Calculations for the noncentral chi distribution

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Abstract

This paper fills in some details in Kent and Hainsworth (1995) on the computation of probability and confidence intervals for the noncentrality parameter of the noncentral chi distribution.

1 Overview

Kent and Hainsworth (1995) discussed several methods for the computation of two-sided confidence intervals for the noncentral chi distribution. However, that paper did not spell out the details. The purpose of this report is to provide sufficient details to construct an R program to implement these intervals.

First some notation. If Y^2 follows a noncentral chi-squared distribution with p degrees of freedom and noncentrality parameter $\lambda^2,\ 0 \le \lambda < \infty$, write

$$Y^2 \sim \chi_p^2(\lambda^2),\ \text{or } Y \sim \chi_p(\lambda). \tag{1.1}$$

The noncentral chi distribution is more convenient for our purposes. In most of the following notation, p is fixed and will not be explicitly stated in the notation. Also, it is useful to set $\nu = (p-2)/2$. The formula for the pdf involves the modified Bessel function $I_\nu(x)$ which has the limiting behaviour

$$I_\nu(x) \sim (x/2)^\nu/\Gamma(\nu+1) \text{ as } x \to 0. \tag{1.2}$$

Denote the cdf of the noncentral chi distribution by $F(y, \lambda)$, with inverse or quantile function $F^{-1}(\beta, \lambda)$ where $\beta \in [0, 1)$ is a probability.

The pdf (with respect to Lebesgue measure) is given by

$$f(y, \lambda) = \lambda^{-\nu} y^{\nu+1} \exp\{-\frac{1}{2}(y^2 + \lambda^2)\} I_\nu(\lambda y), \quad 0 < y < \infty. \tag{1.3}$$

But for our purposes it is more helpful to give the pdf with respect to other measures. The pdf (with respect to a Bessel base $h(y) = y^{\nu+1}\{I_\nu(y^2)\}^{1/2}$) takes the form

$$g_B(y,\lambda) = \lambda^{-\nu}\exp\{-\frac{1}{2}(y^2+\lambda^2)\}I_\nu(\lambda y)/\{I_\nu(y^2)\}^{1/2} \tag{1.4}$$

with limiting values

$$g_B(y,\lambda) = (y/2)^\nu \exp\{-\frac{1}{2}y^2\}/[\{I_\nu(y^2)\}^{1/2}\Gamma(\nu+1)] \text{ for } \lambda = 0,\ y \neq 0; \tag{1.5}$$

$$g_B(y,\lambda) = \exp\{-\frac{1}{2}\lambda^2\}/\{2^\nu\Gamma(\nu+1)\}^{1/2} \text{ for } y = 0. \tag{1.6}$$

The pdf with respect to a radial base $h(y) = y^{2\nu+1}$ takes the form

$$g_R(y,\lambda) = (\lambda y)^{-\nu}\exp\{-\frac{1}{2}(y^2+\lambda^2)\}I_\nu(\lambda y) \tag{1.7}$$

with limiting values

$$g_R(y,\lambda) = \exp\{-\frac{1}{2}(y^2+\lambda^2)\}/\{2^\nu\Gamma(\nu+1)\} \text{ for } \lambda y = 0. \tag{1.8}$$

2 Probability intervals

Let $0 < \alpha < 1$, e.g. $\alpha = 0.05$. We can now derive various probability intervals with exact coverage probability $(1-\alpha)$ for a specified value of λ. In all cases the interval will take the form $[c, d]$ for a suitable choice of c and d (depending on λ and p).

2.1 Central probability intervals

For this interval

$$c = F^{-1}(\alpha/2,\lambda),\ d = F^{-1}(1-\alpha/2,\lambda). \tag{2.1}$$

2.2 Maximum probability intervals

Let $g(y,\lambda)$ denote the pdf with respect to either a Bessel or radial base. If λ is not too small the then the probability interval has an equal pdf value at each endpoint c and d and exact coverage (see (2.3) below). But if λ close to 0, the first equation of (2.3) fails to hold and $c = 0$.

More specifically, set

$$y_{0,\lambda} = F^{-1}(1-\alpha,\lambda).$$

and set

$$g_0 = g(0, \lambda), \quad g_1 = g(y_{0,\lambda}, \lambda).$$

If $g_0 \geq g_1$, then

$$c = 0, \quad d = y_{0,\lambda}. \tag{2.2}$$

However, if $g_0 < g_1$, then we need to solve the simultaneous equations

$$g(c, \lambda) = g(d, \lambda), \quad F(d, \lambda) - F(c, \lambda) = 1 - \alpha \tag{2.3}$$

for c and d.

2.3 Symmetric range probability intervals

If $F(2\lambda, \lambda) < 1 - \alpha$, it is necessary to solve the equation

$$F(\lambda + b, \lambda) - F(\lambda - b, \lambda) = 1 - \alpha \tag{2.4}$$

for $0 < b < \lambda$, and the endpoints of the probability interval take the form

$$c = \lambda - b, \quad d = \lambda + b. \tag{2.5}$$

However, if $F(2\lambda, \lambda) \geq 1 - \alpha$, the interval takes the simple form

$$c = 0, \quad d = y_{0,\lambda}. \tag{2.6}$$

3 Confidence intervals

Again let $0 < \alpha < 1$. In this case y is fixed and we wish to construct confidence intervals by inverting each of the above probability intervals. In each case the interval can be written in the form $[\lambda_L, \lambda_U]$ for a suitable choice of λ_L and λ_U.

3.1 Central confidence intervals

There are three distinct cases.

(a) If $y < F^{-1}(\alpha/2, 0)$, the confidence interval is empty (represented in the R program by setting each endpoint to be NaN — not a number).

(b) If $F^{-1}(\alpha/2, 0) < y \leq F^{-1}(1 - \alpha/2, 0)$, we must solve the equation

$$y = F^{-1}(\alpha/2, \lambda) \tag{3.1}$$

for λ_U, and $\lambda_L = 0$.

(c) If $y > F^{-1}(1-\alpha/2, 0)$, we must solve the two equations

$$y = F^{-1}(1-\alpha/2, \lambda), \quad y = F^{-1}(\alpha/2, \lambda) \tag{3.2}$$

for λ_L and λ_U, respectively.,

3.2 Maximum probability confidence intervals

First define a key quantity λ_y as follows. If $y \leq F^{-1}(1-\alpha, 0)$, then set $\lambda_y = 0$, whereas if $y > F^{-1}(1-\alpha, 0)$, set λ_y to be the solution of

$$F^{-1}(1-\alpha, \lambda_y) = y. \tag{3.3}$$

We can now construct the lower endpoint of the CI under different cases.

(a) If $y \leq F^{-1}(1-\alpha, 0)$, then set $\lambda_L = 0$.

(b) If $y > F^{-1}(1-\alpha, 0)$, then there are two possibilities. First define $g_0 = g(0, \lambda_y)$ and $g_y = g(y, \lambda_y)$.

  (i) If $g_0 \geq g_y$, then set $\lambda_L = \lambda_y$.

  (ii) But if $g_0 > g_y$, solve the pair of simultaneous equations

$$g(c, \lambda) = g(y, \lambda), \quad F(y, \lambda) - F(c, \lambda) = 1 - \alpha \tag{3.4}$$

  for c and λ and set $\lambda_L = \lambda$.

The procedure for λ_U is similar but simpler. In all cases we solve the pair of simultaneous equations

$$g(y, \lambda) = g(d, \lambda), \quad F(d, \lambda) - F(y, \lambda) = 1 - \alpha \tag{3.5}$$

for d and λ and set $\lambda_U = \lambda$.

3.3 Symmetric range confidence

First we construct λ_L under different cases.

(a) If $y \leq F^{-1}(1-\alpha, 0)$, set $\lambda_L = 0$.

(b) Else solve the equation

$$F(y, y-b) - F(\max(y-2b, 0), y-b) = 1 - \alpha \tag{3.6}$$

for $0 < b < y$, and set $\lambda_L = y - b$.

The procdure for λ_U is similar but simpler. In all cases solve the equation

$$F(y+2b, y+b) - F(y, y+b) = 1 - \alpha \tag{3.7}$$

for $b > 0$, and set $\lambda_U = y + b$.

4 R programs

The file `ncc.r` contains a collection of functions to compute probability and confidence intervals and to carry out related calculations. In all cases `lambda` is a noncentrality parameter, `y` is a possible value from the noncentral chi distribution, `p` is the degrees of freedom, and `alpha` is a number strictly between 0 and 1. All of the function arguments and most of the function values below are scalars. The value of each function producing a probability or confidence interval is a vector of length 2. See below for the functions `ncc.ints` and `test`.

In all cases the probability and confidence intervals have exact coverage probability `1 - alpha`.

- `F=function(u,p,lambda)` evaluates the cdf of the noncentral chi distribution at a nonnegative number `u`.
- `Finv=function(prob,p,lambda)` evaluates the quantile function (inverse cdf) of the noncentral chi distribution at a number `prob`, ≥ 0 and < 1.
- `lambound=function(y,p,alpha)` finds an upper bound on `lambda` for which the `alpha` quantile of the noncentral chi distribution equals `y`. Used in `lamfind`.
- `lamfind=function(y,p,alpha)` finds the value of `lambda` for which the `alpha` quantile of the noncentral chi distribution equals `y`.
- `gb=function(u,nu,lambda)` finds the pdf of the noncentral chi pdf with respect to a Bessel base measure at the argument `u` ≥ 0 where `lambda` ≥ 0.
- `gr=function(u,nu,lambda)` finds the pdf of the noncentral chi pdf with respect to a radial base measure at the argument `u` ≥ 0 where `lambda` ≥ 0.
- `ncc.pi.central=function(lambda,p,alpha=0.05)` computes a central probability interval.
- `ncc.pi.mdb=function(lambda,p,alpha=0.05)` computes a maximum density (with respect to a Bessel base) probability interval.

- `ncc.pi.mdr=function(lambda,p,alpha=0.05)` computes a maximum density (with respect to a radial base) probability interval.
- `ncc.pi.sr=function(lambda,p,alpha=0.05)` computes a symmetric range probability interval.
- `ncc.ci.central=function(y,p,alpha=0.05)` computes a central confidence interval.
- `ncc.ci.mdb=function(y,p,alpha=0.05)` computes a maximum density (with respect to a Bessel base) confidence interval.
- `ncc.ci.mdr=function(y,p,alpha=0.05)` computes a maximum density (with respect to a radial base) confidence interval.
- `ncc.ci.sr=function(y,p,alpha=0.05)` computes a symmetric range confidence interval.
- `ncc.ints=function(yl,p,alpha=0.05)` computes an 8×2 matrix of all the probability and confidence intervals, where `yl` plays the role of `lambda` or `y` as appropriate.
- `test=function(lambda,p,alpha=0.05)` is a routine to test some of the computations. Starting with `lambda` the routine produces the prob interval, and using the two endpoints of this interval produces two confidence intervals. All three intervals are printed as a vector of length 6. Column 5, and also column 4 if lambda is not too large, should be equal to lambda.

To use these functions, just source this file, `source("ncc.r")` from within R.

Reference

Kent, T T and Hainsworth, T J (1995) Confidence intervals for the noncentral chi-squared distribution. *J Stat Planning and Inference* **46**, 147–159.