Global magnetorotational instability with inflow
The non-linear regime

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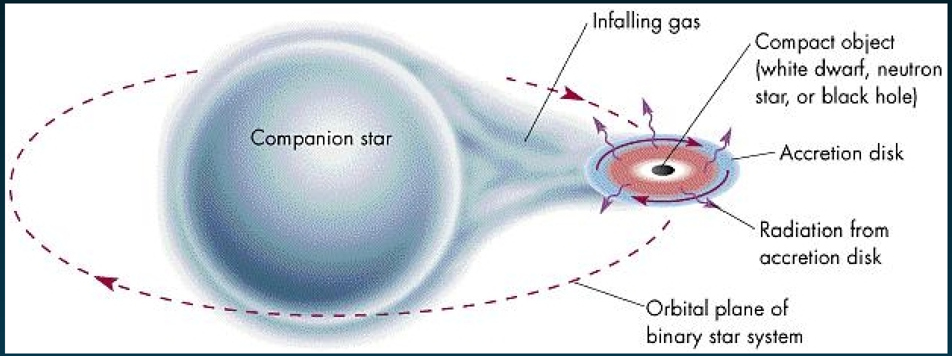
Accretion discs

– Accretion consists in an accumulation of matter onto a massive central body

– Angular momentum conservation leads to a balance between gravity and centrifugal forces and hence to the formation of a disc



– Accretion discs are found in interacting binary stars (neutron star or black hole)
young stars (site of planetary formation)
centre of active galaxies (supermassive black hole)

– Orbiting gas can be accreted if angular momentum is removed by a torque acting on a within the disc

– Shear viscosity fails in transporting angular momentum ⇒ effective shear due to interacting eddies in a turbulent flow needed

Which unstable mechanism can drive turbulence in accretion discs?

Instabilities and turbulent transport in accretion discs

Turbulent transport of angular momentum arises from statistical correlations:
Reynolds stress tensor $\propto \langle \rho U_r \delta U_\varphi \rangle$ and the Maxwell stress tensor $\propto -\langle B_r B_\varphi \rangle$

Onset of turbulence and transport enhancement:

– HD shear instability: Keplerian discs linearly stable $[\mathrm{d}(r^2\Omega)/\mathrm{d}r > 0]$
finite amplitude instabilities still a matter of some controversy

– convective instabilities transport angular momentum inwards

– Papaloizou-Pringle instability (thick discs only): saturation in a strong spiral pressure wave (not in turbulence)

– accretion-ejection instability, magnetic buoyancy, baroclinic & stratification effects

– Magnetorotational instabilities lead to anisotropic turbulence, transport and magnetic field amplification

Magnetorotational instabilities in accretion discs

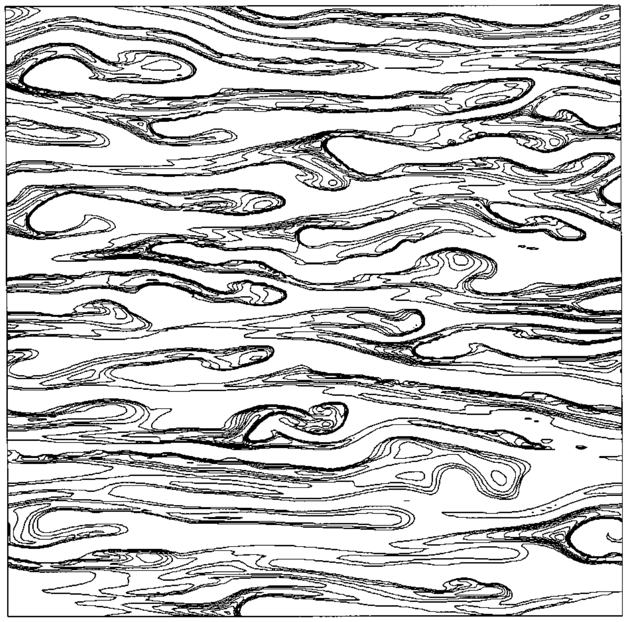
Velikhov 1959, Chandrasekhar 1960, Balbus & Hawley 1991

Local linear analysis:

– Weakly magnetised ($\beta \gg 1$) rotating shearing flows unstable if $\mathrm{d}\Omega/\mathrm{d}r < 0$

– Free energy $\equiv$ differential rotation $\Rightarrow$ MRI extremely powerful $\gamma \sim r|\mathrm{d}\Omega/\mathrm{d}r|$

Fingering instability: small vertical length scale & large radial length scale



Hawley & Balbus 1992

2-D Studies:

– $\langle B_z \rangle \neq 0$: channel flow unstable in 3-D

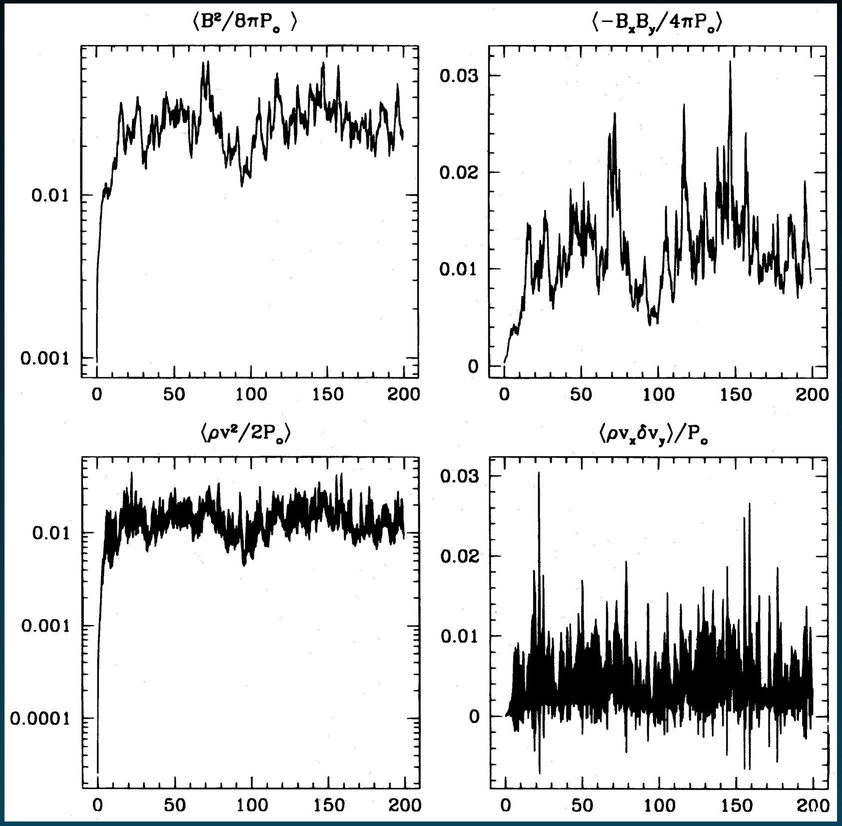
– $\langle B_z \rangle = 0$: turbulence decaying on a resolution dependent timescale

3-D Studies:

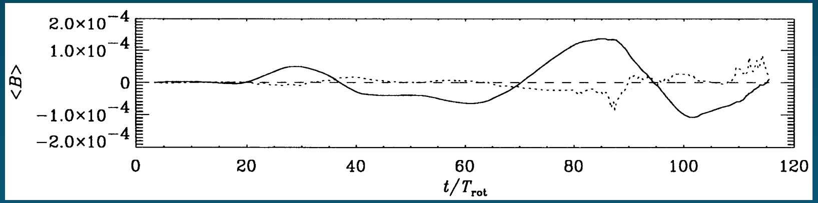
– $\langle B_z \rangle$ or $\langle B_\varphi \rangle \neq 0$ determine the saturation level of turbulence

– no dependence on initial B_z if its mean value is 0 but $\langle B^2 \rangle$ relies on the level of numerical dissipation

Magnetorotational instability & dynamo



Hawley et al. 1996



Brandenburg et al. 1995

Non-linear dynamics:

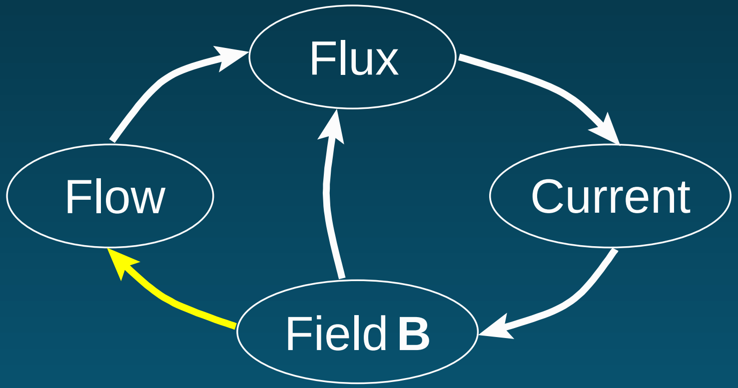
– developed MHD turbulence

– Reynolds & Maxwell stresses compatible with accretion

Dynamo action:

– MHD turbulence-driven (not HD turbulence)

– bootstrapping process



– large scale magnetic field

Numerical Investigation of the MRI

Main studies: local 2D & 3D, vertical stratification, full disc (torus)

Simulations of torus of accretion: – lack of resolution to solve small scale dynamics

– run-down computations (no stationary state)

Shearing-box approximation:

– local & periodic (semi-periodic radially)

– curvature neglected but coriolis & centrifugal forces included

– angular velocity linearized: vorticity gradient not taken into account

– no net transport (radial symmetry)

– periodicity implies strong constrains on the evolution of mean the magnetic field

Turbulence & transport properties rely mostly on numerical dissipation

No strong conclusions regarding $\langle \mathbf{B} \rangle$

Features of the model

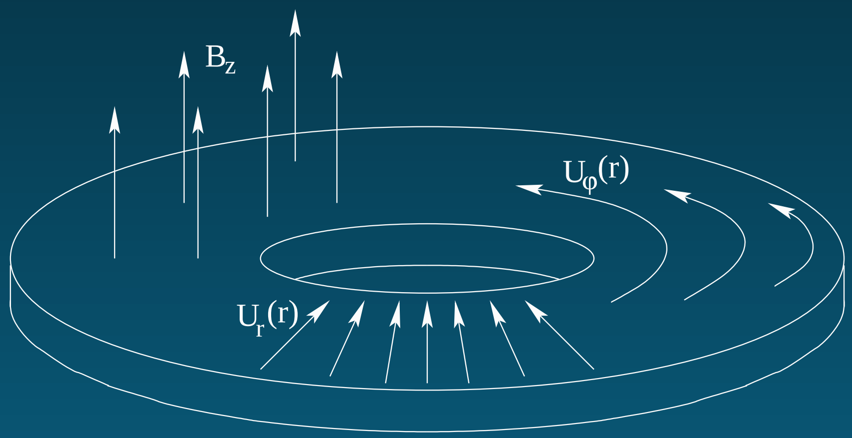
Model to address saturation mechanism of MRIs and nature of dynamo action

Key features of the model:

– global annular section (i.e. including curvature) as opposed to local shearing box

– explicit treatment of dissipative processes (understanding of small scales dynamics)

– permeable radial boundary conditions: permit accretion
ability to reach a steady state



Magnetised Keplerian shearing flow driving accretion flow through viscous torque

Global model with accretion

Evolution equations: incompressible non-ideal MHD

$$(\partial_t + \mathbf{U}\cdot\nabla)\,\mathbf{U} = -\nabla\Phi - \nabla\Pi + \mathbf{B}\cdot\nabla\mathbf{B} + \nu\nabla^2\mathbf{U}$$

$$(\partial_t + \mathbf{U}\cdot\nabla)\,\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{U} + \eta\nabla^2\mathbf{B}$$

$$\nabla\cdot\mathbf{B} = \nabla\cdot\mathbf{U} = 0$$

Basic state: magnetised Keplerian shearing flow with accretion (φ- and z-invariant)

$$U_r = -3\nu/2r, \quad U_\varphi = 1/\sqrt{r}, \quad U_z = 0, \quad B_r = B_\varphi = 0, \quad B_z = B_0, \quad \Pi = \delta - 9\nu^2/8r^2$$

Radial boundary conditions: permeable (no conditions on U_r and B_r)

$$\partial_r(\sqrt{r}U_\varphi) = 0, \quad \partial_r U_z = 0, \quad \Pi = \Pi_0, \quad \partial_r(rB_\varphi) = 0, \quad \partial_r B_z = 0$$

Linear stability theory

Linear evolution equations:

– linearization: put $\{\mathbf{U}, \mathbf{B}, \Pi\} = \{\mathbf{U}_0, \mathbf{B}_0, \Pi_0\} + \varepsilon \boldsymbol{\mathcal{K}}, \varepsilon \ll 1$ and neglect terms of order ε^2

– normal modes: $\boldsymbol{\mathcal{K}}(\mathbf{r}, t) = \boldsymbol{\kappa}(r)\ \exp(\sigma t + im\,\varphi + ik\,z)$

– 10th order linear system: $\sigma\,\boldsymbol{\mathcal{I}}(r)\ \boldsymbol{\kappa}(r) = \boldsymbol{\mathcal{L}}(r)\ \boldsymbol{\kappa}(r)$ (generalised eigenvalue problem)

– Π evolves on much shorter time scales: $\nabla \cdot \mathbf{U} = 0 \Rightarrow \mathcal{I}_\pi = 0$

– linearized boundary conditions:
$\mathrm{d}_r(\sqrt{r} u_\varphi) = 0, \quad \mathrm{d}_r u_z = 0, \quad \pi = 0, \quad \mathrm{d}_r(r b_\varphi) = 0, \quad \mathrm{d}_r b_z = 0$

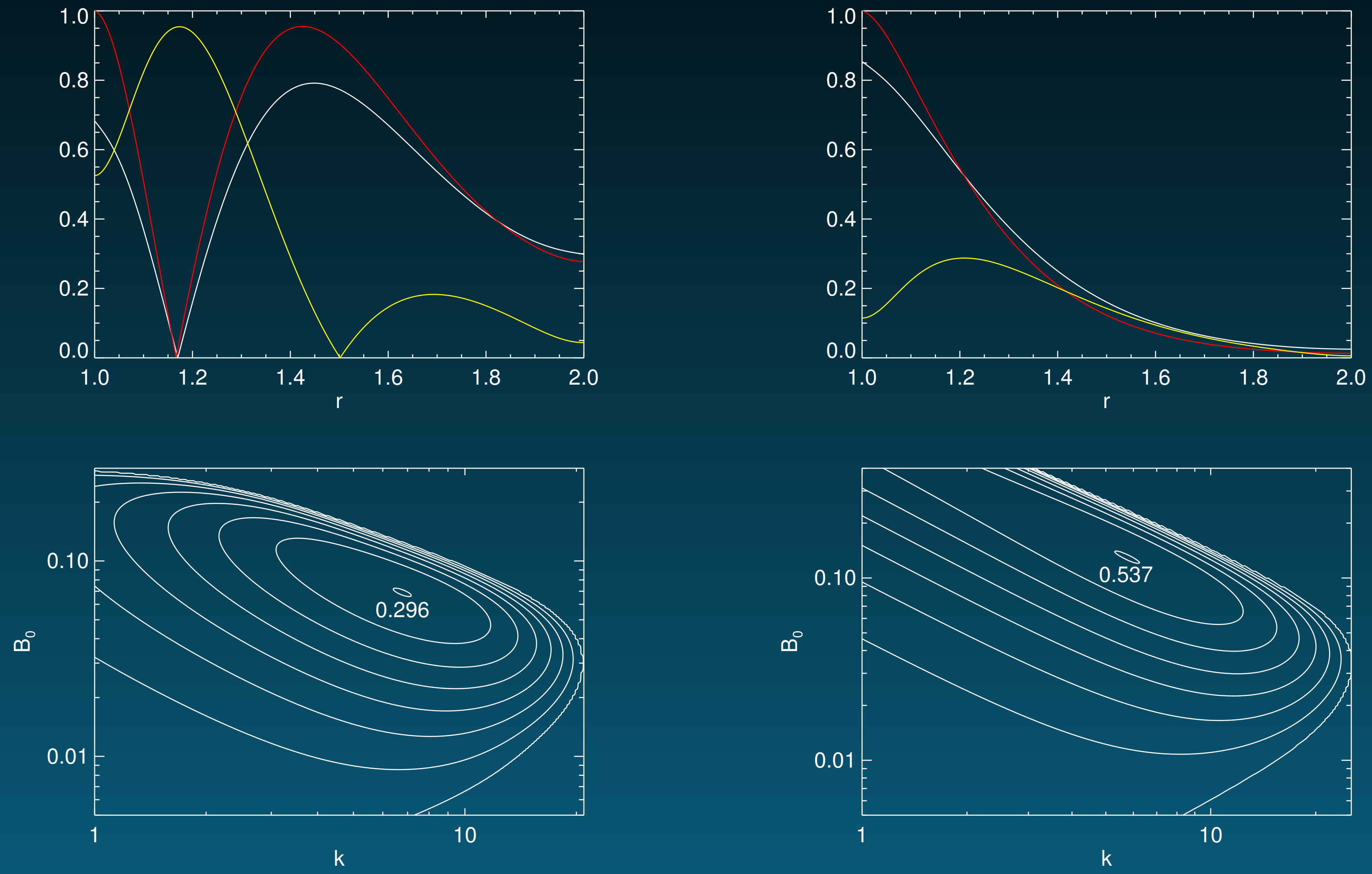
Numerical solution:

– inverse iteration

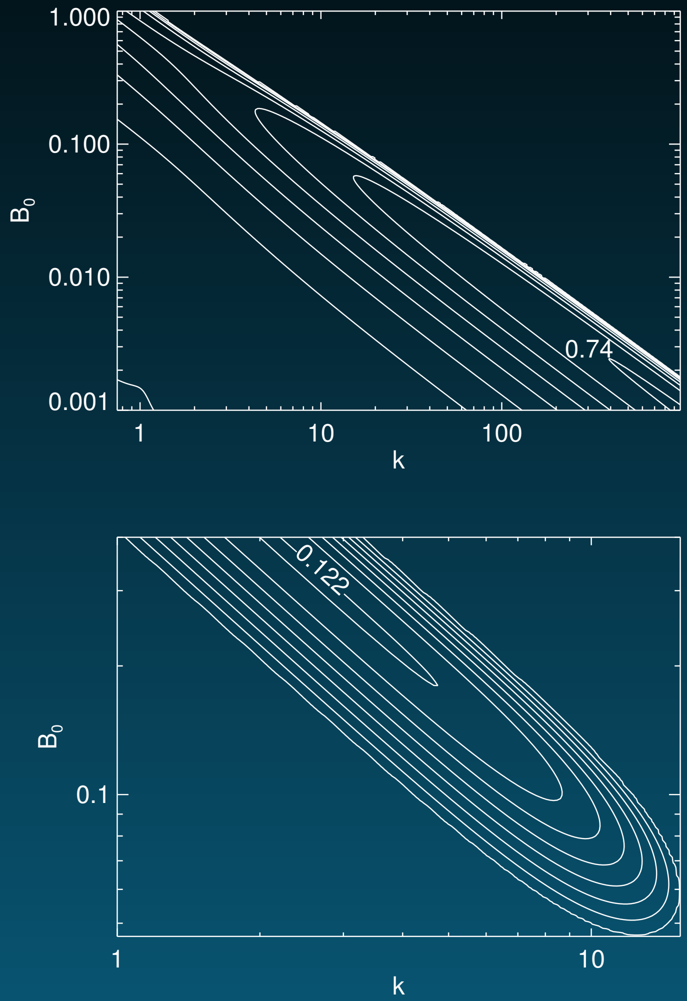
– shooting (double checking)

Unstable modes

Permeable radial boundaries permit the development of wall-modes as well as body-modes



Wall modes



Wall modes properties:

- axisymmetric wall modes are the most unstable of all
- local results recovered in the limit $k \to \infty$
- large range of unstable azimuthal wave-numbers; range shrinks in the presence of large amplitude B_φ
- boundary condition $U_\varphi = U_{\varphi 0}$ has been found to enhance instability artificially
- large parameter space survey

(See Kersalé et al. 2004, ApJ)

Non-linear evolution

Linear theory of prime importance in elucidating:
- instability mechanism
- pertinent regimes of parameter

but complete understanding of MRIs attained only by investigating non-linear evolution

Only a few analytical non-linear results:

Goodman & Xu 1994: 2-D linear modes also solutions of the non-linear equations

Knobloch & Julien 2005: fully non-linear equilibrated solution using asymptotic expansions

Although important, cannot capture all the complexity of non-linear evolution of MRIs

Imperative to make use of numerical computations

Numerical scheme: Space

Requirements: – explicit treatment of dissipative effects
– cylindrical geometry
– permeable boundary conditions

Pseudo-spectral method: accurate & fast

– Spectral decomposition: $X(r,\varphi,z) = \sum_{l,m,k} \mathcal{X}_{lmk} T_l(s)\, e^{im\varphi}\, \{\cos,\sin\}(kz)$,
with $T_l(s) = \cos(l \cos^{-1} s)$ & $2r = [(r_1 - r_2)\, s + r_1 + r_2]$, $s \in [-1,+1]$

– Differentiations performed using properties of trial functions
(Fourier: multiplication & Chebyshev: recurrence relations)

– Non-linear terms computed in configuration space making use of FFTs

Boundary conditions:

– Radial: Chebyshev-Tau (linear relations between the expansion coefficients)

– Azimuthal and vertical: Fourier-Galerkin (satisfied by trial functions)

Numerical scheme: Time

Advance in time: accuracy & stability

– linear terms: Crank-Nicholson (implicit)

– non-linear terms: 4^{th} order Adams-Bashforth with adaptive time step (explicit)

$$\nabla^2 \Gamma^{n+1} = \nabla \cdot \boldsymbol{\mathcal{N}}_u^n$$

$$[\mathcal{I} - \Theta\, \delta t\, \mathcal{L}]\, \mathbf{U}^{n+1} = -\nabla \Gamma^{n+1} + \mathcal{N}_{\mathbf{u}}{}^n + [\mathcal{I} + (1-\Theta)\, \delta t\, \mathcal{L}]\, \mathbf{U}^n$$

$$[\mathcal{I} - \Theta\, \delta t\, \mathcal{L}]\, \tilde{\mathbf{B}}^{n+1} = \mathcal{N}_{\mathbf{b}}{}^n + [\mathcal{I} + (1-\Theta)\, \delta t\, \mathcal{L}]\, \mathbf{B}^n$$

$$\mathbf{B}^{n+1} = \tilde{\mathbf{B}}^{n+1} - \nabla \Psi \quad \text{with} \quad \nabla^2 \Psi = \nabla \cdot \tilde{\mathbf{B}}^{n+1}$$

With $\Theta \in [0,1], \Gamma = \delta t\, \Pi, \mathcal{L} = \nu \nabla^2$ and $\boldsymbol{\mathcal{N}}$ represents the nonlinear terms

– Differential operator $(\mathcal{I} - \Theta\, \delta t\, \mathcal{L})$ non trivial (Chebyshev polynomials in an annulus)

– Divergence-free constraints give boundary conditions to compute U_r and B_r.

Non-linear evolution of axisymmetric wall modes

Numerical investigation:

– basic state: Keplerian shearing flow
viscosity-driven accretion
uniform vertical magnetic field

– non-dimensionalised equations: $R_{\text{in}} = 1$, $\Omega(R_{\text{in}}) = 1$, B in units of the Alfvén speed

– parameter space survey: $R_{\text{out}} = \{2, 4\}$, $H = 0.5$,
$B_0 \simeq 10^{-1}$-10^{-2}, $\nu = \eta \simeq 3.5 \times 10^{-3}$-$10^{-4}$

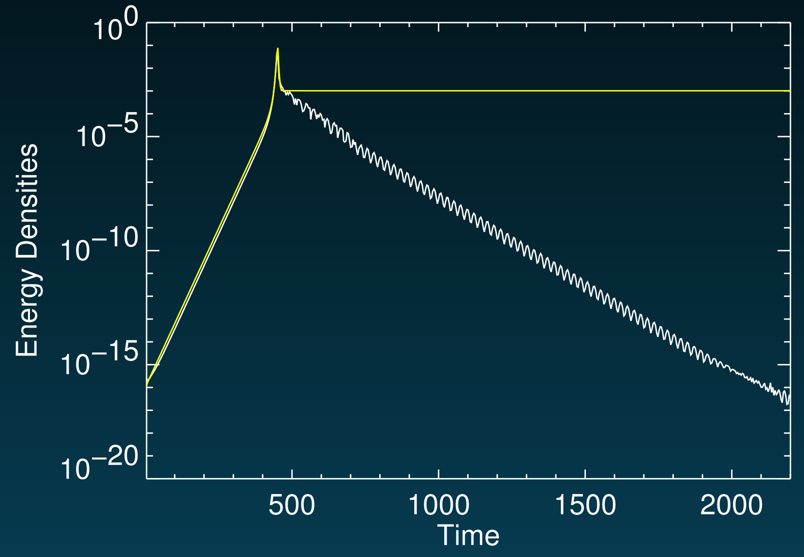
– parameter values chosen such that only one wall mode linearly unstable

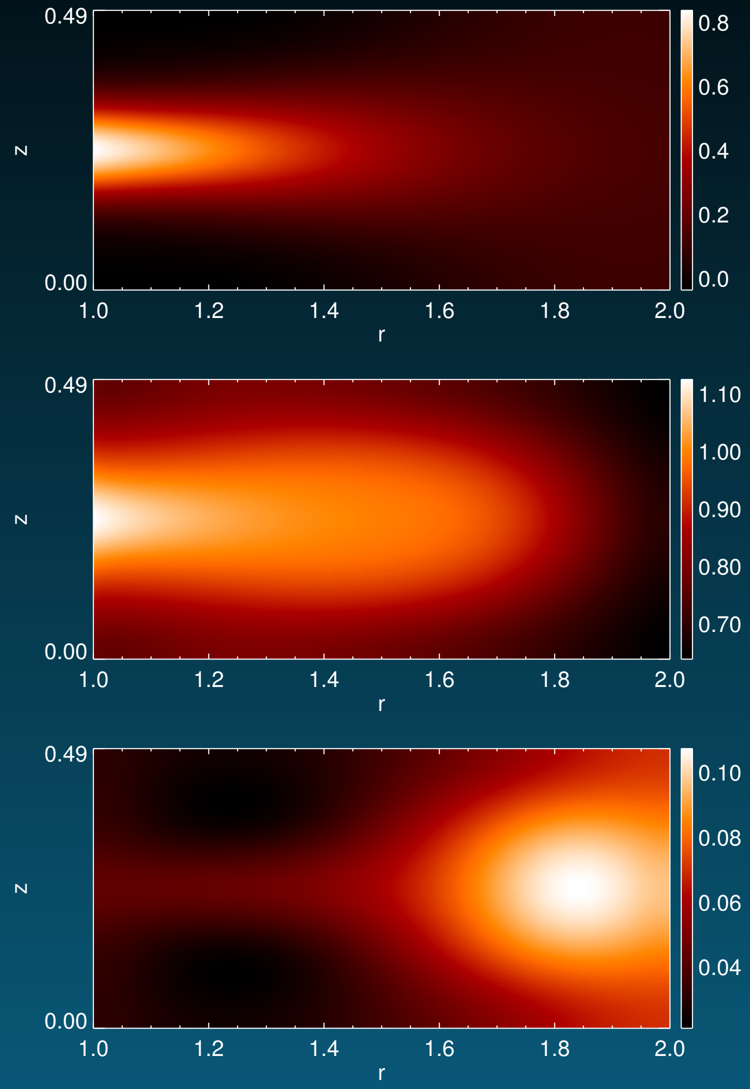
– Instability triggered by random small amplitude perturbations of U_φ

– exponential growth in agreement with linear theory

– non-linear behaviour mediated by coherent structures:
$\longrightarrow$ suppression of MRIs
$\longrightarrow$ cyclic evolution
$\longrightarrow$ relaxation to a non trivial steady state

Self-consistent suppression of the instability





Evolution:

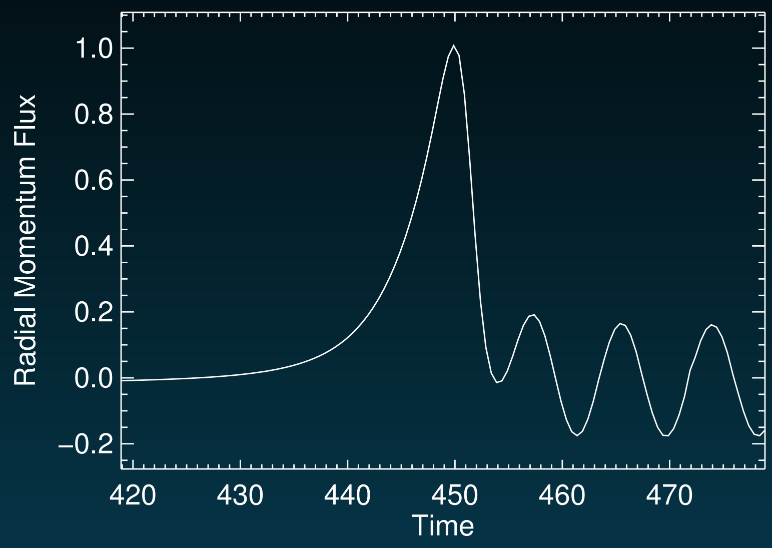
– exponential growth

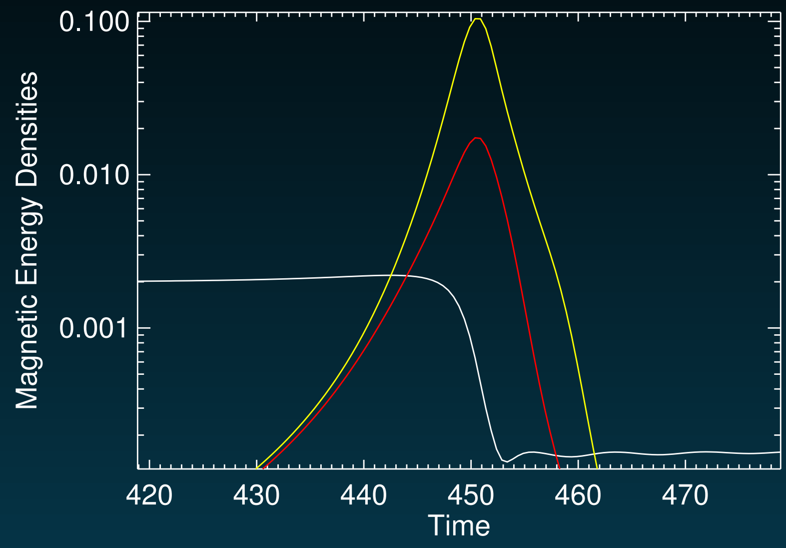
– non-linear readjustments

– relaxation to a new stable Keplerian equilibrium

Coherent radial jet transports magnetic flux outwards

Removing of magnetic flux





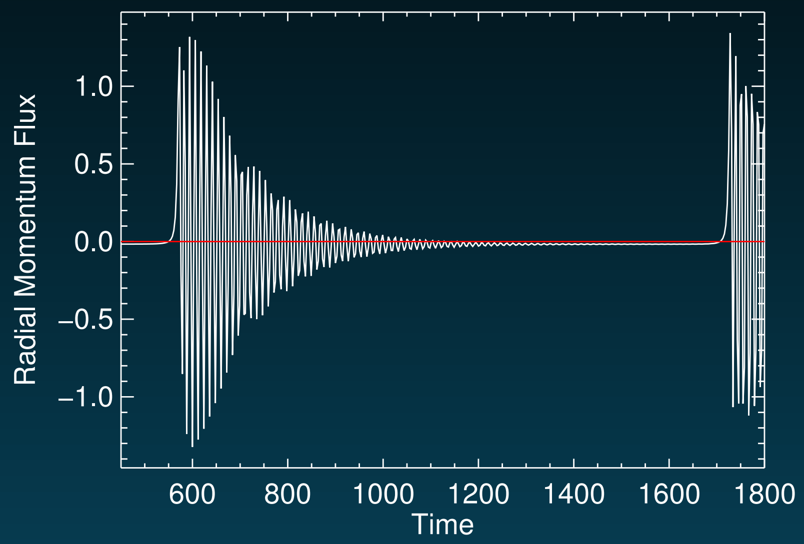
Non linear behaviour:

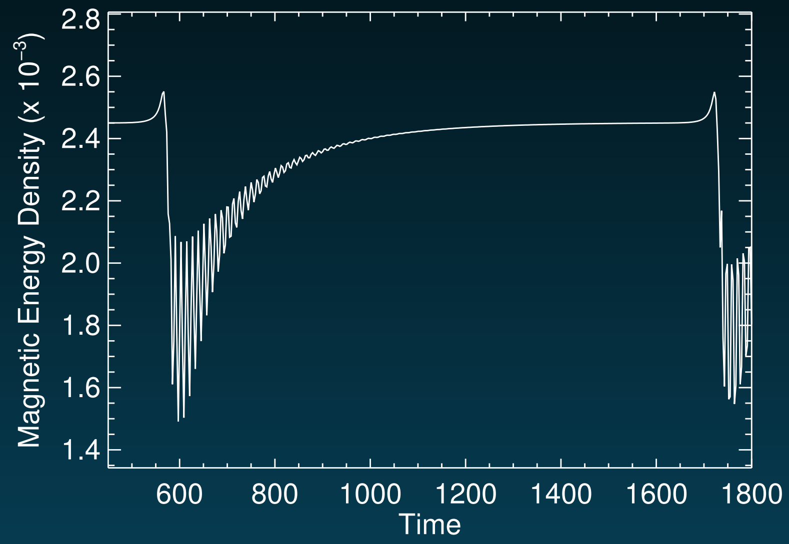
– formation of a fast radial jet: outwards flux of momentum
– super-exponential growth of the magnetic energy
– advection of vertical magnetic field out of the domain
– drop in vertical magnetic field leads to instability suppression
– whole disc undergoes epicyclic oscillations (damped on a dissipative timescale)

Relaxation to a stable equilibrium: initial Keplerian flow (with accretion) with a reduced uniform magnetic field

Radial extension of the disc

Preventing magnetic removal of flux by increasing radial extent of the disc





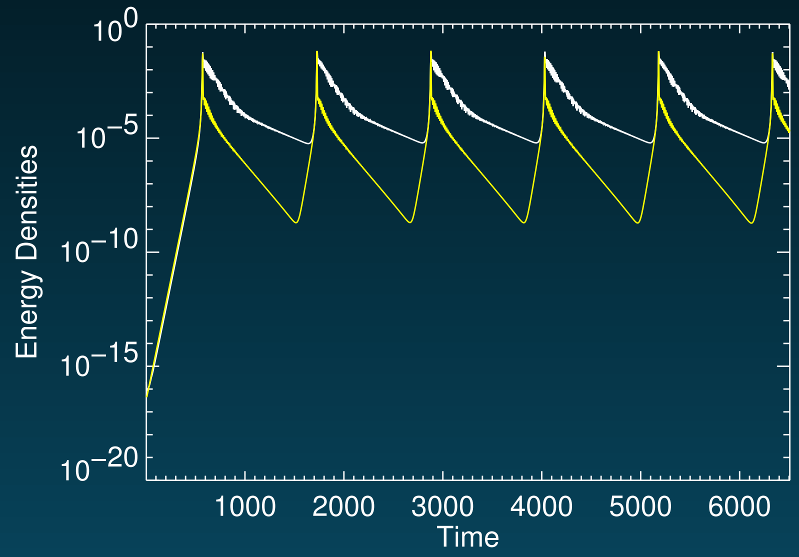
Strong radial jet: – transports B_z-flux outwards; storage in stable part of the disc
– switches-off the instability
– produces large amplitude epicyclic oscillations (viscous damping)

Accretion: – brings B_z-flux towards the centre
– relaxation to initial unstable state

Process leads to a net flux of radial momentum outwards

Relaxation oscillator

Strong radial jet but no B_z-flux expelled: – cyclic behaviour (periodic)
– multiple timescales



Dynamics successively linear and non-linear:

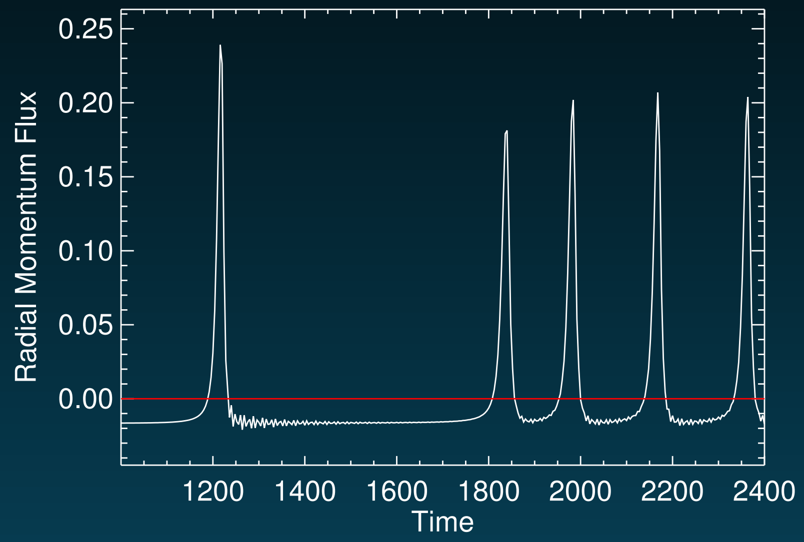
– exponential growth (linear instability)

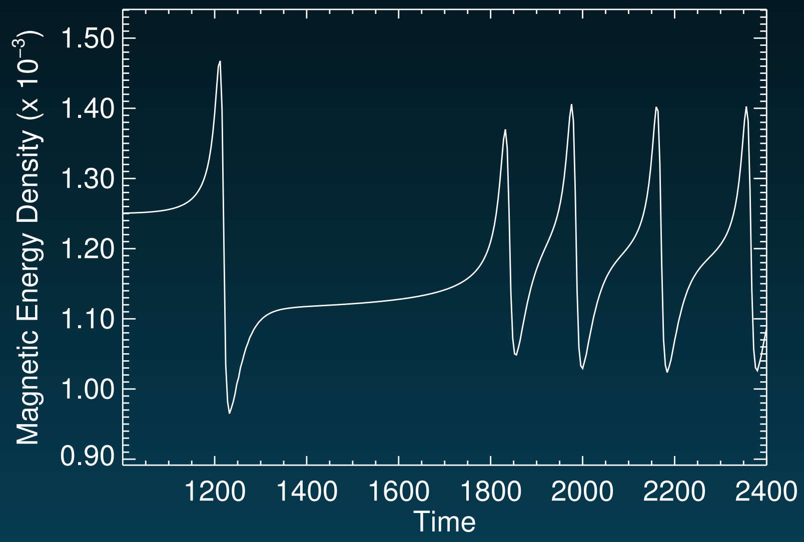
– rapid non-linear readjustments on a dynamical timescale (strong radial jet)

– dissipative relaxation to the initial unstable equilibrium on a viscous timescale

Reducing the strength of the jet

Preventing removal of magnetic flux by decreasing the strength of the non-linear jet





Successive phases — radial jet & accretion: net flux of radial momentum outwards

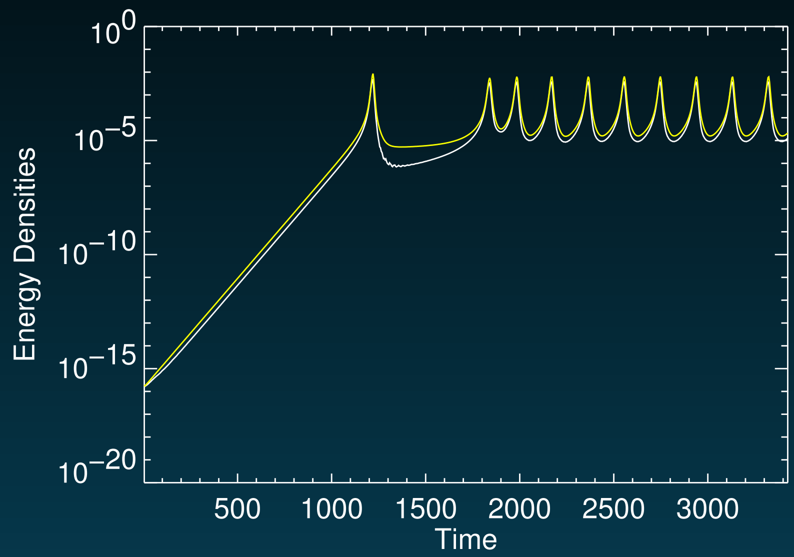
– smoother non-linear events (no large amplitude epicyclic motions)

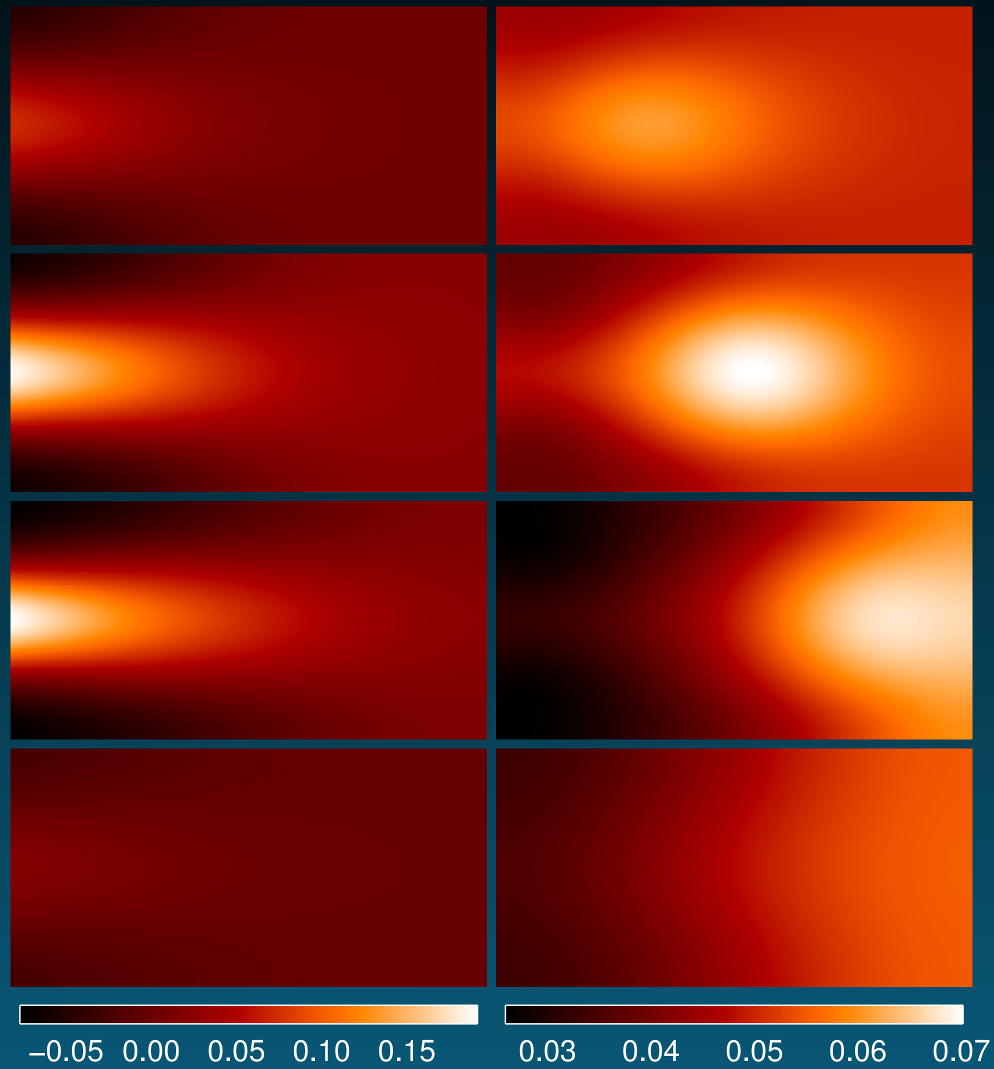
– partial loss of B_z-flux but readjustment of the system (robust process)

– relaxation to a periodic solution

– oscillations around a new equilibrium (not the initial Keplerian state)

Fully non-linear oscillations





Fully non-linear dynamics:

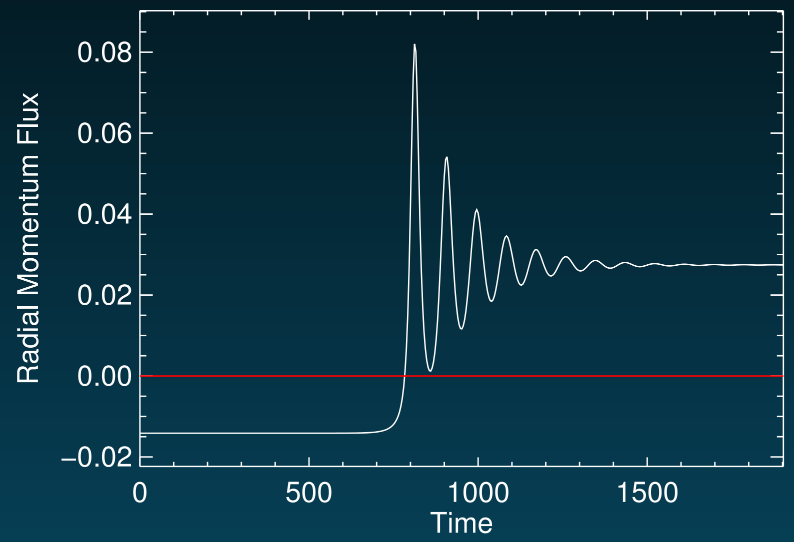
– interval between bursts shorter than a dissipative timescale

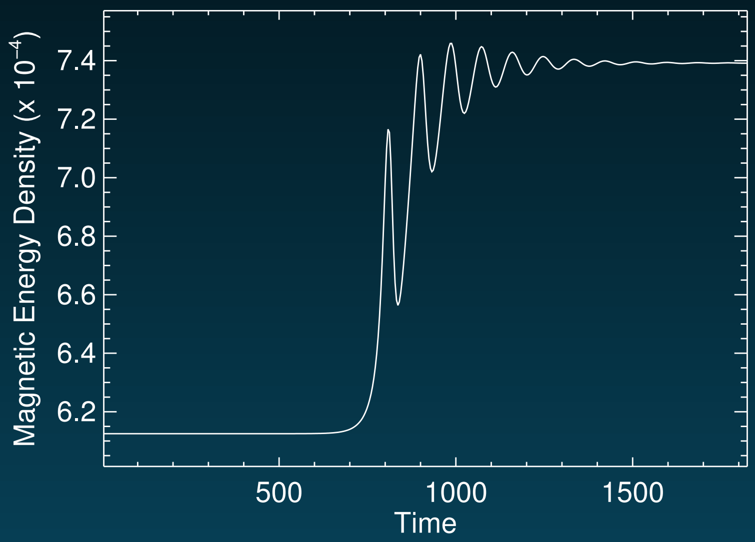
– no relaxation to Keplerian state

– competition between non-linear jet and accretion

Further reduction of jet strength

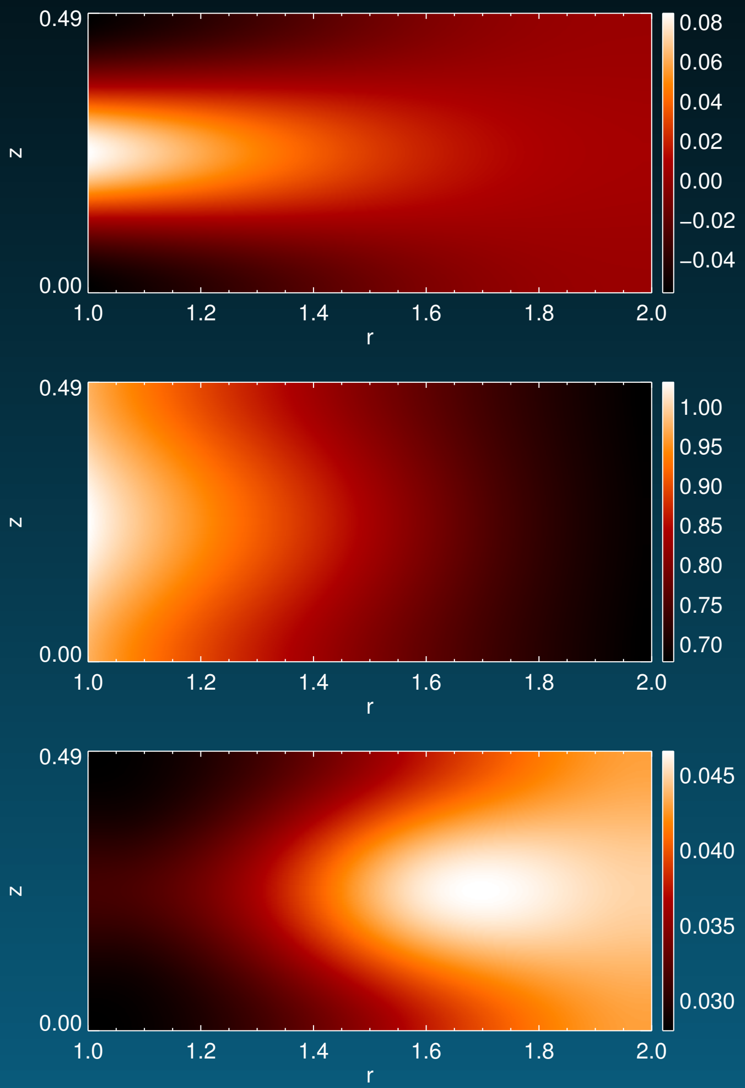
Timescale of non-linear processes sufficiently long to permit
a smooth reorganisation of the system





Evolution of the unstable wall mode:
– linear regime (MRI does not contribute to the transport)
– weak non-linear jet evolving on a dynamical timescale
– transient phase non-linear adjustments on a dissipative timescale
– relaxation to a non-trivial stable equilibrium: increased magnetic energy
outwards flux of radial momentum

Non-trivial stable stationary state



Balance between
accretion & outward MRI-driven jet

Final state:

– radial jet
– non-Keplerian shearing flow
– non-uniform magnetic field (B_r, B_φ, B_z)

Relation with fully non-linear equilibrated solution calculated by *Knobloch & Julien 2005*

Conclusions & Perspectives

Non-linear evolution of MRIs leading to:
– cyclic behaviours
– trivial or non-trivial equilibria

Saturation of the MRI relies on magnetic flux redistribution by non-linear coherent structures

Model requires key features to find the solutions calculated:
- cylindrical geometry (in & out)
- explicit treatment of dissipative processes
- permeable boundary conditions
- ability to reach a steady state (no run down computation)

Correct computation of the evolution of MRIs on very different timescales (dynamical & dissipative)

Transition to less coherent — turbulent — regimes at higher Reynolds numbers:
- 2-D Kelvin-Helmholtz
- 3-D coherent structures unstable to non-axisymmetric perturbations

Stable 2-D structures identified are building blocks: 3-D chaotic flows may well exhibit some properties of 2-D stable solutions (in a statistical sens)