The Evans function review, Part II: Numerical computations

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Outline

1. The Evans function: definition and properties.
2. Numerical computations on phase space.
4. Numerical computations on exterior space.
5. The Magnus method.
The miss-distance function

Consider the Sturm–Liouville problem
\[-u'' + q(x) u = \lambda u \quad \text{for } x \in [a, b]\]
with boundary conditions \(u(a) = u(b) = 0\).

We can rewrite this in first-order form:
\[y' = A(x, \lambda)y \quad \text{where } A(x, \lambda) = \begin{bmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{bmatrix} \quad (\ast)\]
with boundary conditions \(y_1(a) = y_1(b) = 0\).

Denote by \(y^-(x)\) the solution of \((\ast)\) with \(y^-(a) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\).

The miss-distance function is
\[D(\lambda) = y_1^+(b).\]

Eigenvalues correspond to zeros of the miss-distance function.
The matching point $\xi$

We are looking at the Sturm–Liouville problem

$$y' = A(x, \lambda)y \quad \text{where } A(x, \lambda) = \begin{bmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{bmatrix}$$ (*)

with boundary conditions $y_1(a) = y_1(b) = 0$.

Denote by $y^-(x)$ the solution of (*) with $y^-(a) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Denote by $y^+(x)$ the solution of (*) with $y^+(b) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The SLP (*) has a solution if $y^+$ is a multiple of $y^-$.

The miss-distance function, evaluated at $\xi \in [a, b]$, is

$$D(\lambda) = \det \begin{bmatrix} y_1^-(\xi) & y_1^+(\xi) \\ y_2^-(\xi) & y_2^+(\xi) \end{bmatrix}.$$  

Eigenvalues correspond to zeros of the miss-distance function.

For $\xi = b$, we get $D(\lambda) = y_1^-(b)$, as before.
Consider the following Sturm–Liouville problem with $x \in \mathbb{R}$:

$$y' = A(x, \lambda) y$$

where

$$A(x, \lambda) = \begin{bmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{bmatrix}$$

with boundary conditions $y(x) \to 0$ as $x \to \pm \infty$.

Assume that $q(x) \to 0$, then eigenvalues of $A(\pm \infty, \lambda)$ are $\pm \sqrt{-\lambda}$.

Assume that $\lambda \in \mathbb{C} \setminus [0, \infty)$, then $\nu^{-}_u = \begin{bmatrix} 1 \\ \sqrt{-\lambda} \end{bmatrix}$ is the eigenvector corresponding to the unstable eigenvalue $\mu^{-}_u = \sqrt{-\lambda}$ at $x = -\infty$.

Let $y^-$ be the solution of (*) with $e^{\mu^{-}_u x} y^-(x) \to \nu^{-}_u$ as $x \to -\infty$.

Let $y^+$ be the solution of (*) with $e^{\mu^+_s x} y^+(x) \to \nu^+_s$ as $x \to +\infty$.

Define $D(\lambda) = \det \begin{bmatrix} y^-(\xi) & y^+(\xi) \end{bmatrix}$, with $\xi \in \mathbb{R}$ arbitrary. Eigenvalues of (*) correspond to zeros of $D$. 
Increasing the number of dimensions

Consider $y' = A(x, \lambda) y$ with $y(x) \in \mathbb{C}^n$ for $x \in \mathbb{R}$.

We assume that there is a region $\Omega \subset \mathbb{C}$ such that for all $\lambda \in \Omega$

- the matrices $A^\pm(\lambda) = \lim_{x \to \pm \infty} A(x, \lambda)$ exist and are hyperbolic;
- $A^-(\lambda)$ has $k$ unstable eigenvalues $\mu_1^-, \ldots, \mu_k^-$;
- $A^+(\lambda)$ has $n-k$ stable eigenvalues $\mu_1^+, \ldots, \mu_{n-k}^+$.

Let $y_i^-$ be a solution with $e^{\mu_i^- x} y_i^-(x) \to v_i^-$ as $x \to -\infty$.
Let $y_i^+$ be a solution with $e^{\mu_i^+ x} y_i^+(x) \to v_i^+$ as $x \to +\infty$.

The Evans function is defined by

$$D(\lambda) = \det \left[ y_1^-(\xi) \cdots y_k^-(\xi) \ y_1^+(\xi) \cdots y_{n-k}^+(\xi) \right].$$

It is analytic in $\Omega$ and its zeros corresponds to eigenvalues.

(Evans ’75; Alexander, Gardner & Jones ’90; Sandstede ’02)
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Evaluating the Evans function numerically

The definition can be used as a basis for a numerical algorithm:

1. Compute the unstable eigenvectors $v_{1}^{-}, \ldots, v_{k}^{-}$ of $A^{-}$.
2. For $i = 1, \ldots, k$, solve $y' = A(x, \lambda) y$ with initial condition $y(-L) = v_{i}^{-}$ (where $L$ is large) to get $y_{i}^{-}(\xi)$.
3. Compute $y_{i}^{+}(\xi)$ similarly, and calculate the determinant.

The Evans function is analytic, so we can use the argument principle to count the number of eigenvalues in a given region.  
((Evans & Feroe '77)

Alternatively, we can use Newton’s method to solve $D(\lambda) = 0$ and locate the eigenvalues.  
((Pego, Smereka & Weinstein '93)
Problems

- Unstable space \(\implies\) solutions grow exponentially; for example, \(y_i^-\) grows with rate \(\approx \mu_i^-\).

**Solution:** Find \(y_i^-\) by solving \(y' = (A(x, \lambda) - \mu_i^- I)y\).

- Evans function is only analytic if the eigenvectors \(v_i^{\pm}\) are analytic functions in \(\lambda\). This won’t happen automatically, and is impossible if eigenvalues \(\mu_i^{\pm}\) coalesce.

**Solution:** Procrastination.

- If \(\text{Re} \mu_1^- > \text{Re} \mu_2^-\), then \(y_1^-\) grows faster than \(y_2^-\), so any errors in the \(y_1^-\) direction will dominate the \(y_2^-\) solution.

**Solution:** Do not look at the \(y_i^-\) individually, but look at the subspace \(S = \text{span}\{y_1^-, \ldots, y_k^-\}\) and lift the equation \(y' = A(x, \lambda)y\) to \(S' = \ell(A(x, \lambda))S\).
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Exterior product of a vector space

Let $V$ be a vector space with basis $e_1, \ldots, e_n$. The exterior product space $\Lambda^k(V)$ is a vector space with basis

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}.$$ 

For example, $\Lambda^2(\mathbb{C}^4)$ is six-dimensional with basis

$$e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4.$$ 

The wedge product has the following properties:

**Associativity:** $(u \wedge v) \wedge w = u \wedge (v \wedge w)$.

**Linearity:** $u \wedge (v + w) = u \wedge v + u \wedge w$ and $u \wedge (\alpha v) = \alpha (u \wedge v)$.

**Supersymmetry:** $u \wedge v = (-1)^{k\ell} v \wedge u$ if $u \in \Lambda^k(V)$ and $v \in \Lambda^\ell(V)$. 


The Grassmannian manifold $G_k(V)$ is the set of $k$-dimensional subspaces of $V$. We consider the identification

$$\text{span}\{v_1, \ldots, v_k\} \in G_k(V) \leftrightarrow v_1 \wedge \ldots \wedge v_k \in \Lambda^k(V).$$

So we can embed $G_k(V)$ in $\Lambda^k(V)$, or, to be precise, $\mathbb{P}(\Lambda^k(V))$.

A form $w \in \Lambda^k(V)$ is decomposable if it can be written as $w = v_1 \wedge \ldots \wedge v_k$ with $v_i \in V$.

Only decomposable form correspond to subspaces.

Consider for example $\Lambda^2(\mathbb{C}^4)$.

The form $w = e_1 \wedge e_2 + e_3 \wedge e_4$ is not decomposable.

The form $\sum_{1 \leq i < j \leq 4} \alpha_{ij} e_i \wedge e_j$ is decomposable iff

$$\alpha_{12} \alpha_{34} - \alpha_{13} \alpha_{24} + \alpha_{14} \alpha_{23} = 0.$$
**Lifting the differential equation**

A linear differential equation \( y' = A(x) y \) on \( V \) induces an equation \( w' = \ell(A(x)) w \) on \( \Lambda^k(V) \).

Take, for definiteness, \( k = 2 \) and \( V = \mathbb{C}^n \).
Suppose that \( y_1(x) \) and \( y_2(x) \) solve \( y' = A(x) y \).
Then \( w(x) = y_1(x) \wedge y_2(x) \) is a solution of \( w' = \ell(A(x)) w \).

We have the commutative diagram

\[
\begin{array}{ccc}
w' = \ell(A(x)) w & \xrightarrow{\text{solve}} & w(x) = L(\Phi(x)) w(0) \quad \text{in} \quad \Lambda^2(\mathbb{C}^4) \\
\uparrow \text{lift } \ell & & \uparrow \text{lift } L \\
y' = A(x) y & \xrightarrow{\text{solve}} & y(x) = \Phi(x) y(0) \quad \text{in} \quad \mathbb{C}^4 \\
\end{array}
\]

Infinitesimal lift: \( \ell(A)(u \wedge v) = (Au) \wedge v + u \wedge (Av) \).
Finite lift: \( L(A)(u \wedge v) = (Au) \wedge (Av) \).
Lifting the differential equation: An example

Consider \( k = 2 \) and \( n = 3 \) and the equation \( y' = Ay \) with

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}.
\]

Write \( w \in \Lambda^2(\mathbb{C}^3) \) as \( w = w_1(e_1 \wedge e_2) + w_2(e_1 \wedge e_3) + w_3(e_2 \wedge e_3) \).

\[
w_1' = w_1(Ae_1 \wedge e_2 + e_1 \wedge Ae_2)
= w_1\left( (a_{11}e_1 + a_{21}e_2 + a_{23}e_3) \wedge e_2 + e_1 \wedge (a_{12}e_1 + a_{22}e_2 + a_{32}e_3) \right)
= w_1\left( a_{11}e_1 \wedge e_2 + a_{23}e_3 \wedge e_2 + a_{22}e_1 \wedge e_2 + a_{32}e_1 \wedge e_3 \right)
= w_1\left( (a_{11} + a_{22})e_1 \wedge e_2 + a_{32}e_1 \wedge e_3 - a_{23}e_2 \wedge e_3 \right).
\]
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Evaluating the Evans function numerically II

Our first numerical algorithm was:

- Compute the unstable eigenvectors $\nu_1^-, \ldots, \nu_k^-$ of $A^-$.  
- For $i = 1, \ldots, k$, solve $y' = A(x, \lambda) y$ with initial condition $y(-L) = \nu_i^-$ (where $L$ is large) to get $y_i^-(\xi)$.  
- Compute $y_i^+(\xi)$ similarly, and calculate the determinant $D(\lambda) = \det \left[ y_1^-(\xi) \cdots y_k^-(\xi) y_1^+(\xi) \cdots y_{n-k}^+(\xi) \right]$.

We now lift the equation to $\Lambda^k(\mathbb{C}^n)$:

- Compute the unstable eigenvectors $\nu_1^-, \ldots, \nu_k^-$ of $A^-$.  
- Solve $w' = \ell(A(x, \lambda)) w$ with initial condition $w(-L) = \nu_1^- \wedge \ldots \wedge \nu_k^-$ (where $L$ is large) to get $w^-(\xi)$.  
- Compute $w^+(\xi)$ similarly, and compute $D(\lambda) = w^-(\xi) \wedge w^+(\xi)$.  

Evaluating the Evans function numerically III

- Compute the unstable eigenvectors $v_1^- , \ldots , v_k^-$ of $A^-$. 
- Solve $w' = \ell (A(x, \lambda)) w$ with initial condition $w(-L) = v_1^- \wedge \ldots \wedge v_k^-$ (where $L$ is large) to get $w^-(\xi)$. 
- Compute $w^+(\xi)$ similarly, and compute $D(\lambda) = w^-(\xi) \wedge w^+(\xi)$. 

Bridges ('99) notes that the compound matrix method of Davey ('79) and Ng & Reid ('79) is essentially the same. 

The first computations of the Evans function using this algorithm are due to Brin ('00) and Afendikov & Bridges ('01).

An disadvantage is that $\Lambda^k(\mathbb{C}^n)$ has dimension $\binom{n}{k}$. 
The alternative is to use a projection method. 
But this destroys linearity and analyticity.
Problems (rep’d)

- Unstable space $\implies$ solutions grow exponentially; for example, $y_i^-$ grows with rate $\approx \mu_i^-$.  
  **Solution:** Find $y_i^-$ by solving $y' = (A(x, \lambda) - \mu_i^- I)y$.

- Evans function is only analytic if the eigenvectors $v_i^{\pm}$ are analytic functions in $\lambda$. This won't happen automatically, and is impossible if eigenvalues $\mu_i^{\pm}$ coalesce.  
  **Solution:** Procrastination.

- If $\text{Re} \mu_1^- > \text{Re} \mu_2^-$, then $y_1^-$ grows faster than $y_2^-$, so any errors in the $y_1^-$ direction will dominate the $y_2^-$ solution.  
  **Solution:** Do not look at the $y_i^-$ individually, but look at the subspace $S = \text{span}\{y_1^-, \ldots, y_k^-\}$ and lift the equation $y' = A(x, \lambda)y$ to $S' = \mathcal{L}(A(x, \lambda))S$. 

Analytic eigenvectors

The eigenvectors of the lifted matrix $\ell(A_{\pm}(\lambda))$ can be chosen analytically. (Kato ’84)

How to do this in a numerical algorithm?

- Use Kato’s theorem. (Brin & Zumbrun ’02)
- Eigenvectors satisfy a certain analytic ODE; solve this equation. (Bridges, Derks & Gottwald ’02)

Pragmatic approaches:

- If $A^+ = A^-$ and we use the adjoint equation to compute $w^+$, then the computed $w^- \wedge w^+$ will be analytic. (BDG ’02)
- Normalize eigenvectors so that first component is 1.
- Use analytic formula for eigenvectors.
Preservation of the Grassmannian

The flow of $w' = \ell(A(x, \lambda)) w$ leaves the Grassmannian invariant: if we start with a decomposable form (representing a subspace), then the form remains decomposable.

Does the numerical method respect this?

- In the case $k = 2$ and $n = 4$, the Grassmannian is attractive, if we replace $w' = \ell(A(x, \lambda)) w$ by $w' = (\ell(A(x, \lambda)) - \sigma I) w$, where $\sigma$ is the largest eigenvalue, provided the spectrum of $A$ changes not too much as $x$ varies.  
  (Bridges, Derks & Gottwald ’02)

- If $\text{tr } A = 0$, the Grassmannian is a strong quadratic invariant. Gauss–Legendre methods (e.g., implicit midpoint) conserve these.  
  (Allen & Bridges ’02)
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The Magnus series

We need to solve a linear differential equation.

\[ y' = ay \text{ with } y \text{ scalar} \implies y(x) = \exp(ax) y(0). \]
\[ y' = a(x) y \text{ with } y \text{ scalar} \implies y(x) = \exp\left(\int_0^x a(\xi) \, d\xi\right) y(0). \]
\[ y' = Ay \text{ with } y \text{ vector} \implies y(x) = \exp(Ax) y(0). \]
\[ y' = A(x) y \text{ with } y \text{ vector} \implies y(x) = \exp\left(\int_0^x A(\xi) \, d\xi\right) y(0). \]

Instead, we have \( y(x) = \exp(\Omega(x)) y(0) \) where

\[
\Omega(x) = \int_0^x A(\xi) \, d\xi - \frac{1}{2} \int_0^x \int_0^{\xi_1} [A(\xi_2), A(\xi_1)] \, d\xi_2 \, d\xi_1 + \frac{1}{12} \int_0^x \int_0^{\xi_1} \int_0^{\xi_1} [A(\xi_3), [A(\xi_2), A(\xi_1)]] \, d\xi_3 \, d\xi_2 \, d\xi_1 + \cdots
\]

(Magnus '54)
Convergence

\[ \Omega(x) = \int_0^x A(\xi) \, d\xi \]
\[ - \frac{1}{2} \int_0^x \int_0^{\xi_1} [A(\xi_2), A(\xi_1)] \, d\xi_2 \, d\xi_1 \]
\[ + \frac{1}{12} \int_0^x \int_0^{\xi_1} \int_0^{\xi_1} [A(\xi_3), [A(\xi_2), A(\xi_1)]] \, d\xi_3 \, d\xi_2 \, d\xi_1 \]
\[ + \frac{1}{4} \int_0^x \int_0^{\xi_1} \int_0^{\xi_2} [[A(\xi_3), A(\xi_2)], A(\xi_1)] \, d\xi_3 \, d\xi_2 \, d\xi_1 + \cdots \]

Magnus (’54) says that the series converges for “sufficiently” small \( x \). Since then, many convergence results have been found. The latest is:

**Theorem**

The Magnus series converges if \( \int_0^x \|A(\xi)\|_2 \, d\xi < \pi \).

The constant \( \pi \) is sharp.  

(Moan & N.)
Numerical methods based on the Magnus series

The Magnus series can be used to solve \( y' = A(x) \, y \) numerically.

Truncate the Magnus series:

\[
y(x) \approx \exp \left( \int_0^x A(\xi) \, d\xi - \frac{1}{2} \int_0^x \int_0^{\xi_1} [A(\xi_2), A(\xi_1)] \, d\xi_2 \, d\xi_1 \right) y(0).
\]

Replace \( A(x) \) by interpolant at G-L points \( x_{1,2} = \left( \frac{1}{2} \pm \frac{\sqrt{3}}{6} \right) x \):

\[
y(x) \approx \exp \left( \frac{1}{2} x (A(x_1) + A(x_2)) - \frac{\sqrt{3}}{12} x^2 \left[ A(x_1), A(x_2) \right] \right) y(0).
\]

This is a method of order four.  

(Iserles & Nørsett '99)
Preservation of the Grassmannian II

Recall the commutative diagram

\[ w' = \ell(A(x))w \xrightarrow{\text{solve}} w(x) = L(\Phi(x))w(0) \quad \text{in } \Lambda^k(\mathbb{C}^n) \]

\[ y' = A(x)y \xrightarrow{\text{solve}} y(x) = \Phi(x)y(0) \quad \text{in } \mathbb{C}^n \]

The diagram also commutes for Magnus:

\[ w' = \ell(A(x))w \xrightarrow{\text{Magnus}} w(x) = L(\exp(\Omega_*))w(0) \quad \text{in } \Lambda^k(\mathbb{C}^n) \]

\[ y' = A(x)y \xrightarrow{\text{Magnus}} y(x) = \exp(\Omega_*)y(0) \quad \text{in } \mathbb{C}^n \]

In particular, the Magnus method preserves the Grassmannian.
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The autocatalytic model

The autocatalytic reaction \( U + mV \rightarrow (m + 1)V \) is modelled by

\[
\begin{align*}
    u_t &= \delta u_{xx} - uv^m, \\
    v_t &= v_{xx} + uv^m.
\end{align*}
\]

There is a unique travelling wave solution with \((u, v) \rightarrow (0, 1)\) as \(x \rightarrow -\infty\) and \((u, v) \rightarrow (1, 0)\) as \(x \rightarrow \infty\) for any speed \(c \in [c_*, \infty)\).

Aparicio, Malham & Oliver ('05) evaluate the Evans function with the Magnus method to assess the stability of travelling waves with \(c = c_*\).
Stability of the rotating Ekman layer

Linearization of the 3d Navier–Stokes equation in a rotating frame about the Ekman layer coupled to a compliant surface leads to

\[
\phi'''' - b(x)\phi'' - a(x)\phi + 2\psi' = 0, \\
\psi'' + (\gamma^2 - b(x))\psi - i\gamma RV'(x)\phi - 2\phi' = 0,
\]

for \(0 \leq x \leq +\infty\) with compliant surface BCs at \(x = 0\).

Lifting yields an ODE on \(\Lambda^3(\mathbb{C}^6)\), which has dimension \(20\). However, the dimension of the Grassmannian is only \(9\). It is unclear whether the integrator can stay on the Grassmannian. \(\Lambda^3(\mathbb{C}^6)\) is divided in three equivalence classes. (Hitchin ’00)

Notwithstanding all these questions, the algorithm is robust. (Allen & Bridges ’03)
Stability of viscous shock waves

A viscous shock waves is a travelling wave solution to the conservation law

\[ u_t + (f(u))_x = (B(u)u_x)_x. \]

An important technical tool is the Gap Lemma, which allows one to extend the Evans function into the essential spectrum.

As an example, consider the cubic model in MHD

\[ u_t + (|u|^2 u)_x = u_{xx}. \]

(Brin ’00, Brin & Zumbrun ’03)
Edge bifurcations and gap solitary waves

Optical pulses in grated waveguides like the periodic Kerr medium are described by

\[ i(u_t + u_x) + v + (|v|^2 + \rho |u|^2)u = 0, \]
\[ i(v_t + v_x) + u + (|u|^2 + \rho |v|^2)v = 0. \]

This equation has continuous spectrum along the imaginary axis. **Gap solitons** are solitons that “live” in a gap in the spectrum. The usual procedure leads to an equation in \( \Lambda^2(\mathbb{C}^4) \).

There are various interesting bifurcations from the continuous spectrum.

*(Derks & Gottwald '05)*
A. L. Afendikov and T. J. Bridges.
Instability of the Hocking–Stewartson pulse and its implications for three-dimensional Poiseuille flow.

A topological invariant arising in the stability analysis of travelling waves.

L. Allen and T. J. Bridges.
Numerical exterior algebra and the compound matrix method.

L. Allen and T. J. Bridges.
Hydrodynamic stability of the Ekman boundary layer including interaction with a compliant surface: a numerical framework.

N. D. Aparicio, S. J. A. Malham, and M. Oliver.
Numerical evaluation of the Evans function by Magnus integration.
Bibliography II

T. J. Bridges.
The Orr–Sommerfeld equation on a manifold.

T. J. Bridges, G. Derks, and G. Gottwald.
Stability and instability of solitary waves of the fifth-order KdV equation: A numerical framework.

L. Q. Brin.
Numerical testing of the stability of viscous shock waves.

L. Q. Brin and K. Zumbrun.
Analytically varying eigenvectors and the stability of viscous shock waves.

A. Davey.
On the removal of singularities from the Riccati method.
G. Derks and G. Gottwald.
A robust numerical method to study oscillatory instability of gap solitary waves.

J. W. Evans.
Nerve axon equations (iv): The stable and unstable impulse.

J. W. Evans and J. Feroe.
Local stability theory of the nerve impulse.

N. Hitchin.
The geometry of three-forms in six dimensions.

A. Iserles and S. Nørsett.
On the solution of linear differential equations in Lie groups.
T. Kato.

*Perturbation theory for linear operators.*

W. Magnus.

On the exponential solution of differential equations for a linear operator.

B. S. Ng and W. H. Reid.

An initial value method for eigenvalue problems using compound matrices.


Oscillatory instability of traveling waves for a KdV–Burgers equation.

B. Sandstede.

Stability of travelling waves.