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On a new procedure for finding nonclassical symmetries

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Abstract

A new technique for deriving the determining equations of nonclassical symmetries associated with a partial differential equation system is introduced. The problem is reduced to computing the determining equations of the classical symmetries associated with a related equation with coefficients which depend on the nonclassical symmetry operator. As a consequence, all the symbolic manipulation programs designed for the latter task can also be used to find the determining equations of the nonclassical symmetries, without any adaptation of the program. The algorithm was implemented as the MAPLE routine GENDEFNC and uses the MAPLE package DESOLV (authors Carminati and Vu). As an example, we consider the Huxley partial differential equation.

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1. Introduction

The modern approach for finding special solutions of systems of nonlinear partial differential equations (PDEs) was pioneered by Sophus Lie at the end of the nineteenth

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century (Lie, 1929). There is still no general theory for finding analytical solutions to a nonlinear PDE system, but a variety of methods have been developed in the past few years by Ovsyannikov (1982), Ibragimov (1994–1996), and others. Olver and Rosenau (1986) argued that all these methods augment the PDE system with one or more auxiliary differential equations, which they call *side conditions*. Solving the augmented system yields special solutions of the original system. The methods differ in the way in which they find appropriate side conditions.

The *classical Lie method*, originally developed by Lie (1929), leads us to the one-parameter groups of transformations (called *classical symmetries* or *Lie point symmetries*) acting on the space of the independent and dependent variables that leave the PDE system unchanged. Hence, they map the set $\mathcal{S}_{\Delta}$ of all analytical solutions to itself. The classical symmetries are solutions of an over-determined linear PDE system. Introduced by Bluman and Cole (1969), the *nonclassical method* yields the one-parameter group of transformations (called *nonclassical symmetries* or *weak symmetries* (Olver and Rosenau, 1987)) acting on the space of the independent and dependent variables of the system that leave only a subset $\mathcal{S}_{\Delta,\psi}$ of the set $\mathcal{S}_{\Delta}$ of all analytical solutions invariant, where ψ itself is defined by the symmetry. Nonclassical symmetries are solutions to an over-determined nonlinear PDE system. Note that any classical symmetry is a nonclassical symmetry, but not conversely.

Since these two methods are algorithmic procedures, the use of symbolic manipulation programs has become imperative, and hence, a variety of packages have been developed for many computer algebra systems. The first implementation of the classical Lie method is due to Katkov and Kostyukova in 1962–1972 (Ovsyannikov, 1982). A recent survey of other programs can be found in Hereman (1994). For the computer system MAPLE, the programs DESOLV (Carminati and Vu, 2000), SYMMETRY (Hickman, 2001), and RIF (Reid and Wittkopf, 2001) contain routines for generating classical symmetries. At the moment, no such programs implementing the nonclassical method are available for MAPLE, but several packages are available for other computer algebra systems: the interactive program NUSY in REDUCE (Nucci, 1996) and the MUMATH program LIE (Head, 1996). Additionally, the MACSYMA program SYMMGRP.MAX (Champagne et al., 1991) can be adapted for this purpose (Clarkson and Mansfield, 1994). Note that Levi and Winternitz (1989) first implemented the nonclassical method in a symbolic manipulation program.

The aim of this paper is to introduce a new method for finding nonclassical symmetries. The standard procedure, due to Bluman and Cole (1969), starts with the PDE system to be analyzed. This system is augmented with the invariant surface conditions, representing the characteristics of a fixed but arbitrary vector field X. Then the classical Lie method is applied to the augmented system. If this results in the same vector field X as is used to define the invariant surface conditions, then X is a nonclassical symmetry of the original PDE.

Our procedure consists of reducing the augmented PDE system to its involutive form and then applying the classical Lie method to the reduced PDE system, but with an arbitrary symmetry operator W which is not related anymore to the invariant surface condition. Note that we apply the algorithm introduced by Clarkson and Mansfield (1994) to reduce the system to involutive form. In this way, we avoid infinite loops in the reduction procedure, which may occur in a more naive implementation. The determining equations of the symmetry group related to the augmented PDE with the arbitrary operator W are linear. However, we prove that the determining equations of the nonclassical symmetries can be derived from the outcome when we put $W = X$. We thus obtain the nonlinear system of the determining equations of the nonclassical symmetries related to the studied PDE system.

As a consequence of the new method described here, any symbolic manipulation program designed to compute classical symmetries can be used to find nonclassical symmetries. We provide an implementation in MAPLE, called GENDEFNC. It uses the DESOLV package by Carminati and Vu (2000), and the output of our subroutine is the system of the determining equations of nonclassical symmetries. In order to integrate this nonlinear PDE system, one can use the MAPLE packages DIFFGROB2 (Mansfield and Clarkson, 1997) or RIF (Reid and Wittkopf, 2001).

The structure of this paper is as follows. In Section 2 we recall briefly the classical Lie method and the nonclassical method. The new procedure is presented in Section 3, and its MAPLE implementation is explained in Section 4. In the last section we show how the procedure can be applied to the Huxley PDE.

2. Classical and nonclassical symmetries

We give a short introduction to the symmetry theory of PDEs (see Olver (1986) for more details). Consider an nth order PDE system

$$\Delta(x, u^{(n)}) = 0, \tag{1}$$

with p independent variables $x = (x^1, \ldots, x^p) \in \mathcal{X} \subset \mathbf{R}^p$, and q dependent variables $u = (u^1, \ldots, u^q) \in U \subset \mathbf{R}^q$. Here $\mathcal{X}$ is the space of the independent variables and U is the space of the dependent variables associated with the system. Denote by $\Delta = (\Delta_1, \ldots, \Delta_l)$ the function whose components define the equations of the system (1). Finally, $u^{(n)}$ denotes all the partial derivatives of u with respect to x up to order n and the corresponding space is denoted by $U^{(n)}$. For any multi-index $J = (j_1, \ldots, j_k)$ with $1 \le j_i \le n$, we denote

$$u^\alpha_J = \frac{\partial^k u^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}.$$

The system (1) is said to be a *maximal rank system* if

$$\operatorname{rank}\left(\frac{\partial \Delta}{\partial x^i}, \frac{\partial \Delta}{\partial u^\alpha_J}\right) = l \qquad \text{whenever} \qquad \Delta(x, u^{(n)}) = 0.$$

2.1. *The classical Lie method*

Let us consider the submanifold

$$\mathcal{S}_\Delta = \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\} \subset \mathcal{X} \times U^{(n)} \tag{2}$$

defined by the PDE system (1) itself. The set $\mathcal{S}_\Delta$ contains all analytic solutions of the system (1) (for more details see Clarkson (1995) and Olver (1986), p. 96). The *symmetry group* G associated with the PDE system (1) consists of one-parameter groups of transformations acting on an open subset $M \subset \mathcal{X} \times U$ which leave the set $\mathcal{S}_\Delta$ invariant. Let the general infinitesimal generator associated with G be given by

$$X = \sum_{i=1}^{p} \zeta^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{3}$$

Its nth order prolongation, which is defined on the corresponding jet space $M^{(n)} \subset \mathcal{X} \times U^{(n)}$, is given by

$$\mathrm{pr}^{(n)} X = X + \sum_{\alpha=1}^{q}\sum_{J} \Phi_J^\alpha(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{4}$$

where

$$\Phi_J^\alpha(x,u^{(n)}) = D_J\left(\phi^\alpha - \sum_{i=1}^{p}\zeta^i u_i^\alpha\right) + \sum_{i=1}^{p}\zeta^i u_{J,i}^\alpha. \tag{5}$$

The summation in (4) is carried out over all multi-indices J of order k, with $1 \le k \le n$, and D_J denotes the so-called *total derivative*.

Theorem 2.1 *(Criterion for Infinitesimal Invariance). Suppose* (1) *is a PDE system of maximal rank defined over $M \subset \mathcal{X} \times U$. If G is a local group of transformations acting on M, and*

$$\mathrm{pr}^{(n)} X(\Delta)|_{\Delta=0} = 0 \tag{6}$$

is identically satisfied for every infinitesimal generator X of G, then G is a symmetry group of the system (1).

This theorem suggests the following method for finding the symmetry group of a maximal rank PDE system. We substitute the nth order prolongation of the vector field X given by the relations (4) and (5) into the condition (6). Then we reduce the resulting equations by eliminating any dependence between all the partial derivatives u_J^α. Next we demand that the coefficients of the u_J^α be zero. This yields an over-determined linear PDE system for the infinitesimals ζ^i and ϕ^α of the symmetry operator X. This system is called the *determining equations of the symmetry group* of the PDE system (1).

2.2. *The nonclassical method*

One associates with the infinitesimal generator X given by (3) the following first order PDE system

$$\psi^\alpha := \sum_{i=1}^{p}\zeta^i(x,u)u_i^\alpha - \phi^\alpha(x,u) = 0, \qquad \alpha = 1,\ldots,q, \tag{7}$$

which represents the characteristics of the vector field X. In our context, the equations (7) are called the *invariant surface conditions*. For $\psi = (\psi^1,\ldots,\psi^q)$, let us denote by

$$\mathcal{S}_{\Delta,\psi} = \{(x,u^{(n)}) : \Delta(x,u^{(n)}) = 0,\ \psi(x,u^{(1)}) = 0\}$$

the submanifold associated with the PDE system (1) and (7) (for more details see Clarkson (1995)). If the one-parameter group of transformations generated by X leaves $\mathcal{S}_{\Delta,\psi}$ invariant then X is called a *nonclassical operator* associated with the PDE system (1). The nonclassical method is based on the criterion for infinitesimal invariance (see Theorem 2.1), which in this case becomes

$$\begin{aligned} \mathrm{pr}^{(n)} X(\Delta)|_{\Delta=0,\psi=0} &= 0, \\ \mathrm{pr}^{(1)} X(\psi)|_{\Delta=0,\psi=0} &= 0. \end{aligned} \tag{8}$$

Note that the second condition in (8) is satisfied identically, because

$$\mathrm{pr}^{(1)} X(\psi^\alpha) = -\sum_{\beta=1}^{q} \psi^\beta \psi^\alpha_{u^\beta} \qquad \alpha = 1, \ldots, q.$$

This implies that every classical symmetry is also a nonclassical symmetry. We would like to remind the reader that the nonclassical operators do not form a vector space, still less a Lie algebra, as the symmetry operators do.

Clarkson and Mansfield (1994) made the remark that infinite loops may occur in the reduction process in the implementation of the nonclassical method, if a straightforward implementation is used. In order to eliminate this problem, they introduced a new algorithm based on the theory of differential Gröbner bases.

Let us discuss in the following this procedure, referring the reader to (Clarkson and Mansfield, 1994) for further details. If X is a nonclassical operator, then so is λX for any function $\lambda = \lambda(x, u)$; see Hydon (2000, p. 167). Thus, one can assume without loss of generality that $\zeta^p = 1$ (the case $\zeta^p = 0$ needs to be handled separately). In this case, let us denote by Y the new vector field X,

$$Y = \sum_{j=1}^{p-1} \zeta^j(x, u) \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^p} + \sum_{\alpha=1}^{q} \phi^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \tag{9}$$

Its associated invariant surface condition $\psi = 0$ turns into

$$u^\alpha_p = -\sum_{j=1}^{p-1} \zeta^j(x, u) u^\alpha_j + \phi^\alpha(x, u), \qquad \alpha = 1, \ldots, q. \tag{10}$$

Clarkson and Mansfield (1994) now find the determining equations of the nonclassical symmetries by the following two-step procedure. First, the invariant surface conditions (10) and its differential consequences are used to eliminate all the derivatives with respect to x^p from the PDE system (1). This yields a new PDE system, say

$$\Omega(x, u^{[n]}) = 0, \tag{11}$$

for the unknown functions $u^\alpha = u^\alpha(x^1, \ldots, x^{p-1}; x^p)$ of $x^1, \ldots, x^{p-1}$ (here x^p is considered as a parameter). Note that the function $\Omega = (\Omega_1, \ldots, \Omega_l)$ defining the system (11) depends on x and $u^{[n]}$, where by $u^{[n]}$ we denote all the partial derivatives of u^α (as functions only of $x^1, \ldots, x^{p-1}$) up to order n.

If the PDE system (11) is of maximal rank, we can proceed with the second step: apply the classical Lie method to this, with Y given by (9) as a symmetry operator. That is, we require that

$$\mathrm{pr}^{(n)} Y(\Omega)|_{\Omega=0} = 0. \tag{12}$$

Substituting the nth order prolongation of the vector field Y into (12), we get a relation which must be satisfied on the set of the solutions of the system (11). By eliminating any dependence between the partial derivatives of u^α occurring in this relation and (11)—this means that the highest order partial derivative from (11) is eliminated—we get an over-determined nonlinear PDE system for the infinitesimals ζ^i and ϕ^α. This system is called the *determining equations of the nonclassical symmetries* associated with the system (1).

3. A new procedure for finding nonclassical symmetries

In this section, we propose a new procedure for finding the determining equations of the nonclassical symmetries related to a PDE system. Specifically, we give an alternative for the second step of the method of Clarkson and Mansfield (1994). Recall that the nonclassical symmetries of the system (1) are found by seeking the classical symmetries of the system (11), while demanding that the symmetry operator be related to the invariant surface conditions (10), representing its associated characteristics. Now suppose that we drop this requirement, in other words, we consider a symmetry operator

$$W = \sum_{i=1}^{p} L^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} M^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

of (11), whose coefficients are not related anymore to the invariant surface conditions (10).

Our algorithm is based on the following observation.

Let Y be a nonclassical operator of (1) *and W be a symmetry operator of* (11). *Then the determining equations of the nonclassical symmetries associated with* (1) *can be derived by substituting*

$$\begin{aligned} &L^j = \zeta^j \ (\textit{with } j = 1,\dots,p-1),\ L^p = 1,\\ &\quad \textit{and } M^\alpha = \phi^\alpha \ (\textit{with } \alpha = 1,\dots,q) \end{aligned} \tag{13}$$

into the determining equations of the classical symmetries associated with (11).

Indeed, the criterion for infinitesimal invariance under W is

$$\mathrm{pr}^{(n)} W(\Omega)|_{\Omega=0} = 0. \tag{14}$$

The nth order prolongation of the vector field W is given by

$$\mathrm{pr}^{(n)} W = W + \sum_{\alpha=1}^{q}\sum_{J} \mathcal{M}_J^\alpha(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{15}$$

see (4) and (5), where

$$\mathcal{M}_J^\alpha(x, u^{(n)}) = D_J\left(M^\alpha - \sum_{j=1}^{p-1} L^j u_j^\alpha\right) + \sum_{j=1}^{p-1} L^j u_{J,i}^\alpha + L^p u_{J,p}^\alpha - D_J\left(L^p u_p^\alpha\right). \tag{16}$$

Having substituted (15) and (16) into the condition (14), we get the relation

$$\sum_{j=1}^{p-1} L^j \frac{\partial \Omega_k}{\partial x^j} + L^p \frac{\partial \Omega_k}{\partial x^p} + \sum_{\alpha=1}^{q} M^\alpha \frac{\partial \Omega_k}{\partial u^\alpha} + \sum_{\alpha=1}^{q}\sum_{I}\left[D_I\left(M^\alpha - \sum_{j=1}^{p-1} L^j u_j^\alpha\right) + \sum_{j=1}^{p-1} L^j u_{I,j}^\alpha\right]\frac{\partial \Omega_k}{\partial u_I^\alpha} = \sum_{\alpha=1}^{q}\sum_{I}\left[D_I\left(L^p u_p^\alpha\right) - L^p u_{I,p}^\alpha\right]\frac{\partial \Omega_k}{\partial u_I^\alpha}, \quad k = 1, \ldots, l, \tag{17}$$

which must be identically satisfied on the set of the solutions of the system (11). Here, the summation over I is carried out over all multi-indices which do not contain p (remember that the reduced equation (11) does not contain any derivatives with respect to x^p). Note that all derivatives of the form $u_{p,K}^\alpha$ occur on the right-hand side of (17).

To find the determining equations of the symmetry group of the system (11), first we have to eliminate any dependence between the partial derivatives of u^α occurring in (11) and (17) by substituting the highest order partial derivative from (11) into (17). In this process, all derivatives of u^α with respect to x^p are kept on the right-hand side. The determining equations for W to be a symmetry operator of (11) are now found by equating the coefficients of the partial derivatives u_K^α to zero. We now substitute (13) into this system. Since L^p is constant, the coefficients of the partial derivatives u_p^α and $u_{p,K}^\alpha$ vanish, because $D_I(L^p u_p^\alpha) = L^p D_I u_p^\alpha = L^p u_{I,p}^\alpha$. So the equations resulting from the right-hand side of (17) are identically zero. Hence, the equations resulting from the left-hand side of (17) represent the determining equations of the nonclassical symmetries of the system (1), and the symmetry operator W becomes the nonclassical operator Y of the system (1).

The above observation suggests the following procedure.

Step 1. Find the normal form (11) of (1) by reducing the initial system using the invariant surface conditions (7) and its differential consequences.

Step 2. Use any package designed for finding classical symmetries to generate the determining equations of the symmetry group associated with the system (11).

Step 3. Substitute $L^p = 1$, $L^j = \zeta^j$, and $M^\alpha = \phi^\alpha$ into the PDE system resulting from Step 2, thus obtaining the determining equations of the nonclassical symmetries of the original PDE system (1).

4. Implementation in MAPLE

The procedure for finding nonclassical symmetries described above has been implemented as the MAPLE routine GENDEFNC. As the name implies, it is built on top of the GENDEF routine in the DESOLV package (Carminati and Vu, 2000) which calculates *classical* symmetries. The user interface for GENDEFNC is almost identical to that of GENDEF. Three arguments are required: the PDE system (1), the independent variables $x = (x^1, \dots, x^p)$, and the dependent variables $u = (u^1, \dots, u^q)$, respectively. An optional fourth argument may be given to specify which of the independent variables plays the role of x^p; the corresponding infinitesimal will be set to one, see (9). The function GENDEFNC returns the determining equations for the nonclassical symmetries of the given system.

The procedure outlined in the previous section is implemented fairly straightforwardly in GENDEFNC. First, the PDE system given by the user is reduced relative to the invariant surface conditions. Note that it is not necessary to program the whole machinery of differential Gröbner bases (Clarkson and Mansfield, 1994) to achieve this reduction because of the simple form that the invariant surface conditions take. Then, the coefficients of the partial derivatives appearing in the reduced system are replaced by placeholder symbols $\mathcal{A}_i$. This is a well-known trick to reduce intermediate expression swell, and our experience is that it saves time and memory. The resulting system is passed to GENDEF which returns the determining equations for the classical symmetries. Finally, the placeholders $\mathcal{A}_i$ are replaced by the original coefficients, and the substitution (13) is performed. This yields the determining equations for the nonclassical symmetries of the original PDE, which is returned to the user. An example of the use of the GENDEFNC routine can be found in the next section.

5. Application of GENDEFNC routine to the Huxley PDE

Following Hydon (2000), we consider the Huxley equation

$$u_t = u_{xx} + 2u^2(1-u). \tag{18}$$

The symmetry group associated with this equation is generated by

$$X = c_1 X_1 + c_2 X_2, \quad \text{where} \quad X_1 = \partial_x, \quad X_2 = \partial_t,$$

with c_1 and c_2 arbitrary real numbers. In order to apply the nonclassical method, consider a one-parameter group of transformation generated by the vector field (3) which in our case turns into

$$Y = \zeta(x,t,u)\partial_x + \eta(x,t,u)\partial_t + \phi(x,t,u)\partial_u,$$

where we denote $x^1 = x$, $x^2 = t$, $u^1 = u$, $\zeta^1 = \zeta$, $\zeta^2 = \eta$ and $\phi^1 = \phi$. The invariant surface condition (7) becomes

$$\psi^1 = \zeta u_x + \eta u_t - \phi = 0$$

If we seek symmetries with $\eta = 1$, then the relation (10) is written as

$$u_t = \phi - \zeta u_x,$$

and this must be substituted into the PDE (18). This results in

$$u_{xx} + \zeta u_x + 2u^2(1-u)\phi = 0,$$

which can be written in the equivalent form

$$u_{xx} + \mathcal{A}_1(x,t,u)u_x + \mathcal{A}_2(x,t,u) = 0, \tag{19}$$

where $\mathcal{A}_1 = \zeta$ and $\mathcal{A}_2 = 2u^2(1-u)\phi$. Since in (19) the coefficients $\mathcal{A}_i$ can be viewed as arbitrary functions, we can apply the classical Lie method to this equation. If

$$W = L(x,t,u)\partial_x + K(x,t,u)\partial_t + M(x,t,u)\partial_u$$

is the symmetry operator associated with the Eq. (19), then its coefficients are solutions of the following PDE system

$$\begin{aligned}
&K_x = 0,\\
&K_u = 0,\\
&L_{uu} = 0,\\
&M_{uu} - 2L_{xu} + 2L_u\mathcal{A}_1 = 0,\\
&4L_{xu}\mathcal{A}_2 - L_{xx}\mathcal{A}_1 + 2M_{xu}\mathcal{A}_1 - M_{uu}\mathcal{A}_2 - L_u\mathcal{A}_1\mathcal{A}_2 + L_x\mathcal{A}_1^2 + L\mathcal{A}_1\mathcal{A}_{1,x}\\
&\qquad + K\mathcal{A}_1\mathcal{A}_{1,t} + M\mathcal{A}_1\mathcal{A}_{1,u} = 0,\\
&M_{uu}\mathcal{A}_2^2 - 2M_{xu}\mathcal{A}_1\mathcal{A}_2 + M_{xx}\mathcal{A}_1^2 + L_{xx}\mathcal{A}_1\mathcal{A}_2 - 2L_{xu}\mathcal{A}_2^2 - L_x\mathcal{A}_1^2\mathcal{A}_2\\
&\qquad - L_u\mathcal{A}_1\mathcal{A}_2^2 - M_x\mathcal{A}_1^3 - M_u\mathcal{A}_1^2\mathcal{A}_2 - L\mathcal{A}_1(\mathcal{A}_2\mathcal{A}_{1,x} - \mathcal{A}_1\mathcal{A}_{2,x})\\
&\qquad + K\mathcal{A}_1(\mathcal{A}_1\mathcal{A}_{2,t} - \mathcal{A}_2\mathcal{A}_{1,t}) + M\mathcal{A}_1(\mathcal{A}_1\mathcal{A}_{2,u} - \mathcal{A}_2\mathcal{A}_{1,u}) = 0,
\end{aligned} \tag{20}$$

which represents the determining equations of the symmetry group related to (19). Note that this is a linear PDE system for the unknowns L, K and M. If we substitute the coefficients $\mathcal{A}_i$ given above and $L = \zeta$, $K = 1$ and $M = \phi$ into (20), and reduce to triangular form, we get the nonlinear PDE system

$$\begin{aligned}
&\zeta_{uu} = 0\\
&\phi_{uu} - 2\zeta_{xu} + 2\zeta\zeta_u = 0\\
&2\phi_{xu} - \zeta_{xx} - (2\phi + 6u^3 - 6u^2)\zeta_u + 2\zeta\zeta_x + \zeta_t = 0\\
&\phi_{xx} - 2\phi\zeta_x + 2u^2(u-1)(\phi_u - 2\zeta_x) - \phi_t + (4u - 6u^2)\phi = 0,
\end{aligned} \tag{21}$$

which represents the determining equations of the nonclassical symmetries associated with the Huxley equation.

The same result can be obtained with the GENDEFNC routine, described in the previous section. This is achieved by giving the following commands, after loading GENDEFNC and the DESOLV package.

```
PDE := diff(u(x,t),t) = diff(u(x,t),x,x) + 2*u(x,t)^2*(1-u(x,t));
gendefnc(PDE, u, [x,t]);
```

The general solution of the nonlinear PDE system (21) is

$$\zeta = c_1, \qquad \phi = 0, \tag{22}$$

where c_1 is a real number, and

$$\zeta = \pm(3u - 1), \qquad \phi = 3u^2(1 - u). \tag{23}$$

As we can see, the case (22) corresponds to the classical operator $c_1 X_1 + X_2$. At this stage, we do not get all the classical operators, since we assumed that the coefficient of ∂_t is 1. We can retrieve all the classical operators by using that any multiple of a nonclassical operator is again a nonclassical operator, as mentioned in Section 2.2.

From (23) we get the nonclassical operator

$$Y = \pm(3u - 1)\partial_x + \partial_t + 3u^2(1 - u)\partial_u.$$

More details about the group-invariant solutions of the Huxley equation obtained from its classical and nonclassical symmetries can be found in (Hydon, 2000).

6. Conclusions

There are large PDE systems for which it is difficult to determine their nonclassical symmetries due to the limited memory of the system on which the symbolic manipulation program runs. Even finding the determining equations of the nonclassical symmetries can create memory problems. In this paper, we introduce a new algorithm for computing the determining equations of the nonclassical symmetries of a given PDE system with the help of any symbolic manipulation program designed to determine classical symmetries, without any change of the program. This algorithm has been implemented in MAPLE. To our knowledge, it is the first routine available in MAPLE for this task.

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