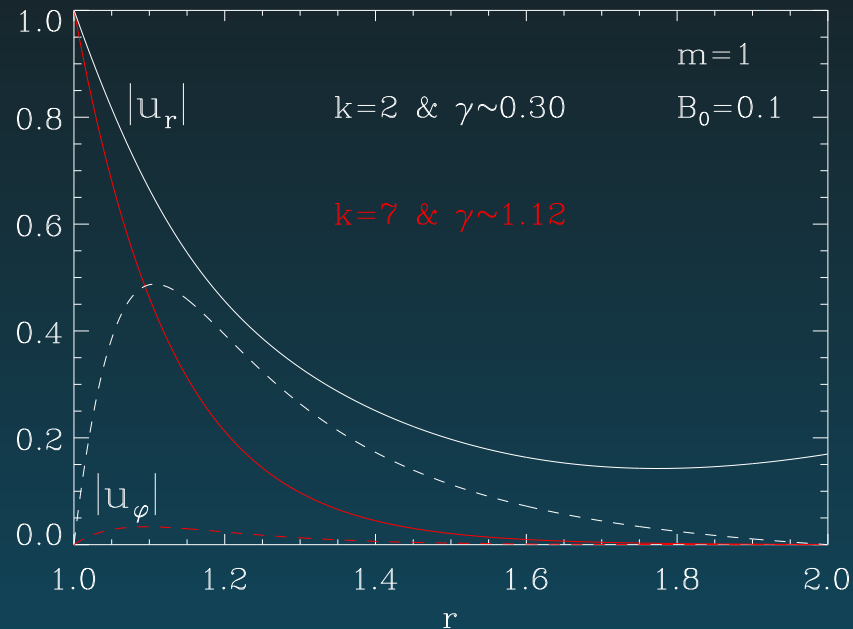
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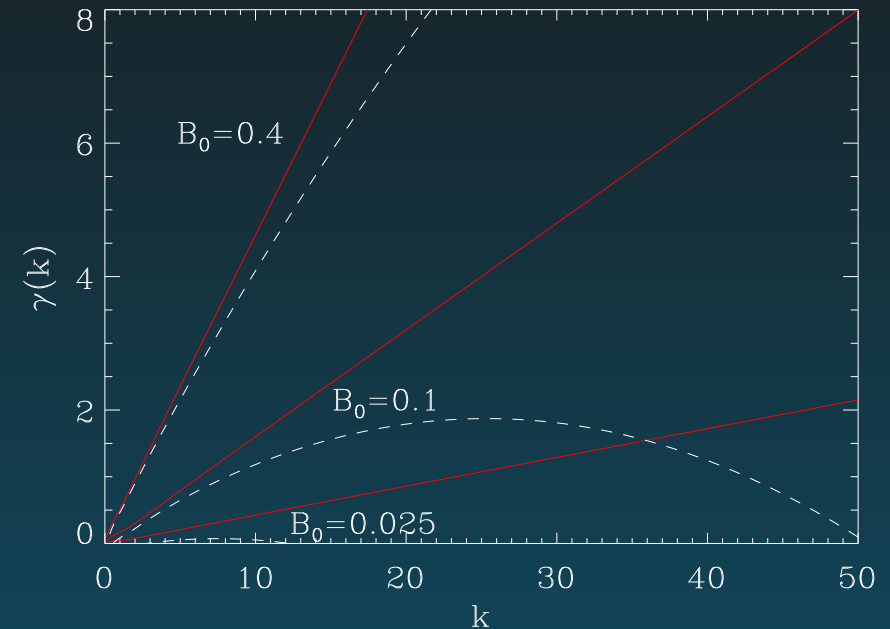
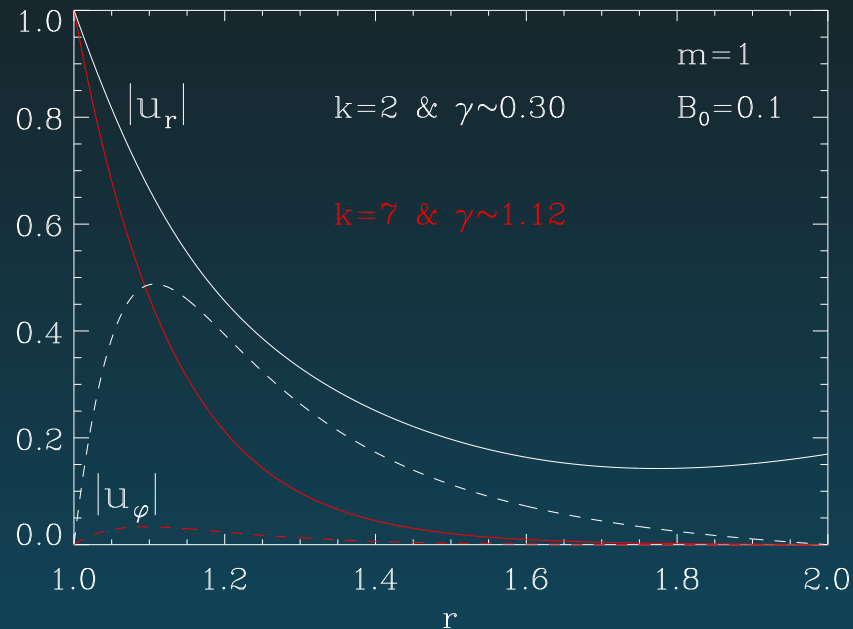
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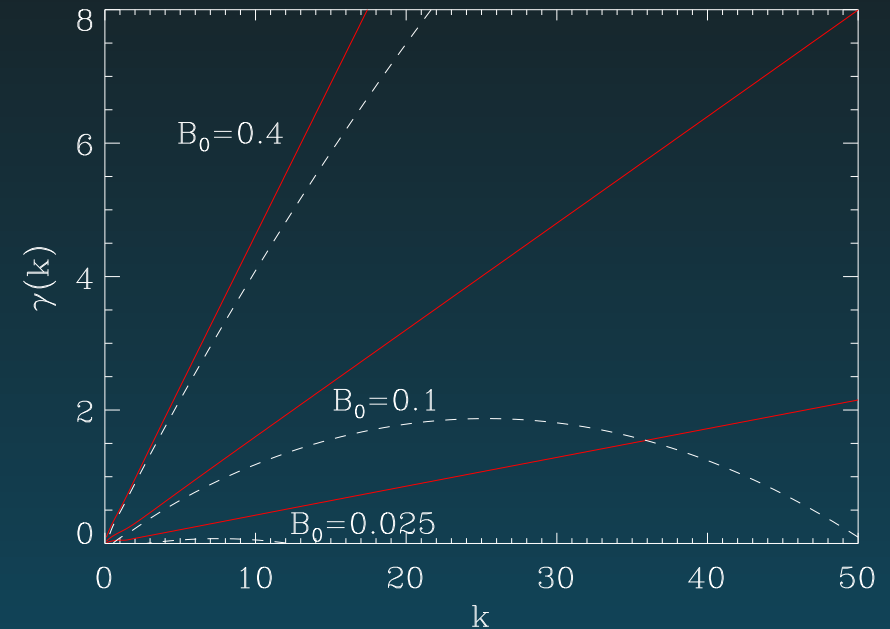
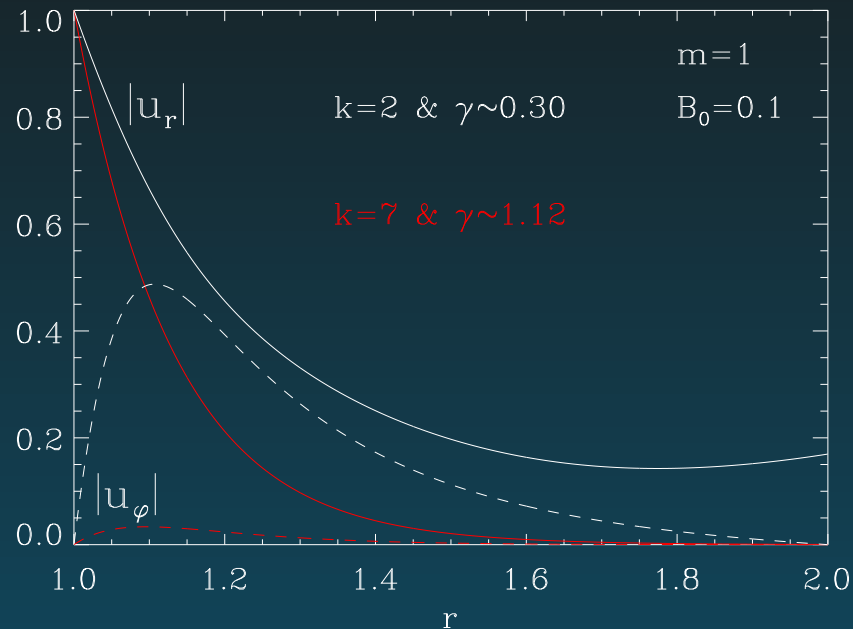
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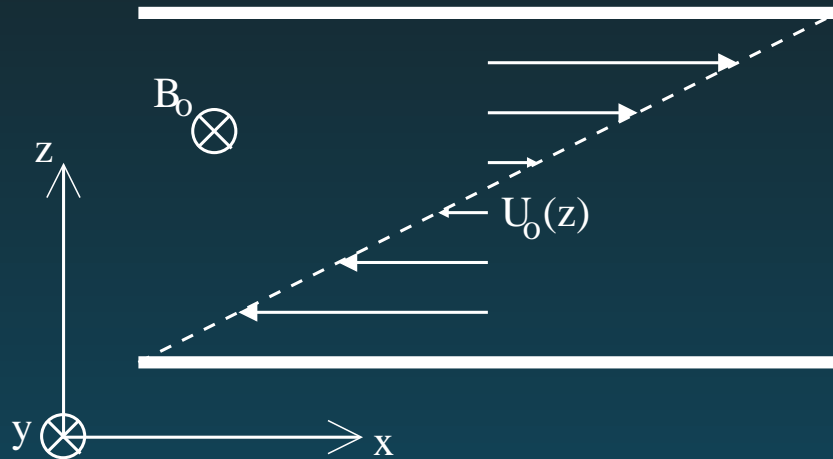
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Wall Modes



- Wall modes solutions of the linear system too
- γ too large for the free energy available and large range of k unstable
- Ideal case: γ scales linearly with k and increases rapidly with B_0
- Significant flux of energy through the boundaries to feed these modes
- Inflow, curvature and Coriolis force non crucial

Cartesian Linear Shearing Flow



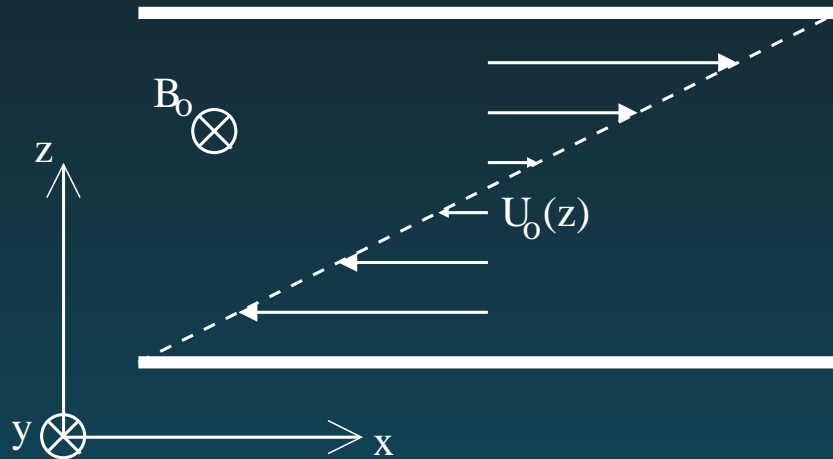
Incompressible, non dissipative basic state:

$$\rho_0 = 1$$

$$\mathbf{U}_0 = z \mathbf{e}_x, \quad z \in [-z_0, +z_0]$$

$$\mathbf{B}_0 = B_0 \mathbf{e}_y$$

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2nd order system of linear ODEs:

$$\chi \mathcal{H} u_x = -U_0' \mathcal{H} u_z - ik_x \pi$$

$$\chi \mathcal{H} u_y = -ik_y \pi$$

$$\chi \mathcal{H} u_z = -\pi'$$

$$0 = ik_x u_x + ik_y u_y + u_z'$$

where,

$$\underline{\mathcal{K}}(\mathbf{x}, t) = \underline{\kappa}(z) \exp(\sigma t + ik_x x + ik_y y)$$

$$\omega_a = kB_0$$

$$\chi = \sigma + ik_x U_0$$

$$\mathcal{H} = (1 + \omega_a^2 / \chi^2)$$

Cartesian Wall Modes: HD Limit

Hydrodynamic limit: $\omega_a = 0$ and $\mathcal{H} = 1$

HD modes are solutions of $\chi \left[u_z'' - \left(k^2 + \frac{\chi''}{\chi} \right) u_z \right] = 0$

Linear shear $\Rightarrow \chi'' = 0$ and $u_z = c_- \exp(-kz) + c_+ \exp(kz)$

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- No discrete mode with the BCs $u_z = 0$, only a continuum of stable modes

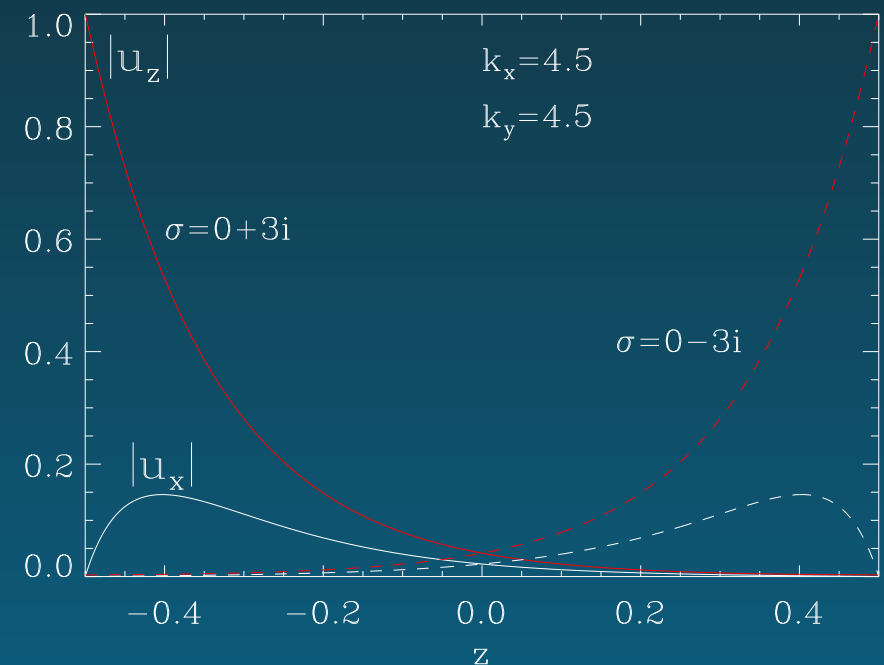
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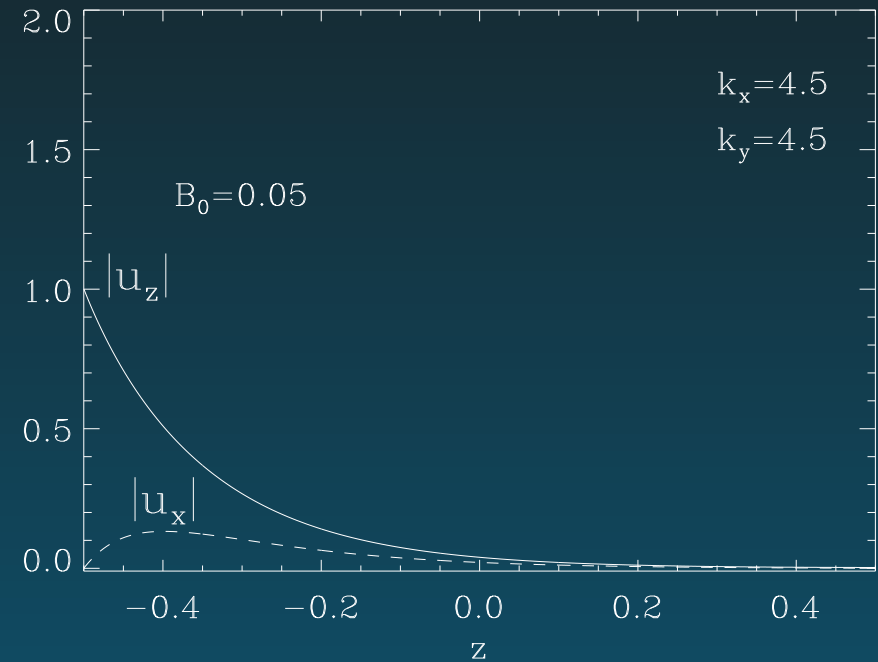
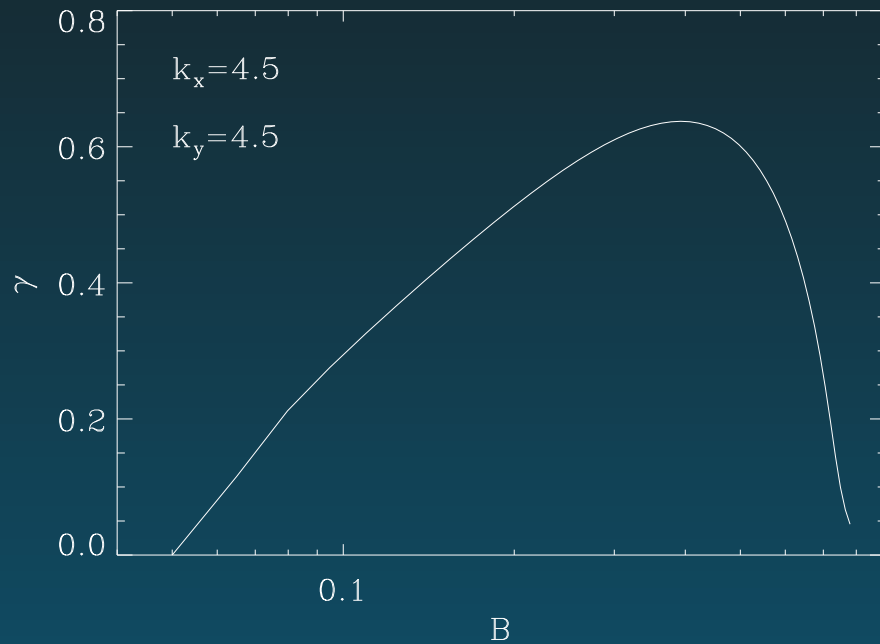
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- **Neutral wall modes solutions with BCs $u_x = 0$**



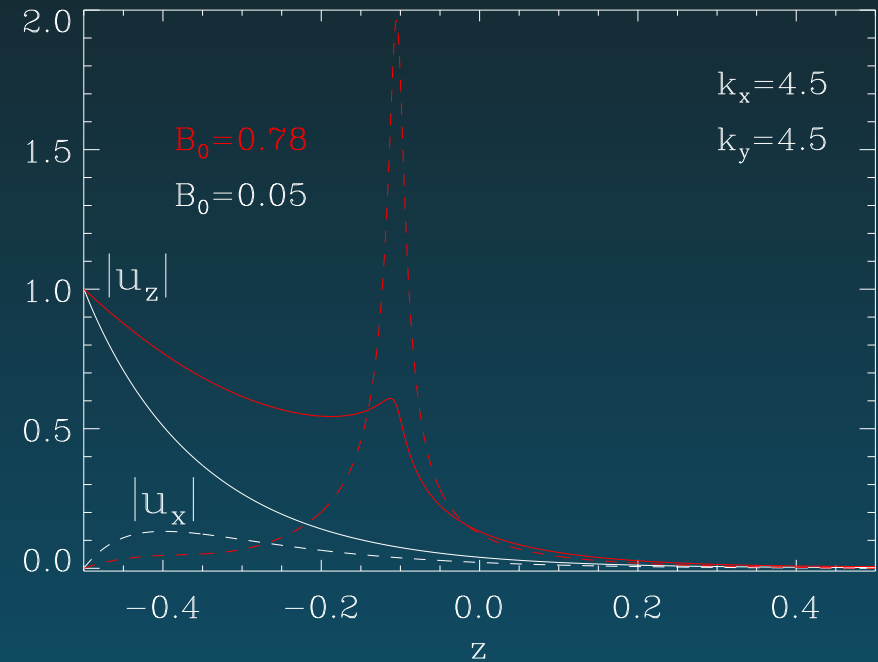
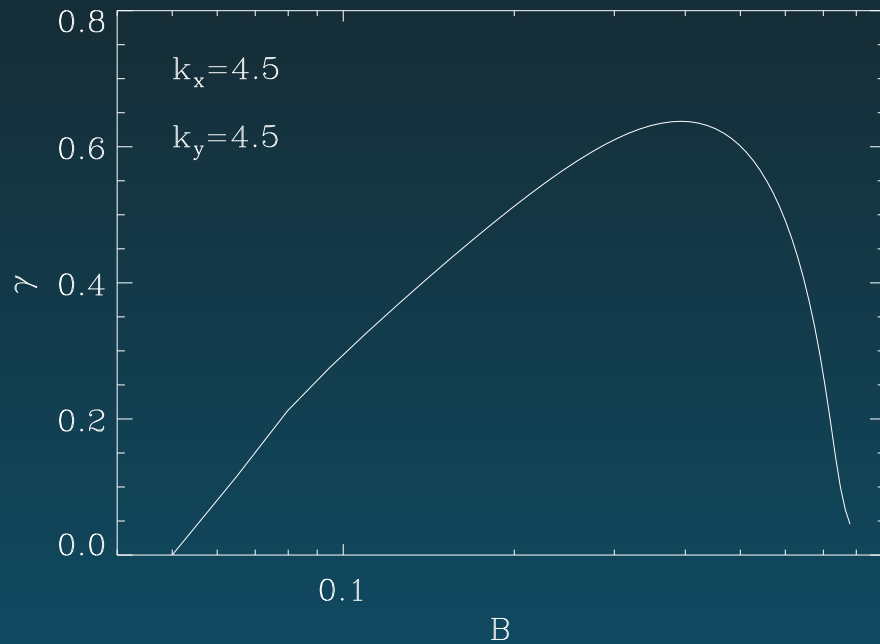
Cartesian Wall Modes in MHD

Magnetic field destabilizes the wall modes



Cartesian Wall Modes in MHD

Magnetic field destabilizes the wall modes



$$u_z'' - 2 \frac{\omega_a^2}{\chi^2 + \omega_a^2} \frac{\chi'}{\chi} u_z' - \left[k^2 + \frac{\chi''}{\chi} - 2 \frac{\omega_a^2}{\chi^2 + \omega_a^2} \left(\frac{\chi'}{\chi} \right)^2 \right] u_z = 0$$

Singularity when $\gamma = 0$ and $\omega = -k_x U_0 \pm \omega_a$

Origin of the Instability

Analyse on the boundaries:

$$\gamma^2 = \frac{(\mathcal{S}^2 U'_0)^2}{(\mathcal{S}^2 U'_0)^2 + \omega_a^2 k^2 k_x^2} \left[(U'_0)^2 \frac{\mathcal{K}^2}{k_x^2} + \omega_a^2 k^2 \mathcal{L}^2 - \omega_a^2 \right]$$

where $\mathcal{S}^2 = |\pi''|/|\pi|$, $\mathcal{K} = |\pi'|/|\pi|$, $\mathcal{L}^2 = |\pi| |\pi''|/|\pi''|^2$ and $k_x = k_y$

B_0 and U'_0 both non zero at either of the boundaries \Rightarrow wall modes unstable

Mechanism:

$$u_z \xrightarrow{k_y B_0} b_z \xrightarrow{U'_0} b_x \xrightarrow{k_y B_0} T_x = k_y B_0 b_x$$

In the vicinity of the boundary $u_x \sim 0 \Rightarrow$ energy from the outside required to balance T_x

Conclusion about Accretion Discs

- Wall modes are solutions of incompressible shearing flows when rigid BCs are relaxed
- B_0 makes them linearly unstable if $u_\varphi = 0$ on the boundaries

Forcing the differential rotation of the boundaries impossible unless Ω'_0 or B_0 locally zero

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or

BCs on the pressure to keep it low in agreement with quasi-keplerian accretion discs...