3.3 Redington immunization (continued)

We continue where we left off last lecture, in which we considered a company that has to pay £10,000 in \(2\frac{1}{2}\) years. We found that, on the basis of \(r = 0.03\), a portfolio of \(X = 49.26\) two-year zero-coupon bonds and \(Y = 50.76\) three-year bonds matches the liability both in present value and duration.

Non-matching assets — effect of convexity

As mentioned before, the change in the present value of a cash flow because of a change in interest rate is given by a Taylor series. If the interest rate \(r\) changes by \(\Delta r\), then the present value of the liability changes to

\[
PV_L + \Delta(PV_L) = PV_L + (\Delta r) \frac{d(PV_L)}{dr} + \frac{1}{2}(\Delta r)^2 \frac{d^2(PV_L)}{dr^2} + \frac{1}{6}(\Delta r)^3 \frac{d^3(PV_L)}{dr^3} + \cdots,
\]

and the present value of the portfolio of assets changes to

\[
PV_A + \Delta(PV_A) = PV_A + (\Delta r) \frac{d(PV_A)}{dr} + \frac{1}{2}(\Delta r)^2 \frac{d^2(PV_A)}{dr^2} + \frac{1}{6}(\Delta r)^3 \frac{d^3(PV_A)}{dr^3} + \cdots.
\]

We constructed our portfolio of bonds such that \(PV_A = PV_L\) and \(d(PV_A)/dr = d(PV_L)/dr\), so the first two terms in the Taylor series for \(PV_L\) and \(PV_A\) are the same. If the second derivatives satisfy \(d^2(PV_A)/dr^2 > d^2(PV_L)/dr^2\) and we ignore the higher-order terms, then

\[
PV_A + \Delta(PV_A) > PV_L + \Delta(PV_L).
\]

This says that the bonds will always have a higher value than the liability, even if the interest rate changes. This means that our portfolio of bonds will always sell for enough money to cover the liability.

The second-derivative condition \(d^2(PV_A)/dr^2 > d^2(PV_L)/dr^2\) is equivalent to \(c_A > c_L\) (the convexity of the bonds is higher than the convexity of the liability). In our example, we have:

\[
c_L = \frac{(2.5)^2 \cdot 10000e^{-2.5-0.03}}{10000e^{-2.5-0.03}} = (2.5)^2 = 6.25,
\]

\[
c_A = \frac{2^2 \cdot X \cdot 100e^{-2.03} + 3^2 \cdot Y \cdot 100e^{-3.03}}{X \cdot 100e^{-2.03} + Y \cdot 100e^{-3.03}} = 6.5,
\]

so the condition \(c_A > c_L\) is indeed satisfied, and we can be fairly confident that the portfolio of bonds covers the liability.

We can come to the same conclusion without any computations, if we use the formula \(c - \nu^2 = \text{Var}_w(t_k)\) from lecture 12. The liability consists of only one payment, so the variance is zero, so \(c_L = \nu_L^2\). The cash flow from the portfolio of bonds consists of two payments, one after two years and one after three years, so the variance is greater than zero, so \(c_A > \nu_A^2\). However, \(\nu_L = \nu_A\) so \(c_A > c_L\).
Redington immunization We say that a company achieves Redington immunization if its liabilities and assets satisfy the following three conditions:

- $PV_A = PV_L$ (at the current interest rate $r$, the present value of the assets equals the present value of the liabilities).
- $\frac{d(PV_A)}{dr} = \frac{d(PV_L)}{dr}$ (equivalently, the volatility or the duration of the assets and liabilities are equal).
- $\frac{d^2(PV_A)}{dr^2} > \frac{d^2(PV_L)}{dr^2}$ (equivalently, the convexity of the assets is greater than the convexity of the liabilities, or the weighted variance of due times of the assets is greater than the weighted variance of due times of the liabilities).

In these conditions, we can also use the annually compounded interest rate $i$ instead of $r$. If the conditions for Redington immunizations are satisfied, then the value of the assets is at least as great as the value of the liabilities, even if the interest rate changes (a little bit).

Issues with Redington immunization This is not the end of the story. The company cannot be absolutely certain that its assets cover its liabilities even if the conditions for Redington immunization are satisfied. Firstly, we ignored third-order and higher-order terms in the Taylor series, so our conclusion that the value of assets exceeds the value of liabilities is only valid for sufficiently small changes in the interest rate. Secondly, the value of our portfolio of assets changes over time, so we need to rebalance the portfolio to continue satisfying the conditions for Redington immunization. Thirdly, the theory only takes fluctuations in the interest rate into account, but in practice there are other factors that influence the payouts of assets (for example, the risk of default for bonds).
Lecture 14 (part 2)

3.4 Example: Question D from the MATH2510 Jan. 2011 exam

A pension company has to pay out £10m to its policy holders in each of the years 10 to 14 (specifically: at times 10, 11, . . . , 14). The yield curve is flat; the annually compounded interest rate is 4%. There is a bond market in which two bullet bonds\(^1\) are traded; a 5-year bond and a 25-year bond. The bullet bonds pay annual coupons of 4%. The current time (now) is 0.

1. Calculate the present value and the Macauley duration of the company’s liabilities.

2. Calculate the arbitrage-free price (per £100 notional) and the Macauley duration of the two bullet bonds.

3. How (if it is at all possible) can an asset portfolio of the two bullet bonds be formed such that the combined asset-liability position is immunised to small interest rate changes?

**Answer to part 1** The present value of the liabilities is

\[
PV_L = \frac{10m}{(1.04)^{10}} + \frac{10m}{(1.04)^{11}} + \frac{10m}{(1.04)^{12}} + \frac{10m}{(1.04)^{13}} + \frac{10m}{(1.04)^{14}} = 31.28\text{m pounds.}
\]

Alternatively, you can use the formulas for annuities from Financial Mathematics 1.

The duration of the liabilities is

\[
D_L = \left( \frac{10 \cdot 10m}{(1.04)^{10}} + \frac{11 \cdot 10m}{(1.04)^{11}} + \frac{12 \cdot 10m}{(1.04)^{12}} + \frac{13 \cdot 10m}{(1.04)^{13}} + \frac{14 \cdot 10m}{(1.04)^{14}} \right) / 31.28\text{m} = 11.92.
\]

**Answer to part 2** The coupon rate of the bonds equals the interest rate. Thus, the bonds are trading at par. This means that their present value equals £100.

For the duration of a bond, we can use the following formula (which was part of the precursor of this module but is not treated anymore):

\[
duration = \frac{D (Ia_n)_{\frac{1}{i}} + Fn v^n}{D a_{\frac{1}{i}} + F v^n},
\]

where \(D = 0.04\) is the coupon rate, \(F = 100\) is the face value, \(n\) is the term, and

\[
v = (1 + i)^{-1}, \quad (Ia_n)_{\frac{1}{i}} = (1 + i) a_{\frac{1}{i}} - n (1 + i)^{-n}, \quad a_{\frac{1}{i}} = \frac{1 - (1 + i)^{-n}}{i}.
\]

Evaluating this for \(n = 5\) and \(n = 25\) results in a duration of 4.63 years for the five-year bond and 16.25 years for the 25-year bond.

\(^1\) A bullet bond is a bond that pays coupons.
**Answer to part 3**  Let $X$ be the number of five-year bonds that are bought, and $Y$ be the number of 25-year bonds. For immunization, the present values must match,

$$100X + 100Y = 31.28m,$$

and the durations must match:

$$\frac{4.63 \cdot 100X + 16.25 \cdot 100Y}{100X + 100Y} = 11.92.$$

From $100X + 100Y = 31.28m$, we find $Y = 0.3128m - X$, and substitution of this relation in the equation for the durations to match yields

$$\frac{508.3m - 1162X}{31.28m} = 11.92 \implies X = \frac{508.3m - 11.92 \cdot 31.28m}{1162} = 116k$$

and

$$Y = 0.3128m - X = 196k.$$

So the portfolio consists of 116,000 five-year bonds and 196,000 25-year bonds.

Since the assets are more spread out / dispersed around the duration than the liabilities, the third condition (on convexity) for immunization is also satisfied.

**Further reading:** CT1 Unit 13, Section 5.5.