Magnetorotational Instabilities in a Disc with Permeable Boundaries

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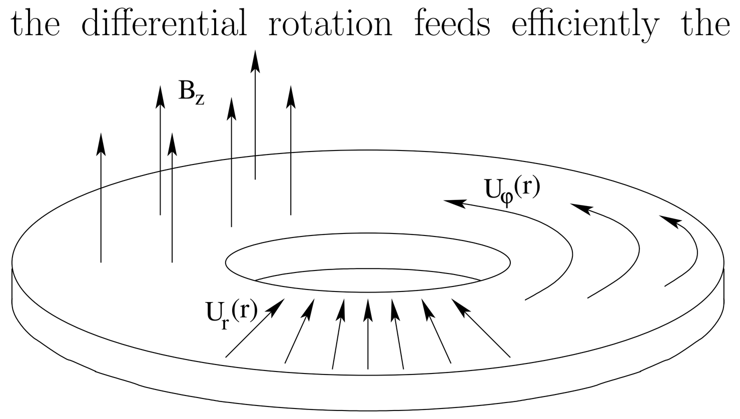
Abstract

We carefully set up a model to investigate numerically the global properties of magnetorotational instabilities (MRIs hereafter) in viscous, resistive, accretion discs with permeable radial boundaries. First, we develop the linear theory which exhibits the existence of two types of unstable modes. Then, we describe the cylindrical pseudo-spectral MHD code developed to simulate the nonlinear evolution of the MRIs. We finally discuss the properties of the coherent structures that emerge in the 2-D axisymmetric computations and we address the issue of the instability suppression.

The Model

MRIs develop in differential rotating flows where $\mathrm{d}\Omega/\mathrm{d}r < 0$. The free energy contained in the differential rotation feeds efficiently the unstable modes with growth rates $\gamma \sim r|\mathrm{d}\Omega/\mathrm{d}r|$.



We adopt a simple model that retains all the essential features required for the linear and nonlinear study of the instability: a differentially rotating cylindrical annulus, composed of an incompressible, resistive, viscous plasma threaded by a magnetic field. One particular feature of our approach is to consider permeable radial boundaries.

We look for solutions of the incompressible non ideal MHD equations

$$(\partial_t + \mathbf{U}\cdot\nabla)\,\mathbf{U} = -\nabla\Phi - \nabla\Pi + \mathbf{B}\cdot\nabla\mathbf{B} + \nu\Delta\mathbf{U},$$
$$(\partial_t + \mathbf{U}\cdot\nabla)\,\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{U} + \eta\Delta\mathbf{B},$$
$$\nabla\cdot\mathbf{B} = \nabla\cdot\mathbf{U} = 0,$$

with the following radial boundary conditions (the later prevents the pressure to balance the gravity):

$$\frac{\partial}{\partial r}(\sqrt{r}U_\varphi) = \frac{\partial U_z}{\partial r} = 0, \quad \frac{\partial}{\partial r}(rB_\varphi) = \frac{\partial B_z}{\partial r} = 0 \quad \text{with} \quad \nabla\cdot U = \nabla\cdot B = 0 \quad \& \quad \Pi = \Pi_0.$$

The basic state considered consists of an axisymmetric magnetised Keplerian rotating flow which drives an accretion flow through the viscosity action

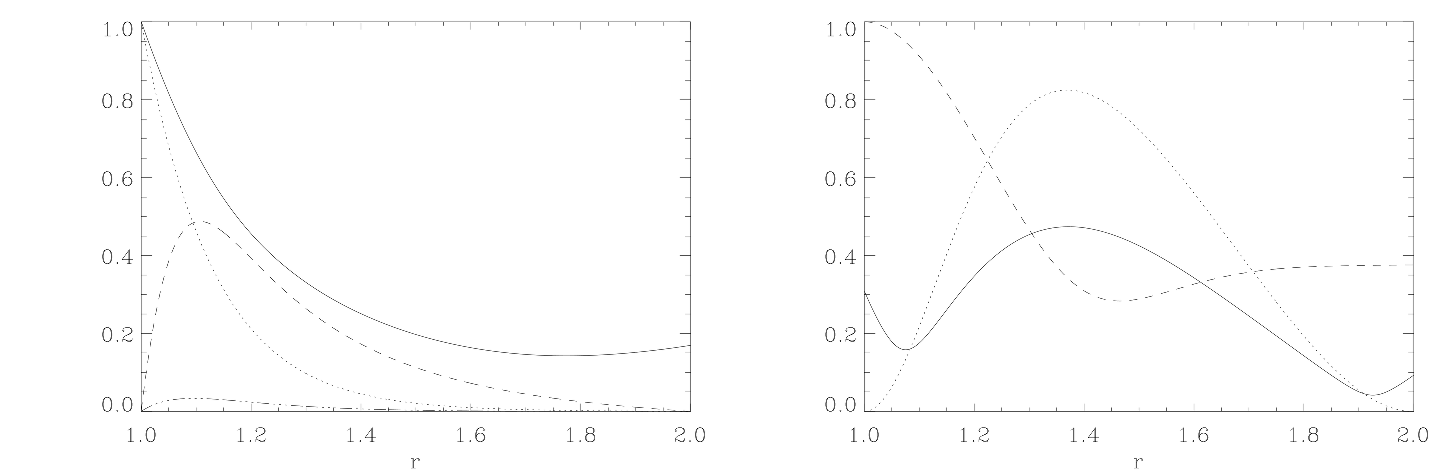
$$\Omega \propto r^{-3/2}, \quad U_r \propto -\nu/r \quad \& \quad B_0 = \text{const.}$$

Linear Theory

We write the linearised MHD equations as a 10^{th} order eigenvalue system

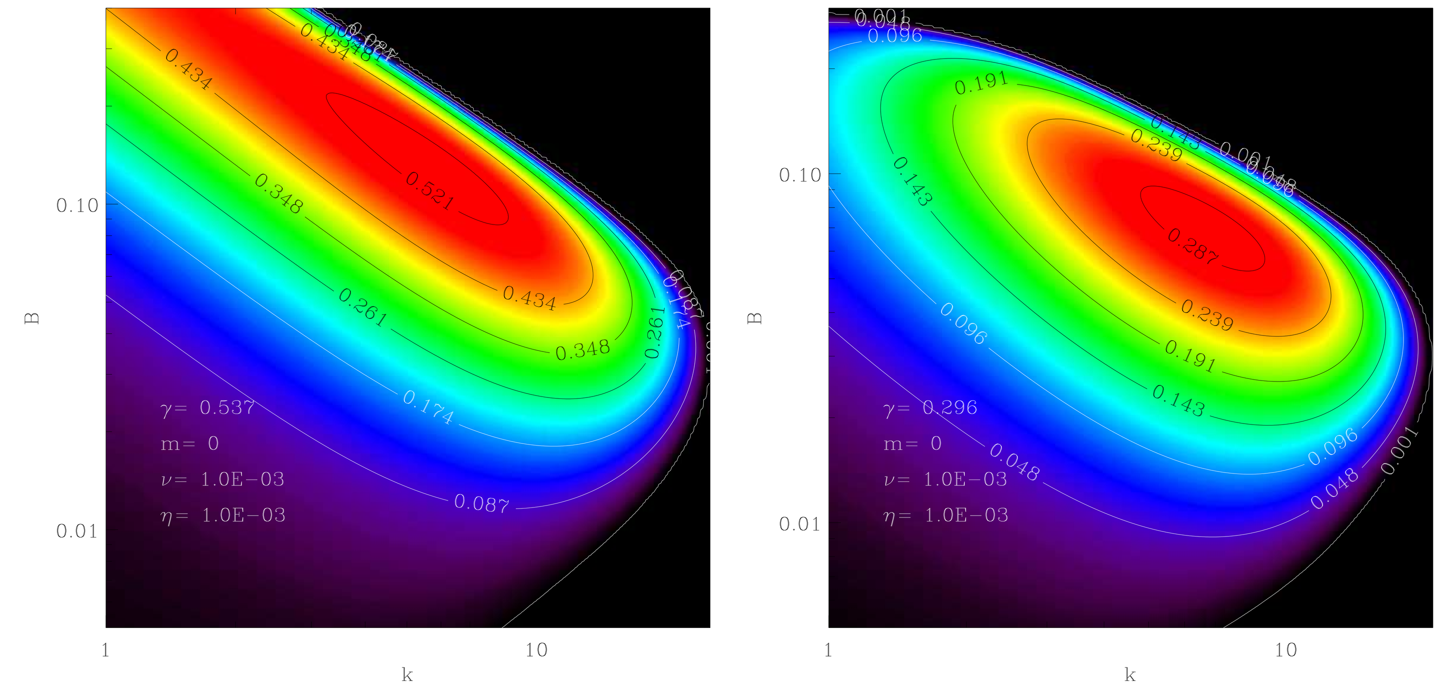
$$\sigma\,\boldsymbol{\mathcal{I}}(\boldsymbol{r})\,\boldsymbol{\kappa}(r) = \boldsymbol{\mathcal{L}}(r)\,\boldsymbol{\kappa}(r)$$

and look for eigenfunctions of the form $\boldsymbol{\mathcal{K}}(\mathbf{r},t) = \boldsymbol{\kappa}(r)\,\exp(\sigma t + im\varphi + ikz)$ where $\boldsymbol{\kappa} = [\mathbf{u}, \pi, \mathbf{b}]$. In incompressible regime, the pressure evolves on infinitely short time scales, so the constrain $\nabla\cdot\mathbf{u} = 0$ implies $\mathcal{I}_\pi = 0$.
We solve numerically the linear system (using "inverse iteration" and "shooting" techniques) and we find that two classes of unstable modes coexist in the disc, namely wall and body modes:



Left: wall-modes (solid & dashed lines represent u_r & u_φ, respectively) ; right: body-modes (solid, dotted and dashed lines represent u_r, u_φ and b_z respectively).

Both classes of unstable modes exhibit the expected properties for MRIs:



Dependence of the growth rate against the vertical magnetic field amplitude and the vertical spatial scale for dissipative wall modes (left) and body modes (right).

See Kersalé et al. (2004) for a discussion on the care required in the choice of the boundary conditions as well as for the description of the dependence of the growth of the wall and body modes against the dissipation, the spatial scales and the amplitude of vertical and azimuthal magnetic fields.

Numerical Methods

The theory developed here above gives an accurate description of linearly unstable magnetorotational modes in the cylindrical and dissipative system considered. In order to follow their nonlinear evolution and to operate a systematic parameter space survey, we have developed a pseudo-spectral numerical code.

Characteristics of the numerical code:

- Solve the non ideal incompressible MHD equations on a thin annulus.
- Dissipative and resistive processes treated exactly (i.e. do not rely on numerical effects).
- Spectral methods in space have been preferred for their fast convergence properties.
- The existence of linearly unstable wall modes suggests to increase the spatial resolution in the vicinity of the boundaries.

The integration in time uses a semi-implicit scheme, Crank-Nicholson and 2^{nd} order Adams-Bashforth, with adaptive time step, for the linear and nonlinear terms respectively. We solve for $\mathbf{U}$, $\mathbf{B}$ and Π the discrete equations:

$$\nabla^2\Gamma^{n+1} = \nabla\cdot\mathcal{N}_{\mathbf{u}}{}^n,$$
$$[\mathcal{I} - \Theta\,\delta t\,\mathcal{L}]\,\mathbf{U}^{n+1} + \nabla\Gamma^{n+1} = \mathcal{N}_{\mathbf{u}}{}^n + [\mathcal{I} + (1-\Theta)\,\delta t\,\mathcal{L}]\,\mathbf{U}^n,$$
$$[\mathcal{I} - \Theta\,\delta t\,\mathcal{L}]\,\tilde{\mathbf{B}}^{n+1} = \mathcal{N}_{\mathbf{b}}{}^n + [\mathcal{I} + (1-\Theta)\,\delta t\,\mathcal{L}]\,\mathbf{B}^n,$$
$$\mathbf{B}^{n+1} = \tilde{\mathbf{B}}^{n+1} - \nabla\Phi \quad \text{with} \quad \nabla^2\Phi = \nabla\cdot\tilde{\mathbf{B}}^{n+1} \quad \text{(projection methods)},$$

where, $\Theta \in [0,1]$, $\Gamma = \delta t\,\Pi$, $\mathcal{L} = \nu\nabla^2$ and $\mathcal{N}$ represents the nonlinear terms.

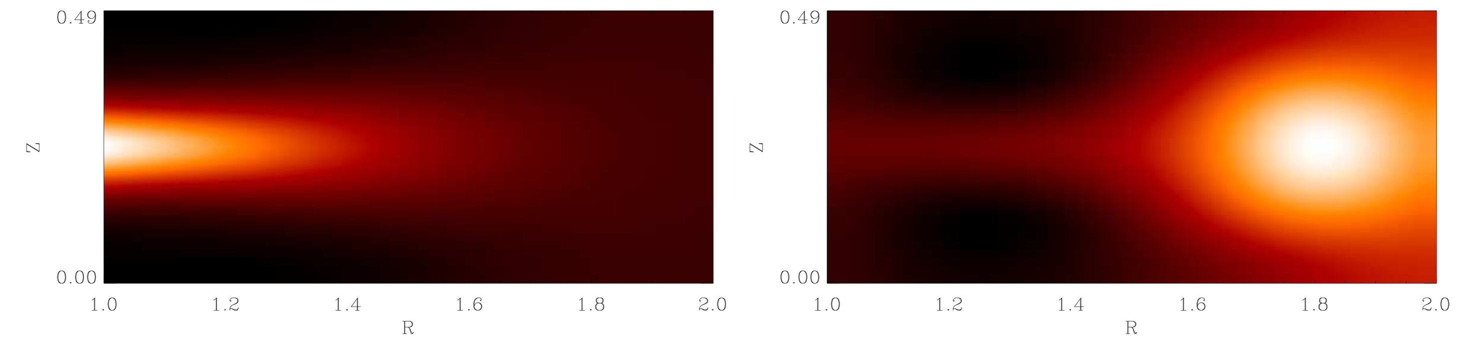
We have implemented pseudo-spectral schemes in space: Chebyshev in r, Fourier in φ and $\{\sin,\cos\}$ in z (Canutto et al., 1988). The dependent variables $\mathbf{U}$, $\mathbf{B}$ and Π are expended on the basis of trial functions,

$$X(r,\varphi,z) = \sum_{l,m,k} \mathcal{X}_{lmk} T_l(s)\,e^{im\varphi}\,\{\cos,\sin\}(kz), \quad \text{where} \quad T_l(s) = \cos(l\cos^{-1}s)$$

with $r = [(r_2 - r_1)\,s + r_1 + r_2]$, $\quad r \in [r_1, r_2]$.
Apart from the nonlinear terms computed in configuration space, we make use of the properties of the trial functions to perform all the computations in spectral space. This approach is straightforward for the Fourier components but requires further algebraic developments for Chebyshev polynomials because of the cylindrical geometry and the presence of rational functions (Gottlieb & Orzsag, 1977; Coutsias et al., 1996).
The azimuthal and vertical boundary conditions are automatically satisfied by the trial functions, whilst we implement a tau-Chebyshev treatment of the encountered radial Dirichlet, Neumann and Robin boundary conditions (Canutto et al., 1988).

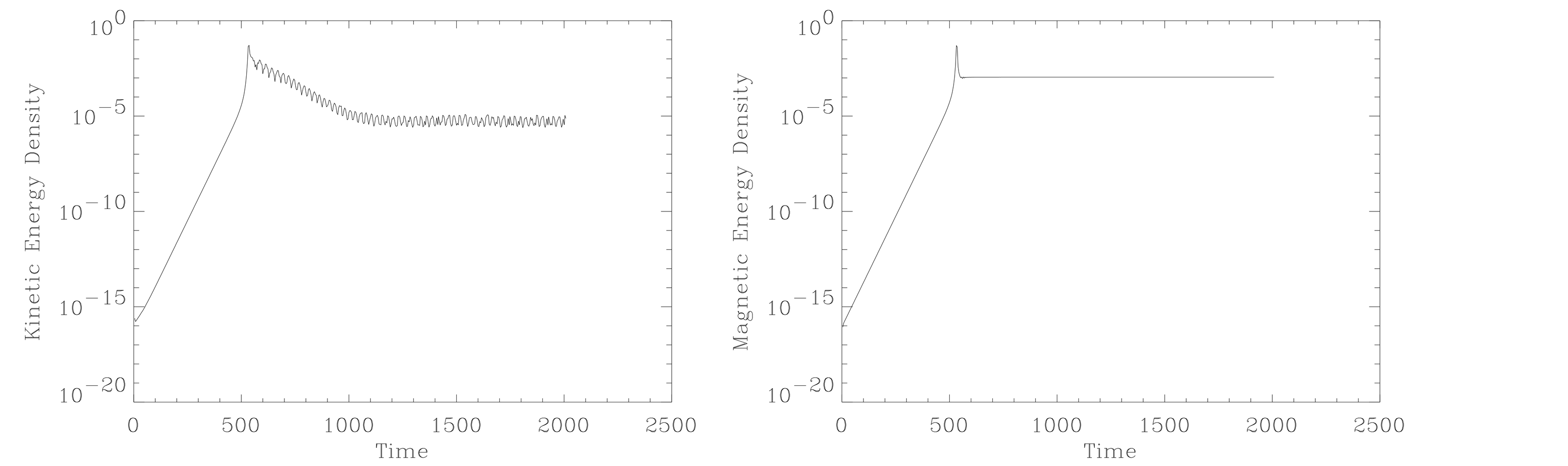
2-D Nonlinear Axisymmetric Evolution

We set-up an axisymmetric numerical experiment such that, according to the linear theory, one mode only is unstable and grows on a very long time scale (compared to the inner angular velocity).



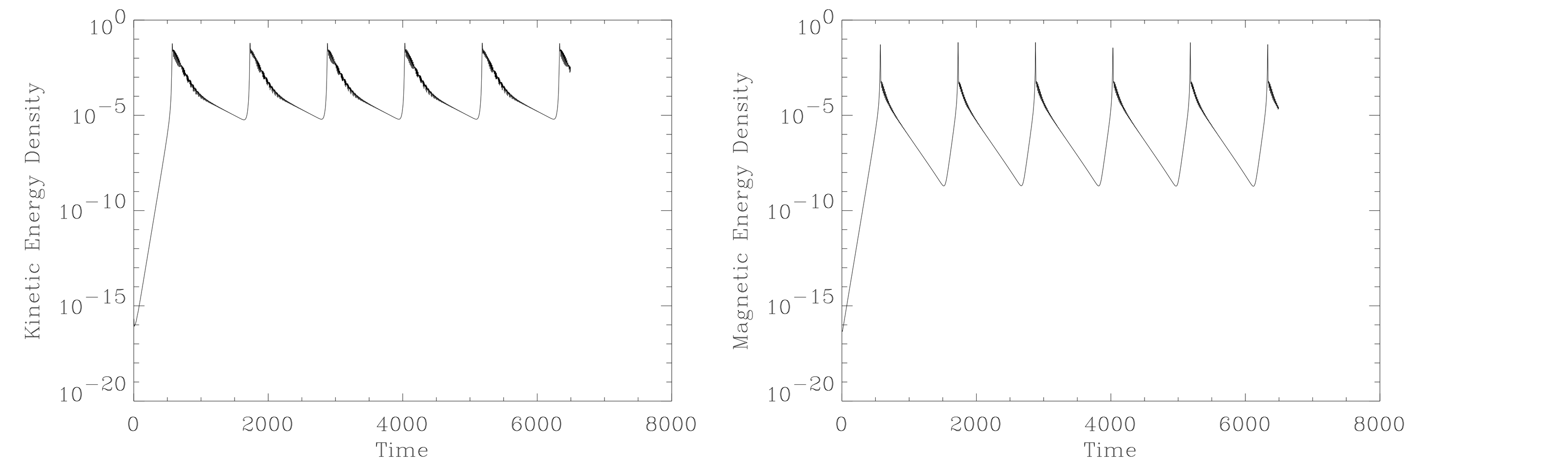
Nonlinear evolution of an unstable wall mode; we represent U_r (left) and B_z (right).

The eigenfunction of the radial velocity evolves, in the nonlinear regime, into a coherent jet-like structure. The angular velocity profile changes completely (i.e. it becomes strongly non Keplerian) and $\max(U_r) \sim U_\varphi$.



Disc extend $R_2 - R_1 = 1$ — Kinetic (left) and magnetic (right) energy density in the perturbations.

When the radial extend of the disc is small enough, the radial jet advects a significant part of the vertical magnetic flux out of the computational domain. This mechanism suppresses the instability and the highly perturbed system relaxes to a new stable Keplerian equilibrium, where the amplitude of the uniform B_z has decreased compared to its initial value.



Disc extend $R_2 - R_1 = 3$ — Kinetic (left) and magnetic (right) energy density in the perturbations.

But, when the radial extend of the disc is sufficiently large the radial jet can only transport and store the vertical magnetic flux into a stable part of the disc. The instability is temporarily suppressed until the magnetic field diffuses away from the stable region into the inner unstable region; so the process can start again.

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