MATH2620: Fluid Dynamics 1

School of Mathematics, University of Leeds

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Office Hours: Open door policy

Assessment: 85% final examination and 15% coursework (10 credits).

Textbooks & Picture books:

- P.S. Bernard, Fluid dynamics, CUP, 2015. (Recommended)
Module summary:
Fluid dynamics is the science of the motion of materials that flow, e.g. liquid or gas. Understanding fluid dynamics is a real mathematical challenge which has important implications in an enormous range of fields in science and engineering, from physiology, aerodynamics, climate, etc., to astrophysics.
This course gives an introduction to fundamental concepts of fluid dynamics. It includes a formal mathematical description of fluid flows (e.g. in terms of ODEs) and the derivation of their governing equations (PDEs), using elementary techniques from calculus and vector calculus. This theoretical background is then applied to a series of simple flows (e.g. bath-plug vortex or stream past a sphere), giving students a feel for how fluids behave, and experience in modelling everyday phenomena.
A wide range of courses, addressing more advanced concepts in fluid dynamics, with a variety of applications (polymers, astrophysical and geophysical fluids, stability and turbulence), follows on naturally from this introductory course.

Objectives:
This course demonstrates the importance of fluid dynamics and how interesting physical phenomena can be understood using rigorous, yet relatively simple, mathematics. But, it also provides students with a general framework to devise models of real-world problems, using relevant theories. Students will learn how to use methods of applied mathematics to derive approximate solutions to given problems and to have a critical view on these results.

Pre-requisites: Calculus, vector calculus, ODEs.

Course Outline:

- Mathematical modelling of fluids.
- Mass conservation and streamfunctions.
- Vorticity.
- Potential flow.
- Euler’s equation.
- Bernoulli’s equation.
- Flow in an open channel.
- Lift forces.
Lectures:

- You should read through and understand your notes before the next lecture... otherwise you will get hopelessly lost.

- Please, do not hesitate to interrupt me whenever you have questions or if I am inaudible, illegible, unclear or just plain wrong. (I shall also stay at the front for a few minutes after lectures in order to answer questions.)

- If you feel that the module is too difficult, or that you are spending too much time on it, please come and talk to me.

- Please, do not wait until the end of term to give a feedback if you are unhappy with some aspects of the module.

Lecture notes:

- Detailed lecture notes can be downloaded from the module’s website. You can print and use them in the lecture if you wish; however, the notes provided should only be used as a supplement, not as an alternative to your personal notes.

- These printed notes are an adjunct to lectures and are not meant to be used independently.

- Please email me (E.Kersale@leeds.ac.uk) corrections to the notes, examples sheets and model solutions.

Example sheets & homework:

- Five example sheets in total to be handed out every fortnight.

- Examples will help you to understand the material taught in the lectures and will give you practice on the types of questions that will be set in the examination. It is very important that you try them before the example classes.

- There will be only two, yet quite demanding, pieces of coursework (mid and end of term deadlines). Your work will be marked and returned to you with a grade from 1-100.

- Model solutions will be distributed once the homework is handed in
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In fluid dynamics, it is crucial to distinguish vectors from scalars. In these lecture notes we shall represent vectors and vector fields using bold fonts, e.g. $\mathbf{A}$ and $\mathbf{u}(x)$. Other commonly used notations for vectors include $\vec{A}$ or $\textbf{A}$ (used in the lecture). A vector $\mathbf{A}$, of components $A_1$, $A_2$ and $A_3$ in the basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, will interchangeably be written as a column or row vector,

$$\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = (A_1, A_2, A_3) = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3.$$ 

Three main systems of coordinates will be used throughout to represent points in a three dimensional space, Cartesian coordinates $(x, y, z)$ with $x$, $y$ and $z$ in $\mathbb{R}$; cylindrical polar coordinates $(r, \theta, z)$ with $r \in [0, \infty)$, $\theta \in [0, 2\pi)$ and $z \in \mathbb{R}$; and spherical polar coordinates $(r, \theta, \varphi)$ with $r \in [0, \infty)$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. Cylindrical polar coordinates reduce to plane polar coordinates $(r, \theta)$ in two dimensions. The vector position $\mathbf{r} \equiv \mathbf{x}$ of a point in a three dimensional space will be written as

$$\mathbf{x} = x \hat{e}_x + y \hat{e}_y + z \hat{e}_z \quad \text{in Cartesian coordinates},$$
$$= r \hat{e}_r + z \hat{e}_z \quad \text{in cylindrical coordinates},$$
$$= r \hat{e}_r \quad \text{in spherical coordinates},$$

using the orthonormal basis $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$, $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_z\}$ and $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$ respectively. Notice that $||\mathbf{x}|| = \sqrt{x^2 + y^2 + z^2}$ in Cartesian coordinates, $||\mathbf{x}|| = \sqrt{r^2 + z^2}$ in cylindrical coordinates and $||\mathbf{x}|| = r$ in spherical coordinates.

The variable $t$ will represent time; for time derivatives (rates of change) we shall use the notation $\frac{df}{dt} \equiv \dot{f}$, where $f$ is a function of $t$.

The velocity of a fluid element, defined by $\mathbf{u} = \frac{d\mathbf{x}}{dt}$, will be written as

$$\mathbf{u} = u \hat{e}_x + v \hat{e}_y + w \hat{e}_z \quad \text{in Cartesian coordinates},$$
$$= u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z \quad \text{in cylindrical coordinates},$$
$$= u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_\varphi \hat{e}_\varphi \quad \text{in spherical coordinates}.$$ 

For the sake of simplicity, we shall write integrals of multivariable functions as single integrals, e.g. we shall use

$$\iiint_V f(\mathbf{x}) \, dV \equiv \int_V f(\mathbf{x}) \, dV \quad \text{or} \quad \iint_S \mathbf{u}(\mathbf{x}) \cdot \mathbf{n} \, dS \equiv \int_S \mathbf{u}(\mathbf{x}) \cdot \mathbf{n} \, dS,$$

for the volume integral of $f$ in $V$ and for the flux of $\mathbf{u}$ through the surface $S$ with normal vector $\mathbf{n}$, respectively.
Unless otherwise stated, we shall use the following naming conventions: \( \psi \) for planar streamfunctions and \( \Psi \) for Stokes’ streamfunctions; \( \phi \) for the velocity potential; \( \Omega \) for angular velocity; \( \omega \) for the vector vorticity and \( \omega \) for its magnitude; \( \Gamma \) for fluid circulation; \( Q \) for the volume flux; \( \mathbf{g} \) for the vector gravity and \( g \) for its magnitude; \( p \) for pressure and \( p_{\text{atm}} \) for atmospheric pressure; \( \mathcal{H} \) for the Bernoulli function.
Chapter 1

Mathematical modelling of fluids

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1.1 Introduction

Fluid dynamics (or fluid mechanics) is the study of the motion of liquids, gases and plasmas (e.g. water, air, interstellar plasma) which have no large scale structure and can be deformed to an unlimited extent (in contrast with solids).

It is an old subject (Newton, Euler, Lagrange) but still a very active research area, e.g.

- astrophysical fluids: galaxies, stars, interstellar medium;
- geophysical fluids: earth’s core, atmosphere and ocean, weather;
- environmental fluids: pollution, water and wind power;
- biological fluids: blood and air flows, swimming organisms;
- Aerodynamics and hydrodynamics: aeroplanes, ships.

These fluids differ widely in their physical properties (e.g. density, temperature, viscosity) and in the time- and length-scales of their motion. However, they are all governed by the same physical laws (e.g. Newton’s law of motion), which can be mathematically formalised in terms of differential equations for scalar and vector fields.

1.1.1 Continuum hypothesis

One cubic centimetre of water contains of the order of $10^{23}$ molecules of typical size $l_m \approx 10^{-10}$ m, in continuous motion — even in still water (thermal agitation).

It is impossible to calculate the motion — velocity and position — of individual particles. Instead, one tries to concentrate on "bulk properties" of fluids, i.e. to look at the motion, mass, etc., of a “blob” of fluid called a fluid particle.

For instance, in the section of a pipe of radius $a \simeq 1$ cm, one calculates the average velocity of all molecules in a test volume $\delta V$ (fluid particle) of length $d$ (mesoscopic scale, i.e. between macroscopic and atomic/molecular scales).
1.1 Introduction

- If $d \gg l_m$ (molecular scale) then $\delta V$ contains many molecules and the fluctuations due to individual motions are averaged out.
- If $d \ll a$ (macroscopic scale) then $\delta V$ is approximately a point in space.

Hence, if $l_m \ll d \ll a$, the average velocity $\bar{u}$ is a smooth function of position, independent of $d$.

**Continuum hypothesis.** Molecular details can be smoothed out by assigning the velocity at a point $P$ to be the average velocity in a fluid element $\delta V$ centred in $P$. Thus, we can define the velocity field $u(x,t)$ as a smooth function (shock waves break this assumption), differentiable and integrable.

Similarly, $\rho(x,t) = \frac{\text{mass in } \delta V}{\delta V}$ is the local density of mass.

1.1.2 Velocity field

The fluid velocity is defined, within the continuum hypothesis, as the vector field $u(x,t)$, function of space and time.

**Example 1.1**

**Shear flow:** flow between two parallel plates when one is moved relative to the other, with a constant velocity $U$.

$$u = \begin{pmatrix} \frac{Uy}{d} \\ 0 \\ 0 \end{pmatrix} = \frac{Uy}{d} \hat{e}_x.$$ 

The direction of the flow is indicated by an arrow and its magnitude by the arrow length.
Stagnation-point flow: flow with a point at which $u = 0$.

Consider the extensional flow

$$\mathbf{u} = \begin{pmatrix} Ex \\ -Ey \\ 0 \end{pmatrix} = Ex \hat{e}_x - Ey \hat{e}_y,$$

where $E$ is a constant. The point $x = 0$ where $u = 0$ is a stagnation point.

Vortex flow: flow in rotation about a central point.

$$\mathbf{u} = \begin{pmatrix} -\frac{y}{x^2 + y^2} \\ \frac{x}{x^2 + y^2} \\ 0 \end{pmatrix} = \frac{y}{r^2} \hat{e}_x - \frac{x}{r^2} \hat{e}_y,$$

where $r^2 = x^2 + y^2$.

This flow is singular at $x = 0$: $||\mathbf{u}|| \sim 1/r \rightarrow \infty$ as $r \rightarrow 0$.

1.2 Kinematics

In the first four chapters of the course we shall concentrate solely on kinematic properties of fluid flows, i.e. on properties of fluid velocity fields, ignoring the causes of motion. Dynamics (effects of forces) will be covered in subsequent chapters.

We shall first answer the question “How can we visualise fluid motion?”

1.2.1 Particle paths

This method consists in following the motion of a “tracer” particle in the flow. Let a particle be released at time $t_0$ and at position $(x_0, y_0, z_0)$ into the velocity field

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} u(x, y, z, t) \\ v(x, y, z, t) \\ w(x, y, z, t) \end{pmatrix}.$$

Since the particle is suspended in the fluid, its velocity will be equal to that of the fluid. Hence the particle position, $\mathbf{x}(t)$, satisfies the system of first order ODEs and Initial Conditions

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \quad \text{with} \quad \mathbf{x} = \mathbf{x}_0 \text{ at } t = t_0,$$

or, equivalently, in Cartesian components form
\[
\frac{dx}{dt} = u(x, y, z, t), \quad \frac{dy}{dt} = v(x, y, z, t), \quad \frac{dz}{dt} = w(x, y, z, t),
\]
(1.2)

with the initial conditions \(x = x_0, y = y_0\) and \(z = z_0\) at \(t = t_0\).

A particle path is a line of equation \(\mathbf{x}(t) = (x(t), y(t), z(t))\), parametrised by the variable time \(t\).

Example 1.2 (Stagnation point flow)
Consider the extensional flow \(u(x) = (Ex, -Ey, 0)\) with constant \(E\). From equation (1.2) one gets

\[
\frac{dx}{dt} = Ex, \quad \frac{dy}{dt} = -Ey, \quad \frac{dz}{dt} = 0,
\]
\[
\Rightarrow x(t) = x_0 e^{Et}, \quad y(t) = y_0 e^{-Et}, \quad z(t) = z_0 \quad \text{if } \mathbf{x} = \mathbf{x}_0 \text{ at } t = 0.
\]

Note that particles at the stagnation point \(x_0 = y_0 = z_0 = 0\) do not move since \(u = 0\).

The time variable, \(t\), can be eliminated to show that particle paths are hyperbolae of equation

\[
y = \frac{x_0 y_0}{x}.
\]

1.2.2 Streamlines
A streamline is a line everywhere tangent to the local fluid velocity, at some instant. At fixed time \(t\), the equation of the streamline \(\mathbf{x}(s, t) = (x(s, t), y(s, t), z(s, t))\), parametrised by a parameter \(s\) ("distance" along the streamline, say), is

\[
\frac{dx}{ds} = u(x, t) \Leftrightarrow \frac{dx}{ds} = u, \quad \frac{dy}{ds} = v, \quad \frac{dz}{ds} = w,
\]
(1.3)
in Cartesian components form or, equivalently,

\[
\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad (= ds),
\]
(1.4)
since equations (1.3) are separable as \(u(x, t)\) is not explicitly function of \(s\). (Notice that \(dx/ds\) is the vector tangent to the line \(\mathbf{x}(s)\).)

Example 1.3 (Stagnation point flow)
Calculate the streamline passing through \((x_0, y_0, 0)\) in the stagnation point flow \(u = \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix}\).

From equation (1.4), in 2D,

\[
\frac{dx}{x} = \frac{dy}{-y} \Rightarrow \ln |x| = -\ln |y| + C \Rightarrow xy = A = x_0 y_0,
\]

where \(C\) is an integration constant and \(|A| = e^C\). So, streamlines are hyperbolae of equation

\[
y = \frac{x_0 y_0}{x}.
\]
Example 1.4
Consider the time dependent flow \( u(t) = \begin{pmatrix} U_0 \\ kt \\ 0 \end{pmatrix} \), where \( U_0 \) and \( k \) are constants.

- **Particle paths:**
  \[
  \frac{dx}{dt} = U_0, \quad \frac{dy}{dt} = kt, \quad \frac{dz}{dt} = 0,
  \]
  \[
  \Rightarrow x(t) = U_0 t + x_0, \quad y(t) = \frac{k}{2} t^2 + y_0, \quad z(t) = z_0, \quad \text{if } x = x_0 \text{ at } t = t_0.
  \]

  Eliminating \( t \), the equation of the particle path through \( x_0 \) becomes
  \[
  \left( \frac{x - x_0}{U_0} \right)^2 = \frac{2}{k} (y - y_0) \quad \text{(parabola)}.
  \]

- **Streamlines:** the streamline through \( x_0 \) is defined by
  \[
  \frac{dx}{ds} = U_0, \quad \frac{dy}{ds} = kt \Rightarrow \begin{cases} 
  x = U_0 s + x_0 \Rightarrow s = \frac{x - x_0}{U_0}, \\
  y = kts + y_0.
  \end{cases}
  \]

  At \( t = 0 \), streamlines are horizontal lines \( y = y_0 \); at later time \( t > 0 \), they are straight lines of gradient \( kt/U_0 \).

1.2.3 Streaklines

Instead of releasing a single particle at \( x_0 = (x_0, y_0, z_0) \) at \( t = t_0 \), say (case of particle path tracing), release a continuous stream of dye at that point.

The dye will move around and define a curve given by \( \{x(t_0, t)\} \), the position of fluid elements that passed through \( x_0 \) at some time \( t_0 \), prior to the current time \( t \).

A streakline is the line \( x(t_0, t) \) for which

\[
\frac{\partial x}{\partial t} = u(x, t) \quad \text{with } x = x_0 \text{ at } t = t_0.
\]

At \( t \) fixed, streaklines are lines parametrised by \( t_0 \). The point where the parameter \( t_0 = t \) corresponds to \( x_0 \), the locus of the source of dye.
Steady flows: For time-independent flows, which therefore satisfy \( \frac{\partial u}{\partial t} = 0 \), particle paths, streamlines and streaklines are identical. This is not true for unsteady flows in general.

Example 1.5

- **Steady flow**:

  \[
  \mathbf{u}(x) = \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \Rightarrow \frac{\partial x}{\partial t} = x, \quad \frac{\partial y}{\partial t} = -y, \quad \frac{\partial z}{\partial t} = 0,
  \]

  \[
  \Rightarrow x(t_0, t) = A(t_0) e^t, \quad y(t_0, t) = B(t_0) e^{-t}, \quad z(t_0, t) = C(t_0).
  \]

  At \( t = t_0 \), \( x = x_0 = (x_0, y_0, z_0) \), so

  \[
  x_0 = A e^{t_0}, \quad y_0 = B e^{-t_0}, \quad z_0 = C,
  \quad \Rightarrow A = x_0 e^{-t_0}, \quad B = y_0 e^{t_0}, \quad C = z_0.
  \]

  Hence,

  \[
  x(t_0, t) = x_0 e^{t-t_0}, \quad y(t_0, t) = y_0 e^{-t-t_0}, \quad z(t_0, t) = z_0.
  \]

  Again, we can eliminate the parameter \( t_0 \) to obtain the equation \( xy = x_0 y_0 \) as for particle paths and streamlines.

- **Unsteady flow**:

  \[
  \mathbf{u}(t) = \begin{pmatrix} U_0 \\ k t \\ 0 \end{pmatrix} \text{ with } U_0 \text{ and } k \text{ constants},
  \]

  \[
  \Rightarrow x(t_0, t) = x_0 + U_0(t - t_0),
  \]

  and \( y(t_0, t) = y_0 + \frac{k}{2} (t^2 - t_0^2) \).

  At \( t = 0 \), \( x(t_0, 0) = x_0 - U_0 t_0 \) and \( y(t_0, 0) = y_0 - \frac{k}{2} t_0^2 \).

  Eliminating \( t_0 \), one obtains the equation of a parabola with negative curvature

  \[
  y - y_0 = -\frac{k}{2U_0^2} (x - x_0)^2.
  \]

1.2.4 Time derivatives

Let \( f(x, t) \) be some quantity of interest (e.g. density, temperature, one component of the velocity, etc.).

**Eulerian description of fluids.** (At a fixed position in space.)

The partial derivative \( \frac{\partial f}{\partial t} \) is the rate of change of \( f \) at fixed position \( x \).

E.g., \( \frac{\partial \rho}{\partial t} \) (scalar) is the rate of change of mass density; \( \frac{\partial \mathbf{u}}{\partial t} \) (vector) is not the acceleration of a fluid particle.
Lagrangian description of fluids. (Following fluid particles.)

The convective derivative (also Lagrangian derivative, or material derivative) \( \frac{D}{Dt} f(x, t) \) is the rate of change of \( f \) when \( x \) is the position of a fluid particle (i.e. \( x \) travels with the fluid along particle paths).

So, \( \frac{Df}{Dt} = 0 \) implies that \( f \) remains constant along particle paths.

**Example 1.6**

For the flow \( \mathbf{u}(x) = \begin{pmatrix} x \\ y \\ -2z \end{pmatrix} \) a particle path is given by

\[
x(t) = x_0 e^t, \quad y(t) = y_0 e^t \quad \text{and} \quad z(t) = z_0 e^{-2t}, \quad \text{if} \quad x = x_0 \quad \text{at} \quad t = 0.
\]

So, along the particle path, \( \mathbf{u}_{pp} = \begin{pmatrix} x_0 e^t \\ y_0 e^t \\ -2z_0 e^{-2t} \end{pmatrix} \) and

\[
\frac{Du}{Dt} \equiv \frac{d}{dt} \mathbf{u}_{pp} = \begin{pmatrix} x_0 e^t \\ y_0 e^t \\ 4z_0 e^{-2t} \end{pmatrix} \neq 0 \quad \text{whereas} \quad \frac{\partial u}{\partial t} = 0.
\]

**Relation between \( D/Dt \) and \( \partial/\partial t \).**

There is no need to go through particle paths calculations to evaluate \( D/Dt \). Consider a particle path \( x(t) = (x(t), y(t), z(t)) \) defined by \( \frac{dx}{dt} = \mathbf{u} \).

Using the chain rule,

\[
\frac{Df}{Dt} = \frac{df}{dt}(x(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.
\]

\[
\Rightarrow \quad \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f.
\]

Hence, \( \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho \) is the Lagrangian derivative of the fluid density and the acceleration of a fluid particle is

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u}.
\]

**Example 1.7**

Again, consider the steady flow \( \mathbf{u} = \begin{pmatrix} x \\ y \\ -2z \end{pmatrix} \).

Since \( \frac{\partial \mathbf{u}}{\partial t} = 0 \) (the velocity field \( \mathbf{u} \) does not depend on time \( t \) for a steady flow),

\[
\frac{Du}{Dt} = (\mathbf{u} \cdot \nabla) \mathbf{u} = u \frac{\partial \mathbf{u}}{\partial x} + v \frac{\partial \mathbf{u}}{\partial y} + w \frac{\partial \mathbf{u}}{\partial z} = \begin{pmatrix} \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \end{pmatrix},
\]

\[
= x \frac{\partial \mathbf{u}}{\partial x} + y \frac{\partial \mathbf{u}}{\partial y} - 2z \frac{\partial \mathbf{u}}{\partial z} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} - 2z \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 4z \end{pmatrix},
\]

as before (see example 1.6).
**Steady flows.** In steady flows (i.e. flows such that $\partial \mathbf{u}/\partial t = 0$), the rate of change of $f$ following a fluid particle becomes

$$ \frac{Df}{Dt} = (\mathbf{u} \cdot \nabla) f. $$

Furthermore, since particle paths and streamlines are identical for time-independent flows,

$$ \frac{Df}{Dt} = \|\mathbf{u}\| (\hat{e}_s \cdot \nabla) f = \|\mathbf{u}\| \frac{df}{ds} $$

where $s$ denotes the distance along the streamlines $\mathbf{x}(s)$ and $\hat{e}_s$ is the unit vector parallel to the streamlines, in the direction of the flow.

So, in steady flows, $(\mathbf{u} \cdot \nabla) f = 0$ implies that $f$ is constant along streamlines — the constant can take different values for different streamlines however.
Chapter 2

Mass conservation & streamfunction

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2.1 Conservation of mass: the continuity equation

In any situation the mass of a fluid must be conserved. For continuous media, such as a fluids, this fundamental principle is expressed mathematically in the form of the continuity equation.

Consider a volume \( V \), fixed in space, with surface \( S \) and outward normal \( \mathbf{n} \).

The total mass in \( V \) is

\[ M_V = \int_V \rho \, dV, \]

where \( \rho \) is the density of mass (mass per unit volume).

\( M_V \) can only change if mass is carried inside or outside the volume by the fluid. The mass flowing through the surface per unit time (i.e. the mass flux) is

\[ -\int_S \rho \mathbf{u} \cdot \mathbf{n} \, dS = \frac{dM_V}{dt}. \]

So,

\[ -\int_S \rho \mathbf{u} \cdot \mathbf{n} \, dS = \frac{d}{dt} \int_V \rho \, dV = \int_V \frac{\partial \rho}{\partial t} \, dV, \quad \text{since } V \text{ is fixed}. \]

Applying the divergence theorem,

\[ \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot (\rho \mathbf{u}) \, dV \leftrightarrow \int_V \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \, dV = 0. \]
2.2 Incompressible fluids

Since $V$ is arbitrary, this equation must hold for all volume $V$. Thus, the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

holds at all points in the fluid. Expand the divergence as $\nabla \cdot (\rho \mathbf{u}) = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho$ to derive the Lagrangian form of the continuity equation

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{2.2}
\]

The density of a fluid particle, which moves with the fluid, only changes if there is an expansion (i.e. divergence such that $\nabla \cdot \mathbf{u} > 0$) or a contraction (i.e. convergence such that $\nabla \cdot \mathbf{u} < 0$) of the flow.

\[
\nabla \cdot \mathbf{u} < 0 \quad \nabla \cdot \mathbf{u} > 0
\]

2.2 Incompressible fluids

In an incompressible fluid, the density of each fluid particle (i.e. fluid element following the motion of the fluid) remains constant, so that $D\rho/Dt = 0$. Thus, the continuity equation (2.2) reduces to

\[
\rho \nabla \cdot \mathbf{u} = 0.
\]

So, incompressible flows must satisfy the constraint

\[
\nabla \cdot \mathbf{u} = 0, \tag{2.3}
\]

which means that the fluid velocity can be expressed in the form

\[
\mathbf{u} = \nabla \times \mathbf{S}, \tag{2.4}
\]

for some vector field $\mathbf{S}(x,t)$. Indeed, since $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any vector field $\mathbf{F}$, the flow defined by equation (2.4) satisfies the continuity equation (2.3).

For the rest of this course we shall assume that $\rho$ is constant. (So, the fluid is incompressible and therefore $\nabla \cdot \mathbf{u} = 0$.) This is a good approximation in many circumstances, e.g. water and very subsonic air flows.

2.3 Two-dimensional flows

Simplifications arise in the mathematical modelling of fluid flows when one considers systems which possess symmetries. We shall first consider flows confined to a plane, expressed in Cartesian coordinates as

\[
\mathbf{u} = \begin{pmatrix} u(x, y, t) \\ v(x, y, t) \\ 0 \end{pmatrix} = u(x, y, t) \hat{\mathbf{e}}_x + v(x, y, t) \hat{\mathbf{e}}_y.
\]
2.3.1 Streamfunctions

For two-dimensional incompressible flows, we define

\[ S = \psi(x, y, t) \hat{e}_z, \]  

(2.5)

where \( \psi \) is a (scalar) streamfunction, such that, from equation (2.4),

\[ \mathbf{u} = \nabla \times (\psi \hat{e}_z) \Rightarrow u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \]  

(2.6)

Clearly the incompressibility condition is then automatically satisfied:

\[ \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0. \]

Streamfunctions and streamlines. A key property of streamfunctions comes from considering

\[ \mathbf{u} \cdot \nabla \psi = u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} = 0, \]

which shows that the gradient of the streamfunction is orthogonal to the velocity field, implying that \( \psi \) remains constant in the direction of flow. So, the streamfunction \( \psi \) is constant along streamlines.

Conversely, consider a parametric curve \( \mathbf{x}(s) = (x(s), y(s)) \) of constant \( \psi \), so that

\[ \frac{d\psi}{ds} = 0 \Leftrightarrow \frac{\partial \psi}{\partial x} \frac{dx}{ds} + \frac{\partial \psi}{\partial y} \frac{dy}{ds} = -v \frac{dx}{ds} + u \frac{dy}{ds} = \mathbf{u} \times \frac{dx}{ds} = 0, \]

where \( dx/ds \) is a vector tangent to the curve \( \mathbf{x}(s) \). So, \( \mathbf{u} \) is tangent to the curve where \( \psi \) remains constant which is therefore a streamline.

Example 2.1

Consider the 2-D flow \( \mathbf{u} = \left( \begin{array}{c} U_0 t \\ k \end{array} \right) \), with \( U_0 \) and \( k \) constants. Since \( \nabla \cdot \mathbf{u} = 0 \), one can define the streamfunction

\[ \psi(x, y, t) = U_0 y - ktx + C, \]

where \( C \) is an arbitrary constant of integration. The value assigned to \( C \) does not affect the flow, so we shall usually take \( C = 0 \) without loss of generality.

The streamfunction is constant along the lines \( \psi = U_0 y - ktx = C \), where \( C \) is a constant identifying streamlines. So, the streamlines are lines of equation

\[ y = \frac{k}{U_0} x + \frac{C}{U_0} \quad \text{with gradient} \quad \frac{dy}{dx} = \frac{k}{U_0}. \]

Example 2.2

The flow defined as \( \mathbf{u} = \left( \begin{array}{c} \frac{y}{x^2 + y^2} \\ -x \\ \frac{-x}{x^2 + y^2} \end{array} \right) \) is incompressible since

\[ \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} + \frac{(-2y)(-x)}{(x^2 + y^2)^2} = 0. \]
So, $\frac{\partial \psi}{\partial y} = u = \frac{y}{x^2 + y^2} = \frac{1}{2} \frac{2y}{x^2 + y^2} \Rightarrow \psi(x, y) = \frac{1}{2} \ln(x^2 + y^2) + \alpha(x)$.

Then $v = -\frac{\partial \psi}{\partial x} = -\frac{x}{x^2 + y^2} + \frac{d\alpha}{dx} \Rightarrow \frac{d\alpha}{dx} = 0$. So, $\alpha$ is constant and

$$\psi(x, y) = \frac{1}{2} \ln(x^2 + y^2) \text{ (choosing } \alpha = 0).$$

### 2.3.2 Polar coordinates

We now change from Cartesian coordinates, $(x, y)$, in the $(X, Y)$-plane to polar coordinates, $(r, \theta)$, defined by

$$\begin{align*}
\{ & x(r, \theta) = r \cos \theta, \\
& y(r, \theta) = r \sin \theta,
\end{align*}$$

and consider again a two-dimensional flow, $u$, independent of $z$, with $u_z = 0$, such that

$$u(r, \theta) = u_r(r, \theta) \hat{e}_r + u_\theta(r, \theta) \hat{e}_\theta.$$ 

So, in plane polar coordinates, substituting $S = \psi(r, \theta, t) \hat{e}_z$ in $u = \nabla \times S$ for 2-D incompressible flows gives

$$u = \nabla \times (\psi \hat{e}_z) \Rightarrow u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}.$$ (2.7)

**Example 2.3**

For the streamfunction $\psi = \ln \sqrt{x^2 + y^2} = \ln r$, $u_r = 0$ and $u_\theta = -1/r$. The streamlines are circles about the origin with $|u_\theta|$ decreasing as $r$ increases. This is a reasonable model for a bath-plug vortex.

### 2.3.3 Physical significance of the streamfunction

We noted earlier that the streamfunction is constant on streamlines. So, we consider the two streamlines defined by $\psi(x, y) = \psi_P$ and $\psi(x, y) = \psi_T$.

The flow rate or the volume flux through an arbitrary curve $C : (x(s), y(s))$, parametrised by $s \in [0, 1]$ and connecting $P$ and $T$, is

$$Q = \int_0^1 u \cdot n \, ds.$$ (2.8)
Let \( dl = dx \hat{e}_x + dy \hat{e}_y = ds \left( \frac{dx}{ds} \hat{e}_x + \frac{dy}{ds} \hat{e}_y \right) \) be an infinitesimal displacement along the curve \( C \). The infinitesimal vector normal to \( dl \) is therefore
\[
nds = dy \hat{e}_x - dx \hat{e}_y = \left( \frac{dy}{ds} \hat{e}_x - \frac{dx}{ds} \hat{e}_y \right) ds.
\]
So,
\[
Q = \int_0^1 \left( \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx \right) ds = \int_0^1 \frac{d\psi}{ds} ds = \int_{\psi_p}^{\psi_T} d\psi = \psi_T - \psi_P.
\]
Hence, the flux of fluid flowing between two streamlines is equal to the difference in the streamfunction. Consequently, if streamlines are close to one another the flow must be fast. Equivalently, the equation \( \|u\| = \|\nabla \psi\| \) shows that the speed of the flow increases with the gradient of the streamfunction, that is when the distance between two streamlines decreases.

2.4 Axisymmetric flows

So far, we have considered flows confined to a plane, with velocity fields of the form \( u(x, y, t) = \nabla \times (\psi(x, y, t) \hat{e}_z) \), invariant along the \( z \)-axis, with no \( z \)-component.

Similarly, axisymmetric flows, such as
\[
u(r, z) = u_r(r, z) \hat{e}_r + u_z(r, z) \hat{e}_z \tag{2.9}
\]
in cylindrical polar coordinates, have only two non-zero components and two effective coordinates.

A flow in a circular pipe or a flow past a sphere are examples of flows with axial symmetry.

2.4.1 Stokes streamfunctions

For axisymmetric incompressible flows, we define
\[
S = \frac{1}{r} \Psi(r, z, t) \hat{e}_\theta,
\tag{2.10}
\]
where \( \Psi \) is a (scalar) Stokes streamfunction, such that, from equation (2.4),
\[
u = \nabla \times \left( \frac{1}{r} \Psi \hat{e}_\theta \right) \Rightarrow u_z = \frac{1}{r} \frac{\partial \Psi}{\partial r} \text{ and } u_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z}. \tag{2.11}
\]
(We use \( \Psi \) to distinguish Stokes streamfunctions from planar streamfunctions denoted \( \psi \).)

Clearly the incompressibility condition is then again automatically satisfied:
\[
\nabla \cdot \nu = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial z} = -\frac{1}{r} \frac{\partial^2 \Psi}{\partial r \partial z} + \frac{1}{r} \frac{\partial^2 \Psi}{\partial z^2} + \frac{1}{r} \frac{\partial^2 \Psi}{\partial z \partial r} = 0.
\]
2.4.2 Properties of Stokes streamfunctions

The Stokes streamfunctions have properties analogous to planar streamfunctions.

i. $\Psi$ is constant on streamlines.

$$\mathbf{u} \cdot \nabla \Psi = u_r \frac{\partial \Psi}{\partial r} + u_z \frac{\partial \Psi}{\partial z} = \frac{1}{r} \left( ru_r \frac{\partial \Psi}{\partial r} + ru_z \frac{\partial \Psi}{\partial z} \right),$$

$$\frac{1}{r} \left( - \frac{\partial \Psi}{\partial z} \frac{\partial \Psi}{\partial r} + \frac{\partial \Psi}{\partial r} \frac{\partial \Psi}{\partial z} \right) = 0.$$

Thus, $\Psi$ is constant in the direction of the flow.

For axisymmetric flows it is useful to think of streamtubes: surface of revolution spanned by all the streamlines through a circle about the axis of symmetry.

ii. Relation between volume flux and streamtubes.

The volume flux, or fluid flow, between two streamtubes with $\Psi = \Psi_i$ and $\Psi = \Psi_o$ is

$$Q = \int_S \mathbf{u} \cdot \mathbf{n} \, dS = 2\pi (\Psi_o - \Psi_i). \tag{2.12}$$

Proof.

$$Q = \int_S \mathbf{u} \cdot \mathbf{n} \, dS = \int_S \nabla \times \left( \frac{1}{r} \mathbf{e}_\theta \right) \cdot \mathbf{n} \, dS, \quad \text{(definition of $\Psi$)}$$

$$= \oint_{C_0} \frac{1}{r} \mathbf{e}_\theta \cdot d\mathbf{l} + \oint_{C_i} \frac{1}{r} \mathbf{e}_\theta \cdot d\mathbf{l}, \quad \text{(Stokes’ theorem)}$$

$$= \Psi_o \oint_{C_o} \frac{1}{r} \mathbf{e}_\theta \cdot d\mathbf{l} + \Psi_i \oint_{C_i} \frac{1}{r} \mathbf{e}_\theta \cdot d\mathbf{l}, \quad \text{(}\Psi \equiv \Psi_{(a,i)} \text{ onto } C_{(a,i)})$$

$$= \Psi_o \int_0^{2\pi} d\theta + \Psi_i \int_0^\theta d\theta = 2\pi (\Psi_o - \Psi_i) \quad \text{(recall, } d\mathbf{l} = dr \mathbf{e}_r + r d\theta \mathbf{e}_\theta).$$

Example 2.4

For a uniform flow parallel to the axis, $u_r = 0$ and $u_z = U$,

$$\frac{\partial \Psi}{\partial r} = rU \quad \text{and} \quad \frac{\partial \Psi}{\partial z} = 0 \Rightarrow \Psi(r) = \frac{1}{2} Ur^2.$$
(We choose the integration constant such that $\Psi = 0$ on the axis, at $r = 0$).

Now consider a streamtube of radius $a$.

The volume flux

$$Q = \int_S \mathbf{u} \cdot \mathbf{n} \, dS = \int_S \mathbf{u} \cdot \hat{\mathbf{e}}_z \, dS = \int_S u_z \, dS = U \int_S dS = \pi U a^2.$$ 

Also,

$$2\pi (\Psi_o - \Psi_i) = 2\pi (\Psi(a) - \Psi(0)) = 2\pi \left( \frac{1}{2} U a^2 - 0 \right) = \pi U a^2 \text{ as required.}$$

**Example 2.5**

Consider a flow in a long pipe a radius $a$:

$$u_r = 0, \quad u_z = \frac{U}{a^2} \left( a^2 - r^2 \right) \quad \text{with} \quad \{ \begin{align*} u_z &= 0 \text{ on } r = a, \\ u_z &= U \text{ at } r = 0. \end{align*}$$

Hence,

$$\Psi(r) = \frac{U r^2}{4 a^2} (2a^2 - r^2) \quad \text{(choose } C \text{ such that } \Psi(0) = 0).$$

So, $\Psi(0) = 0$ and $\Psi(a) = U a^2 / 4$, and the volume flux

$$Q = \int_S \mathbf{u} \cdot \mathbf{n} \, dS = 2\pi (\Psi(a) - \Psi(0)) = \frac{\pi}{2} U a^2.$$ 

Indeed,

$$\int_S \mathbf{u} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^a u_z r \, dr \, d\theta = \frac{2\pi U}{a^2} \int_0^a (a^2 r - r^3) \, dr, \quad \text{(since } dS = r d\theta dr)$$

$$= \frac{2\pi U}{a^2} \left[ \frac{a^2 r^2}{2} - \frac{r^4}{4} \right]_0^a = \frac{\pi U a^2}{2}, \quad \text{as required.}$$
2.4 Axisymmetric flows
Chapter 3
Vorticity

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3.1 Definition

We have already considered \( \nabla \cdot u \), which is a measure of the local expansion or contraction of
the fluid. (\( \nabla \cdot u = 0 \) for incompressible fluids.)

The vorticity,
\[
\omega = \nabla \times u,
\]
(3.1)
is a measure of the local rotation — or spin — in a flow. It is a concept of central importance
in fluid dynamics.

3.2 Physical meaning

In the simplest case of a 2-D flow, \( u(x, y, t) = \begin{pmatrix} u(x, y, t) \\ v(x, y, t) \\ 0 \end{pmatrix} \), the vorticity \( \omega = \nabla \times u = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix} \) is perpendicular to the plane of motion; its magnitude is \( \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \).

Consider the two short fluid line-elements \( AB \) and \( AC \). The vertical differential velocity is
3.2 Physical meaning

\[ \delta v = v_C - v_A = v(x + \delta x, y) - v(x, y) \simeq \delta x \frac{\partial v}{\partial x} \]  
(Taylor theorem).

So, \( \frac{\partial v}{\partial x} \) is the angular velocity of the fluid line-element AC.

Similarly, the horizontal differential velocity

\[ \delta u = u_B - u_A = u(x, y + \delta y) - u(x, y) \simeq \delta y \frac{\partial u}{\partial y}. \]

So, \( -\frac{\partial u}{\partial y} \) is the angular velocity of the fluid line element AB.

Thus, \( \frac{\omega}{2} = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \) represents the average angular velocity of the two fluid line-elements AB and AC.

This could be measured using a crossed pair of small vanes that float with the fluid.

Example 3.1 (Solid body rotation)
For \( u = \Omega \times r \), with constant angular velocity \( \Omega \) and \( r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \),

\[ \omega = \nabla \times (\Omega \times r) = \Omega (\nabla \cdot r) - (\Omega \cdot \nabla) r, \]

\[ = 3\Omega - \Omega = 2\Omega. \]

So, the vorticity is twice the local rotation rate — which is also global here.

Example 3.2 (Shear flow)
For \( u = ky \), \( v = 0 \), the vorticity \( \omega = \begin{pmatrix} 0 \\ 0 \\ -k \end{pmatrix} \) with \( \omega = -k \).

The vorticity is not a measure of global rotation: a shear flow has no global rotation but a non-zero vorticity.

Example 3.3 (Line vortex flow)
Let \( u_r = 0 \), \( u_\theta = k/r \) and \( u_z = 0 \) where \( k \) is a positive constant. (This is a crude model for a bath-plug vortex — see examples 2.2 and 2.3)
Streamfunction: \( \frac{\partial \psi}{\partial r} = -\frac{k}{r} \Rightarrow \psi = -k \ln r \) (const. = 0). So, the streamlines are circles.

However, from

\[
\omega = \left[ \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} u_r \right] \hat{e}_z = \frac{1}{r} \frac{\partial k}{\partial r} \hat{e}_z = 0,
\]

one finds that the vorticity is zero everywhere in the flow except at \( r = 0 \) where the functions \( u \) and \( \omega \) are not defined. (In fact \( \omega \) can be defined as a Dirac delta distribution.)

Although the flow is rotating globally, there is no local rotation. Crossed vanes placed in the flow would move in a circle, but without spinning.

### 3.3 Streamfunction and vorticity

In a two-dimensional flow, the vorticity \( \omega = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \) where \( \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \) in Cartesian coordinates. If, in addition, the fluid is incompressible, \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \), so that

\[
\omega = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\nabla^2 \psi.
\] (3.2)

More generally (Cartesian or plane polar coordinates),

\[
\begin{align*}
\mathbf{u} = \nabla \times (\psi \hat{e}_z) \Rightarrow \omega &= \nabla \times (\nabla \times (\psi \hat{e}_z)) \\
&= \nabla(\nabla \cdot (\psi \hat{e}_z)) - \nabla^2 (\psi \hat{e}_z) \\
&= \nabla \left( \frac{\partial}{\partial z} \psi \right) - \nabla^2 \psi \hat{e}_z.
\end{align*}
\]

Since \( \frac{\partial \psi}{\partial z} = 0 \) for a 2-D flow, one finds again

\[
\omega = -\nabla^2 \psi \hat{e}_z
\] (3.3)

### 3.4 The Rankine vortex

Example 3.3, with \( u_\theta = k/r \), is a crude model for the bath-plug vortex: infinite vorticity concentrated in \( r = 0 \) (singularity). In real vortices the vorticity is spread over a small area.

Consider an azimuthal flow, \( u_r = u_z = 0 \) and \( u_\theta = f(r) \), in cylindrical coordinates, such that

\[
\omega = \begin{cases} 
\Omega & \text{if } r \leq a \quad (\Omega \text{ constant}), \\
0 & \text{if } r > a
\end{cases}
\]
Hence, \( \psi \equiv \psi(r) \) and \( \nabla^2 \psi = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = \begin{cases} -\Omega & \text{if } r \leq a, \\ 0 & \text{if } r > a. \end{cases} \)

- \( r \leq a: \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = -r \Omega \Rightarrow \frac{d\psi}{dr} = -\frac{\Omega}{2} r^2 + \frac{B}{r} = -u_\theta. \)

We require \( u_\theta \) to be bounded at \( r = 0 \). So, \( B = 0 \) and, for \( r \leq a \),

\[
\begin{align*}
\frac{d\psi}{dr} &= -\frac{\Omega}{2} r, \\
\psi &= \frac{\Omega}{4} r^2.
\end{align*}
\]

- \( r > a: \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = 0 \Rightarrow u_\theta = -\frac{d\psi}{dr} = \frac{A}{r}. \)

The continuity of \( u_\theta \) at \( r = a \) implies that \( A = \frac{\Omega}{2} a^2. \)

So, \( \frac{d\psi}{dr} = -\frac{\Omega a^2}{2r} \Rightarrow \psi = -\frac{\Omega a^2}{4} \ln r + D, \) and applying the continuity of \( \psi \) at \( r = a \) now gives \( \psi = -\frac{\Omega a^2}{4} \left( 1 + 2 \ln \frac{r}{a} \right). \)

So,

\[
\psi = \begin{cases} \frac{-\Omega}{4} r^2 & \text{if } r \leq a, \\ \frac{-\Omega a^2}{4} \left( 1 + 2 \ln \frac{r}{a} \right) & \text{if } r > a. \end{cases}
\]

and

\[
u_\theta = \begin{cases} \frac{\Omega}{2} r & \text{if } r \leq a : \text{solid body rotation,} \\ \frac{\Omega a^2}{2r} & \text{if } r > a : \text{bath-plug flow.} \end{cases}
\]

There is still a discontinuity in vorticity but the flow is quite adequate for predicting the shape of the water surface.

### 3.5 Circulation

Consider a closed curve \( C \) in the flow. The circulation around \( C \) is the line integral of the tangential velocity around \( C \):

\[
\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l}. \quad (3.4)
\]
By Stokes theorem, for any surface $S$ spanning the curve $C$,
\[
\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} \, dS = \int_S \mathbf{\omega} \cdot \mathbf{n} \, dS.
\] (3.5)

So, the circulation around $C$ is equal to the vorticity flux through the surface $S$: it is the strength of the vortex tube.

**Example 3.4 (The Rankine vortex)**

The circulation around a circle of radius $r$ is given by
\[
\Gamma = \int_0^{2\pi} u_\theta(r) \, r \, d\theta = 2\pi r u_\theta(r) = \begin{cases} 
\pi \Omega r^2 & \text{if } r \leq a, \\
\pi \Omega a^2 & \text{if } r > a.
\end{cases}
\]

When the vorticity is concentrated in thin filaments (tubes) as it is in the Rankine vortex, it is useful to think in term of vortex.

### 3.6 Examples of vortex lines (vortices)

1. Bath-plug vortex.

   The shape of the free surface of water can be modelled using the Rankin vortex. Note that the sense of rotation is not determined by the rotation of the Earth!

2. Vortices behind aeroplanes.
3.6 Examples of vortex lines (vortices)

The characteristic vapour trails left by aircraft are vortex lines shed from the wing tips. (The vortices have low pressure, so vapour water condenses there.) These vortices decay very slowly and are a danger for small aircraft flying behind large ones.

iii. Horseshoe vortex & downwash behind chimneys.

Vortex lines in shear flows above ground level travel with the air and can be stretched and bent by tall buildings and chimneys. This results in a downwards flow behind chimneys, dragging pollutant down to ground level.

iv. Vortex rings.

Smoke rings and underwater bubble rings are examples of vortex rings.
v. von Kármán vortex street.

In certain conditions, a flow past an obstacle (e.g. a cylinder, an island) produces a series of line vortices.
3.6 Examples of vortex lines (vortices)
Chapter 4

Potential flows

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4.1 Velocity potential

We shall now consider the special case of irrotational flows, i.e. flows with no vorticity, such that

\[ \omega = \nabla \times \mathbf{u} = 0. \]  

(4.1)

If a velocity field \( \mathbf{u} \) is irrotational, that is if \( \nabla \times \mathbf{u} = 0 \), then there exists a velocity potential \( \phi(x, t) \) defined by

\[ \mathbf{u} = \nabla \phi. \]  

(4.2)

This is a result from vector calculus; the converse is trivially true since \( \forall \phi, \nabla \times \nabla \phi \equiv 0 \).

If in addition the flow is incompressible, the velocity potential \( \phi \) satisfies Laplace’s equation

\[ \nabla^2 \phi = 0. \]  

(4.3)

Indeed, for incompressible irrotational flows one has \( \nabla \cdot \mathbf{u} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \).

Hence, incompressible irrotational flows can be computed by solving Laplace’s equation (4.3) and imposing appropriate boundary conditions (or conditions at infinity) on the solution. (Notice that for 2-D incompressible irrotational flows, both velocity potential, \( \phi \), and stream-function, \( \psi \), are solutions to Laplace’s equation, \( \nabla^2 \psi = -\omega = 0 \) and \( \nabla^2 \phi = 0 \); boundary conditions on \( \psi \) and \( \phi \) are different however.)
4.2 Kinematic boundary conditions

Consider a flow past a solid body moving at velocity \( U \). If \( n \) is the unit vector normal to the surface of the solid, then, locally, the surface advances (i.e. moves in the direction of \( n \)) at the velocity \((U \cdot n) n\).

Since the fluid cannot penetrate into the solid body, its velocity normal the surface, \((u \cdot n) n\), must locally equal that of the solid,

\[
u \cdot n = U \cdot n.
\]

So, since \( u = \nabla \phi \),

\[
n \cdot \nabla \phi = \frac{\partial \phi}{\partial n} = U \cdot n; \tag{4.4}
\]

the velocity potential satisfies Neumann boundary conditions at the solid body surface.

4.3 Elementary potential flows

4.3.1 Source and sink of fluid

Line source/sink

Consider an axisymmetric potential \( \phi \equiv \phi(r) \). From Laplace’s equation in plane polar coordinates,

\[
\nabla^2 \phi = \frac{1}{r} \frac{d}{dr} \left( r \frac{d \phi}{dr} \right) = 0 \iff \frac{d \phi}{dr} = \frac{m}{r},
\]

one finds

\[
\phi(r) = m \ln r + C, \tag{4.5}
\]

where \( m \) and \( C \) are integration constants. This potential produces the planar radial velocity

\[
u = \nabla \phi = \frac{m}{r} \hat{e}_r
\]

corresponding to a source \((m > 0)\) or sink \((m < 0)\) of fluid of strength \( m \). Notice that the constant \( C \) in \( \phi \) is arbitrary and does not affect \( \nu \). By convention the constant \( m = Q/2\pi \) where \( Q \) is the flow rate. This flow could be produced approximately using a perforated hose.
Point source/sink

Consider a *spherically* symmetric potential $\phi \equiv \phi(r)$. From Laplace’s equation in spherical polar coordinates,

$$\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 0 \Leftrightarrow \frac{d\phi}{dr} = \frac{m}{r^2},$$

one finds

$$\phi(r) = -\frac{m}{r} + C, \quad (4.6)$$

where $m$ and $C$ are integration constants. This potential produces the three-dimensional radial velocity

$$u = \nabla \phi = \frac{m}{r^2} \mathbf{\hat{e}}_r,$$

corresponding to a source ($m > 0$) or sink ($m < 0$) of fluid of strength $m$. By convention the constant $m = Q/4\pi$ where $Q$ is the flow rate or volume flux.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{source_sink.png}
  \caption{source and sink}
  \label{fig:source_sink}
\end{figure}

4.3.2 Line vortex

For the potential $\phi(\theta) = k\theta$, solution to Laplace’s equation in plane polar coordinates, one has

$$u_r = \frac{\partial \phi}{\partial r} = 0 \quad \text{and} \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{k}{r},$$

where the strength of the flow $k$ is a constant. By convention $k = \Gamma/2\pi$ if $\Gamma$ is the circulation of the flow.

This represents a rotating fluid (bath-plug vortex) around a line vortex at $r = 0$; it has zero vorticity but is singular at the origin.

4.3.3 Uniform stream

For a uniform flow along the $z$-axis, $u = (0, 0, U)$, the velocity potential

$$\phi(z) = Uz.$$

(The integration constant is set to zero.)

4.3.4 Dipole (doublet flow)

Since Laplace’s equation is linear we can add two solutions together to form a new one. A dipole is the superposition of a sink and an source of equal but opposite strength next to each other.
Three-dimensional flow

Consider a point sink of strength $-m$ at the origin and a point source of strength $m$ at the position $(0, 0, \delta)$.

\[
\text{Sink Source}
\]

\[
\delta
\]

\[
O
\]

\[
z
\]

The velocity potential of the flow is formed by adding the potentials of the source and sink,

\[
\phi = \frac{m}{\sqrt{x^2 + y^2 + z^2}} - \frac{m}{\sqrt{x^2 + y^2 + (z - \delta)^2}},
\]

where $m > 0$ is constant and $r = (x^2 + y^2 + z^2)^{1/2}$. Expanding the potential to first order in $\delta$,

\[
\phi = \frac{m}{r} - \frac{m}{r} \left( 1 + \frac{z}{r^2} \delta + O(\delta^2) \right),
\]

and taking the limit $\delta \to 0$, leads to the potential of a dipole

\[
\phi = -m\delta \frac{z}{r^3},
\]

where $m \to \infty$ as $\delta \to 0$ so that the strength of the dipole $\mu = m\delta$ remains finite. Thus,

\[
\phi = -\frac{\mu \cdot r}{r^3} = \mu \cdot \nabla \left( \frac{1}{r} \right),
\]

where $\mu = \mu \hat{e}_z$ and $r \equiv x$ is the vector position. The three components of the fluid velocity, $u = \nabla \phi$, are for a dipole of strength $\mu$,

\[
\begin{align*}
    u_x &= 3\mu \frac{xz}{r^3}, \\
    u_y &= 3\mu \frac{yz}{r^3}, \\
    u_z &= -\frac{\mu}{r^3} \left( 1 - 3\frac{z^2}{r^2} \right).
\end{align*}
\]

Planar flow

Similarly, combining a line sink at the origin with a line source of equal but opposite strength at $(\delta, 0)$ gives

\[
\phi = -\frac{m}{2} \left[ \ln(x^2 + y^2) - \ln((x - \delta)^2 + y^2) \right] = \frac{m}{2} \ln \left( \frac{(x - \delta)^2 + y^2}{x^2 + y^2} \right), \quad m > 0.
\]

As in the three-dimensional case, we consider the limit $\delta \to 0$, with $\mu = m\delta$ fixed. The expression of the potential for a two-dimensional dipole of strength $\mu$ then becomes

\[
\phi = -\frac{\mu \cdot r}{r^2} = -\mu \cdot \nabla \ln r,
\]

where $\mu = \mu \hat{e}_x$ and $r \equiv x$ is the vector position.
4.4 Properties of Laplace’s equation

4.4.1 Identity from vector calculus

Let \( f(x) \) be a function defined in a simply connected domain \( V \) with boundary \( S \). From vector calculus,

\[
\nabla \cdot (f \nabla f) = f \nabla^2 f + |\nabla f|^2
\]

\[
\Rightarrow \int_V \nabla \cdot (f \nabla f) \, dV = \int_V f \nabla^2 f \, dV + \int_V |\nabla f|^2 \, dV.
\]

So, using the divergence theorem

\[
\int_S f(\nabla f) \cdot \mathbf{n} \, dS = \int_V f \nabla^2 f \, dV + \int_V |\nabla f|^2 \, dV.
\]

(4.7)

4.4.2 Uniqueness of solutions of Laplace’s equation

Given the value of the normal component of the fluid velocity, \( \mathbf{u} \cdot \mathbf{n} \), on the surface \( S \) (i.e. the boundary condition), there exists a unique flow satisfying both \( \nabla \cdot \mathbf{u} = 0 \) and \( \nabla \times \mathbf{u} = 0 \) (i.e. incompressible and irrotational).

**Proof.** Suppose there exists two distinct solutions to the boundary value problem, \( \mathbf{u}_1 = \nabla \phi_1 \) and \( \mathbf{u}_2 = \nabla \phi_2 \). Let \( f = \phi_1 - \phi_2 \), then

\[
\nabla^2 f = \nabla^2 \phi_1 - \nabla^2 \phi_2 = 0
\]

in the domain \( V \) and

\[
(\nabla f) \cdot \mathbf{n} = (\nabla \phi_1) \cdot \mathbf{n} - (\nabla \phi_2) \cdot \mathbf{n} = \mathbf{u}_1 \cdot \mathbf{n} - \mathbf{u}_2 \cdot \mathbf{n} = 0
\]

on the boundary \( S \). Hence, from identity (4.7), \( \int_V |\nabla \phi_1 - \nabla \phi_2|^2 \, dV = 0 \). However, since \( |\nabla \phi_1 - \nabla \phi_2|^2 \geq 0 \) one must have \( \nabla \phi_1 = \nabla \phi_2 \) everywhere. Therefore \( \mathbf{u}_1 = \mathbf{u}_2 \) and the solution to the boundary value problem is unique. \( \square \)

4.4.3 Uniqueness for an infinite domain

The proof above holds for flows in a finite domain. What about flows in an infinite domain — e.g. flow around an obstacle?

The above argument holds by considering the volume \( V \) as shown and letting \( S' \to \infty \). (See, e.g. Patterson p. 211.)
4.4.4 Kelvin’s minimum energy theorem

Of all possible fluid motions satisfying the boundary condition for \( \mathbf{u} \cdot \mathbf{n} \) on the surface \( S \) and \( \nabla \cdot \mathbf{u} = 0 \) in domain \( V \), the potential flow is the flow with the smallest kinetic energy,

\[
K = \frac{1}{2} \int_V \rho |\mathbf{u}|^2 \, dV.
\]

**Proof.** Let \( \mathbf{u}' \) be another incompressible but non vorticity-free flow such that \( \mathbf{u} \cdot \mathbf{n} = \mathbf{u}' \cdot \mathbf{n} \) on \( S \) and \( \nabla \cdot \mathbf{u}' = 0 \) in \( V \) but with \( \nabla \times \mathbf{u}' \neq 0 \).

The fluid flow \( \mathbf{u} \) is potential, so let \( \mathbf{u} = \nabla \phi \) such that

\[
\int_V \rho |\mathbf{u}|^2 \, dV = \int_V \rho |\nabla \phi|^2 \, dV = \rho \int_V |\nabla \phi|^2 \, dV,
\]

\[
= \rho \int_S \phi \mathbf{u} \cdot \mathbf{n} \, dS \quad \text{(by identity (4.7) with } f = \phi),
\]

\[
= \rho \int_S \phi \mathbf{u}' \cdot \mathbf{n} \, dS \quad \text{(boundary condition)},
\]

\[
= \rho \int_V \nabla \cdot (\phi \mathbf{u}') \, dV \quad \text{(divergence theorem)},
\]

\[
= \rho \int_V \nabla \cdot \nabla \phi \, dV \quad (\nabla \cdot \mathbf{u}' = 0),
\]

\[
= \rho \int_V \mathbf{u}' \cdot \nabla \phi \, dV.
\]

So,

\[
\int_V \rho (\mathbf{u} - \mathbf{u}')^2 \, dV = \int_V (\rho |\mathbf{u}|^2 - 2\rho \mathbf{u} \cdot \mathbf{u}' + \rho |\mathbf{u}'|^2) \, dV,
\]

\[
= \int_V (\rho |\mathbf{u}'|^2 - \rho |\mathbf{u}|^2) \, dV \quad \text{(from (4.8))}.
\]

Therefore, since \( (\mathbf{u} - \mathbf{u}')^2 \geq 0 \),

\[
\int_V \rho |\mathbf{u}'|^2 \, dV = \int_V \rho |\mathbf{u}|^2 \, dV + \int_V \rho (\mathbf{u} - \mathbf{u}')^2 \, dV \geq \int_V \rho |\mathbf{u}|^2 \, dV.
\]

4.5 Flow past an obstacle

Since the solution to Laplace’s equation for given boundary conditions is unique, if we find a solution, we have found the solution. (This is only true if the domain is simply-connected; if the domain is multiply connected, multiple solutions become possible.)

One technique to calculate non elementary potential flows involves adding together simple known solutions to Laplace’s equation to get the solution that satisfies the boundary conditions.

4.5.1 Flow around a sphere

We seek an axisymmetric flow of the form \( \mathbf{u} = u_r \hat{e}_r + u_z \hat{e}_z \) in cylindrical polar coordinates \((r, \theta, z)\).
At large distances from the sphere of radius $a$ the flow is asymptotic to a uniform stream, $u_r = 0$, $u_z = U$, and at the sphere’s surface, $r = a$, the fluid velocity must satisfy $\mathbf{u} \cdot \mathbf{n} = 0$ since the solid body forms a non-penetrable boundary.

The unit vector normal to surface of the sphere is

$$\mathbf{n} = n_r \hat{e}_r + n_z \hat{e}_z \quad \text{with} \quad n_r = \frac{r}{a} \quad \text{and} \quad n_z = \frac{z}{a}.$$  

So, the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ implies that

$$u_r \frac{r}{a} + u_z \frac{z}{a} = 0 \Leftrightarrow ru_r + zu_z = 0$$

at the spherical surface of equation $r^2 + z^2 = a^2$.

At large distances, the flow is essentially uniform along the $z$-axis,

$$\phi \simeq U z, \quad \text{for} \ ||\mathbf{r}|| \gg a.$$  

Now, add to the uniform stream a dipole velocity field of strength $\mu = \mu \hat{e}_z$ at the origin,

$$\phi(r, z) = U z - \frac{\mu z}{(r^2 + z^2)^{3/2}}$$

so that

$$u_r = \frac{\partial \phi}{\partial r} = \frac{3\mu r z}{(r^2 + z^2)^{5/2}} \quad \text{and} \quad u_z = \frac{\partial \phi}{\partial z} = U + \frac{\mu}{(r^2 + z^2)^{3/2}} \left( \frac{3z^2}{r^2 + z^2} - 1 \right).$$

Thus, at the sphere’s surface,

$$\mathbf{u} \cdot \mathbf{n} = u_r \frac{r}{a} + u_z \frac{z}{a} = \frac{z}{a} \left( U + \frac{3\mu(r^2 + z^2)}{(r^2 + z^2)^{5/2}} - \frac{\mu}{(r^2 + z^2)^{3/2}} \right),$$

since $r^2 + z^2 = a^2$. Hence the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ at the sphere’s surface determines the strength of the dipole,

$$\mu = -\frac{U a^3}{2}.$$  

The velocity potential for a uniform flow past a stationary sphere is therefore given by

$$\phi(r, z) = U z \left( 1 + \frac{a^3}{2(r^2 + z^2)^{3/2}} \right) \quad \text{(4.9)}$$

The corresponding Stokes streamfunction is given by

$$\Psi(r, z) = \frac{U r^2}{2} \left( 1 - \frac{a^3}{(r^2 + z^2)^{3/2}} \right) \quad \text{(4.10)}$$

Outside the sphere $\Psi > 0$, but we also obtain a solution inside the sphere with $\Psi < 0$. This flow is not real; it is a “virtual flow” that allows for fluid velocity to be consistent with the boundary condition on a solid sphere.
4.5 Flow past an obstacle

4.5.2 Rankine half-body

Suppose that, in the velocity potential of a flow past a sphere, we replace the dipole with a point source \((m > 0)\), so that

\[
\phi(r, z) = Uz - \frac{m}{(r^2 + z^2)^{1/2}} \Rightarrow u = \nabla \phi = \left( \frac{mr}{(r^2 + z^2)^{3/2}}, U + \frac{mz}{(r^2 + z^2)^{3/2}} \right).
\]

This flow has a single stagnation point \(u_r = u_z = 0\) at \(r = 0\) and \(z = -\sqrt{m/U}\).

To find the streamlines of the flow we calculate the Stokes streamfunction using

\[
u_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z} \quad \text{and} \quad u_z = \frac{1}{r} \frac{\partial \Psi}{\partial r}.
\]

Thus,

\[
\frac{\partial \Psi}{\partial r} = Ur + \frac{mrz}{(r^2 + z^2)^{3/2}} \Rightarrow \Psi = \frac{Ur^2}{2} - \frac{mz}{(r^2 + z^2)^{1/2}} + \alpha(z),
\]

\[
\Rightarrow \frac{1}{r} \frac{\partial \Psi}{\partial z} = -\frac{m}{r (r^2 + z^2)^{1/2}} + \frac{mz^2}{r (r^2 + z^2)^{3/2}} + \frac{\alpha'(z)}{r} - \frac{mrz}{r (r^2 + z^2)^{3/2}} + \frac{\alpha'(z)}{r},
\]

\[
= -u_r = -\frac{m}{(r^2 + z^2)^{3/2}}.
\]

So, since \(\alpha'(z) = 0\), \(\alpha\) is a constant (set to zero). The Stokes streamfunction is therefore

\[
\Psi(r, z) = \frac{Ur^2}{2} - \frac{mz}{(r^2 + z^2)^{1/2}}.
\]

At the stagnation point \((r = 0, z = -\sqrt{m/U})\), \(\Psi = m\). Hence, the equation of the streamline, or streamtube, passing through this stagnation point is

\[
\Psi(r, z) = m \Leftrightarrow \frac{Ur^2}{2} = m \left( 1 + \frac{z}{(r^2 + z^2)^{1/2}} \right).
\]

Notice that the straight line \(r = 0\) with \(z < 0\) satisfies the equation of the streamline \(\Psi = m\).

For large positive \(z\), the equation of the streamtube \(\Psi = m\) becomes

\[
\frac{Ur^2}{2} \approx 2m \Rightarrow r \approx 2 \sqrt{\frac{m}{U}}.
\]

Thus, the velocity potential and the Stokes streamfunction

\[
\phi(r, z) = U \left( z - \frac{a^2}{4 (r^2 + z^2)^{1/2}} \right) \quad \text{and} \quad \Psi(r, z) = \frac{U}{2} \left( r^2 - \frac{a^2 z}{2 (r^2 + z^2)^{1/2}} \right)
\]

provide a model for a long slender body of radius \(a = 2\sqrt{m/U}\).
4.6 Method of images

In previous examples we introduced flow singularities (e.g. sources and dipoles) outside of the domain of fluid flow in order to satisfy boundary conditions at a solid surface. This technique can also be used to calculate the flow produced by a singularity near a boundary; it is then called method of images.

Example 4.1 (Point source near a wall)

Consider a point-source of fluid placed at the position $(d, 0, 0)$ (Cartesian coordinates) near a solid wall at $x = 0$.

In free space (no wall), the potential of the source is

$$
\phi_\infty = -\frac{m}{\sqrt{(x-d)^2 + y^2 + z^2}},
$$

$$
\Rightarrow u_\infty = \frac{\partial \phi_\infty}{\partial x} = \frac{m(x-d)}{[(x-d)^2 + y^2 + z^2]^{3/2}}.
$$

So that, at $x = 0$,

$$
u_\infty = -\frac{md}{(d^2 + y^2 + z^2)^{3/2}} \neq 0,
$$

which is inconsistent with the boundary condition $\mathbf{u} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{\hat{e}}_x = u = 0$ at the wall.

To rectify this problem, (i.e. for the flow to satisfy the boundary condition at the wall), we add a source of equal strength $m$ outside the domain, at $(-d, 0, 0)$. By symmetry, this source will produce an equal but opposite velocity field at $x = 0$, so that the boundary condition for the combined flow can be satisfied. The velocity potential for both sources becomes

$$
\phi = -\frac{m}{\sqrt{(x-d)^2 + y^2 + z^2}} - \frac{m}{\sqrt{(x+d)^2 + y^2 + z^2}},
$$

and the velocity field along the $x$-axis,

$$
u = \frac{\partial \phi}{\partial x} = \frac{m(x-d)}{[(x-d)^2 + y^2 + z^2]^{3/2}} + \frac{m(x+d)}{[(x+d)^2 + y^2 + z^2]^{3/2}}.
$$

Clearly, at $x = 0$, now $u = 0$ as required.

The fluid can slip along the wall however as, for $x = 0$,

$$
v = \frac{2my}{(d^2 + y^2 + z^2)^{3/2}},
$$

$$
w = \frac{2mz}{(d^2 + y^2 + z^2)^{3/2}}.
$$

4.7 Method of separation of variables

This is a standard method for solving linear partial differential equations with compatible boundary conditions.

We shall seek separable solutions to Laplace’s equations, of the form $\phi(x, y) = f(x)g(y)$ in Cartesian coordinates or $\phi(r, \theta) = f(r)g(\theta)$ in polar coordinates.
4.7 Method of separation of variables

Plane polar coordinates. We substitute a potential of the form \( \phi(r, \theta) = f(r)g(\theta) \) in Laplace’s equation expressed in plane polar coordinates,

\[
\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0,
\]

\[
\Rightarrow g \frac{d}{dr} \left( r \frac{df}{dr} \right) + f \frac{d^2 g}{d\theta^2} = 0,
\]

\[
\Rightarrow r \frac{d}{dr} \left( r \frac{df}{dr} \right) + \frac{1}{r^2} \frac{d^2 g}{d\theta^2} = 0, \quad \text{(division by } f(r)g(\theta)/r^2\text{)}
\]

\[
\Rightarrow r \frac{d}{dr} \left( r \frac{df}{dr} \right) = -\frac{1}{g} \frac{d^2 g}{d\theta^2}.
\]

Since the terms on the left and right sides of the equation are functions of independent variables, \( r \) and \( \theta \) respectively, they must take a constant value, \( k^2 \) say. Thus we have transformed a partial differential equation for \( \phi \) into two ordinary differential equations for \( f \) and \( g \),

\[
r \frac{d}{dr} \left( r \frac{df}{dr} \right) = k^2 \Rightarrow r \frac{d}{dr} \left( r \frac{df}{dr} \right) - k^2 f = 0,
\]

\[
\Rightarrow 1 \frac{d^2 g}{d\theta^2} = -k^2 \Rightarrow \frac{d^2 g}{d\theta^2} + k^2 g = 0.
\]

Thus, \( g(\theta) = A \cos(k\theta) + B \sin(k\theta) \). For a \( 2\pi \)-periodic function \( g \), such that \( g(\theta) = g(\theta + 2\pi) \), \( k \) must be integer. So \( g(\theta) = A \cos(n\theta) + B \sin(n\theta) \), \( n \in \mathbb{Z} \),

and \( f \) is solution to

\[
r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} - n^2 f = 0.
\]

Substituting nontrivial functions of the form \( f = ar^\alpha \) gives,

\[
[\alpha(\alpha - 1) + \alpha - n^2] ar^\alpha = 0 \Leftrightarrow \alpha^2 = n^2.
\]

The two independent solutions have \( \alpha = \pm n \); the general separable solution to Laplace’s equation in plane polar coordinates is therefore

\[
\phi(r, \theta) = (Ar^n + Br^{-n}) \cos(n\theta) + (Cr^n + Dr^{-n}) \sin(n\theta), \quad n \in \mathbb{Z},
\]

(4.11)

where \( A, B, C \) and \( D \) are constants to be determined by the boundary conditions.

Separable solutions to Laplace’s equation in spherical polar coordinates can be obtained in a similar manner, but involves Legendre polynomials \( P_l(\cos(\theta)) \).

Example 4.2 (Cylinder in an extensional flow)

Consider the velocity potential

\[
\phi(r, \theta) = (Ar^2 + Br^{-2}) \cos(2\theta)
\]

corresponding to a particular solution to Laplace’s equation of the form (4.11), with \( n = 2 \). The radial velocity of this flow is

\[
u_r = \frac{\partial \phi}{\partial r} = 2r \left( A - \frac{B}{r^4} \right) \cos(2\theta).
\]
It vanishes at the surface of a solid cylinder of radius $a$ placed at the origin if $B = a^4A$. Therefore the velocity field

$$u_r = 2Ar \left( 1 - \frac{a^4}{r^4} \right) \cos(2\theta) \quad \text{and} \quad u_\theta = -2Ar \left( 1 + \frac{a^4}{r^4} \right) \sin(2\theta)$$

produced by the potential

$$\phi(r, \theta) = Ar^2 \left( 1 + \frac{a^4}{r^4} \right) \cos(2\theta)$$

represents a fluid flow past a solid cylinder placed in an extensional flow.

Notice that at large distances, i.e. if $r \gg a$, the fluid velocity is that of an extensional flow

$$u_r \simeq 2Ar \cos(2\theta) \quad \text{and} \quad u_\theta \simeq -2Ar \sin(2\theta),$$

in polar coordinates, or equivalently

$$u \simeq 2Ax \quad \text{and} \quad v \simeq -2Ay,$$

in Cartesian coordinates.
4.7 Method of separation of variables
Chapter 5

Euler’s equation

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5.1 Fluid momentum equation

So far, we have discussed some kinematic properties of the velocity fields for incompressible
and irrotational fluid flows.

We shall now study the dynamics of fluid flows and consider changes
in motion due to forces acting on a fluid.

We derive an evolution equation for the fluid momentum by considering
forces acting on a small blob of fluid, of volume $V$ and surface $S$,
containing many fluid particles.

5.1.1 Forces acting on a fluid

The forces acting on the fluid can be divided into two types.

**Body forces**, such as gravity, act on all the particles throughout $V$,

$$F_V = \int_V \rho g \, dV.$$  

**Surface forces** are caused by interactions at the surface $S$. For the rest of this course we
shall only consider the effect of fluid pressure.
Collisions between fluid molecules on either sides of the surface $S$ produce a flux of momentum across the boundary, in the direction of the normal $\mathbf{n}$.

The force exerted on the fluid into $V$ by the fluid on the other side of $S$ is, by convention, written as

$$
\mathbf{F}_s = \int_S -p \mathbf{n} \, dS,
$$

where $p(\mathbf{x}) > 0$ is the fluid pressure.

### 5.1.2 Newton’s law of motion

*Newton’s second law of motion* tells that the sum of the forces acting on the volume of fluid $V$ is equal to the rate of change of its momentum. Since $D\mathbf{u}/Dt$ is the acceleration of the fluid particles, or fluid elements, within $V$, one has

$$
\int_V \rho \frac{D\mathbf{u}}{Dt} \, dV = \int_S -p \mathbf{n} \, dS + \int_V \rho \mathbf{g} \, dV.
$$

We now apply the divergence theorem,

$$
\int_V \rho \frac{D\mathbf{u}}{Dt} \, dV = \int_V (-\nabla p + \rho \mathbf{g}) \, dV,
$$

and notice that both integrands must be identical, since $V$ is arbitrary.

So, the evolution of fluid momentum is governed by *Euler’s equation*

$$
\rho \frac{D\mathbf{u}}{Dt} = \rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p + \rho \mathbf{g}.
$$

(5.1)

This equation neglects viscous effects (tangential surface forces due to velocity gradients) which would otherwise introduce an extra term, $\mu \nabla^2 \mathbf{u}$, where $\mu$ is the viscosity of the fluid, as in the Navier-Stokes equation

$$
\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}.
$$

(5.2)

For the rest of the course we shall only consider *perfect fluids* which are idealised fluids, inviscid and incompressible with constant mass density.

### 5.2 Hydrostatics

We first consider the case of a *fluid at rest*, such that $\mathbf{u} = 0$. Euler’s equation is then reduced to the *equation of hydrostatic balance*,

$$
\nabla p = \rho \mathbf{g} \Leftrightarrow p(\mathbf{x}) = \rho \mathbf{g} \cdot \mathbf{x} + C,
$$

(5.3)

where $C$ is a constant.

**Example 5.1**

The density of mass in the ocean can be considered as constant, $\rho_0$, and the gravity $\mathbf{g} = -g \hat{\mathbf{e}}_z$. (The coordinate $z$ is the upward distance from sea-level.)
From Euler’s equation one has
\[
\frac{dp}{dz} = -\rho_0 g \Rightarrow p(z) = p_0 - \rho_0 gz.
\]
Hence the pressure increases linearly with depth \((z < 0)\).

Taking typical values for the physical constant, \(g \simeq 10 \text{ m s}^{-2}\), \(\rho_0 \simeq 10^3 \text{ kg m}^{-3}\) and a pressure of one atmosphere at sea-level, \(p_0 \simeq 10^5 \text{ Pa} = 10^5 \text{ N m}^{-2}\) gives \(p(z) \simeq 10^5(1 - 0.1z)\); the pressure increases by one atmosphere every 10 m.

\[\text{(Notice that the change in pressure force on a surface } S, \text{ between the ocean surface } z = 0 \text{ and a given depth } z = -d, \text{ is equal to } (p - p_0)S = \rho_0 g dS, \text{ which is the weight of a column of water of height } d \text{ and section } S.)\]

### 5.3 Archimedes’ theorem

The force on a body in a fluid is an upthrust equal to the weight of fluid displaced.

Consider a solid body of volume \(V\) and surface \(S\) totally submerged in a fluid of density \(\rho_0\). The total force on the body caused by the fluid surrounding it is
\[
F = -\int_S p \mathbf{n} \, dS,
\]
where \(p\) is the fluid pressure.

(Notice that the pressure distribution on the surface \(S\) is the same whether the fluid contains a solid or not.) So, using successively the divergence theorem and the equation of hydrostatic balance, \(\nabla P = \rho g\), we find
\[
F = -\int_V \nabla p \, dV = -\int_V \rho_0 g \, dV = -\rho_0 V g.
\]

The buoyancy force is equal the weight of the mass of fluid displaced, \(M = \rho_0 V\), and points in the direction opposite to gravity.

If the fluid is only partially submerged, then we need to split it into parts above and below the water surface, and apply Archimedes’ theorem to the lower section only.

Consider a solid body of volume \(V\) and density \(\rho_s\) partially submerged in a fluid of density \(\rho_0 > \rho_s\). Let \(V_1\) be the volume of solid above the fluid surface and \(V_2\) the volume underneath. Since, the solid is in equilibrium, its weight is balanced by the buoyancy force, so that \(\rho_s V g = \rho_0 V_2 g\).

Hence, the fractions of the volume of solid immersed in the fluid and not immersed are
\[
\frac{V_2}{V} = \frac{\rho_s}{\rho_0} \quad \text{and} \quad \frac{V_1}{V} = 1 - \frac{V_2}{V} = \frac{\rho_0 - \rho_s}{\rho_0},
\]
respectively. For an iceberg of density \(\rho_s \simeq 0.915 \text{ kg m}^{-3}\) floating in salted water of density \(\rho_s \simeq 1.025 \text{ kg m}^{-3}\), \(V_2/V \simeq 89.3\%\) and \(V_1/V \simeq 10.7\%\).

**Question:** A glass of water with an ice cube in it is filled to rim. What happens as ice melts?
5.4 The vorticity equation

In the expression of the acceleration of a fluid particle,

\[ \frac{D}{Dt} \mathbf{u} = \frac{\partial}{\partial t} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}, \]

the nonlinear term can be rewritten using the vector identity

\[ \mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla \left( \frac{\|\mathbf{u}\|^2}{2} \right) - (\mathbf{u} \cdot \nabla) \mathbf{u} \Rightarrow (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left( \frac{\|\mathbf{u}\|^2}{2} \right) - \mathbf{u} \times \boldsymbol{\omega}, \]

where \( \|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} \). So, we can rewrite Euler’s equation (5.1) as

\[ \frac{\partial}{\partial t} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\rho} \nabla p + \mathbf{g}, \]

\[ \frac{\partial}{\partial t} \mathbf{u} + \nabla \left( \frac{\|\mathbf{u}\|^2}{2} \right) - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left( \frac{p}{\rho} \right) + \mathbf{g} \quad (\rho \text{ constant}), \]

and take its curl

\[ \frac{\partial}{\partial t} \boldsymbol{\omega} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = 0, \]

\[ \frac{\partial}{\partial t} \mathbf{\omega} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + (\nabla \cdot \mathbf{u}) \boldsymbol{\omega} - (\nabla \cdot \boldsymbol{\omega}) \mathbf{u} = 0. \]

For incompressible flows \( \nabla \cdot \mathbf{u} = 0 \) and, in addition, \( \nabla \cdot \boldsymbol{\omega} = 0 \) as \( \boldsymbol{\omega} = \nabla \times \mathbf{u} \). Hence,

\[ \frac{D}{Dt} \boldsymbol{\omega} = \frac{\partial}{\partial t} \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u}. \]  (5.4)

This is the vorticity equation. It shows that the vorticity of a fluid particle changes because of gradients of \( \mathbf{u} \) in the direction of \( \boldsymbol{\omega} \).

Properties of the vorticity equation.

i. If \( \boldsymbol{\omega} = 0 \) everywhere initially, then \( \boldsymbol{\omega} \) remains zero. Thus, flows that start off irrotational remain so.

ii. In a two-dimensional planar flow, \( \mathbf{u} = (u(x, y), v(x, y), 0) \), the vector vorticity has only one non-zero component, \( \boldsymbol{\omega} = (\partial v/\partial x - \partial u/\partial y) \hat{e}_z \), so that

\[ (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} = \omega \frac{d}{dz} u(x, y) = 0. \]

Hence, the vorticity equation, reduced to

\[ \frac{D}{Dt} \mathbf{\omega} = \frac{\partial}{\partial t} \mathbf{\omega} + (\mathbf{u} \cdot \nabla) \mathbf{\omega} = 0, \]  (5.5)

shows that the vorticity of a fluid particle remains constant. If, in addition the flow is steady, \( \partial \mathbf{\omega}/\partial t = 0 \) then the vorticity is constant along streamlines.
iii. Vortex stretching.

The stretching of a vortex leads to the increase of its vorticity.

Consider, for example, an incompressible steady flow in a converging cone, function of the radial distance only in spherical polar coordinates,

\[ \mathbf{u}(r) = u_r(r) \hat{e}_r + u_\varphi(r) \hat{e}_\varphi. \]

The component \( u_r \) represents a radial inflow and \( u_\varphi \) the swirling of the fluid. Since \( \nabla \times (u_r(r) \hat{e}_r) = 0 \), only the swirling motion contributes to a non-zero vorticity

\[ \mathbf{\omega}(r) = \nabla \times \mathbf{u} = \nabla \times (u_\varphi(r) \hat{e}_\varphi) = \omega_r \hat{e}_r + \omega_\theta \hat{e}_\theta. \]

From mass conservation in an incompressible spherically symmetric inflow, we find

\[ \nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{du_r}{dr} (r^2 u_r) = 0 \Leftrightarrow u_r = -\frac{k}{r^2}, \]

where \( k > 0 \) is constant. Since the flow is steady, \( \partial \mathbf{\omega}/\partial t = 0 \), the evolution equation for the radial component of the vorticity, \( \omega_r \), becomes

\[ \frac{D\omega_r}{Dt} = u_r \frac{d\omega_r}{dr} + \omega_r \frac{du_r}{dr} \Leftrightarrow \frac{d}{dr} \ln \left| \frac{\omega_r}{u_r} \right| = 0 \Leftrightarrow \frac{\omega_r}{u_r} = \alpha, \text{ constant.} \]

Thus,

\[ \omega_r = \alpha u_r = -\frac{\alpha k}{r^2}, \]

which demonstrates that the vorticity \( \omega_r \) increases as \( u_r \) increases; the initial vortex is stretched by the inflow.

This is the reason for the bath-plug vortex. A small amount of background vorticity is amplified by the flow converging into a small hole. (This mechanism can be interpreted as the conservation of angular momentum of fluid particles.)

5.5 Kelvin’s circulation theorem

The circulation around a closed material curve remains constant — in an inviscid fluid of uniform density, subject to conservative forces. Hence,

\[ \frac{d\Gamma}{dt} = \frac{d}{dt} \oint_{C(t)} \mathbf{u} \cdot d\mathbf{l} = 0, \quad (5.6) \]

if \( C(t) \) is a closed curve formed of fluid particles following the flow.

Proof. Let \( C(t) \) be a closed material curve, hence formed of fluid particles, of parametric representation \( \mathbf{x}(s,t) \) with \( s \in [0,1] \), say. Using this parametric representation, the rate of
change of the circulation around $C(t)$ can be written as

$$
\frac{d\Gamma}{dt} = \frac{d}{dt} \int_0^1 u(x(s,t), t) \cdot \frac{\partial}{\partial s} x(s,t) \, ds = \int_0^1 \frac{d}{dt} \left[ u(x(t), t) \cdot \frac{\partial x}{\partial s} \right] \, ds,
$$

$$
= \int_0^1 \left\{ \frac{\partial u}{\partial t} + \left( \frac{\partial x}{\partial t} \cdot \nabla \right) u \right\} \cdot \frac{\partial x}{\partial s} + u \cdot \frac{\partial^2 x}{\partial s \partial t} \, ds,
$$

$$
= \int_0^1 \left\{ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right\} \cdot \frac{\partial x}{\partial s} + u \cdot \frac{\partial u}{\partial s} \, ds,
$$

since $\frac{\partial x}{\partial t} = u$ is the velocity of the fluid particle at $x(s,t)$. So, using Euler’s equation in the form

$$
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = \nabla \left( -\frac{p}{\rho} + g \cdot x \right),
$$

we find

$$
\frac{d\Gamma}{dt} = \int_0^1 \left\{ \nabla \left( -\frac{p}{\rho} + g \cdot x \right) \right\} \cdot \frac{\partial x}{\partial s} + \frac{\partial}{\partial s} \left( \frac{\|u\|^2}{2} - \frac{p}{\rho} + g \cdot x \right) \, ds = \int_0^1 \frac{\partial}{\partial s} \left( \frac{\|u\|^2}{2} - \frac{p}{\rho} + g \cdot x \right) \, ds.
$$

Thus, since the curve $C(t)$ is closed,

$$
\frac{d\Gamma}{dt} = \oint_{C(t)} u \cdot dl = \frac{d}{dt} \int_{S(t)} \omega \cdot n \, dS = 0;
$$

as required. \hfill \square

Recall that the circulation around a closed curve $C$ is equal to the flux of vorticity through an arbitrary surface $S$ that spans $C$. So, from Kelvin’s circulation theorem,

$$
\frac{d\Gamma}{dt} = \frac{d}{dt} \oint_{C(t)} u \cdot dl = \frac{d}{dt} \int_{S(t)} \omega \cdot n \, dS = 0;
$$

this demonstrates that the flux of vorticity through a surface that spans a material curve is constant. Thus, many properties of the vorticity equation could equivalently be derived from the circulation theorem (e.g. vortex stretching, persistence of irrotationality.)

### 5.6 Shape of the free surface of a rotating fluid

The surface of a rotating liquid placed in a container is not flat but dips near the axis of rotation. This phenomenon, which can be observed when stirring coffee in a mug, results from a radial pressure-gradient balancing the centrifugal force acting within the fluid.

Consider, for example, a cylindrical container partially filled with fluid and mounted on a horizontal turntable. When the flow reaches a steady state, $\frac{\partial u}{\partial t} = 0$, the fluid rotates uniformly, with a constant angular velocity $\Omega$ about the vertical $z$-axis. (The fluid rotates with the container as a solid body.)

In order to calculate the height of the free surface of fluid, $z = h(r)$, we shall solve Euler’s equation,

$$
\rho \left( \frac{\partial u}{\partial t} - u \times \omega \right) = -\rho u \times \omega = -\nabla \left( p + \frac{1}{2} \rho \|u\|^2 \right) + \rho g,
$$

(5.7)
in cylindrical polar coordinates. The velocity and vorticity fields of the fluid in uniform rotation are \( \mathbf{u} = u_\theta \hat{e}_\theta \) and \( \mathbf{\omega} = \omega_z \hat{e}_z \), where \( u_\theta = r\Omega \) and \( \omega_z = 2\Omega \). Hence, the nonlinear term in Euler’s equation

\[
\mathbf{u} \times \mathbf{\omega} = u_\theta \omega_z \hat{e}_r = 2r\Omega^2 \hat{e}_r.
\]

The radial component of the vector equation (5.7) (i.e. its scalar product with \( \hat{e}_r \)) is therefore

\[
-2pr\Omega^2 = -\frac{\partial p}{\partial r} - \frac{\rho}{2} \frac{du_\theta^2}{dr} = -\frac{\partial p}{\partial r} - \rho r\Omega^2,
\]

\[\Leftrightarrow \frac{\partial p}{\partial r} = \rho r\Omega^2. \quad (5.8)\]

This shows that the pressure must vary with the radius inside the fluid, in order to balance the centrifugal force. Moreover, the pressure must also satisfy the vertical hydrostatic balance (i.e. balance between pressure and gravity).

From the vertical component of the vector equation (5.7) (i.e. its scalar product with \( \hat{e}_z \)) we find

\[
0 = -\frac{\partial p}{\partial z} - \rho g \Rightarrow p(r, z) = -\rho g z + A(r),
\]

where \( A(r) \) is a function to be determined using appropriate boundary conditions. At the free surface the fluid pressure should match the atmospheric pressure. Hence, \( p = p_{\text{atm}} \) at \( z = h(r) \), so that

\[
p_{\text{atm}} = -\rho gh(r) + A(r) \Leftrightarrow A(r) = p_{\text{atm}} + \rho gh(r).
\]

So, substituting the expression of the pressure,

\[
p(r, z) = p_{\text{atm}} + \rho g (h(r) - z),
\]

in equation (5.8) leads to

\[
\frac{\partial p}{\partial r} = \rho g \frac{dh}{dr} = \rho r^2 \Omega^2 \Rightarrow h(r) = h_0 + \frac{r^2 \Omega^2}{2g},
\]

where \( h_0 \) is the height of the free surface at \( r = 0 \), on the rotation axis. This result shows that the free surface of a uniformly rotating fluid in a cylindrical container is a paraboloid.

Such rotating cylindrical containers filled with highly-reflecting liquids (e.g. mercury or ionic liquid coated with silver) have been used to build mirrors with a large (up to 6 m) smooth reflecting paraboloid surface, which could be used as primary mirrors for telescopes.
5.6 Shape of the free surface of a rotating fluid
In section (5.4) we obtained the momentum equation for ideal fluids (i.e. inviscid and with constant density) in the form
\[ \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \mathbf{\omega} + \nabla \left( \frac{1}{2} \| \mathbf{u} \|^2 \right) = -\nabla \left( \frac{p}{\rho} \right) + \mathbf{g}. \]

So, since the constant gravity \( \mathbf{g} = \nabla (\mathbf{g} \cdot \mathbf{x}) \), one has
\[ \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \mathbf{\omega} + \nabla \mathcal{H} = 0, \quad (6.1) \]

where
\[ \mathcal{H}(\mathbf{x}, t) = \frac{p}{\rho} + \frac{1}{2} \| \mathbf{u} \|^2 - \mathbf{g} \cdot \mathbf{x} \quad (6.2) \]
is called the Bernoulli function.\(^1\)

**6.1 Bernoulli’s theorem for steady flows**

In the case of steady flows, i.e. when \( \partial \mathbf{u} / \partial t = 0 \), taking the scalar product of equation (6.1) with the fluid velocity, \( \mathbf{u} \), gives the Bernoulli equation
\[ (\mathbf{u} \cdot \nabla) \mathcal{H} = 0, \quad (6.3) \]
since \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{\omega}) \equiv 0 \).

Hence, for an ideal fluid in steady flow,
\[ \mathcal{H}(\mathbf{x}) = \frac{p}{\rho} + \frac{1}{2} \| \mathbf{u} \|^2 - \mathbf{g} \cdot \mathbf{x} \quad (6.4) \]
is constant along a streamline.

\(^1\)Not to be mistaken for Bernoulli’s polynomials.
So, if a streamfunction $\psi(x)$ can be defined, $\mathcal{H}$ is a function of $\psi$ ($\mathcal{H}(x) \equiv \mathcal{H}(\psi)$).

**Example 6.1 (The Venturi effect.)**

Consider a flow through a narrow constriction of cross-section area $A_2$; upstream and downstream the cross-sectional area is $A_1$.

(Three narrow vertical tubes, (a), (b) and (c), are used to measure the pressure at different points.)

The fluid velocity is assumed uniform on cross sections, $S$. Upstream the fluid velocity is $V_1$.

Mass conservation implies $\int_S \rho u \cdot n \, dS = \text{constant}$ for any cross-section $S$, so

$$\int_{S_1} \rho u \cdot n \, dS = \int_{S_2} \rho u \cdot n \, dS \Rightarrow \int_{S_1} u \cdot n \, dS = \int_{S_2} u \cdot n \, dS \quad (\rho = \text{constant}),$$

$$\Rightarrow A_1 V_1 = A_2 V_2 \Rightarrow V_2 = \frac{A_1}{A_2} V_1 > V_1 \quad (\text{since } A_1 > A_2).$$

Neglecting gravity, we apply Bernoulli’s equation to any streamline,

$$\frac{p_1}{\rho} + \frac{1}{2} V_1^2 = \frac{p_2}{\rho} + \frac{1}{2} V_2^2 \Rightarrow p_2 = p_1 - \frac{\rho}{2} \left( V_2^2 - V_1^2 \right),$$

$$\Rightarrow p_2 = p_1 - \frac{\rho V_1^2}{2 A_2^2} (A_1^2 - A_2^2) < p_1.$$

Thus, in the constriction the speed of the flow increases (conservation of mass) and its pressure decreases (Bernoulli’s equation).

This can be measured by the thin tubes where there is fluid but no flow (i.e. fluid in hydrostatic equilibrium). If $h_1$ is the height of fluid in the tube (a) then

$$p_1 = p_0 + \rho gh_1 \quad (p_0 \equiv p_{\text{atm}}).$$

If $h_2$ is the height of fluid in the tube (b) then

$$p_2 = p_0 + \rho gh_2 \Rightarrow h_2 = \frac{p_2 - p_0}{\rho g} = \frac{p_1 - p_0}{\rho g} - \frac{V_1^2}{2g A_2^2} (A_1^2 - A_2^2),$$

$$\Rightarrow h_2 = h_1 - \frac{V_1^2}{2g A_2^2} (A_1^2 - A_2^2) < h_1.$$

In tube (c), $V_3 = V_1$ since $A_3 = A_1$ (mass conservation). So, Bernoulli’s equation gives

$$p_3 + \frac{1}{2} \rho V_3^2 = p_1 + \frac{1}{2} \rho V_1^2 \Rightarrow p_3 = p_1,$$

and so $h_3 = h_1$. (In practice, $h_3$ will be slightly less than $h_1$ due to viscosity but the effect is small.)
Example 6.2 (Flow down a barrel.)
How fast does fluid flow out of a barrel?

Let \( h \) be the height of fluid level in the barrel above the outlet, which has cross-sectional area \( a \). If \( a \ll A(h) \), then the flow can be treated as approximately steady.

**Mass conservation:** \(-A \frac{dh}{dt} = aU\) (with \( U > 0 \)). So, if \( a \ll A \) then \( \left| \frac{dh}{dt} \right| \ll |U| \).

**Bernoulli’s theorem:** consider a streamline from the surface of the fluid to the outlet,

\[
p + \frac{1}{2} \rho ||u||^2 + \rho gz = \text{const.}
\]

At \( z = 0 \): \( p = p_{\text{atm}} \) and \( u = U \); at \( z = h \): \( p = p_{\text{atm}} \) and \( u = \frac{dh}{dt} \). So,

\[
p_{\text{atm}} + \frac{1}{2} \rho U^2 = p_{\text{atm}} + \frac{1}{2} \rho \left( \frac{dh}{dt} \right)^2 + \rho gh,
\]

\[
\Rightarrow U^2 = \left( \frac{dh}{dt} \right)^2 + 2gh \Rightarrow U \simeq \sqrt{2gh} \quad \text{since} \quad \left| \frac{dh}{dt} \right| \ll |U|.
\]

We could have guessed this result from conservation of energy

with \( KE \simeq 0 \) and \( PE = \rho gh \) at \( z = h \)

and \( KE = \frac{1}{2} \rho U^2 \) and \( PE = 0 \) at \( z = 0 \)

\[
\Rightarrow \frac{1}{2} \rho U^2 \simeq \rho gh.
\]

Example 6.3 (Siphon.)
A technique for removing fluid from one vessel to another without pouring is to use a siphon tube.
To start the siphon we need to fill the tube with fluid, but once it is going, the fluid will continue to flow from the upper to the lower container.

In order to calculate the flow rate, we can use Bernoulli’s equation along a streamline from the surface to the exit of the pipe.

At point A: \( p = p_{\text{atm}} \), \( z = 0 \). We shall assume that the container’s cross-sectional area is much larger than that of the pipe. So, \( U_A \approx 0 \) (from mass conservation; see example 6.2 \( -A \, dh/dt = aU \)).

At point C: \( p = p_{\text{atm}} \), \( z = -H \), \( u = U_c \equiv U \).

Bernoulli’s equation:

\[
\frac{p_{\text{atm}}}{\rho} + \frac{1}{2} U_A^2 = \frac{p_{\text{atm}}}{\rho} + \frac{1}{2} U^2 - gH \Rightarrow U \approx \sqrt{2gH}.
\]

If B is the highest point: \( (U_B = U_C \equiv U \) from mass conservation)

\[
\frac{p_B}{\rho} + \frac{1}{2} U^2 + gL = \frac{p_{\text{atm}}}{\rho} + \frac{1}{2} U^2 - gH \Rightarrow p_B = p_{\text{atm}} - \rho g (L + H) < p_{\text{atm}}.
\]

For \( p_B > 0 \), we need \( H + L < \frac{p_{\text{atm}}}{\rho g} \approx \frac{10^5}{10^3 \times 10} = 10m.\)

### 6.2 Bernoulli’s theorem for potential flows

In this section we shall extend Bernoulli’s theorem to the case of irrotational flows.

Recall that Euler’s equation can written in the form

\[
\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \mathbf{\omega} = -\nabla H \quad \text{where} \quad H(\mathbf{x}, t) = \frac{p}{\rho} + \frac{1}{2} \| \mathbf{u} \|^2 - \mathbf{g} \cdot \mathbf{x}.
\]

If the fluid flow is irrotational, i.e. if \( \mathbf{\omega} = \nabla \times \mathbf{u} = 0 \), then \( \mathbf{u} \times \mathbf{\omega} = 0 \) and \( \mathbf{u} = \nabla \phi \); so, the equation above becomes

\[
\nabla \left( \frac{\partial \phi}{\partial t} + H \right) = 0,
\]

since \( \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \nabla \phi}{\partial t} = \nabla \left( \frac{\partial \phi}{\partial t} \right) \).

Thus, for irrotational flows,

\[
\frac{\partial \phi}{\partial t} + H = \frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2} \| \nabla \phi \|^2 - \mathbf{g} \cdot \mathbf{x} \equiv f(t) \quad (6.5)
\]

is a function of time, independent of the position, \( \mathbf{x} \).

If, in addition, the flow is steady,

\[
H = \frac{p}{\rho} + \frac{1}{2} \| \nabla \phi \|^2 - \mathbf{g} \cdot \mathbf{x}, \quad (6.6)
\]

is constant; \( H \) has the same value on all streamlines.
Example 6.4 (Shape of the free surface of a fluid near a rotating rod)

We consider a rod of radius \( a \), rotating at constant angular velocity \( \Omega \), placed in a fluid. Assuming a potential, axisymmetric and planar fluid flow, \( (u_r(r), u_\theta(r)) \) in cylindrical polar coordinates, we wish to calculate the height of the free surface of the fluid near to the rod, \( h(r) \). We also assume that the solid rod is an impenetrable surface on which the fluid does not slip, so that the boundary conditions for the velocity field are

\[
\begin{align*}
    u_r &= 0 \quad \text{and} \quad u_\theta = a \Omega \quad \text{at} \quad r = a.
\end{align*}
\]

From mass conservation, one has

\[
\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{d}{dr} (ru_r) = 0 \iff u_r(r) = \frac{C}{r},
\]

where \( C \) is a constant of integration. However, the boundary condition \( u_r = C/a = 0 \) at \( r = a \) implies that \( C = 0 \). So, \( u_r = 0 \) and the fluid motion is purely azimuthal.

As we assume an irrotational flow,

\[
\nabla \times \mathbf{u} = \frac{1}{r} \frac{d}{dr} (ru_\theta) \hat{e}_z = 0 \iff u_\theta(r) = \frac{k}{r},
\]

where \( k \) is an integration constant to be determined using the second boundary condition. At \( r = a \), \( u_\theta = k/a = a \Omega \) which implies that \( k = a^2 \Omega \). So, the fluid velocity near to the rod is

\[
\begin{align*}
    u_r &= 0 \quad \text{and} \quad u_\theta = \frac{a^2 \Omega}{r}.
\end{align*}
\]

Notice that the velocity potential, function of \( \theta \), can be determined using

\[
\mathbf{u} = \nabla \phi \Rightarrow \frac{1}{r} \frac{d}{d\theta} \frac{\partial \phi}{\partial \theta} = a^2 \Omega \quad \Rightarrow \quad \phi(\theta) = a^2 \Omega \theta.
\]

By applying Bernoulli’s theorem for steady potential flows to the free surface (which is not a streamline, as streamlines are circles about the rod axis) we obtain,

\[
\mathcal{H} = \frac{p_{\text{atm}}}{\rho} + \frac{1}{2} u_\theta^2(r) + gh(r) = \frac{p_{\text{atm}}}{\rho} + gh_\infty,
\]

where the constant pressure \( p = p_{\text{atm}} \) is the atmospheric pressure and \( \lim_{r \to \infty} h(r) = h_\infty \). (Notice also that \( u_\theta \propto 1/r \to 0 \) as \( r \to \infty \).)

Thus, the height of the free surface is

\[
h(r) = h_\infty - \frac{1}{2g} u_\theta^2(r) = h_\infty - \frac{a^4 \Omega^2}{2gr^2}, \quad (6.7)
\]

which shows that the free surface dips as \( 1/r^2 \) near to the rotating rod.

Alternatively, Euler’s equation could be solved directly (i.e. without involving Bernoulli’s theorem) as in § 5.6 with an azimuthal flow, now potential, of the form \( u_\theta = a^2 \Omega/r \). We
can then explain the result (6.7) in terms of centripetal acceleration; since the fluid particles move in circles, there must be an inwards central force producing the necessary centripetal acceleration (i.e. balancing the centrifugal force). Indeed, from the radial component of the momentum equation, one has

\[ -\rho \frac{u_r^2}{r} = -\frac{\partial p}{\partial r} \Rightarrow \frac{\partial p}{\partial r} = \rho \frac{a^4 \Omega^2}{r^3}. \]

However, since the fluid is in vertical hydrostatic equilibrium, the pressure satisfies

\[ \frac{\partial p}{\partial z} = -\rho g \Rightarrow p(r, z) = p_{\text{atm}} - \rho g(z - h(r)). \]

Hence, we have

\[ \frac{\partial p}{\partial r} = \rho g \frac{dh}{dr} = \rho \frac{a^4 \Omega^2}{r^3} \Rightarrow h(r) = h_\infty - \frac{a^4 \Omega^2}{2gr^2}, \]

as in equation (6.7).

### 6.3 Drag force on a sphere

We wish to calculate the pressure force exerted by a steady fluid flow on a solid sphere.

In § 4.5.1 we obtained the velocity potential of a uniform stream, \( U\hat{e}_z \), past a stationary sphere of radius \( a \),

\[ \phi(r, z) = U z \left( 1 + \frac{a^3}{2(r^2 + z^2)^{3/2}} \right), \]

in cylindrical polar coordinates \((r, \theta, z)\). In spherical polar coordinates, \((r, \theta, \varphi)\), this velocity potential becomes

\[ \phi(r, \theta) = U \cos \theta \left( r + \frac{a^3}{2r^2} \right). \tag{6.8} \]

The non-zero components of the fluid velocity, \( \mathbf{u} = \nabla \phi \), are then

\[ u_r = \frac{\partial \phi}{\partial r} = U \cos \theta \left( 1 - \frac{a^3}{r^2} \right) \quad \text{and} \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U \sin \theta \left( 1 + \frac{a^3}{2r^3} \right). \tag{6.9} \]

Hence, at \( r = a \), on the solid sphere’s surface, \( u_r = 0 \) as required by the kinematic boundary conditions and

\[ u_\theta(\theta)|_{r=a} = -\frac{3}{2} U \sin \theta. \]

To express the pressure force on the sphere in terms of the fluid velocity, we use Bernoulli’s theorem for steady potential flows, \( H = p/\rho + \| \mathbf{u} \|^2/2 = \) constant, ignoring gravity. At \( r = a \) the fluid pressure, \( p(\theta) \), therefore satisfies

\[ \frac{p(\theta)}{\rho} + \frac{1}{2} u_\theta^2|_{r=a} = \frac{p_\infty}{\rho} + \frac{1}{2} U^2, \]

where \( p_\infty \) is the pressure as \( r \to \infty \).
Thus, the pressure distribution on the sphere is
\[ p(\theta) = p_\infty + \frac{1}{2} \rho U^2 \left( 1 - \frac{9}{4} \sin^2 \theta \right), \]
(6.10)
and the total pressure force is the surface integral of \( p(\theta) \) on the sphere \( r = a \),
\[ \mathbf{F} = -\int_S p \mathbf{n} \, dS = -\int_0^\pi \int_0^{2\pi} p(\theta) \hat{e}_r \, a^2 \sin \theta \, d\varphi \, d\theta, \]
(6.11)
where \( \hat{e}_r = \sin \theta \cos \varphi \hat{e}_x + \sin \theta \sin \varphi \hat{e}_y + \cos \theta \hat{e}_z \).

As the flow is axisymmetric, the only non-zero component of the force should be in the axial direction, \( z \). Indeed,
\[ F_x = \mathbf{F} \cdot \hat{e}_x = -a^2 \int_0^{2\pi} \cos \varphi \, d\varphi \int_0^\pi p(\theta) \sin^2 \theta \, d\theta = 0, \]
and
\[ F_y = \mathbf{F} \cdot \hat{e}_y = -a^2 \int_0^{2\pi} \sin \varphi \, d\varphi \int_0^\pi p(\theta) \sin^2 \theta \, d\theta = 0. \]

However, after substituting for \( p(\theta) \) in
\[ F_z = \mathbf{F} \cdot \hat{e}_z = -2\pi a^2 \int_0^\pi p(\theta) \sin \theta \cos \theta \, d\theta, \]
we find that
\[ F_z = -2\pi a^2 \left[ (p_\infty + \frac{1}{2} \rho U^2) \int_0^\pi \sin \theta \cos \theta \, d\theta - \frac{9}{8} \rho U^2 \int_0^\pi \sin^3 \theta \cos \theta \, d\theta \right] = 0, \]
so that the total drag force on the sphere, due to the fluid flow around it, is zero!

D’Alembert’s paradox: it can be demonstrated that the drag force on any 3-D solid body moving at uniform speed in a potential flow is zero (see, e.g., Paterson, § XI.9, p. 240).

This is not true in reality of course, as flows past 3-D solid bodies are not potential.

We can see why a potential flow past a sphere gives zero drag by looking at the streamlines.

![Streamlines of a potential flow past a sphere](image)

The flow is clearly fore-aft symmetric (symmetry about \( z = 0 \)); the front \( (S_1) \) and the back \( (S_2) \) of the sphere are stagnation points at equal pressure, \( P_{S_1} = P_{S_2} = p_\infty + \frac{1}{2} \rho U^2 \). At the side, \( u_r = 0 \) and \( u_\theta^2 > 0 \), so from Bernoulli’s theorem, the pressure there is lower than at the stagnation points but it must have the same symmetry as the flow. Notice that, from Bernoulli’s theorem, the pressure does not depend on the direction of the flow, but on its speed \( ||\mathbf{u}|| \) only.

However, the real flow past a sphere is not symmetric and, as a consequence, the fluid exerts a net drag force on the sphere.
6.4 Separation

The pressure distribution on the surface a solid sphere placed is a uniform stream,

\[ p(\theta) = p_\infty + \frac{1}{2} \rho U^2 \left( 1 - \frac{9}{4} \sin^2 \theta \right), \]

reaches its minimum, \( p_{\min} = p_\infty - \frac{5}{8} \rho U^2 \), at \( \theta = \pm \pi/2 \). So, the pressure gradient in the direction of the flow, \( (\mathbf{u} \cdot \nabla) p \), is a positive from \( \theta = 0 \) to \( \theta = \pm \pi/2 \) and negative beyond.

An adverse pressure gradient, \( (\mathbf{u} \cdot \nabla) p > 0 \) (i.e. pressure increasing in the direction of the flow along the surface), is “bad news” and causes the flow to separate, leaving a turbulent wake behind the sphere.

Very roughly one can estimate the pressure difference upstream and downstream as \( \frac{1}{2} \rho U^2 \), so that the drag force \( F \propto \frac{1}{2} \rho U^2 \times A \), where \( A \) is the cross-sectional area.

The ratio

\[ C_D = \frac{F}{\frac{1}{2} \rho U^2 A} \]  \hspace{1cm} (6.12)

is called drag coefficient and depends, e.g., on the shape of the body (see Acheson §4.13, p. 150).

The way to reduce drag (i.e. resistance) is to reduce separation:

- **Streamlining**: separation occurs because of adverse pressure gradients on the surface of solid bodies. These can be reduced by using more “streamlined” shapes, that avoid diverging streamlines (e.g., aerodynamic bike helmets (time trial cyclist), ships, aeroplanes and cars).

- **Surface roughness**: paradoxically, a rough surface can reduce drag by reducing separation (e.g. dimple pattern of golf balls and shining of cricket ball on one side).
6.5 Unsteady flows

6.5.1 Flows in pipes

In example 6.2 we consider a flow out of a barrel through a small hole. Now, consider a flow out of a narrowing tube, opened to the atmosphere at both ends, where the exit is not much smaller than the cross-section (i.e. the fluid flow cannot be assumed steady).

Let \( A(z) \) be the smoothly varying cross-sectional area of the pipe at height \( z \), such that \( A \to A_\infty \) as \( z \to \infty \) and \( A(0) = a \).

We assume that the flow is potential and purely in the \( z \)-direction, \( u_z = \partial \phi / \partial z \equiv w \).

By conservation of mass the volume flux, \( Q(t) = -w(z,t)A(z) \), must be independent of height. Hence,

\[
\frac{\partial \phi}{\partial z} = w(z,t) = -\frac{Q(t)}{A(z)} \Rightarrow \phi(z,t) = \phi(0,t) - \frac{Q(t)}{A(\mu)} \int_0^z \frac{d\mu}{A(\mu)}.
\]

(Note that we could set \( \phi(0,t) = 0 \) without loss of generality.) Applying Bernoulli’s theorem for potential flows, \( \frac{p}{\rho} + \frac{\|u\|^2}{2} + \frac{\partial \phi}{\partial t} - g \cdot \mathbf{x} = F(t) \), at the free surface and the exit gives,

at \( z = 0 \), \( \frac{p_{\text{atm}}}{\rho} + \frac{1}{2} \frac{Q^2(t)}{a^2} + \frac{d}{dt} \phi(0,t) = F(t) \),

and at \( z = h \), \( \frac{p_{\text{atm}}}{\rho} + \frac{1}{2} \left( \frac{dh}{dt} \right)^2 + \frac{d}{dt} \phi(0,t) - \frac{dQ}{dt} \int_0^h \frac{dz}{A(z)} + gh = F(t) \).

Equating both expressions gives

\[
\frac{1}{2} \left( \frac{dh}{dt} \right)^2 - \frac{Q^2(t)}{a^2} - \frac{dQ}{dt} \int_0^h \frac{dz}{A(z)} + gh = 0,
\]

\[
\Leftrightarrow \frac{1}{2} \left[ 1 - \frac{A^2(h)}{a^2} \right] \left( \frac{dh}{dt} \right)^2 + A(h) \frac{d^2h}{dt^2} \int_0^h \frac{dz}{A(z)} + gh = 0 \quad \text{since} \quad Q(t) = -A(h) \frac{dh}{dt}.
\]

The fluid height, \( h(t) \), is then solution to the nonlinear second order ordinary differential equation

\[
\left( A(h) \int_0^h \frac{dz}{A(z)} \right) \frac{d^2h}{dt^2} + \frac{1}{2} \left[ 1 - \frac{A^2(h)}{a^2} \right] \left( \frac{dh}{dt} \right)^2 + gh = 0. \tag{6.13}
\]

Far from the exit this equation becomes approximately

\[
h \ddot{h} + \frac{1}{2} \left( 1 - \frac{A_\infty^2}{a^2} \right) \dot{h}^2 + gh = 0,
\]

since, as \( h \to \infty \),

\[
A(h) \sim A_\infty \quad \text{and} \quad \int_0^h \frac{dz}{A(z)} \sim \int_0^h \frac{dz}{A_\infty} = \frac{h}{A_\infty}.
\]
Using the chain rule, \( h = \frac{dh}{dt} = \frac{dh}{dh} \frac{dh}{dt} = h \frac{dh}{dt} \), one finds
\[
h \frac{dh}{dt} + \frac{1}{2} \left(1 - \frac{A_{\infty}^2}{a^2}\right) h^2 + gh = 0 \iff \frac{1}{2} \frac{d\dot{h}^2}{dh} + \frac{1}{2} \left(1 - \frac{A_{\infty}^2}{a^2}\right) \frac{\dot{h}^2}{h} + g = 0
\]
which can be written as a linear differential equation for \( Z = \frac{\dot{h}^2}{2} \),
\[
\frac{dZ}{dh} + \left(1 - \frac{A_{\infty}^2}{a^2}\right) \frac{Z}{h} + g = 0.
\]

### 6.5.2 Bubble oscillations

The sound of a “babbling brook” is due to the oscillation (compression/expansion) of air bubbles entrained into the stream. The pitch of the sound depends on the size of the bubbles.

Consider a bubble of radius \( a(t) \); the velocity of the fluid at the bubble surface, \( u_r = \frac{da}{dt} \equiv \dot{a} \).

We can model the oscillations of the bubble of air using a potential flow due to a point source/sink of fluid at the centre of the bubble,
\[
\phi(r, t) = -\frac{k(t)}{r} \Rightarrow u_r = \frac{\partial \phi}{\partial r} = \frac{k}{r^2}.
\]

The boundary condition at the bubble’s surface, \( r = a \), is \( u_r = \frac{k}{a^2} = \dot{a} \). So,
\[
k = \dot{a} a^2 \Rightarrow u_r = \frac{\dot{a} a^2}{r^2} \text{ and } \phi = -\frac{\dot{a} a^2}{r} \Rightarrow \frac{\partial \phi}{\partial t} = -\frac{\dot{a} a^2}{r} - 2 \frac{\dot{a} a^2}{r}
\]

Applying Bernoulli’s theorem (ignoring gravity) as \( r \to \infty \),
\[
\frac{p}{\rho} + \frac{1}{2} \|\nabla \phi\|^2 + \frac{\partial \phi}{\partial t} = F(t) = \frac{p_{\infty}}{\rho} \quad (\text{as } r \to \infty, \phi \to 0 \text{ and } \|\mathbf{u}\| \to 0: \text{ the fluid is stationary}).
\]

At the bubble’s surface,
\[
\frac{p(a)}{\rho} + \frac{1}{2} \frac{\dot{a}^2}{a} - \frac{\dot{a} a^2}{a} - 2 \frac{\dot{a} a^2}{a} = \frac{p(a)}{\rho} - \ddot{a} a - \frac{3}{2} \dot{a}^2 = F(t).
\]

Combining the two expressions above, one gets
\[
\frac{p(a) - p_{\infty}}{\rho} = \ddot{a} a + \frac{3}{2} \dot{a}^2, \quad (6.14)
\]

where \( p(a) \) is the fluid pressure at the bubble’s surface. Now, if the gas inside the bubble of mass \( m \) is subject to adiabatic changes, its equation of state is
\[
p_g = K \rho_g^\gamma \quad \text{where} \quad \rho_g = \frac{3m}{4\pi a^3},
\]
and $K$ is a constant to determine — the adiabatic index $\gamma$ depends on the gas considered. Moreover, since the bubble of gas is in balance with the surrounding fluid, continuity of pressure $p_g = p(a)$ must be satisfied at the surface $r = a(t)$.

Now, for a bubble in equilibrium, such that $a = a_0$ and $\dot{a} = \ddot{a} = 0$, equation (6.14) gives $p = p_\infty$ and, imposing pressure continuity $p_g = p$ at $r = a_0$, one gets

$$p_g = K\rho \gamma^\gamma = K \left( \frac{3m}{4\pi a_0^3} \right)^\gamma = p_\infty \Rightarrow K = p_\infty \left( \frac{4\pi a_0^3}{3m} \right)^\gamma.$$

So, pressure continuity at the bubble’s surface $r = a(t)$ implies

$$p(a) = p_g = K\rho \gamma^\gamma = p_\infty \left( \frac{4\pi a_0^3}{3m} \right)^\gamma \left( \frac{3m}{4\pi a^3} \right)^\gamma = p_\infty \left( \frac{a_0}{a} \right)^{3\gamma}.$$

Then, equation (6.14) becomes

$$\frac{p_\infty}{\rho} \left( \frac{a_0^{3\gamma}}{a^{3\gamma}} - 1 \right) = \ddot{a} + \frac{3}{2} \dot{a}^2.$$

For small amplitude oscillations about the equilibrium $a(t) = a_0 + \epsilon(t)$ where $|\epsilon| \ll a_0$, so that $\dot{a} = \dot{\epsilon}$, $\ddot{a} = \ddot{\epsilon}$ and $\dot{a}^2 = \epsilon^2 \simeq 0$; the nonlinear terms are negligible at first approximation. Thus,

$$a_0 \ddot{\epsilon} = \frac{p_\infty}{\rho} \left( \frac{a_0^{3\gamma}}{a^{3\gamma}} \left( 1 + \frac{\epsilon}{a_0} \right)^{3\gamma} - 1 \right) \simeq -3\gamma \frac{p_\infty}{\rho} \frac{\epsilon}{a_0},$$

$$\Rightarrow \ddot{\epsilon} + \frac{3\gamma p_\infty}{\rho a_0^3} \epsilon = 0.$$

The bubble undergo periodic small amplitude oscillations with frequency $\omega = \left( \frac{3\gamma p_\infty}{\rho a_0^3} \right)^{1/2}$.

Note that the frequency scales with the inverse of the (mean) radius of the bubbles. E.g. for $\gamma = 3/2$, $p_\infty = 10^5$ Pa and $\rho = 10^3$ kg m$^{-3}$,

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi a_0} \sqrt{\frac{3\gamma p_\infty}{\rho}} \Rightarrow f \times a_0 \simeq 3$ kHz mm.$$

For bubbles of size $a_0 = 0.2$ mm, $f \simeq 15$ kHz (G9).

### 6.6 Acceleration of a sphere

We have already shown that a sphere moving with a steady velocity under a potential flow has no drag force. What about an accelerating sphere?

The velocity potential for a sphere of radius $a$ moving with velocity $U$ in still water is

$$\phi = -\frac{Ua^3}{2r^2} \cos \theta.$$

(This flow satisfies the following boundary conditions: $u = \nabla \phi \to 0$ as $r \to \infty$ together with $u_r = U \cos \theta \hat{e}_r$ at $r = a$.)
Rather than calculating the pressure via Bernoulli’s theorem, we calculate the work done by the forces acting on the sphere as it moves at speed $U$, function of time, through the fluid.

The total kinetic energy of the system sphere of mass $m$ plus fluid is

$$T = \frac{1}{2}mU^2 + \int_V \frac{1}{2} \rho (\nabla \phi)^2 dV,$$

$$= \frac{1}{2}mU^2 + \frac{1}{2} \rho \int_V \left[ \nabla \cdot (\phi \nabla \phi) - \phi \nabla^2 \phi \right] dV, \quad \text{(using } \nabla \cdot (fA) = A \cdot \nabla f + f \nabla \cdot A \text{)}$$

$$= \frac{1}{2}mU^2 + \frac{1}{2} \rho \int_S \phi \nabla \phi \cdot n dS, \quad \text{by divergence theorem.}$$

Here $S$ is the surface of the sphere of radius $a$. So $n = -\hat{e}_r$ and $dS = a^2 \sin \theta d\theta d\phi$, such that

$$T = \frac{1}{2}mU^2 - \frac{1}{2} \rho \int_0^\pi \phi |_{r=a} \left. \frac{\partial \phi}{\partial r} \right|_{r=a} \ 2\pi a^2 \sin \theta d\theta,$$

$$= \frac{1}{2}mU^2 + \frac{\pi a^3}{2} \rho U^2 \int_0^\pi \cos^2 \theta \sin \theta d\theta,$$

$$= \frac{1}{2}mU^2 + \frac{\pi a^3}{3} \rho U^2 \quad \text{since } \int_0^\pi \cos^2 \theta \sin \theta d\theta = -\frac{1}{3} \int_0^\pi \frac{d\cos^3 \theta}{d\theta} d\theta = \frac{2}{3}.$$

So $T = \frac{1}{2}(m + M)U^2$, where $M = \frac{2}{3} \pi a^3 \rho$ is called the added mass and represents the mass of fluid that must be accelerated along with the sphere.

The rate of working of the forces $F$ acting on the sphere equals the change of kinetic energy,

$$FU = \frac{dT}{dt} = (m + M)U \frac{dU}{dt}.$$

Hence, the force required to accelerate the sphere is given by

$$F = (m + M) \frac{dU}{dt}.$$

Thus, the acceleration of a bubble (mass $m$ and radius $a$) rising under gravity (see §5.3 on Archimedes theorem) satisfies

$$F = \frac{4}{3} \pi a^3 \rho g - mg = (2M - m)g = (m + M) \frac{dU}{dt},$$

$$\Rightarrow \frac{dU}{dt} = \frac{2M - m}{M + m} g = \frac{4\pi a^3 \rho - 3m}{2\pi a^3 \rho + 3m} g.$$

As mass density is much less for a gas than for a liquid, we can assume $m \ll M$, so that $\frac{dU}{dt} \approx 2g$.

Alternatively: Consider a bubble of mass $m$ rising under gravity with speed $U = \frac{dz}{dt}$. 
At height $z$ the potential energy is

$$V = mgz - \frac{4}{3}\pi a^3 \rho gz.$$  

In absence of dissipative processes the total energy remains constant; hence,

$$T + V = \frac{1}{2}(m + M)U^2 + mgz - \frac{4}{3}\pi a^3 \rho gz = \text{const.}$$

Differentiating this expression with respect to time gives

$$(m + M)U \frac{dU}{dt} + \left(m - \frac{4}{3}\pi a^3 \rho\right) gU = 0,$$

$$\Rightarrow \frac{dU}{dt} = \frac{2M - m}{M + m} g = \frac{4\pi a^3 \rho - 3m}{2\pi a^3 \rho + 3m} g.$$  

Again, for a bubble of gas in a liquid $M \gg m$, so $\frac{dU}{dt} \simeq 2g$; the bubble accelerates at twice the gravitational acceleration.
6.6 Acceleration of a sphere
In this chapter we shall use conservation of mass and Bernoulli’s equation to study simplified models of smooth steady flows in open channels, such as rivers and channels with weirs.

7.1 Conservation of mass

Consider a channel of large width which can be approximated as a two dimensional flow in the \( (x, z) \)-plane and whose base lies on \( z = Z(x) \) (topography), with water of depth \( h(x) \). We define \( \xi(x) = Z(x) + h(x) \), the height of the free surface.

For a steady flow (\( \partial h/\partial t = 0 \)), the net mass flux through a volume \( \delta V \) between \( x \) and \( x + \delta x \) (i.e. the mass flows in and out of \( \delta V \)) must be zero. This leads to

\[
\rho \int_{Z(x+\delta x)}^{\xi(x+\delta x)} u \, dz - \rho \int_{Z(x)}^{\xi(x)} u \, dz = 0 \Rightarrow \rho \frac{d}{dx} \left( \int_{Z(x)}^{\xi(x)} u \, dz \right) = 0.
\]

So the volume flux

\[
Q = \int_{Z(x)}^{\xi(x)} u \, dz
\]

is constant in space (and time).
Let us further assume that $u$ is independent of $z$, so that

$$Q = u(x)[\xi(x) - Z(x)] = u(x)h(x) \quad (7.1)$$

is independent of $x$.

### 7.2 Bernoulli’s theorem

We can apply Bernoulli’s theorem to the free surface where the pressure $p = p_{\text{atm}}$. Recall that, for steady flows,

$$\mathcal{H}(x, t) = \frac{p}{\rho} + \frac{1}{2}\|u\|^2 - g \cdot x$$

is constant along a streamline.

As the free surface is a streamline, $\mathcal{H} = \frac{p_{\text{atm}}}{\rho} + \frac{1}{2}(u^2 + w^2) + g\xi$ is independent of $x$.

Let $dl = \left(\frac{1}{d\xi/dx}\right) dx$ be an infinitesimal line element along the free surface. Since $u$ is parallel to streamlines

$$u \times dl = \left(\frac{1}{w}\right) \times \left(\frac{1}{d\xi/dx}\right) dx = 0 \Rightarrow w = \frac{d\xi}{dx}.$$  

This is similarly derived from the equation of streamlines $dx/ds = u \Leftrightarrow d\xi/w = dx/u$.

(Alternative derivation: the free surface is defined by the equation $z = \xi(x)$ which is equivalent to $H(x, z) = z - \xi(x) = 0$. Hence, $\nabla H = \left(-\frac{d\xi}{dx} \quad 1\right)$ must be perpendicular the isosurface $H = 0$, i.e. to the free surface. So $u \cdot \nabla H = 0$, leading again to

$$-u \frac{d\xi}{dx} + w = 0 \Rightarrow w = \frac{d\xi}{dx},$$

on the free surface.)

Thus, if the free surface is smooth, i.e. if $|d\xi/dx| \ll 1$, then $w \ll u$ can be neglected and, from Bernoulli’s equation, we obtain

$$\frac{1}{2}u^2 + g\xi = \frac{1}{2}u^2 + g(h + Z) = \text{const.} \quad (7.2)$$

Equivalently,

$$gh \left(\frac{F^2}{2} + 1 + \frac{Z}{h}\right) = \text{const.}, \quad (7.3)$$

where the Froude number $F = \frac{u}{\sqrt{gh}}$ has no dimension and determines how flows react to disturbances. (Recall that $Z = 0$ when the topography is flat.)
7.3 Flow over a hump

Suppose that the fluid flowing along a channel with base at \( z = 0 \) encounters a smooth hump of height \( Z(x) \). We assume \( |dZ/dx| \ll 1 \), so that the flow is approximately unidirectional (the argument for the free surface holds for the base of the channel) and \( |d\xi/dx| \ll 1 \).

Let \( H \) be the depth of water and \( U \) its velocity far upstream. Hence, the upstream Froude number \( F = \frac{U}{\sqrt{gH}} \) is given.

Conservation of mass gives

\[
Q = u(x) h(x) = UH \Rightarrow u = \frac{UH}{h},
\]

and Bernoulli’s equation

\[
\frac{1}{2} u^2(x) + g(h(x) + Z(x)) = \frac{1}{2} U^2 + gH.
\]

Substituting for \( u^2 = U^2 \frac{H^2}{h^2} \) leads to

\[
\frac{1}{2} U^2 \frac{H^2}{h^2} + g(h(x) + Z(x)) = \frac{1}{2} U^2 + gH,
\]

which can be rearranged as an equation for \( Z \),

\[
Z = \frac{H}{2} \frac{U^2}{gH} \left( 1 - \frac{H^2}{h^2} \right) + H - h,
\]

\[
\Rightarrow \frac{Z}{H} = \frac{F^2}{2} \left( 1 - \frac{H^2}{h^2} \right) + 1 - \frac{h}{H},
\]

where \( F = \frac{U}{\sqrt{gH}} \) is the upstream Froude number. This gives the relationship between the height of the base of the channel, \( Z \), and the depth of water, \( h \), when \( F \) is fixed.

For simplicity, let \( \tilde{Z} = \frac{Z}{H} \) and \( \tilde{h} = \frac{h}{H} \). (\( \tilde{Z} \) and \( \tilde{h} \) are nondimensional measures of the bump height and depth of water respectively.) Hence,

\[
\tilde{Z} = f(\tilde{h}) \quad \text{where} \quad f(\tilde{h}) = \frac{F^2}{2} \left( 1 - \frac{1}{\tilde{h}^2} \right) + 1 - \tilde{h}. \tag{7.4}
\]

The function \( f(\tilde{h}) \to -\infty \), both as \( \tilde{h} \to 0 \) and as \( \tilde{h} \to +\infty \). Its derivative is

\[
\frac{df}{d\tilde{h}} = \frac{F^2}{\tilde{h}^3} - 1,
\]

\[
\Rightarrow \frac{df}{d\tilde{h}} = 0 \quad \text{at} \quad \tilde{h}_c = F^{2/3} \quad \text{with} \quad \frac{d^2f}{d\tilde{h}^2} = -3\frac{F^2}{\tilde{h}^4} < 0.
\]
So, \( f \) has a unique maximum at \( \tilde{h}_c = F^{2/3} \) — or equivalently at \( h_c = HF^{2/3} \); its values is given by

\[
\tilde{Z}_c = \max (f) = f(\tilde{h}_c) = \frac{F^2}{2} \left( 1 - F^{-4/3} \right) + 1 - F^{2/3} = \frac{F^2}{2} + 1 - \frac{3}{2} F^{2/3},
\]

\[
= \frac{1}{2} \left( F^{2/3} - 1 \right)^2 \left( F^{2/3} + 2 \right) > 0.
\]

(7.5)

The function \( f \) reaches its maximum \( \tilde{Z}_c \) at \( \tilde{h}_c = F^{2/3} \). (Note that \( \tilde{Z}_c \) and \( \tilde{h}_c \) are determined by the upstream Froude number, i.e. by the upstream properties of the fluid flow.)

\[\begin{array}{c}
\tilde{Z} \\
\tilde{Z}_c \\
\tilde{Z}_c \\
\tilde{Z}_c \\
\tilde{h}_c \\
\tilde{h}_1 \quad \tilde{h}_c \quad \tilde{h}_2 \quad \tilde{h}
\end{array}\]

The equation \( f(\tilde{h}) = 0 \) has two solutions, \( \tilde{h}_1 \) and \( \tilde{h}_2 \), one of which is \( \tilde{h}_c = 1 \) (i.e. \( h = H \)); indeed, far upstream \( Z/H = f(1) = 0 \).

Note also that, using equation (7.4), we find that the height of the free surface \( \xi = H\tilde{\xi} = Z + h \) is given by

\[\tilde{\xi} = \tilde{Z} + \tilde{h} = 1 + \frac{F^2}{2} \left( 1 - \frac{1}{\tilde{h}^2} \right).\]

Thus \( \tilde{\xi} > 1 \) if \( \tilde{h} > 1 \) and \( \tilde{\xi} < 1 \) if \( \tilde{h} < 1 \); the free surface goes up when the depth of water \( \tilde{h} \) increases and goes down when the depth of water decreases.

Let the size of the hump (i.e. its maximum height) be less than the critical height, \( \tilde{Z}_{\text{max}} < \tilde{Z}_c \).

- If \( F > 1 \), then \( \tilde{h}_1 = 1 \) (since \( \tilde{h}_c = F^{2/3} > 1 \)).

\[\begin{array}{c}
\tilde{Z} \\
\tilde{Z}_c \\
\tilde{Z}_c \\
\tilde{Z}_c \\
\tilde{Z}_c \\
\tilde{h}_c \quad \tilde{h}_c \\
\tilde{h}_c \quad \tilde{h}
\end{array}\]

Starting from \( \tilde{h} = 1 \), the depth of water first increases then decreases with \( \tilde{Z} \) and returns to \( \tilde{h} = 1 \) (i.e. \( h = H \)). This is called a supercritical flow.
• If $F < 1$, then $\tilde{h}_2 = 1$ (since $\tilde{h}_c = F^{2/3} < 1$).

Here, as $\tilde{Z}$ increases the water surface goes down and when $\tilde{Z}$ returns to zero, the depth of water returns to $\tilde{h} = 1$ (i.e. $h = H$). This is called a subcritical flow.

Why these two different behaviours? The equation $\frac{u^2}{2} + g(h + Z) = \text{const.}$ expresses the conservation of energy (kinetic energy, KE, and gravitational potential energy, PE). In order to flow over the hump, the fluid has two choices. (i) To increase its KE by reducing its PE, i.e. decreasing $h$; (ii) to increase its PE by reducing its KE, i.e. increasing $h$. The Froude number, $F = \frac{U}{\sqrt{gH}}$ is the ratio of kinetic to potential energy. If $F > 1$ then KE > PE upstream, so that it is easier to reduce KE as the fluid flows over the bump; on the contrary, if $F < 1$ then PE > KE upstream and conversion of PE into KE is preferred.

7.4 Critical flows

In both previous cases (supercritical and subcritical flows), the fluid flow returns to its original height and speed after passing over the bump.

However, if the maximal height of the hump $Z_{\text{max}} = Z_c$ the flow can move across from one branch of the solution to the other.

Stable transitions occur only in one direction, from subcritical to supercritical.
In this case, a subcritical flow upstream \((F < 1)\) is transformed smoothly into a supercritical flow downstream \((F > 1)\). The condition for this to happen, \(Z_{\text{max}} = Z_c\), leads to \(h = h_c = HF^{2/3}\) at the top of the hump.

From conservation of mass \(uh = UH \Rightarrow u = UH/h\). So, the local Froude number \(f\) satisfies

\[
f^2 = \frac{u^2}{gh} = \frac{U^2}{gH} \left( \frac{H}{h} \right)^3 = F^2 \left( \frac{H}{h} \right)^3.
\]

At the top of the bump, \(h_c/H = F^{2/3}\), so that \(f = 1\). The local Froude number must be equal to 1 at the top of the bump. (Note that the local \(f\) is a continuous function of \(x\), less than 1 upstream, greater than 1 downstream and equal to 1 at the top of the hump.)

**Example 7.1**

A flow along a uniform channel encounters a bump of height \(Z(x)\). Downstream the height of fluid \(h_0\) is a half of the height upstream. Find the upstream and downstream fluid velocities and the height of the bump.

Let \(U_1\) and \(U_2\) be the upstream and downstream fluid velocities respectively.

From conservation of mass,

\[
Q = u(x)h(x) = 2h_0U_1 = h_0U_2 \Rightarrow U_2 = 2U_1;
\]

and from Bernoulli’s equation,

\[
\frac{1}{2} u^2 + g(h + Z) = \frac{1}{2} U_1^2 + 2gh_0 = \frac{1}{2} U_2^2 + gh_0.
\]

Combining the two equations above leads to

\[
\frac{1}{2} U_1^2 + 2gh_0 = 2U_1^2 + gh_0 \Rightarrow \frac{3}{2} U_1^2 = gh_0 \Rightarrow U_1 = \sqrt{\frac{2}{3} gh_0}.
\]

The upstream Froude number \(F_1^2 = \frac{U_1^2}{2gh_0} = \frac{1}{3} < 1\) (subcritical flow); so, \(U_2 = 2\sqrt{\frac{2}{3} gh_0}\) and the downstream Froude number \(F_2^2 = \frac{U_2^2}{gh_0} = \frac{8}{3} > 1\) (supercritical flow). The flow is critical as it is smoothly transformed from subcritical upstream to supercritical downstream.

To find the height of the bump, \(Z_{\text{max}}(= Z_c)\), for a critical flow, use \(f^2 = 1\) (local Froude number) at the top of the bump.

We get \(Z_{\text{max}}\) from Bernoulli’s equation, using

\[
f^2 = \frac{u_c^2}{gh_c} \Rightarrow u_c^2 = gh_c.
\]
Indeed
\[
\frac{1}{2} U_1^2 + 2gh_0 = \frac{1}{2} u_c^2 + g(h_c + Z_{\text{max}}) = \frac{gh_c}{2} + g(h_c + Z_{\text{max}}),
\]
\[\Rightarrow gZ_{\text{max}} = \frac{1}{2} U_1^2 + 2gh_0 - \frac{3}{2} gh_c \quad \text{but} \quad U_1^2 = \frac{2}{3} gh_0,
\]
\[\Rightarrow Z_{\text{max}} = \frac{7}{3} h_0 - \frac{3}{2} h_c.
\]
Next, \(h_c\) can be eliminated using mass conservation, \(u_c h_c = 2U_1 h_0\), together with \(u_c^2 = gh_c\) and \(U_1^2 = \frac{2}{3} gh_0\):
\[
u_c^2 h_c^2 = gh_c^3 = 4U_1^2 h_0^2 = \frac{8}{3} gh_0^3 \quad \Rightarrow \quad h_c = \left(\frac{8}{3}\right)^{1/3} h_0 = \frac{2}{3^{1/3}} h_0 = 2h_0 F_1^{2/3}.
\]
So, the size of the hump (i.e. its maximal height) is
\[Z_{\text{max}} = Z_c = \left(\frac{7}{3} - 3^{2/3}\right) h_0 \approx 0.253 h_0.
\]

### 7.5 Flow through a constriction

An alternative to varying the height of the base of the channel is to vary its breadth \(b(x)\).

What does happen to the height of water as \(b\) varies?

**Conservation of mass** gives
\[Q = u(x)b(x)h(x) = UBH = \text{const.},\]
and **Bernoulli’s theorem**,
\[\frac{1}{2} u^2(x) + gh(x) = \frac{1}{2} U^2 + gH = \text{const.}\]

So,
\[u^2 = U^2 + 2g(H - h),\]
\[\Rightarrow \quad [U^2 + 2g(H - h)] b^2 h^2 = U^2 B^2 H^2. \quad (7.6)\]
7.5 Flow through a constriction

Dividing by \( gH^3B^2 \),

\[
\left( \frac{U^2}{gH} + 2\left(1 - \frac{h}{H}\right) \right) \frac{h^2}{B^2H^2} = \frac{U^2}{gH} \Rightarrow \frac{h^2}{H^2} \left( F^2 + 2\left(1 - \frac{h}{H}\right) \right) = \frac{U^2}{b^2},
\]

where \( F = \frac{U}{\sqrt{gH}} \) is the upstream Froude number.

Again, let us make use of the nondimensional variables \( \tilde{h} = \frac{h}{H} \) and \( \tilde{b} = \frac{b}{B} \):

\[
\frac{\tilde{h}^2}{F^2} \left( F^2 + 2\left(1 - \tilde{h}\right) \right) = \frac{1}{\tilde{b}^2}.
\]

Now, let

\[
\frac{1}{\tilde{b}^2} - 1 \equiv K(\tilde{h}) \quad \text{where} \quad K(\tilde{h}) = \frac{\tilde{h}^2}{F^2} \left( F^2 + 2\left(1 - \tilde{h}\right) \right) - 1. \tag{7.7}
\]

The function \( K \sim -\frac{2}{F^2} \tilde{h}^3 \to -\infty \) as \( \tilde{h} \to +\infty \) and \( K(0) = -1 \). Its derivative

\[
\frac{dK}{d\tilde{h}} = 2\frac{\tilde{h}}{F^2} \left( F^2 + 2 - 3\tilde{h} \right) = 0
\]

if either \( \tilde{h} = 0 \) or \( \tilde{h} = \tilde{h}_c = \frac{F^2 + 2}{3} \). Furthermore,

\[
\frac{d^2K}{d\tilde{h}^2} = \frac{2}{F^2} \left( F^2 + 2 - 6\tilde{h} \right);
\]

so, \( K \) has a local minimum at \( \tilde{h} = 0 \) since \( \frac{d^2K}{d\tilde{h}^2} = \frac{2F^2 + 4}{F^2} > 0 \) at \( \tilde{h} = 0 \); and \( K \) has a local maximum at \( \tilde{h} = \tilde{h}_c \) since \( \frac{d^2K}{d\tilde{h}^2} = \frac{2F^2 + 4}{F^2} < 0 \) at \( \tilde{h} = \tilde{h}_c \).

The equation \( K(\tilde{h}) = 0 \) has two solutions, \( \tilde{h}_1 \) and \( \tilde{h}_2 \), one of which is \( \tilde{h} = \frac{h}{H} = 1 \). Note also that \( \tilde{h}_c = \frac{F^2 + 2}{3} > 1 \) if \( F > 1 \) and \( \tilde{h}_c < 1 \) if \( F < 1 \).

- If \( F > 1 \) (supercritical), then \( \tilde{h}_1 = 1 \) and \( \tilde{h} = h/H \) increases as \( \tilde{b} = b/B \) decreases (i.e. \( K \) increases). The height of the free surface rises through the constriction.
- If \( F < 1 \) (subcritical), then \( \tilde{h}_2 = 1 \) and \( \tilde{h} = h/H \) decreases as \( \tilde{b} = b/B \) decreases.
Once again, a smooth transition from a subcritical flow to a supercritical flow can occur if the narrowest point in the constriction reaches a critical breadth, \( \tilde{b}_c \), defined such that

\[
\frac{1}{\tilde{b}_c^2} - 1 = \max \left( \frac{1}{b^2} - 1 \right) = K(\tilde{h}_c) = \frac{\tilde{h}_c^2}{F^2} \left[ F^2 + 2 \left( 1 - \tilde{h}_c \right) \right] - 1,
\]

where \( \tilde{h}_c = \frac{F^2 + 2}{3} \). In this case the local Froude number \( f^2 = \frac{u^2}{g \tilde{h}_c} \) satisfies

\[
f^2 = \frac{U^2 + 2g(H - \tilde{h}_c)}{gh} = \frac{U^2 + 2g(H - \tilde{h}_c)}{gH} \frac{H}{\tilde{h}_c},
\]

using equation (7.6). So,

\[
f^2 = \left( \frac{U^2}{gH} + 2 - \frac{2\tilde{h}_c}{H} \right) \frac{H}{\tilde{h}_c} = \left( F^2 + 2 - \frac{2\tilde{h}_c}{H} \right) \frac{H}{\tilde{h}_c},
\]

\[
eq \left( F^2 + 2 - \frac{2F^2 + 2}{3} \right) \frac{1}{F^2/3 + 2/3} = \frac{F^2/3 + 2/3}{F^2/3 + 2/3} = 1.
\]

Thus, the local Froude number \( f = 1 \) at the narrowest point in the constriction for a critical flow.

### 7.6 Transition caused by a sluice gate

Another way to generate a transition between a subcritical and a supercritical flow is via a sluice gate.

Using conservation of mass and Bernoulli’s equation for the free surface we obtain

\[
U_1 H_1 = U_2 H_2 \quad \text{and} \quad \frac{1}{2} U_1^2 + gH_1 = \frac{1}{2} U_2^2 + gH_2,
\]

dividing by \( gH_1 \) \( \Rightarrow \frac{1}{2} \left( F_1^2 + 2 \right) = \frac{1}{2} \left( F_2^2 + 2 \right) \frac{H_2}{H_1} \).

Then, since

\[
\frac{H_2^2}{H_1^2} = \frac{U_1^2}{U_2^2} = \frac{F_1^2 H_1}{F_2^2 H_2} \Rightarrow \frac{H_2}{H_1} = \frac{F_1^2}{F_2^2},
\]

one has

\[
\left( F_1^2 + 2 \right)^3 = \left( F_2^2 + 2 \right)^3 \frac{F_1^2}{F_2^2} \Leftrightarrow \left( F_1^2 + 2 \right)^3 \frac{F_1^2}{F_2^2} = \left( F_2^2 + 2 \right)^3.
\]

Thus, \( G(F_1) = G(F_2) \) where \( G(f) = \left( f^2 + 2 \right)^3 f^2 \). The function \( G(f) \to +\infty \) as \( f \to 0 \) and as \( f \to \infty \). Its derivative

\[
\frac{dG}{df} = 6 f^2 \left( f^2 + 2 \right)^2 - 2 \left( f^2 + 2 \right)^3 = 0 \Rightarrow \left( f^2 + 2 \right)^2 \left( 6 f^2 - 2 f^2 - 4 \right) = 0 \Rightarrow f^2 = 1.
\]
The equation \( G(f) = \frac{(f^2 + 2)^3}{f^2} = C \) (where \( C > 27 \)) has two solutions, one corresponding to a subcritical flow and one corresponding to a supercritical flow.
Chapter 8

Lift forces

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How do aeroplanes fly? An Airbus A380 weights 560 tonnes (5.6 × 10⁵ kg) at take-off and so requires a lift force in excess of 5.6 × 10⁶ N. The lift force is provided by the wings (span ≈ 80 m, area ≈ 845 m²) and is generated by aerodynamic forces.

In this section we shall give a brief discussion of the lift forces on aerofoils.

8.1 Two-dimensional thin aerofoils

Consider a 2-D flow past a thin aerofoil and assume that the flow does not separate and can be modelled as a potential flow (i.e. aerofoil is smooth).

We can use Bernoulli’s theorem to calculate the pressure — recall that there is no drag force acting on a solid body placed in a potential flow.

\[ F = \int (p_B - p_T) \, dx. \]

For a flat aerofoil, the force in the upward vertical direction will be the difference between pressure forces on the bottom and on the top of the aerofoil. This force per unit length is

\[ F = \frac{\rho}{2} \int (u_T^2 - u_B^2) \, dx = \frac{\rho}{2} \int (u_T + u_B)(u_T - u_B) \, dx. \]
For a thin aerofoil, both $u_T$ and $u_B$ will be close to $U$ (the free stream velocity), so that
\[ u_T + u_B \simeq 2U \Rightarrow F \simeq \rho U \int (u_T - u_B) \, dx = -\rho U \oint_C \mathbf{u} \cdot d\mathbf{l}, \]
where $C$ is the curve around the aerofoil.

Thus, the force acting on the aerofoil,
\[ F = -\rho U \Gamma \quad \text{where} \quad \Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l}, \]
is proportional to the circulation around the wing.

### 8.2 Kutta-Joukowski theorem

The above result is an example of a general exact general result of inviscid irrotational flow theory.

**Theorem 8.1 (Kutta-Joukowski)**

Any 2-D body in relative motion to the ambient fluid with velocity $U$ has a lift force, perpendicular to $U$, of magnitude
\[ F = -\rho U \Gamma \quad \text{where} \quad \Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l}. \quad (8.1) \]

For a flow around a flat plate, $\Gamma = -\pi UL \sin \alpha$ (using conformal mapping).

Nose up ($\alpha \uparrow$) leads to more lift and nose down ($\alpha \downarrow$) to less lift.

### 8.3 Lift produced by a spinning cylinder

\[
\begin{align*}
\phi &= U \left( r + \frac{a^2}{r} \right) \cos \theta + \frac{\Gamma}{2\pi} \cos \theta, \\
u_r &= U \left( 1 - \frac{a^2}{r^2} \right) \cos \theta, \\
u_\theta &= \frac{\Gamma}{2\pi r} - U \left( 1 + \frac{a^2}{r^2} \right) \sin \theta.
\end{align*}
\]

On $r = a$, $u_r = 0$ and $u_\theta = \frac{\Gamma}{2\pi a} - 2U \sin \theta$; the circulation is $\Gamma = \int_0^{2\pi} a u_\theta |_{r=a} \, d\theta$. So,
\[
\left. u_\theta^2 \right|_{r=a} = \left( \frac{\Gamma}{2\pi a} - 2U \sin \theta \right)^2 = \left( \frac{\Gamma}{2\pi a} \right)^2 - 2\frac{\Gamma U}{\pi a} \sin \theta + 4U^2 \sin^2 \theta.
\]

The pressure at the cylinder surface can now be calculated using Bernoulli’s theorem,
\[
p = p_\infty + \frac{1}{2} \rho U^2 - \frac{\rho \Gamma^2}{8\pi^2 a^2} + \frac{\rho \Gamma U}{\pi a} \sin \theta - 2\rho U^2 \sin^2 \theta.
\]
The pressure force per unit length

\[ F = \oint -p \, n \, dl = \int_0^{2\pi} -p \, n \, a \, d\theta, \quad \text{where} \quad n = \hat{e}_r = \cos \theta \, \hat{e}_x + \sin \theta \, \hat{e}_y. \]

The force can be decomposed into its components parallel and perpendicular to the free stream velocity (in the x direction): \( F = F_\parallel \hat{e}_x + F_\perp \hat{e}_y \), with \( F_\parallel = 0 \) (no drag force) and the lift force \( F_\perp = -\rho \Gamma U \) (Kutta-Joukowski theorem). This is called the “Magnus effect” (e.g. football, tennis, table tennis).

### 8.4 Origin of circulation around a wing

When the plane is stationary on the runway, there is no circulation around the wings. In § 5.4, we showed that vorticity cannot be created in an initially vorticity free fluid, in the absence of viscosity. Thus the flow should remain vortex free. (Recall that the circulation is equal to the flux of vorticity.)

A potential flow past an inclined wing is of the form:

However, small viscous effects allow the aerofoil to shed a vortex off the trailing edge, so that downstream separation occurs at the trailing edge.

This vortex, called the starting vortex, remains behind on the runway. Its circulation is equal and opposite to the circulation around the wing.

Note: the greater the angle of inclination (or attack angle), the greater the circulation and hence the lift (e.g. flat plate \( \Gamma = -2\pi UL \sin \alpha \)). This is true up to a point: if the angle is too steep, the flow separates so the drag force on the aeroplane increases significantly and it partially loses its lift force. This is called a stall.

### 8.5 Three-dimensional aerofoils

No wings is infinitely long (i.e. 2-D). Special care needs to be taken with wings tips.
Since $p_B > p_T$, there is a pressure gradient driving a flow around the edge of the wing. This leads to a vortex from the edge of the wing.

These trailing vortices are parts of a single vortex tube formed by the wings, the trailing vortices and the starting vortex. (Vortex tubes must be closed as they cannot start or end in an inviscid fluid.)
Appendix A

Vector calculus

We shall only consider the case of three-dimensional spaces.

A.1 Definitions

A physical quantity is a scalar when it is only determined by its magnitude and a vector when it is determined by its magnitude and direction. It is crucial to distinguish vectors from scalars; standard notations for vectors include $\vec{u} \equiv u \equiv \mathbf{u}$. A unit vector, commonly denoted by $\hat{i} \equiv \hat{i}$, has magnitude one. The coordinates of a vector $\mathbf{a}$ are the scalars $a_1, a_2$ and $a_3$ such that

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \equiv (a_1, a_2, a_3) \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. If $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ are orthonormal (i.e. $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$) the magnitude of the vector is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Scalar or dot product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3$$

is a scalar.

Vector or cross product:

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{e}_1 + (a_3 b_1 - a_1 b_3) \mathbf{e}_2 + (a_1 b_2 - a_2 b_1) \mathbf{e}_3.$$

is a vector with magnitude $|\mathbf{a}||\mathbf{b}| \sin \theta$ and a direction perpendicular to both vectors $\mathbf{a}$ and $\mathbf{b}$ in a right-handed sens.

Triple scalar product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a}$$

is a scalar.

Triple vector product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

is a vector.
A.2 Suffix notation

It is often very convenient to write vector equations using the suffix notation. Any suffix may appear once or twice in any term in an equation; a suffix that appears just once is called a free suffix and a suffix that appears twice is called a dummy suffix.

Summation convention.

Dummy suffices are summed over from 1 to 3 whilst free suffices take the values 1, 2 and 3. Hence, free suffices must be the same on both sides of an equation whereas the names of dummy suffices are not important (e.g. \(a_i b_i c_k = a_j b_j c_k\)).

Tensors used in suffix notation.

Kronecker Delta:
\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j,
\end{cases} 
\Rightarrow (\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}.
\]

The Kronecker Delta is symmetric, \(\delta_{ij} = \delta_{ji}\), and \(\delta_{ij}a_j = a_i\).

Alternating Tensor:
\[
\epsilon_{ijk} = \begin{cases} 
0 & \text{if any of } i, j \text{ or } k \text{ are equal;} \\
1 & \text{if } (i, j, k) = (1, 2, 3), (2, 3, 1) \text{ or } (3, 1, 2); \\
-1 & \text{if } (i, j, k) = (1, 3, 2), (3, 2, 1) \text{ or } (2, 1, 3).
\end{cases}
\]

The Alternating Tensor is antisymmetric, \(\epsilon_{ijk} = -\epsilon_{jik}\), and it is invariant under cyclic permutations of the indices, \(\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}\).

The tensors \(\delta_{ij}\) and \(\epsilon_{ijk}\) are related to each other by \(\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\).

Examples.

Standard algebraic operations on vectors can be written in a compact form using the suffix notation.

A scalar product can be written as \(a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_j b_j\) and the \(i^{th}\) component of a vector product as \((a \times b)_i = \epsilon_{ijk} a_j b_k\).

A.3 Vector differentiation

A.3.1 Differential operators in Cartesian coordinates

We consider scalar and vector fields, \(f(\mathbf{x})\) and \(\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}))\) respectively, where \(\mathbf{x} = (x_1, x_2, x_3) \equiv (x, y, z)\) are Cartesian coordinates in the orthonormal basis \(\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \equiv \{\hat{i}, \hat{j}, \hat{k}\}\). We also define the vector differential operator
\[
\nabla \equiv \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right).
\]

(\nabla\) is pronounced grad, nabla or del.)

- The gradient of a scalar field \(f(x_1, x_2, x_3)\) is given by the vector field
\[
\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) = \frac{\partial f}{\partial x_1} \hat{e}_1 + \frac{\partial f}{\partial x_2} \hat{e}_2 + \frac{\partial f}{\partial x_3} \hat{e}_3.
\]

\(\nabla f\) is the vector field with a direction perpendicular to the isosurfaces of \(f\) with a magnitude equal to the rate of change of \(f\) in that direction.
• The divergence of a vector field \( \mathbf{F} \) is given by the scalar field

\[
\text{div} \, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.
\]

A vector field \( \mathbf{F} \) is solenoidal if \( \nabla \cdot \mathbf{F} = 0 \) everywhere.

• The curl of a vector field \( \mathbf{F} \) is given by the vector field

\[
\text{curl} \, \mathbf{F} = \nabla \times \mathbf{F} = \left( \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \hat{e}_3.
\]

A vector field \( \mathbf{F} \) is irrotational if \( \nabla \times \mathbf{F} = 0 \) everywhere.

• The directional derivative \((\mathbf{F} \cdot \nabla)\) is a differential operator which calculates the derivative of scalar or vector fields in the direction of \( \mathbf{F} \). (It is not to be confused with the scalar \( \nabla \cdot \mathbf{F} \).)

The directional derivative of a scalar field is given by the scalar field

\[
(\mathbf{F} \cdot \nabla) f = \mathbf{F} \cdot \nabla f = F_1 \frac{\partial f}{\partial x_1} + F_2 \frac{\partial f}{\partial x_2} + F_3 \frac{\partial f}{\partial x_3}.
\]

If \( \hat{n} \) is a unit vector, \((\hat{n} \cdot \nabla)f\) gives the rate of change of \( f \) in the direction of \( \hat{n} \).

The directional derivative of a vector field \( \mathbf{G} \) is given by the vector field

\[
(\mathbf{F} \cdot \nabla) \mathbf{G} = ((\mathbf{F} \cdot \nabla) G_1, (\mathbf{F} \cdot \nabla) G_2, (\mathbf{F} \cdot \nabla) G_3) = \left( F_1 \frac{\partial G_1}{\partial x_1} + F_2 \frac{\partial G_1}{\partial x_2} + F_3 \frac{\partial G_1}{\partial x_3}, F_1 \frac{\partial G_2}{\partial x_1} + F_2 \frac{\partial G_2}{\partial x_2} + F_3 \frac{\partial G_2}{\partial x_3}, F_1 \frac{\partial G_3}{\partial x_1} + F_2 \frac{\partial G_3}{\partial x_2} + F_3 \frac{\partial G_3}{\partial x_3} \right).
\]

• The Laplacian \( \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \) is a differential operator which can act on scalar or vector fields:

\[
\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}
\]

is a scalar field and

\[
\nabla^2 \mathbf{F} = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)
\]

is a vector field.

• The Lagrangian derivative of scalar and vector fields are

\[
\frac{\text{D}}{\text{Dt}} f(\mathbf{x}, t) = \frac{\text{d}}{\text{dt}} f(\mathbf{x}(t), t) = \frac{\partial}{\partial t} f(\mathbf{x}, t) + (\mathbf{u} \cdot \nabla) f(\mathbf{x}, t)
\]
and
\[
\frac{D}{Dt} F(x, t) = \frac{d}{dt} F(x(t), t) \frac{\partial}{\partial t} F(x, t) + (u \cdot \nabla) F(x, t),
\]
respectively, with \( u(x, t) = \frac{dx}{dt} \) and where \( \nabla \) acts upon the variables \((x_1, x_2, x_3)\) only.

Notice that if \( F = F(t) \) then \( \frac{dF}{dt} = (\frac{dF_1}{dt}, \frac{dF_2}{dt}, \frac{dF_3}{dt}) \).

Differential operators in Cartesian coordinates using suffix notation
\[
(\text{grad} \, f)_i = (\nabla f)_i = \frac{\partial f}{\partial x_i},
\]
\[
\text{div} \, F = \nabla \cdot F = \frac{\partial F_j}{\partial x_j},
\]
\[
(\text{curl} \, F)_i = (\nabla \times F)_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j},
\]
\[
(F \cdot \nabla) f = F_j \frac{\partial f}{\partial x_j}.
\]

Notice that the operator \( \frac{\partial}{\partial x_j} \) cannot be moved around as it acts on everything that follows it.

A.3.2 Differential operators polar coordinates

The differential operators defined above in Cartesian coordinates take different forms for different systems of coordinates, e.g. in cylindrical polar coordinates or spherical polar coordinates.

Cylindrical polar coordinates

Let \( p \) and \( u = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z \) be scalar and vector fields respectively, functions of \((r, \theta, z)\).

\[
\nabla p = \frac{\partial p}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \hat{e}_\theta + \frac{\partial p}{\partial z} \hat{e}_z,
\]
\[
\nabla \cdot u = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},
\]
\[
\nabla \times u = \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \hat{e}_r + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{e}_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} (ru_\theta) - \frac{\partial u_r}{\partial \theta} \right) \hat{e}_z.
\]
\[
(u \cdot \nabla) p = u_r \frac{\partial p}{\partial r} + u_\theta \frac{\partial p}{\partial \theta} + u_z \frac{\partial p}{\partial z},
\]
\[
\nabla^2 p = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}.
\]
Spherical polar coordinates

Let \( p \) and \( \mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_\phi \mathbf{e}_\phi \) be scalar and vector fields respectively, functions of \((r, \theta, \phi)\).

\[
\nabla p = \frac{\partial p}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} \mathbf{e}_\phi, \\
\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 u_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( u_\theta \sin \theta \right) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi}, \\
\nabla \times \mathbf{u} = \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \theta} \left( u_\phi \sin \theta \right) - \frac{\partial u_\theta}{\partial \phi} \right) \mathbf{e}_r + \frac{1}{r} \left( \frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r} \left( ru_\phi \right) \right) \mathbf{e}_\theta \\
\quad + \frac{1}{r} \left( \frac{\partial}{\partial r} \left( ru_\theta \right) - \frac{\partial u_r}{\partial \theta} \right) \mathbf{e}_\phi, \\
(u \cdot \nabla) p = \frac{\partial p}{\partial r} + \frac{u_\theta}{r} \frac{\partial p}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial p}{\partial \phi}, \\
\nabla^2 p = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial p}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial p}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta \sin \phi} \frac{\partial^2 p}{\partial \phi^2}.
\]

A.3.3 Vector differential identities

The simple vector identities that follow can all be proved with suffix notation. Note that these vector identities are true for all systems of coordinates.

Let \( \mathbf{F} \) and \( \mathbf{G} \) be vector fields and \( \varphi \) and \( \psi \) be scalar fields.

\[
\nabla \cdot (\nabla \varphi) = \nabla^2 \varphi, \\
\nabla \cdot (\nabla \times \mathbf{F}) = 0, \\
\nabla \times (\nabla \varphi) = 0, \\
\n\nabla (\varphi \psi) = \varphi \nabla \psi + \psi \nabla \varphi, \\
\n\nabla \cdot (\varphi \mathbf{F}) = \varphi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \varphi, \\
\n\nabla \times (\varphi \mathbf{F}) = \varphi \nabla \times \mathbf{F} + \mathbf{F} \times \nabla \varphi, \\
\n\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}, \\
\n\nabla (\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F}, \\
\n\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \times (\nabla \times \mathbf{F}) - \mathbf{F} \times (\nabla \times \mathbf{G}), \\
\n\n\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F} (\nabla \cdot \mathbf{G}) - \mathbf{G} (\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G}, \\
\n\n\nabla^2 \mathbf{F} = \nabla (\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}), \\
\n\mathbf{F} \times (\nabla \times \mathbf{F}) = \frac{1}{2} \nabla (\mathbf{F} \cdot \mathbf{F}) - (\mathbf{F} \nabla) \mathbf{F}.
\]

A.4 Vector integral theorems

A.4.1 Alternative definitions of divergence and curl

- An alternative definition of divergence is given by

\[
\nabla \cdot \mathbf{F} = \lim_{\delta V \to 0} \frac{1}{\delta V} \oiint_{\delta S} \mathbf{F} \cdot \mathbf{n} \, dS,
\]

where \( \delta V \) is a small volume bounded by a surface \( \delta S \) which has a normal \( \mathbf{n} \), pointing outwards.
A.4 Vector integral theorems

• An alternative definition of curl is given by

\[ \mathbf{n} \cdot \nabla \times \mathbf{F} = \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \mathbf{F} \cdot d\mathbf{x}, \]

where \( \delta S \) is a small open surface bounded by a curve \( \delta C \) which is oriented in a right-handed sense.

A.4.2 Physical interpretation of divergence and curl

• The divergence of a vector field gives a measure of how much expansion and contraction there is in the field.

• The curl of a vector field gives a measure of how much rotation or twist there is in the field.

A.4.3 The Divergence and Stokes’ Theorems

• The divergence theorem states that

\[ \iiint_{V} \nabla \cdot \mathbf{F} = \iint_{S} \mathbf{F} \cdot \mathbf{n} dS, \]

where \( S \) is the closed surface enclosing the volume \( V \) and \( \mathbf{n} \) is the outward-pointing normal from the surface.

• Stokes’ theorem states that

\[ \iint_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_{C} \mathbf{F} \cdot d\mathbf{x}, \]

where \( C \) is the closed curve enclosing the open surface \( S \) and \( \mathbf{n} \) is the normal from the surface.

A.4.4 Conservative vector fields, line integrals and exact differentials

• The following five statements are equivalent in a simply-connected domain:

  i. \( \nabla \times \mathbf{F} = 0 \) at each point in the domain.
  ii. \( \mathbf{F} = \nabla \phi \) for some scalar \( \phi \) which is single-valued in the region.
  iii. \( \mathbf{F} \cdot d\mathbf{x} \) is an exact differential.
  iv. \( \int_{P}^{Q} \mathbf{F} \cdot d\mathbf{x} \) is independent of the path of integration from \( P \) to \( Q \).
  v. \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \) around every closed curve in the region.

• If \( \nabla \cdot \mathbf{F} = 0 \) then \( \mathbf{F} = \nabla \times \mathbf{A} \) for some \( \mathbf{A} \). (This vector potential \( \mathbf{A} \) is not unique.)
Appendix B

Ordinary differential equations

B.1 First order equations

Separable equations

If an equation is of the form
\[ \frac{dy}{dx} = f(y)g(x), \]
we can separate variables to find
\[ \int \frac{1}{f(y)} \, dy = \int g(x) \, dx, \]
for which each side can now be integrated independently.

Linear equations

An equation of the form
\[ a(x) \frac{dy}{dx} + b(x)y = f(x) \]
can be integrated using an integrating factor. First put the equation into standard form by dividing through by \( a(x) \),
\[ \frac{dy}{dx} + \frac{b(x)}{a(x)} y = \frac{f(x)}{a(x)}. \]
The integrating factor is then
\[ p(x) = \exp \int \frac{b(x)}{a(x)} \, dx \]
which can be calculated. Multiply through by the integrating factor and if everything has been done correctly the equation can now be written
\[ \frac{d}{dx} (p(x) y) = \frac{p(x)g(x)}{a(x)} \]
which can be integrated up with respect to \( x \).
B.2 Second order equations

Equations with constant coefficients

Equations of the form
\[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x), \]
where \(a\), \(b\) and \(c\) are constants can be solved using the complimentary function and particular integral method.

First, consider the homogeneous equation by setting the RHS to zero:
\[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0. \]

Seeking solutions of the form \( y_1 = e^{\lambda x} \) leads to the quadratic auxiliary equation
\[ a\lambda^2 + b\lambda + c = 0. \]

So, \( y_1 = Ae^{\lambda_1 x} + Be^{\lambda_2 x} \) where \( \lambda_1 \) and \( \lambda_2 \) are the roots of the quadratic and \( A \) and \( B \) constants. The solution becomes \( y_1 = (A + Bx)e^{\lambda x} \) if \( \lambda \) is a double root. (Notice that solutions can be written in terms of sine and cosine functions when roots have complex values.)

Next, find a particular integral — a special case \( y_2 \) that gives the correct RHS — by trying functions \( y_2 \) that look like the desired RHS.

The solution sought is the sum of the general solution to the homogeneous equation with the particular integral, \( y = y_1 + y_2 \).

Cauchy equations

Equations of the general form
\[ ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = 0. \]
can be solved by seeking solutions of the form \( y = x^\lambda \), giving, as above, an algebraic auxiliary equation
\[ a\lambda^2 + (b - a)\lambda + c = 0. \]

B.3 System of coupled equations

In calculating the path of a fluid particle we have to solve a set of differential equations of the form,
\[ \frac{dx}{dt} = f(x, y, z, t), \]
\[ \frac{dy}{dt} = g(x, y, z, t), \]
\[ \frac{dz}{dt} = h(x, y, z, t). \]

The method of solution depends upon how the equations are coupled.
• Example 1:

\[
\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = -y.
\]

Here the equation for \(\frac{dx}{dt}\) depends upon knowing \(y(t)\), however since \(\frac{dy}{dt}\) does not contain \(x(t)\) we can compute \(y(t)\) first. This has the general solution, \(y(t) = y_0 e^{-t}\). We can now substitute this expression for \(y(t)\) to give

\[
\frac{dx}{dt} = x + y_0 e^{-t}.
\]

This is a linear equation so it can be solved using an integrating factor \(p(t) = e^{-t}\) to give

\[
\frac{d}{dt} (xe^{-t}) = y_0 e^{-2t},
\]

so that

\[
x(t) = \frac{y_0}{2} e^{-t} + Ce^t.
\]

• Example 2

\[
\frac{dx}{dt} = 2x + y, \quad \frac{dy}{dt} = -x.
\]

Here, \(\frac{dx}{dt}\) depends on \(y(t)\) and \(\frac{dy}{dt}\) depends on \(x(t)\) therefore, we can’t solve either equation directly as it depends on the solution to other equation, which we don’t know. Instead, we can eliminate \(y(t)\) to form a second-order equation for \(x(t)\). Differentiating \(\frac{dx}{dt} = 2x + y\) with respect to \(t\) on both sides shows that

\[
\frac{d^2 x}{dt^2} = 2 \frac{dx}{dt} + \frac{dy}{dt}.
\]

Then, substituting \(\frac{dy}{dt}\) by its expression results in

\[
\frac{d^2 x}{dt^2} = 2 \frac{dx}{dt} - x.
\]

This is a constant coefficient second-order differential equation and can be solved via an auxiliary equation, to give

\[
x(t) = (At + B)e^t.
\]

Once \(x(t)\) has been found, this can be plugged into the equation for \(y(t)\) which can then be solved to find

\[
y(t) = \frac{dx}{dt} - 2x = (A + At + B)e^t - 2(At + B)e^t = (A - B - At)e^t.
\]