Chapter 4

Potential flows

Contents

4.1 Velocity potential

We shall now consider the special case of *irrotational flows*, i.e. flows with no vorticity, such that

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0. \tag{4.1}$$

If a velocity field $\mathbf{u}$ is irrotational, that is if $\nabla \times \mathbf{u} = 0$, then there exists a *velocity potential* $\phi(\mathbf{x}, t)$ defined by

$$\mathbf{u} = \nabla\phi. \tag{4.2}$$

This is a result from vector calculus; the converse is trivially true since $\forall\phi$, $\nabla \times \nabla\phi \equiv 0$.

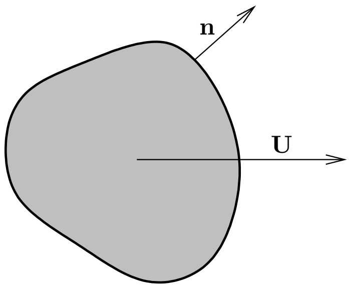
If in addition the flow is incompressible, the velocity potential ϕ satisfies Laplace's equation

$$\nabla^2\phi = 0. \tag{4.3}$$

Indeed, for incompressible irrotational flows one has $\nabla \cdot \mathbf{u} = \nabla \cdot \nabla\phi = \nabla^2\phi = 0$.

Hence, incompressible irrotational flows can be computed by solving Laplace's equation (4.3) and imposing appropriate *boundary conditions* (or conditions at infinity) on the solution. (Notice that for 2-D incompressible irrotational flows, both velocity potential, ϕ, and streamfunction, ψ, are solutions to Laplace's equation, $\nabla^2\psi = -\omega = 0$ and $\nabla^2\phi = 0$; boundary conditions on ψ and ϕ are different however.)

4.2 Kinematic boundary conditions



Consider a flow past a solid body moving at velocity $\mathbf{U}$. If $\mathbf{n}$ is the unit vector normal to the surface of the solid, then, locally, the surface advances (i.e. moves in the direction of $\mathbf{n}$) at the velocity $(\mathbf{U}\cdot\mathbf{n})\,\mathbf{n}$.

Since the fluid cannot penetrate into the solid body, its velocity normal the surface, $(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}$, must locally equal that of the solid,

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{U}\cdot\mathbf{n}.$$

So, since $\mathbf{u} = \nabla\phi$,

$$\mathbf{n}\cdot\nabla\phi = \frac{\partial\phi}{\partial n} = \mathbf{U}\cdot\mathbf{n}; \tag{4.4}$$

the velocity potential satisfies Neumann boundary conditions at the solid body surface.

4.3 Elementary potential flows

4.3.1 Source and sink of fluid

Line source/sink

Consider an *axisymmetric* potential $\phi \equiv \phi(r)$. From Laplace's equation in plane polar coordinates,

$$\nabla^2\phi = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = 0 \Leftrightarrow \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{m}{r},$$

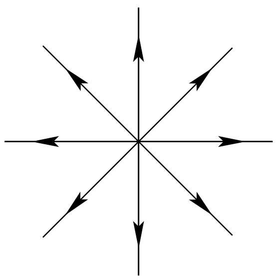
one finds

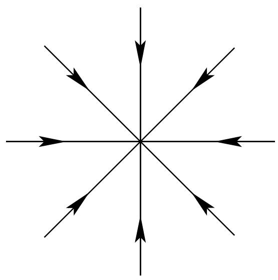
$$\phi(r) = m\ln r + C, \tag{4.5}$$

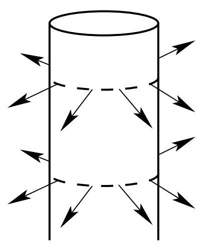
where m and C are integration constants. This potential produces the planar radial velocity

$$\mathbf{u} = \nabla\phi = \frac{m}{r}\hat{\mathbf{e}}_{\boldsymbol{r}}$$

corresponding to a source ($m > 0$) or sink ($m < 0$) of fluid of strength m. Notice that the constant C in ϕ is arbitrary and does not affect $\mathbf{u}$. By convention the constant $m = Q/2\pi$ where Q is the flow rate. This flow could be produced approximately using a perforated hose.


Source ($m > 0$)


Sink ($m < 0$)



Point source/sink

Consider a *spherically* symmetric potential $\phi \equiv \phi(r)$. From Laplace's equation in spherical polar coordinates,

$$\nabla^2\phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = 0 \Leftrightarrow \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{m}{r^2},$$

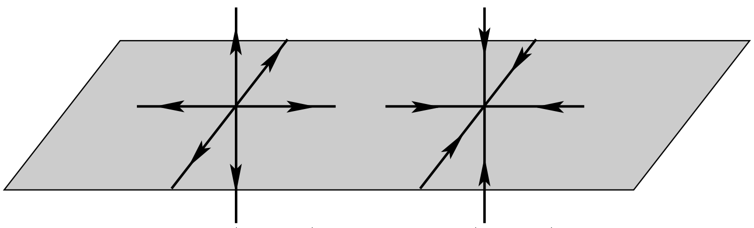
one finds

$$\phi(r) = -\frac{m}{r} + C, \tag{4.6}$$

where m and C are integration constants. This potential produces the three-dimensional radial velocity

$$\mathbf{u} = \nabla\phi = \frac{m}{r^2}\hat{\mathbf{e}}_{\boldsymbol{r}}.$$

corresponding to a source ($m > 0$) or sink ($m < 0$) of fluid of strength m. By convention the constant $m = Q/4\pi$ where Q is the flow rate or volume flux.

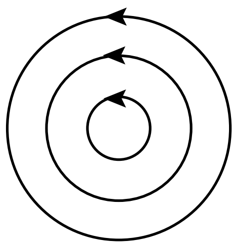


Source ($m > 0$) Sink ($m < 0$)

4.3.2 Line vortex

For the potential $\phi(\theta) = k\theta$, solution to Laplace's equation in plane polar coordinates, one has

$$u_r = \frac{\partial\phi}{\partial r} = 0 \quad \text{and} \quad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{k}{r},$$



where the strength of the flow k is a constant. By convention $k = \Gamma/2\pi$ if Γ is the circulation of the flow.
This represents a rotating fluid (bath-plug vortex) around a line vortex at $r = 0$; it has zero vorticity but is singular at the origin.

4.3.3 Uniform stream

For a uniform flow along the z-axis, $\mathbf{u} = (0, 0, U)$, the velocity potential

$$\phi(z) = Uz.$$

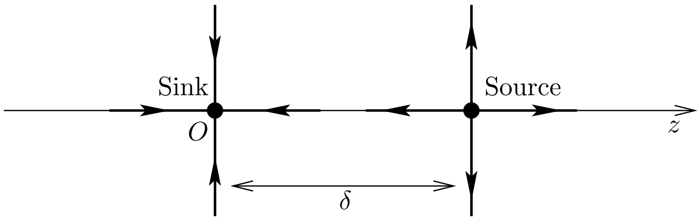
(The integration constant is set to zero.)

4.3.4 Dipole (doublet flow)

Since Laplace's equation is linear we can add two solutions together to form a new one. A dipole is the superposition of a sink and an source of equal but opposite strength next to each other.

Three-dimensional flow

Consider a point sink of strength $-m$ at the origin and a point source of strength m at the position $(0, 0, \delta)$.



The velocity potential of the flow is formed by adding the potentials of the source and sink,

$$\begin{aligned}\phi &= \frac{m}{\sqrt{x^2+y^2+z^2}} - \frac{m}{\sqrt{x^2+y^2+(z-\delta)^2}},\\ &= \frac{m}{r} - \frac{m}{\sqrt{r^2-2z\delta+\delta^2}},\end{aligned}$$

where $m > 0$ is constant and $r = (x^2+y^2+z^2)^{1/2}$. Expanding the potential to first order in δ,

$$\phi = \frac{m}{r} - \frac{m}{r}\left(1 + \frac{z}{r^2}\delta + O(\delta^2)\right),$$

and taking the limit $\delta \to 0$, leads to the potential of a *dipole*

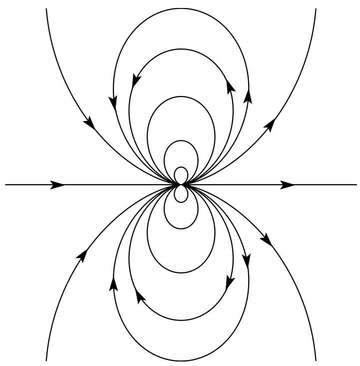
$$\phi = -m\delta\frac{z}{r^3},$$

where $m \to \infty$ as $\delta \to 0$ so that the strength of the dipole $\mu = m\delta$ remains finite. Thus,

$$\phi = -\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^3} = \boldsymbol{\mu}\cdot\nabla\left(\frac{1}{r}\right),$$

where $\boldsymbol{\mu} = \mu\hat{\mathbf{e}}_{\boldsymbol{z}}$ and $\mathbf{r} \equiv \mathbf{x}$ is the vector position. The three components of the fluid velocity, $\mathbf{u} = \nabla\phi$, are for a dipole of strength μ,

$$\begin{aligned}u_x &= 3\mu\frac{xz}{r^5},\\ u_y &= 3\mu\frac{yz}{r^5},\\ u_z &= -\frac{\mu}{r^3}\left(1 - 3\frac{z^2}{r^2}\right).\end{aligned}$$



Planar flow

Similarly, combining a line sink at the origin with a line source of equal but opposite strength at $(\delta, 0)$ gives

$$\phi = -\frac{m}{2}\left[\ln(x^2+y^2) - \ln\left((x-\delta)^2+y^2\right)\right] = \frac{m}{2}\ln\left[\frac{(x-\delta)^2+y^2}{x^2+y^2}\right], \quad m > 0.$$

As in the three-dimensional case, we consider the limit $\delta \to 0$, with $\mu = m\delta$ fixed. The expression of the potential for a two-dimensional dipole of strength μ then becomes

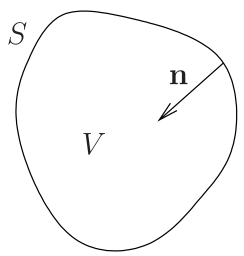
$$\phi = -\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^2} = -\boldsymbol{\mu}\cdot\nabla\ln r,$$

where $\boldsymbol{\mu} = \mu\hat{\mathbf{e}}_{\boldsymbol{x}}$ and $\mathbf{r} \equiv \mathbf{x}$ is the vector position.

4.4 Properties of Laplace's equation

4.4.1 Identity from vector calculus

Let $f(\mathbf{x})$ be a function defined in a simply connected domain V with boundary S. From vector calculus,



$$\nabla \cdot (f\nabla f) = f\nabla^2 f + |\nabla f|^2$$

$$\Rightarrow \int_V \nabla \cdot (f\nabla f)\,\mathrm{d}V = \int_V f\nabla^2 f\,\mathrm{d}V + \int_V |\nabla f|^2\,\mathrm{d}V.$$

So, using the divergence theorem

$$\int_S f(\nabla f)\cdot \mathbf{n}\,\mathrm{d}S = \int_V f\nabla^2 f\,\mathrm{d}V + \int_V |\nabla f|^2\,\mathrm{d}V. \qquad (4.7)$$

4.4.2 Uniqueness of solutions of Laplace's equation

Given the value of the normal component of the fluid velocity, $\mathbf{u}\cdot\mathbf{n}$, on the surface S (i.e. the boundary condition), there exists a unique flow satisfying both $\nabla\cdot\mathbf{u} = 0$ and $\nabla\times\mathbf{u} = 0$ (i.e. incompressible and irrotational).

Proof. Suppose there exists two distinct solutions to the boundary value problem, $\mathbf{u_1} = \nabla\phi_1$ and $\mathbf{u_2} = \nabla\phi_2$. Let $f = \phi_1 - \phi_2$, then

$$\nabla^2 f = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$$

in the domain V and

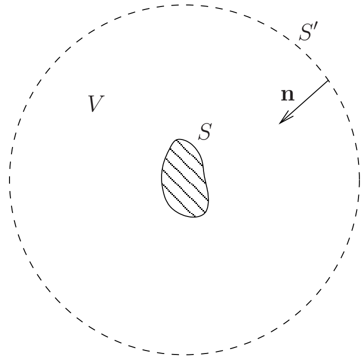
$$(\nabla f)\cdot\mathbf{n} = (\nabla\phi_1)\cdot\mathbf{n} - (\nabla\phi_2)\cdot\mathbf{n} = \mathbf{u_1}\cdot\mathbf{n} - \mathbf{u_2}\cdot\mathbf{n} = 0$$

on the boundary S. Hence, from identity (4.7), $\int_V |\nabla\phi_1 - \nabla\phi_2|^2\,\mathrm{d}V = 0$. However, since $|\nabla\phi_1 - \nabla\phi_2|^2 \geq 0$ one must have $\nabla\phi_1 = \nabla\phi_2$ everywhere. Therefore $\mathbf{u_1} = \mathbf{u_2}$ and the solution to the boundary value problem is unique. □

4.4.3 Uniqueness for an infinite domain

The proof above holds for flows in a finite domain. What about flows in an infinite domain — e.g. flow around an obstacle?

The above argument holds by considering the volume V as shown and letting $S' \to \infty$. (See, e.g. Patterson p. 211.)



4.4.4 Kelvin's minimum energy theorem

Of all possible fluid motions satisfying the boundary condition for $\mathbf{u}\cdot\mathbf{n}$ on the surface S and $\nabla\cdot\mathbf{u}=0$ in domain V, the potential flow is the flow with the smallest *kinetic energy*,

$$K=\frac{1}{2}\int_V \rho\,|\mathbf{u}|^2\,\mathrm{d}V.$$

Proof. Let $\mathbf{u}'$ be another incompressible but non vorticity-free flow such that $\mathbf{u}\cdot\mathbf{n}=\mathbf{u}'\cdot\mathbf{n}$ on S and $\nabla\cdot\mathbf{u}'=0$ in V but with $\nabla\times\mathbf{u}'\neq 0$.
The fluid flow $\mathbf{u}$ is potential, so let $\mathbf{u}=\nabla\phi$ such that

$$\begin{aligned}\int_V \rho\,|\mathbf{u}|^2\,\mathrm{d}V &= \int_V \rho|\nabla\phi|^2\,\mathrm{d}V=\rho\int_V |\nabla\phi|^2\,\mathrm{d}V,\\ &=\rho\int_S \phi\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S \quad \text{(by identity (4.7) with } f=\phi),\\ &=\rho\int_S \phi\,\mathbf{u}'\cdot\mathbf{n}\,\mathrm{d}S \quad \text{(boundary condition)},\\ &=\rho\int_V \nabla\cdot(\phi\,\mathbf{u}')\,\mathrm{d}V \quad \text{(divergence theorem)},\\ &=\rho\int_V \mathbf{u}'\cdot\nabla\phi\,\mathrm{d}V \quad (\nabla\cdot\mathbf{u}'=0),\\ &=\rho\int_V \mathbf{u}'\cdot\mathbf{u}\,\mathrm{d}V. \end{aligned}\tag{4.8}$$

So,

$$\begin{aligned}\int_V \rho\,(\mathbf{u}-\mathbf{u}')^2\,\mathrm{d}V &= \int_V \left(\rho\,|\mathbf{u}|^2-2\rho\mathbf{u}\cdot\mathbf{u}'+\rho\,|\mathbf{u}'|^2\right)\,\mathrm{d}V,\\ &=\int_V \left(\rho\,|\mathbf{u}'|^2-\rho\,|\mathbf{u}|^2\right)\,\mathrm{d}V \quad \text{(from (4.8))}.\end{aligned}$$

Therefore, since $(\mathbf{u}-\mathbf{u}')^2\geq 0$,

$$\int_V \rho\,|\mathbf{u}'|^2\,\mathrm{d}V=\int_V \rho\,|\mathbf{u}|^2\,\mathrm{d}V+\int_V \rho\,(\mathbf{u}-\mathbf{u}')^2\,\mathrm{d}V\geq\int_V \rho\,|\mathbf{u}|^2\,\mathrm{d}V.$$

□

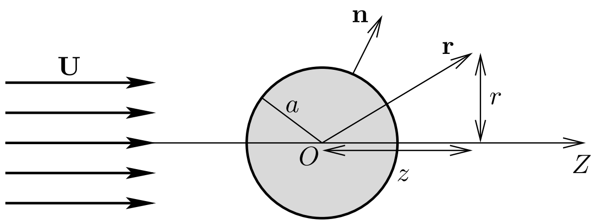
4.5 Flow past an obstacle

Since the solution to Laplace's equation for given boundary conditions is unique, if we find *a* solution, we have found *the* solution. (This is only true if the domain is simply-connected; if the domain is multiply connected, multiple solutions become possible.)

One technique to calculate non elementary potential flows involves adding together simple known solutions to Laplace's equation to get the solution that satisfies the boundary conditions.

4.5.1 Flow around a sphere

We seek an axisymmetric flow of the form $\mathbf{u}=u_r\hat{\mathbf{e}}_{\boldsymbol{r}}+u_z\hat{\mathbf{e}}_{\boldsymbol{z}}$ in cylindrical polar coordinates (r,θ,z).



At large distances from the sphere of radius a the flow is asymptotic to a uniform stream, $u_r = 0$, $u_z = U$, and at the sphere's surface, $r = a$, the fluid velocity must satisfy $\mathbf{u} \cdot \mathbf{n} = 0$ since the solid body forms a non-penetrable boundary.

The unit vector normal to surface of the sphere is

$$\mathbf{n} = n_r \hat{\mathbf{e}}_{\boldsymbol{r}} + n_z \hat{\mathbf{e}}_{\boldsymbol{z}} \quad \text{with} \quad n_r = \frac{r}{a} \quad \text{and} \quad n_z = \frac{z}{a}.$$

So, the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ implies that

$$u_r \frac{r}{a} + u_z \frac{z}{a} = 0 \Leftrightarrow r u_r + z u_z = 0$$

at the spherical surface of equation $r^2 + z^2 = a^2$.

At large distances, the flow is essentially uniform along the z-axis,

$$\phi \simeq Uz, \quad \text{for } \|\mathbf{r}\| \gg a.$$

Now, add to the uniform stream a dipole velocity field of strength $\boldsymbol{\mu} = \mu \hat{\mathbf{e}}_{\boldsymbol{z}}$ at the origin,

$$\phi(r, z) = Uz - \frac{\mu z}{(r^2 + z^2)^{3/2}},$$

so that

$$u_r = \frac{\partial \phi}{\partial r} = \frac{3\mu r z}{(r^2 + z^2)^{5/2}} \quad \text{and} \quad u_z = \frac{\partial \phi}{\partial z} = U + \frac{\mu}{(r^2 + z^2)^{3/2}} \left(\frac{3z^2}{r^2 + z^2} - 1 \right).$$

Thus, at the sphere's surface,

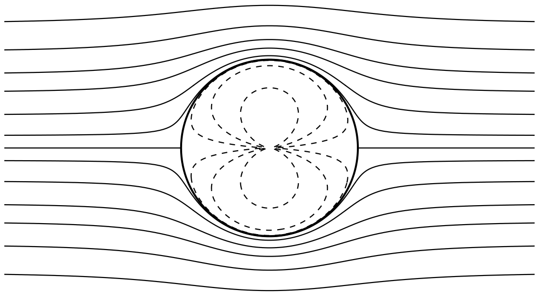
$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} = u_r \frac{r}{a} + u_z \frac{z}{a} &= \frac{z}{a} \left(U + \frac{3\mu(r^2 + z^2)}{(r^2 + z^2)^{5/2}} - \frac{\mu}{(r^2 + z^2)^{3/2}} \right), \\ &= \frac{z}{a} \left(U + \frac{2\mu}{(r^2 + z^2)^{3/2}} \right) = \frac{z}{a} \left(U + \frac{2\mu}{a^3} \right), \end{aligned}$$

since $r^2 + z^2 = a^2$. Hence the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ at the sphere's surface determines the strength of the dipole,

$$\mu = -\frac{Ua^3}{2}.$$

The velocity potential for a uniform flow past a stationary sphere is therefore given by

$$\phi(r, z) = Uz \left(1 + \frac{a^3}{2(r^2 + z^2)^{3/2}} \right). \tag{4.9}$$



The corresponding Stokes streamfunction is given by

$$\Psi(r, z) = \frac{Ur^2}{2} \left(1 - \frac{a^3}{(r^2 + z^2)^{3/2}} \right). \tag{4.10}$$

Outside the sphere $\Psi > 0$, but we also obtain a solution inside the sphere with $\Psi < 0$. This flow is not real; it is a "virtual flow" that allows for fluid velocity to be consistent with the boundary condition on a solid sphere.

4.5.2 Rankine half-body

Suppose that, in the velocity potential of a flow past a sphere, we replace the dipole with a point source ($m > 0$), so that

$$\phi(r,z) = Uz - \frac{m}{(r^2+z^2)^{1/2}} \Rightarrow \mathbf{u} = \nabla\phi = \left(\frac{mr}{(r^2+z^2)^{3/2}}, U + \frac{mz}{(r^2+z^2)^{3/2}}\right).$$

This flow has a single stagnation point $u_r = u_z = 0$ at $r = 0$ and $z = -\sqrt{m/U}$.

To find the streamlines of the flow we calculate the Stokes streamfunction using

$$u_r = -\frac{1}{r}\frac{\partial\Psi}{\partial z} \quad \text{and} \quad u_z = \frac{1}{r}\frac{\partial\Psi}{\partial r}.$$

Thus,

$$\begin{aligned}
\frac{\partial\Psi}{\partial r} &= Ur + \frac{mrz}{(r^2+z^2)^{3/2}} \Rightarrow \Psi = \frac{Ur^2}{2} - \frac{mz}{(r^2+z^2)^{1/2}} + \alpha(z),\\
\Rightarrow \frac{1}{r}\frac{\partial\Psi}{\partial z} &= -\frac{m}{r\,(r^2+z^2)^{1/2}} + \frac{mz^2}{r\,(r^2+z^2)^{3/2}} + \frac{\alpha'(z)}{r},\\
&= -\frac{m}{r}\frac{\left(r^2+z^2-z^2\right)}{(r^2+z^2)^{3/2}} + \frac{\alpha'(z)}{r} = -\frac{mr}{(r^2+z^2)^{3/2}} + \frac{\alpha'(z)}{r},\\
&= -u_r = -\frac{mr}{(r^2+z^2)^{3/2}}.
\end{aligned}$$

So, since $\alpha'(z) = 0$, α is a constant (set to zero). The Stokes streamfunction is therefore

$$\Psi(r,z) = \frac{Ur^2}{2} - \frac{mz}{(r^2+z^2)^{1/2}}.$$

At the stagnation point ($r = 0$, $z = -\sqrt{m/U}$), $\Psi = m$. Hence, the equation of the streamline, or streamtube, passing through this stagnation point is

$$\Psi(r,z) = m \Leftrightarrow \frac{Ur^2}{2} = m\left(1 + \frac{z}{(r^2+z^2)^{1/2}}\right).$$

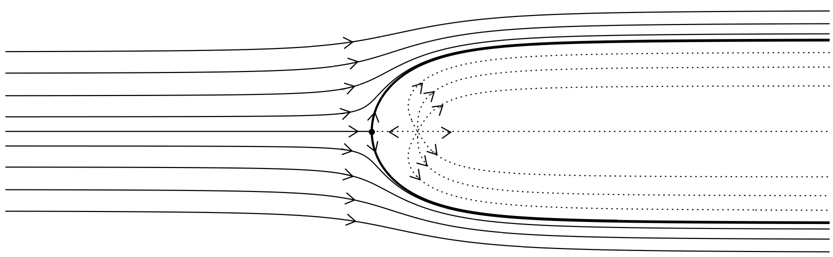
Notice that the straight line $r = 0$ with $z < 0$ satisfies the equation of the streamline $\Psi = m$. For large positive z, the equation of the streamtube $\Psi = m$ becomes

$$\frac{Ur^2}{2} \simeq 2m \Rightarrow r \simeq 2\sqrt{\frac{m}{U}}.$$

Thus, the velocity potential and the Stokes streamfunction

$$\phi(r,z) = U\left(z - \frac{a^2}{4\,(r^2+z^2)^{1/2}}\right) \quad \text{and} \quad \Psi(r,z) = \frac{U}{2}\left(r^2 - \frac{a^2 z}{2\,(r^2+z^2)^{1/2}}\right)$$

provide a model for a long slender body of radius $a = 2\sqrt{m/U}$.



4.6 Method of images

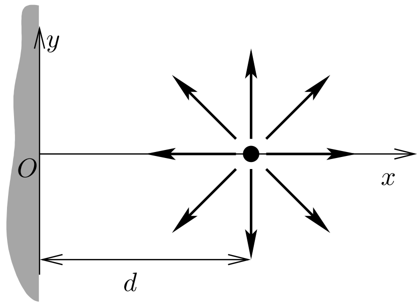
In previous examples we introduced flow singularities (e.g. sources and dipoles) outside of the domain of fluid flow in order to satisfy boundary conditions at a solid surface.

This technique can also be used to calculate the flow produced by a singularity near a boundary; it is then called *method of images*.

Example 4.1 (Point source near a wall)
Consider a point-source of fluid placed at the position $(d, 0, 0)$ (Cartesian coordinates) near a solid wall at $x = 0$.

In free space (no wall), the potential of the source is

$$\phi_\infty = -\frac{m}{\sqrt{(x-d)^2 + y^2 + z^2}},$$
$$\Rightarrow u_\infty = \frac{\partial \phi_\infty}{\partial x} = \frac{m(x-d)}{[(x-d)^2 + y^2 + z^2]^{3/2}}.$$



So that, at $x = 0$,

$$u_\infty = -\frac{md}{(d^2 + y^2 + z^2)^{3/2}} \neq 0,$$

which is inconsistent with the boundary condition $\mathbf{u} \cdot \mathbf{n} = \mathbf{u} \cdot \hat{\mathbf{e}}_{\boldsymbol{x}} = u = 0$ at the wall.

To rectify this problem, (i.e. for the flow to satisfy the boundary condition at the wall), we add a source of equal strength m outside the domain, at $(-d, 0, 0)$. By symmetry, this source will produce an equal but opposite velocity field at $x = 0$, so that the boundary condition for the combined flow can be satisfied. The velocity potential for both sources becomes

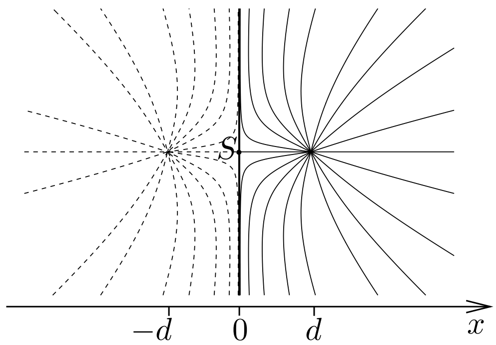
$$\phi = -\frac{m}{\sqrt{(x-d)^2 + y^2 + z^2}} - \frac{m}{\sqrt{(x+d)^2 + y^2 + z^2}},$$

and the velocity field along the x-axis,

$$u = \frac{\partial \phi}{\partial x} = \frac{m(x-d)}{[(x-d)^2 + y^2 + z^2]^{3/2}} + \frac{m(x+d)}{[(x+d)^2 + y^2 + z^2]^{3/2}}.$$

Clearly, at $x = 0$, now $u = 0$ as required. The fluid can slip along the wall however as, for $x = 0$,

$$v = \frac{2my}{(d^2 + y^2 + z^2)^{3/2}},$$
$$w = \frac{2mz}{(d^2 + y^2 + z^2)^{3/2}}.$$



4.7 Method of separation of variables

This is a standard method for solving linear partial differential equations with compatible boundary conditions.

We shall seek separable solutions to Laplace's equations, of the form $\phi(x, y) = f(x)g(y)$ in Cartesian coordinates or $\phi(r, \theta) = f(r)g(\theta)$ in polar coordinates.

Plane polar coordinates. We substitute a potential of the form $\phi(r,\theta) = f(r)g(\theta)$ in Laplace's equation expressed in plane polar coordinates,

$$\begin{aligned}
\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} &= 0,\\
\Rightarrow \frac{g}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) + \frac{f}{r^2}\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} &= 0,\\
\Rightarrow \frac{r}{f}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) + \frac{1}{g}\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} &= 0, \quad \text{(division by } f(r)g(\theta)/r^2\text{)}\\
\Rightarrow \frac{r}{f}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) &= -\frac{1}{g}\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2}.
\end{aligned}$$

Since the terms on the left and right sides of the equation are functions of independent variables, r and θ respectively, they must take a constant value, k^2 say. Thus we have transformed a partial differential equation for ϕ into two ordinary differential equations for f and g,

$$\begin{aligned}
\frac{r}{f}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) = k^2 &\Rightarrow r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) - k^2 f = 0,\\
\frac{1}{g}\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} = -k^2 &\Rightarrow \frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} + k^2 g = 0.
\end{aligned}$$

Thus, $g(\theta) = A\cos(k\theta) + B\sin(k\theta)$. For a 2π-periodic function g, such that $g(\theta) = g(\theta + 2\pi)$, k must be integer. So

$$g(\theta) = A\cos(n\theta) + B\sin(n\theta), \quad n \in \mathbb{Z},$$

and f is solution to

$$r^2\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r\frac{\mathrm{d}f}{\mathrm{d}r} - n^2 f = 0.$$

Substituting nontrivial functions of the form $f = ar^\alpha$ gives,

$$\left[\alpha(\alpha-1) + \alpha - n^2\right] ar^\alpha = 0 \Leftrightarrow \alpha^2 = n^2.$$

The two independent solutions have $\alpha = \pm n$; the general separable solution to Laplace's equation in plane polar coordinates is therefore

$$\phi(r,\theta) = \left(Ar^n + Br^{-n}\right)\cos(n\theta) + \left(Cr^n + Dr^{-n}\right)\sin(n\theta), \quad n \in \mathbb{Z}, \tag{4.11}$$

where A, B, C and D are constants to be determined by the boundary conditions.

Separable solutions to Laplace's equation in spherical polar coordinates can be obtained in a similar manner, but involves Legendre polynomials $P_l(\cos(\theta))$.

Example 4.2 (Cylinder in an extensional flow)
Consider the velocity potential

$$\phi(r,\theta) = \left(Ar^2 + Br^{-2}\right)\cos(2\theta)$$

corresponding to a particular solution to Laplace's equation of the form (4.11), with $n = 2$. The radial velocity of this flow is

$$u_r = \frac{\partial\phi}{\partial r} = 2r\left(A - \frac{B}{r^4}\right)\cos(2\theta).$$

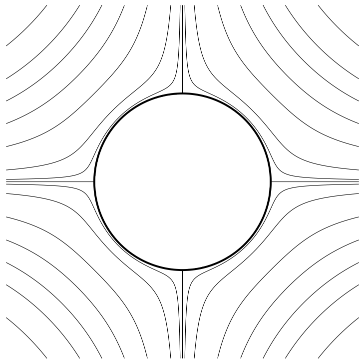
It vanishes at the surface of a solid cylinder of radius a placed at the origin if $B = a^4 A$. Therefore the velocity field

$$u_r = 2Ar\left(1 - \frac{a^4}{r^4}\right)\cos(2\theta) \quad \text{and} \quad u_\theta = -2Ar\left(1 + \frac{a^4}{r^4}\right)\sin(2\theta)$$

produced by the potential

$$\phi(r,\theta) = Ar^2\left(1 + \frac{a^4}{r^4}\right)\cos(2\theta)$$

represents a fluid flow past a solid cylinder placed in an extensional flow.



Notice that at large distances, i.e. if $r \gg a$, the fluid velocity is that of an extensional flow

$$u_r \simeq 2Ar\cos(2\theta) \quad \text{and} \quad u_\theta \simeq -2Ar\sin(2\theta),$$

in polar coordinates, or equivalently

$$u \simeq 2Ax \quad \text{and} \quad v \simeq -2Ay,$$

in Cartesian coordinates.