The Evans function and the stability of travelling waves

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Travelling wave solutions

Consider a PDE like the reaction–diffusion equation

$$y_t = K y_{xx} + F(y)$$

where $y(t,x) \in \mathbb{R}^d$ with $x \in \mathbb{R}$.

Travelling waves are solutions to the equation that move with constant speed c while maintaining their shape.

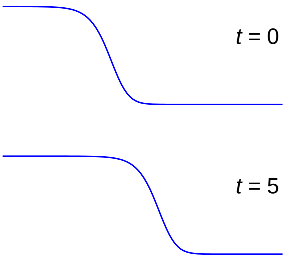
Example: The Fisher equation is

$$y_t = y_{xx} + y(1-y).$$

It models the propagation of genes.
It admits the travelling wave solution

$$y(t,x) = \left(1 + \exp\left(\frac{x-ct}{\sqrt{6}}\right)\right)^{-2}$$

with $c = 5/\sqrt{6}$.



The KdV equation

Formalism described here is not restricted to reaction–diffusion, but also applicable to other equations like Korteweg–De Vries (KdV):

$$u_t + u_{xxx} + 6uu_x = 0.$$

The KdV equation admits soliton solutions:

$$u(t,x) = \tfrac{1}{2}c\,\mathrm{sech}^2\big(\tfrac{1}{2}\sqrt{c}(x-ct)\big).$$

Recreation of soliton wave in Edinburgh

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Stability of travelling waves

Recall: The RD equation is $y_t = Ky_{xx} + F(y)$.
In the moving frame $\xi = x - ct$, the PDE is

$$y_t = Ky_{\xi\xi} + cy_\xi + F(y).$$

Travelling wave solutions are of the form $y(t, \xi) = \hat{y}(\xi)$.
Hence, they satisfy $K\hat{y}_{\xi\xi} + c\hat{y}_\xi + F(\hat{y}) = 0$.

Given a travelling wave $\hat{y}$, what is the fate of solutions whose initial conditions are small perturbations of $\hat{y}$? If any such solution stay close to $\hat{y}$, we say that the travelling wave is stable.

In context of RD, use space of bounded uniformly continuous functions with sup norm.

Orbital stability: If $\hat{y}(\xi)$ is a travelling wave solution, then so is its translate $\hat{y}(\xi - \xi_0)$. So, we require for stability that small perturbations of $\hat{y}(\xi)$ stay close to the family $\{\hat{y}(\xi - \xi_0) : \xi_0 \in \mathbf{R}\}$.

Linear stability analysis

A natural approach to stability analysis is to linearize the equation about the travelling wave.

Suppose that $y(t,\xi) = \hat{y}(\xi) + \tilde{y}(t,\xi)$ is a perturbation of the travelling wave at $t = 0$.

y solves the full PDE:	$Ky_{\xi\xi} + cy_\xi + f(y) = y_t$
Travelling wave $\hat{y}$ solves:	$K\hat{y}_{\xi\xi} + c\hat{y}_\xi + f(\hat{y}) = 0$
Subtracting:	$K\tilde{y}_{\xi\xi} + c\tilde{y}_\xi + f(\hat{y}+\tilde{y}) - f(\tilde{y}) = \tilde{y}_t$
Linearized stability analysis:	$K\tilde{y}_{\xi\xi} + c\tilde{y}_\xi + Df(\hat{y})\,\tilde{y} = \tilde{y}_t$
Set $\tilde{y}(\xi,t) = \mathrm{e}^{\boldsymbol{\lambda} t}\bar{y}(\xi)$ to get:	$K\bar{y}_{\xi\xi} + c\bar{y}_\xi + Df(\hat{y})\,\bar{y} = \boldsymbol{\lambda}\bar{y}$

So, we need to study the spectrum of the differential operator

$$\mathcal{L} = K\partial_{\xi\xi} + c\partial_\xi + DF(\hat{y}).$$

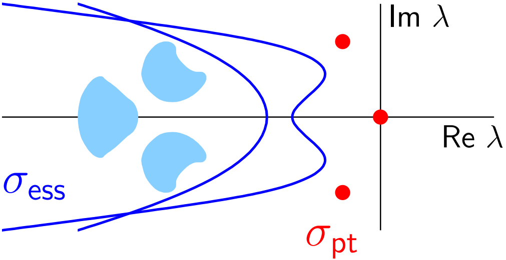
If $\mathcal{L}$ has an eigenvalue $\boldsymbol{\lambda}$ with $\mathrm{Re}\,\boldsymbol{\lambda} \geq 0$, then the wave is unstable.

The spectrum of $\mathcal{L}$

The spectrum of $\mathcal{L}$ can be decomposed in two parts.

The point spectrum σ_{pt} consists of eigenvalues λ, for which $\mathcal{L}\bar{y} = \lambda\bar{y}$ has a solution. We have $0 \in \sigma_{\text{pt}}$.

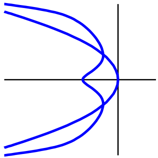
The essential spectrum σ_{ess} typically contains open sets, and is asymptotically (as $\lambda \to \infty$) inside the parabolic region $\{\lambda : (\operatorname{Im}\lambda)^2 \leq -C\operatorname{Re}\lambda\}$.



Nonlinear stability vs linear stability

If the spectrum of $\mathcal{L}$ is contained in the left half of the complex plane and bounded away from the imaginary axis, except for a simple eigenvalue at the origin, then the travelling wave is (nonlinearly) stable. **(Henri '81)**

If the essential spectrum touches the imaginary axis at the origin, then one typically has to introduce weighted norms to prove stability.



For many equations, the essential spectrum is easily computed. Hence, we now assume that σ_{ess} is in the left half-plane, and we concentrate on the point spectrum. **(Sandstede '02)**

Solving the eigenvalue problem

The usual approach for solving the eigenvalue problem

$$K\bar{y}_{\xi\xi} + c\bar{y}_{\xi} + Df(\hat{y})\,\bar{y} = \lambda\bar{y}, \qquad \bar{y} \to 0 \text{ as } \xi \to \pm\infty$$

is to discretize the differential operator $\mathcal{L}$ by finite differences, finite elements, or some spectral method, and solve the resulting (huge) matrix eigenvalue problem.

The discretization may create spurious eigenvalues, but sometimes one can prove that the spectrum of the matrix converges to the spectrum of $\mathcal{L}$ as the grid size $\Delta\xi \to 0$.

Secondly, the infinite domain needs to be truncated to $\xi \in [-L, L]$. The exact asymptotic boundary conditions are nonlinear; replacing them by linear BCs leads again to spurious eigenvalues.

(Bridges, Derks & Gottwald '02)

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The miss-distance function

Consider the Sturm–Liouville problem

$$-u'' + q(x)\,u = \lambda u \qquad \text{for } x \in [a,b]$$

with boundary conditions $u(a) = u(b) = 0$.

We can rewrite this in first-order form:

$$y' = A(x,\lambda)y \qquad \text{where } A(x,\lambda) = \begin{bmatrix} 0 & 1 \\ q(x)-\lambda & 0 \end{bmatrix} \qquad (*)$$

with boundary conditions $y_1(a) = y_1(b) = 0$.

Denote by $y^-(x)$ the solution of $(*)$ with $y^-(a) = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

The miss-distance function is

$$D(\lambda) = y_1^-(b).$$

Eigenvalues correspond to zeros of the miss-distance function.

The matching point ξ

We are looking at the Sturm–Liouville problem

$$y' = A(x,\lambda)y \qquad \text{where } A(x,\lambda) = \begin{bmatrix} 0 & 1 \\ q(x)-\lambda & 0 \end{bmatrix} \qquad (*)$$

with boundary conditions $y_1(a) = y_1(b) = 0$.

Denote by $y^-(x)$ the solution of $(*)$ with $y^-(a) = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.
Denote by $y^+(x)$ the solution of $(*)$ with $y^+(b) = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

The SLP $(*)$ has a solution if y^+ is a multiple of y^-.

The miss-distance function, evaluated at $\xi \in [a,b]$, is

$$D(\lambda) = \det \begin{bmatrix} y_1^-(\xi) & y_1^+(\xi) \\ y_2^-(\xi) & y_2^+(\xi) \end{bmatrix}.$$

Eigenvalues correspond to zeros of the miss-distance function.

For $\xi = b$, we get $D(\lambda) = y_1^-(b)$, as before.

Problems on an infinite interval

Consider the following Sturm–Liouville problem with $x \in \mathbb{R}$:

$$y' = A(x,\lambda)y \qquad \text{where } A(x,\lambda) = \begin{bmatrix} 0 & 1 \\ q(x)-\lambda & 0 \end{bmatrix} \qquad (*)$$

with boundary conditions $y(x) \to 0$ as $x \to \pm\infty$.

Assume that $q(x) \to 0$, then eigenvalues of $A(\pm\infty,\lambda)$ are $\pm\sqrt{-\lambda}$.

Assume that $\lambda \in \mathbb{C} \setminus [0,\infty)$, then $v_u^- = \left[\begin{smallmatrix} 1 \\ \sqrt{-\lambda} \end{smallmatrix}\right]$ is the eigenvector corresponding to the unstable eigenvalue $\mu_u^- = \sqrt{-\lambda}$ at $x = -\infty$.

Let y^- be the sol-n of $(*)$ with $\mathrm{e}^{\mu_u^- x} y^-(x) \to v_u^-$ as $x \to -\infty$.
Let y^+ be the sol-n of $(*)$ with $\mathrm{e}^{\mu_s^+ x} y^+(x) \to v_s^+$ as $x \to +\infty$.

Define $D(\lambda) = \det\big[\, y^-(\xi)\ y^+(\xi) \,\big]$, with $\xi \in \mathbb{R}$ arbitrary.
Eigenvalues of $(*)$ correspond to zeros of D.

Increasing the number of dimensions

Consider $y' = A(x, \lambda)\, y$ with $y(x) \in \mathbb{C}^n$ for $x \in \mathbb{R}$.

We assume that there is a region $\Omega \subset \mathbb{C}$ such that for all $\lambda \in \Omega$

- the matrices $A^{\pm}(\lambda) = \lim_{x \to \pm\infty} A(x, \lambda)$ exist and are hyperbolic;
- $A^{-}(\lambda)$ has k unstable eigenvalues $\mu_1^-, \ldots, \mu_k^-$;
- $A^{+}(\lambda)$ has $n - k$ stable eigenvalues $\mu_1^+, \ldots, \mu_{n-k}^+$.

Let y_i^- be a solution with $\mathrm{e}^{\mu_i^- x} y_i^-(x) \to v_i^-$ as $x \to -\infty$.
Let y_i^+ be a solution with $\mathrm{e}^{\mu_i^+ x} y_i^+(x) \to v_i^+$ as $x \to +\infty$.

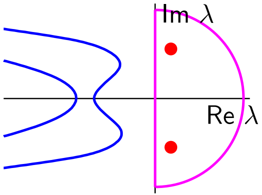
The Evans function is defined by

$$D(\lambda) = \det\big[\, y_1^-(\xi) \cdots y_k^-(\xi)\ y_1^+(\xi) \cdots y_{n-k}^+(\xi) \,\big].$$

It is analytic in Ω and its zeros corresponds to eigenvalues.
(Evans '75; Alexander, Gardner & Jones '90; Sandstede '02)

A topological construction

Evans function is analytic, so by argument principle winding number $W(D(\lambda))$ around curve $\partial\Omega$ equals number of eigenvalues in $\Omega \subset \mathbb{C}$. **(Evans & Feroe '77)**



As above, EVP is $y' = A(x, \lambda)\, y$ and we assume that $A^{\pm}(\lambda)$ is hyperbolic, $A^{-}(\lambda)$ has k unstable eigenvalues and $A^{+}(\lambda)$ has $n - k$ stable eigenvalues for $\lambda \in \Omega$.

Let $S = \text{span}\{y_1^-, \ldots, y_k^-\} \subset \mathbb{C}^n$ be unstable subspace.
S evolves according to lifted equation $S' = \ell(A(x, \lambda))\, S$.

Compactify $x \in \mathbb{R}$ to $\tilde{x} \in [-1, 1]$ with tanh transformation, and consider capped cylinder $C = \{-1, 1\} \times \text{int}(\Omega) \cup [-1, 1] \times \Omega$.

A topological construction II

Recall: $S(\tilde{x}, \lambda) \subset \mathbb{C}^n$ is k-dimensional unstable subspace; cylinder $C = \{-1, 1\} \times \text{int}(\Omega) \cup [-1, 1] \times \Omega$ with $\Omega \subset \mathbb{C}$.

Use $S : C \to \mathbb{C}^k$ to build a k-dimensional bundle over C. This is the augmented unstable bundle $\mathcal{E}$.

Winding number $W(D(\lambda))$ equals first Chern number of $\mathcal{E}$. Before: Winding number $W(D(\lambda))$ counts eigenvalues inside Ω.

This construction is well-suited for singularly perturbed RD equations like "FitzHugh–Nagumo"

$$u_t = u_{xx} + u(u - a)(1 - u) - v, \quad v_t = \delta v_{xx} + \epsilon(u - \gamma v)$$

For small δ, the system splits in a fast and slow system. This induces a splitting of $\mathcal{E}$ in Whitney sum $\mathcal{E}_s \oplus \mathcal{E}_f$. Now use $c_1(\mathcal{E}_s \oplus \mathcal{E}_f) = c_1(\mathcal{E}_s) + c_1(\mathcal{E}_f)$ to count eigenvalues and thus determine stability of travelling waves.

(Alexander, Gardner & Jones '90)

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Evaluating the Evans function numerically

The definition can be used as a basis for a numerical algorithm:

- Compute the unstable eigenvectors $v_1^-, \ldots, v_k^-$ of A^-.
- For $i = 1, \ldots, k$, solve $y' = A(\xi, \lambda)\, y$ with initial condition $y(-L) = v_i^-$ (where L is large) to get $y_i^-(\xi)$.
- Compute $y_i^+(\xi)$ similarly, and calculate the determinant.

The Evans function is analytic, so we can use the argument principle to count the number of eigenvalues in a given region.

Alternatively, we can use Newton's method to solve $D(\lambda) = 0$ and locate the eigenvalues. **(Pego, Smereka & Weinstein '93)**

If $D'(0)$ and $D(\lambda)$ for large real λ have opposite signs, then there is at least one eigenvalue on positive real axis.

Problems

- Unstable space $\implies$ solutions grow exponentially; for example, y_i^- grows with rate $\approx \mu_i^-$.

  Solution: Find y_i^- by solving $y' = \big(A(\xi,\lambda) - \mu_i^- I\big)y$.

- Evans function is only analytic if the eigenvectors $v_i^\pm$ are analytic functions in λ. This won't happen automatically, and is impossible if eigenvalues $\mu_i^\pm$ coalesce.

  Solution: See **(Brin '02)** and **(Bridges et al. '02)** for the first part, and below for the second part.

- If $\operatorname{Re}\mu_1^- > \operatorname{Re}\mu_2^-$, then y_1^- grows faster than y_2^-, so any errors in the y_1^- direction will dominate the y_2^- solution.

  Solution: Do not look at the y_i^- individually, but look at the subspace $S = \operatorname{span}\{y_1^-, \dots, y_k^-\}$ and lift eq'n $y' = A(\xi,\lambda)\,y$ to $S' = \ell(A(\xi,\lambda))\,S$ (as in topological construction).

How to represent subspaces?

We need to evolve the k-dimensional subspace $S(\xi,\lambda) = \operatorname{span}\{y_1^-,\dots,y_k^-\} \subset \mathbb{C}^n$ using $S' = \ell(A(\xi,\lambda))\,S$.

Can represent S by exterior product $y_1^- \wedge \dots \wedge y_k^- \in \Lambda^k(\mathbb{C}^n)$. Theoretically nice, determinant is exterior product, but $\dim \Lambda^k(\mathbb{C}^n) = \binom{n}{k}$ is large. **(Brin '00, Allen & Bridges '02)**

Can represent S by k-frame $Y = (y_1^-,\dots,y_k^-) \in \mathbb{C}^{nk}$ (Stiefel manifold) and apply continuous orthogonalization for stability. This destroys analyticity, but can compensate for this.

(Humpherys & Zumbrun '06)

Use local coordinates for Grassmannian: another basis for S brings Y in the form $\left[\begin{smallmatrix} I \\ Z \end{smallmatrix}\right]$. Dimension is $n(n-k)$, but issues when S leaves coordinate patch = Schubert cell.

(Ledoux, Malham & Thümmler '10)

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Integral equations

The Evans function approach can be applied more generally: Travelling waves in integral(-differential) equations, periodic travelling waves, multipulses, rotating and spiral waves.

Coombes & Owen ('04) analysed the neural field equation

$$u(x,t) = \int_{-\infty}^{\infty} w(y) \int_0^{\infty} \eta(s) f(u(x-y, t-s-|y|)) \, \mathrm{d}s \, \mathrm{d}y.$$

With $\eta(t) = \alpha \mathrm{e}^{-\alpha t}$, we can write this as

$$\frac{1}{\alpha} \partial_t u(x,t) + u(x,t) = \int_{-\infty}^{\infty} w(y) f(u(x-y,t)) \, \mathrm{d}y.$$

If $f(u) = \text{Heaviside}(u-h)$, then we can look for travelling pulses with $u > h$ on left and $u < h$ on right. An Evans function can be defined to assess its stability.

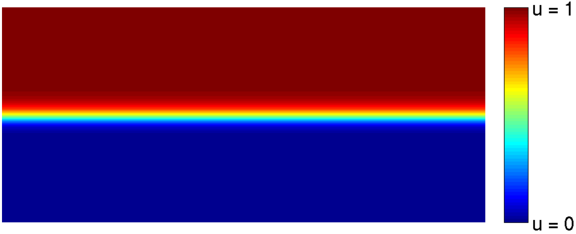
Model can be extended to allow for recovery.

Planar fronts

Planar fronts in two dimensions were treated by **Terman ('90)**.
The 1d front of the cubic autocatalysis equation

$$u_t = \delta u_{xx} - uv^2, \qquad v_t = v_{xx} + uv^2.$$

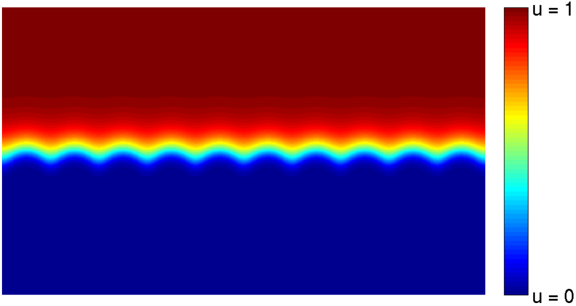
extends naturally to a 2d front (here $\delta = 2$):



Genuinely two-dimensional settings

Recent work includes generalization of Evans function to genuine 2d, e.g., wrinkled front that develops out of planar front when it becomes unstable to transverse perturbations at higher δ:

(Malevanets, Careta & Kapral '95)



Fourier decomposition

As before, we want to solve $K\Delta\bar{y} + c\bar{y}_\xi + Df(\hat{y})\,\bar{y} = \lambda\bar{y}$, where now Δ is 2d Laplacian.

Use that travelling wave is $\hat{y}$ periodic in transverse direction: Expand $\bar{y}$ and $DF(\tilde{U})$ in a Fourier series over transverse variable. Substitute this in the eigenvalue problem and truncate high wave numbers ($|k| > K$). We are left with a (big) system of ODEs, and the usual Evans function techniques apply.

(Ledoux, Malham, N. & Thümmler '09)

This converges as the cut-off $K \to \infty$.

(Gesztesy, Latushkin & Zumbrun '08)

A different approach (perhaps more 2d but up to now theoretical) is taken by **Deng & Nii ('06)** and **Deng & Jones ('10)**.

Conclusion

- The Evans function is a construction which can be used to find eigenvalues and thus assess stability of travelling waves and other coherent structures.
- Topological aspects mean that it is often possible to get enough information analytically to decide on stability (Chern numbers, parity argument).
- Numerical computations provide an alternative to the usual approach of discretization and solving a matrix eigenvalue problem (Evans function is clean but slow).
- Recently, Evans function has been making inroads into two-dimensional problems.