Lecture 2: 27 September 2012

1 Forward contracts and other derivatives

A derivative is a financial instrument whose value depends on the value of some other asset (the underlying asset). A simple derivative is a forward contract, which is an agreement to buy a specific asset (e.g., copper) for a specified price at a specific time in the future.

The main goal of this chapter is the valuation of forward contracts and other derivatives. The “no-arbitrage” principle is the tool which allows us to achieve this.

1.1 The “no-arbitrage” principle

The idea behind the “no-arbitrage” principle can be summarized in a few words: there is no such thing as a free lunch. In other words, the financial markets will not allow you to get a risk-free profit.

First example of arbitrage A simple example of arbitrage is when a foreign currency is traded on two markets with different exchange rates. Suppose that you can exchange euros to pounds in London at a rate of \( \mathbf{\text{€}}1.15 \) per pound, and in Paris at \( \mathbf{\text{€}}1.16 \) per pound. Then it would be possible to get a guaranteed profit by exchanging one pound for \( \mathbf{\text{€}}1.16 \) in Paris and then exchanging \( \mathbf{\text{€}}1.16 \) for \( \frac{1.16}{1.15} \approx 1.01 \) pounds in London, thus getting one penny of profit. That does not sound very impressive, but by starting with a million pounds, you could get a profit of £8695.65 without any risk.

The consequence of this arbitrage opportunity is an increased supply of euros in London which pushes the exchange rate up there. In Paris, the exchange rate will drop, until the rates in London and Paris are equal. In well-developed liquid markets, any arbitrage opportunity will disappear quickly. The assumption of no arbitrage is that all arbitrage opportunities have disappeared.

Definition Arbitrage opportunities may be more complicated and involve trades at different times. In that case, we need to take into account that future prices are uncertain. An arbitrage opportunity is a deal that either gives an initial profit with no risk of future loss or that has zero initial cost, no risk of future loss and a non-zero probability of a future profit.

Second example of arbitrage Consider a market with two securities, A and B. It is possible to buy A now (at \( t = 0 \)) for \( P_A^0 = 6 \) pounds. The prize of B is \( P_B^0 = 11 \). Both securities have a term of one year and make a single pay-out at the end of the year (at \( t = 1 \)), but the pay-out is random. There are two possibilities. Either the market goes up, in which case security A pays \( P_A^1(u) = 7 \) and security B pays \( P_B^1(u) = 14 \). Or the market goes down, in which case security A pays \( P_A^1(d) = 5 \) and security B pays \( P_B^1(d) = 10 \).
This market has an arbitrage opportunity. If you buy one unit of security B for £11 and sell two units of security A for £6 each you make a profit of £1 at \( t = 0 \). Furthermore, you will not make a future loss (at \( t = 1 \)), whatever happens. If the market goes up, security B pays out £14, which covers the pay-out you have to make on the two units of security A (2 times £7). If the market goes down, the pay-out of security B also exactly covers your obligations.

In this example, two units of A has the same pay-out as one unit of B. We say that these portfolios replicate each other. The no-arbitrage principle says that replicating portfolios should have the same price. This is sometimes called the law of one price. The arbitrage opportunity disappears when security B costs twice as much as security A.

**Third example of arbitrage** This example illustrates the phrase “non-zero probability” in the definition of arbitrage. The situation is similar to the second example but with different numbers:

<table>
<thead>
<tr>
<th>Security</th>
<th>( P_0 ) (price at time 0)</th>
<th>( P_1(u) ) (pay-out if market goes up)</th>
<th>( P_1(d) ) (pay-out if market goes down)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

In this situation, an arbitrageur can buy A and sell B. This does not cost anything at \( t = 0 \). At \( t = 1 \), the arbitrageur either makes a profit of £1 (if the market goes down) or breaks even (if the market goes up). Thus, there is zero initial cost, no risk of a future loss and a non-zero probability of a profit (assuming the probability that the market goes down is not zero); this is an arbitrage opportunity.

In the second example, we found that the no-arbitrage principle implies that \( P^B_0 = 2P^A_0 \). The principle is not as strong in the situation here. It only gives a range for the price of B, given the price of A. The underlying reason is that we cannot construct replicating portfolios in this situation if A and B are the only securities on the market.

Assume that \( P^A_0 = 6 \) (as before) but that \( P^B_0 \) is not fixed. Then a deal with zero initial cost is to buy \( P^B_0 \) units of A and to sell 6 units of B. This leads to a pay-out of \( P^B \cdot P^A_{1}(u) - 6 \cdot P^B_{1}(u) = 7P^B_0 - 42 \) if the market goes up and \( P^B \cdot P^A_{1}(d) - 6 \cdot P^B_{1}(d) = 5P^B_0 - 24 \) if the market goes down. The no-arbitrage principle means that one of them must be negative (or both zero), so either \( 7P^B_0 - 42 < 0 \) or \( 5P^B_0 - 24 < 0 \). It follows that either \( P^B_0 < 42/7 = 6 \) or \( P^B_0 < 24/5 = 4.8 \), so the price of B has to be below £6.

Another deal with zero initial cost is to sell \( P^B_0 \) units of A and to buy 6 units of B. A similar reasoning yields that the no-arbitrage principle implies the price of B has to be above £4.80. Thus, the price of B has to be between £4.80 and £6.00.